SECTION 1.1 CONSTRUCTING A NUMBER LINE

“What is Algebra?” Rather than give you an incomplete answer now, we hope that through learning, you will soon be able to answer this question yourself. As a preview, let’s look at some questions that algebra can help us answer that we could not have answered before:

• If I drop a marble off a two-story building, how long will it take the marble to hit the ground?
• If I have $10 and go into a candy shop, where chocolate costs $.50 and licorice costs $.65, how many of each could I buy?

We begin by reviewing numbers and the ways you manipulate and represent them. We will develop collections of numbers in stages, building up smaller groups of numbers until we get all numbers.

We first encounter numbers as children by counting, starting with one, two, and three. We call the numbers that we use in counting the natural numbers, or sometimes the counting numbers. They include the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, . . ., where the “. . .” means that they go on forever in the same way. These numbers describe how many of something, for example, how many brothers or sisters you have, how many days in a week, the number of people in your town and even the number of grains of sand on a beach.

EXPLORATION 1

Make a number line on a large piece of paper. Put the number 1 in the middle of the line. Locate and label the first 20 natural numbers.

Including the number 0 in this set of numbers gives us the whole numbers. The whole numbers take care of many situations, but
if we want to talk about the temperature, there are places on Earth that routinely have temperatures below zero, like $-5 \, ^\circ C$ or $-20 \, ^\circ C$. We must expand our idea of number to include the negatives of the natural numbers. This larger collection of numbers is called the integers, it is denoted by the symbol $\mathbb{Z}$ and includes the whole numbers. Notice that every integer is either positive, zero or negative. The natural numbers are positive integers and denoted sometimes by $\mathbb{Z}^+$. 

In this book we will try to be very precise in our wording, because we want our mathematics and words to be clear. Just as you learn new words in English class to express complicated concepts, we must learn new words and symbols in mathematics. We have discussed 3 collections of numbers so far: the integers, the whole numbers, and the natural numbers. In mathematics, we call collections of numbers (or other objects) sets. A set is defined by its members. We call these members elements. In order to write out what a set is, we want to describe its elements in set notation. This is done for example by listing the elements of a set inside braces. For example, the natural numbers are $\mathbb{Z}^+ = \{1, 2, 3, 4, \ldots \}$ and the integers could be written $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$. Not all sets go on forever, for example, the set of even natural numbers between 4 and 14 is $\{6, 8, 10, 12\}$. 

Every natural number is also an integer. In our more precise language, this could be described as ”every number in the set of natural numbers is also in the set of integers”. For this reason, we call the natural numbers a subset of the integers. We will call a set a subset of another set when any element in the first set is also in the second. We use such precise wording in order to be clear in our discussion of mathematics. The use of precise language is more important in mathematics than it is in everyday life.

**EXPLORATION 2**

Continue to work on the number line from Exploration 1. Using a red marker, plot and label the negative integers from $-1$ to $-20$. What properties does the set of integers have that the set of whole numbers did not?
It does not take long to see the need for numbers that are not integers. You might hear in a weather report that it rained \(2\frac{1}{2}\) inches or know that a person's normal body temperature is 98.6°Fahrenheit. So sometimes we need to talk about parts of whole numbers called fractions. This expanded set of numbers that includes fractions is called the set of rational numbers.

**EXPLORATION 3**

Using a different colored marker, plot and label 3 fractions between each of the following pairs of integers:

- 2 and 3
- 4 and 5
- \(-1\) and 0
- 3 and \(-2\)

A rational number is the quotient of 2 integers and the denominator can not be zero. For example, both \(\frac{3}{7}\) and \(\frac{9}{4}\) are rational numbers. They are called rational numbers because they are the ratio of 2 integers. A rational number can be represented as quotient in more than one way. Also, every rational number can be written in decimal form. For example, \(\frac{1}{2}\) is equivalent to \(\frac{2}{4}\) and to 0.5, \(\frac{3}{4}\) is the same as 0.75 and \(\frac{2}{5}\) is equal to 2.4 and \(\frac{12}{5}\).

We asked you to find two fractions between 2 and 3. Could you find two fractions between the fractions you just found? How about two fractions between these two?

**PROBLEM 1**

How many fractions are there between 0 and 1? How many fractions are there between 2 and 3?

Notice that every integer is a rational number. There are, however, rational numbers that are not integers. This means that the set of integers is a subset of the set of rational numbers, but the set of rational numbers is not a subset of the set of integers.
List 3 examples of rational numbers that are not integers and list 3 examples of integers that are not whole numbers. Locate these numbers on your number line.

**EXAMPLE 1**

Create a Venn Diagram to show the relationship between the following sets of numbers:

- rational numbers
- whole numbers
- integers
- natural numbers

**Solution** The sets of numbers are nested. For example, every integer is a rational number, but not every rational number is an integer.

![Venn Diagram](image)

**Operations on the Number Line**

One advantage of representing numbers on the number line is that it shows a natural order among its members. The location of the points representing 2 numbers reflects a relationship between those 2 numbers that we define as greater than, equal to or less than. You can also examine distances between numbers and model the operations of addition, subtraction, multiplication and division on the number line. In fact, we will frequently use examples on the number line to illustrate algebraic ideas.
Let’s review how to do arithmetic with integers using a linear model, that is, by representing numbers on a number line. We will use the number line in discussing algebraic concepts. Let’s get familiar with using this line.

EXPLORATION 4

1. Use the number line to illustrate the sum $3 + (-4)$ and the difference $3 - 4$. Explain how you arrived at your answer and location for each problem. Then, using the same pattern, explain how you compute the sum $38 + (-63)$ and the difference $38 - 63$ without a detailed number line.

2. Use the number line to illustrate the difference $3 - (-5)$ and sum $3 + 5$. Then explain how you compute the difference $38 - (-63)$ without a detailed number line.

3. Summarize the rules for addition and subtraction of integers.

4. Use the number line to illustrate the product $3(-4)$ and $-3(4)$. Explain how you arrived at your answer and location for each problem. Then using the same pattern, explain how you compute the products $18(-6)$ and $-5(12)$ without a detailed number line.

5. Use the number line to illustrate the product $-3(-4)$. Explain how you arrived at your answer and location for each problem. Then using the same pattern, explain how you compute the product $-28(-3)$.

6. Summarize the rules for multiplication of integers.

The number line is also useful for thinking about operations with rational numbers and exploring the relationship between numbers.
EXPLORATION 5

1. Use the number line to illustrate the sums $1\frac{3}{4} + 2\frac{3}{4}$ and $\frac{4}{5} + \frac{3}{5}$.

2. Starting at the point representing 3, determine and locate on the number line the following numbers. Explain how you arrived at your answer.
   a. The number that is 5 more than this number.
   b. The number that is 5 less than this number.
   c. The number that is 3 times this number.
   d. The number that is half as big as this number.

3. Locate and label three numbers that are greater than $-5$. Locate and label three numbers that are less than $-6$.

Distance on the Number Line

Another important concept to study on the number line is the distance between points.

EXPLORATION 6

Make a new number line from $-15$ to 15, labeling all of the integers between them. Locate the points 6 and 13 on the new number line. Determine the distance between 6 and 13.

1. What is the distance from 12 to 4? Explain how did you got your answer.
2. What is the distance from $-3$ to $-11$? From $-9$ to $-2$? How did you get your answers?
3. What is the distance from $-7$ to 4? From 5 to $-7$? Explain.
4. Find the distance between $\frac{1}{2}$ and $3\frac{1}{2}$.
5. Find the distance between $\frac{1}{2}$ and $\frac{3}{4}$.
6. Find the distance between $\frac{3}{4}$ and $3\frac{1}{2}$.
7. What is the distance from $-\frac{1}{2}$ to $\frac{7}{8}$?
8. What is the distance between $4\frac{2}{3}$ and $1\frac{1}{2}$?
One way you might have found the distance between two points representing integers on a number line is to “count up” from the left most number until you reach the one on the right or to “count down” from the right most number until you reach the one on the left. For example from 6 you might have counted up and noted that it took 7 units to arrive at 13 and so concluded that the distance between 6 and 13 is 7. Or in the second question asking for the distance between 12 and 4, you might have counted down from 12 until you reached 4 and noted that it took 8 units, to conclude that the distance between 12 and 4 is 8. However, you might also have noticed that $12 - 4 = 8$ and $13 - 6 = 7$. The distance between two numbers is the difference of the lesser from the greater.

In part 6, you might want to break the distance from $\frac{3}{4}$ to $3 \frac{1}{2}$ into three parts:

- The distance from $\frac{3}{4}$ to 1 is $\frac{1}{4}$,
- The distance from 1 to 3 is 2,
- The distance from 3 to $3 \frac{1}{2}$ is $\frac{1}{2}$.

These parts add up to $\frac{1}{4} + 2 + \frac{1}{2} = 2 \frac{3}{4}$.

The absolute value of a number is the distance from 0. We have a special symbol to represent absolute value. For example, we write $|6|$ and read it as absolute value of 6. We write $|-6|$ and read it as absolute value of $-6$. Since 6 and $-6$ are both 6 units from 0, we see that $|6| = |-6| = 6$. Since the absolute value is a distance, it is never negative. We often use absolute value when computing or representing distances between numbers. For example, if we want to compute the distance between $-5$ and 3, we can either subtract the lesser number from the greater number $3 - (-5) = 8$. Or we can take the absolute value of the difference, $|-5 - 3| = |-8| = 8$. The advantage of using the absolute value is that we can compute
the difference in either order. Why is this true?

PROBLEM 3

Compute the distance between the following pairs of numbers.

1. $-12$ and $6$
2. $-52$ and $27$
3. $-23$ and $-35$
4. $1.75$ and $-1.25$
5. $\frac{3}{4}$ and $-\frac{1}{3}$

EXERCISES

1. Compute the following sums or differences.
   a. $45 - 64$
   b. $42 + (-36)$
   c. $19 - (-33)$
   d. $17 - (-25)$
   e. $-13 + 26$
   f. $\frac{2}{3} + \frac{1}{5}$
   g. $\frac{3}{5} + \frac{2}{3}$
   h. $\frac{4}{5} - \frac{2}{3}$
   i. $\frac{5}{7} + \frac{1}{3}$
   j. $2\frac{3}{4} + 3\frac{1}{5}$
   k. $5\frac{3}{4} - 2\frac{2}{3}$
   l. $5\frac{1}{4} - 2\frac{2}{3}$
2. Compute the following products and quotients.
   a. \(-2 \cdot 7\)
   b. \(5 \cdot (-5)\)
   c. \(-11 \cdot (-6)\)
   d. \(-24 \div 6\)
   e. \(-33 \div (-5.5)\)
   f. \(\frac{2}{3} \cdot \left(-\frac{4}{5}\right)\)
   g. \(-\frac{5}{7} \div \left(-\frac{15}{16}\right)\)
   h. \(-6 \div \frac{3}{5}\)
   i. \(3 \frac{1}{2} \cdot 2 \frac{2}{5}\)

3. Evaluate the following expressions.
   a. \(5 + 6 \cdot (-3)\)
   b. \(6 \cdot 7 - (-3) \cdot 7\)
   c. \(9 \cdot (-14 + 5)\)
   d. \(-13 - (-6 - 29)\)
   e. \(\frac{-2 + 20}{-3}\)
   f. \(\frac{28 - 21}{-3 - 10}\)

4. Compute the distance between each of the following pairs of numbers.
   a. 8 and \(-3\)
   b. 4 and \(-5\)
   c. 1.1 and .9
   d. 3.4 and 2.95
   e. .26 and .3
   f. \(\frac{2}{4}\) and 2
   g. \(\frac{2}{3}\) and \(\frac{1}{2}\)
   h. \(\frac{3}{4}\) and \(\frac{2}{5}\)
   i. 3.01 and 2.9
   j. 3.01 and 2.99
   k. 3.1 and 2.9
   l. 3.1 and 2.99

5. Using words, describe 3 subsets of whole numbers that are each infinite. Describe another infinite subset of whole numbers that is a subset of one of your first 3 subsets.
6. Copy the Venn Diagram from Example 1. For each condition given below, find a number that satisfies the condition and then place it on the Venn diagram.
   a. A whole number that is not a natural number.
   b. An integer that is not a whole number.
   c. A rational number that is not an integer.
   d. A rational number that is an integer, but not a whole number.

7. Find 3 numbers between each of the following pairs of numbers. Sketch a number line and plot the numbers on it.
   a. \( \frac{2}{3} \) and 1
   b. \( \frac{4}{5} \) and \( \frac{1}{3} \)
   c. \( 2 \frac{3}{4} \) and \( 3 \frac{1}{5} \)
   d. \( \frac{15}{7} \) and \( \frac{5}{2} \)

8. Compute the distance between each pair of the following list of numbers. Explain which pair is closest and which pair is the greatest distance apart.

\[
\begin{align*}
1.39 \text{ and } 2.4 & \quad 1.41 \text{ and } 3.1 & \quad 1\frac{5}{6} \text{ and } 3\frac{1}{3} & \quad \frac{7}{4} \text{ and } \frac{11}{3}
\end{align*}
\]

9. In each of the following problems, 3 numbers are given. Draw a number line and mark and label the 3 numbers. Pay attention to the distances between the numbers. Your picture should give an approximate sense of where the 3 numbers lie in relation to each other.
   a. 1, 4, and 7
   b. 5, 19, and 23
   c. 2, 4, and \(-5\)
   d. \(-2\), \(-7\), and \(-12\)
   e. \(-10\), 20, and 30
   f. 6, 8, and \(-97\)
In mathematics, we try to solve problems. Sometimes we need to find or compute an unknown quantity. It is useful to have a name or symbol for this unknown value. Algebra is a language that uses symbols to describe problems mathematically. A symbol used to represent an unknown value is called a variable. A variable can denote a number that either changes or whose value is unknown in a given problem. We call them variables because the number they represent can vary from problem to problem.

In this section, we will use a letter, like $a$, $b$, $x$, $S$ etc., to represent an unknown number on the number line. We treat these variables just like numbers. We can do anything with variables that we can do with numbers. Combining variables with numbers and arithmetic operations, we can form an algebraic expression. For example, we could start with the variable $a$, and write the expressions $2 \cdot a$, $-a$, $a + 2$. These expressions have the same meaning they would have if $a$ was a number instead of a letter, that is: $2a$ represents twice the number $a$, $-a$ represents the negative (or opposite) of $a$, and $a + 2$ represents 2 more than the number $a$. You can use what you know about the number line to visualize what these expressions mean. In other problems, you will be given some extra information and you can use what you know about the number line to determine what value the variable must have. In the next section, we talk about how to create expressions in solving real problems.

Variables and Expressions on a Number Line

On the number line, each point represents a number and every number is represented by exactly one point on the line. In the first exploration, assume the variable $a$ represents the number located on the number line. The number 0 is also marked on the number line.
EXPLORATION 1

The number $a$ is located on the number line below. Locate and label the points that represent the given algebraic expressions involving $a$. Use string or a ruler to help locate these points. Plot a point that represents each of the following:

$$2a, \ 3a, \ -a, \ -2a, \ \frac{3a}{2}, \ \frac{a}{3}.$$ 

EXPLORATION 2

Suppose $b$ is a number that is located on the number line as seen below. Plot and label the points that represents each of the following:

$$2b, \ 3b, \ -b, \ -2b, \ \frac{3b}{2}, \ \frac{b}{3}.$$ 

Is $b$ a positive or negative number? Is $-b$ a positive or negative number? How can you tell? Compare the results from Explorations 1 and 2. How are the results similar? In what ways are they different?

On the number line, $-5$ is a point that is the same distance from 0 as $+5$, but on the opposite side of 0. The same is true of 3 and $-3$: both are a distance 3 from 0, but on opposite sides. The same is true for 234 and $-234$. Whatever number we take, the opposite of that number is the same distance from 0 and located on the other side of 0. With our language of algebra, we can write this more concisely as the opposite of $n$ is $-n$.

Note that $3+(-3) = 0$. Similarly, $5+(-5) = 0$ and $234+(-234) = 0$ and so on for every number. With the help of variables, we can
state “and so on for every number” in a more precise way: for any number \( n \), \( n + (-n) = 0 \). We call \(-n\) the *additive inverse* of \( n \).

**ADDITIVE INVERSE**

<table>
<thead>
<tr>
<th>For any number ( n ),</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n + (-n) = 0 ).</td>
</tr>
</tbody>
</table>

| For example, \( 3 + (-3) = 0 \). |

**PROBLEM 1**

On the number line below, the numbers \( a \) and \( b \) are marked. Locate and plot the point that represents each of the following:

\[-a, \ (-(-a)), \ (-b), \ (-(-b)).\]

Is \( b \) a positive or negative number? Is \(-b\) a positive or negative number? Is \(-(-b)\) a positive or negative number? How can you tell?

From the results of Problem 2, we discover the following rule:

**THEOREM 1.1: DOUBLE OPPOSITE**

<table>
<thead>
<tr>
<th>For every number ( n ), (-(-n) = n).</th>
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</thead>
<tbody>
<tr>
<td>For example, (-(-3) = 3).</td>
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</tbody>
</table>

This is the formal way of saying that the opposite of the opposite of a number is the original number.
Adding and Subtracting on the Number Line

We can use variables on the number line to develop general properties of addition and subtraction.

EXPLORATION 3

On the number line below, the numbers \( a \) and \( b \) are marked.

1. Use the number line model to locate and plot the point \( a - b \).
2. Locate and plot the point \( -b \).
3. Locate and plot the point \( a + (-b) \). What do you notice?

PROBLEM 2

Consider the number line from Problem 1:

1. Use the number line model to locate and plot the point \( a - b \).
2. Locate and plot the point \( -b \).
3. Locate and plot the point \( a + (-b) \).

The results of Problem 2 and Exploration 3 tell us how to rewrite any subtraction problem as an addition problem.

<table>
<thead>
<tr>
<th>ADDITION OF THE OPPOSITE OF A NUMBER</th>
</tr>
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<tbody>
<tr>
<td>For every pair of numbers ( a ) and ( b ), ( a - b = a + (-b) ).</td>
</tr>
<tr>
<td>For example, ( 3 - 5 = 3 + (-5) = -2 ) and ( 3 - (-5) = 3 + 5 = 8 ).</td>
</tr>
</tbody>
</table>
PROBLEM 3

Rewrite each of the following subtraction problems as an addition problem and then compute the sum.

1. $10 - 3$
2. $5 - (-2)$
3. $-2 - (-4)$
4. $-14 - 5$

Locating Expressions on a Number Line

In each of the following explorations, the numbers 0 and 1 are labeled on the number line. This means the length of 1 is marked. Use it to locate other points on the number line.

In the next exploration, you will be given the location of a variable like $x$ and asked to locate related quantities that are expressions of $x$, like $2x - 1$ and $2x + 1$. Use the straight edge of a piece of paper as a ruler to locate these numbers. Be as accurate as you can be.

To use the straight edge, place the edge of your paper on the number line in such a way that you see the ticks and labels. Copy the ticks at numbers 0 and 1 from the number line to your paper. By moving one of the ticks to $x$, the other will show you where the number one less than $x$ or the number one greater than $x$ is located. You can repeat this to add or subtract larger integers.

EXPLORATION 4

Both $x$ and $y$ are numbers located on the number line below. Plot a point that represents each of the following expressions on the number line below:

$x + 1$, $x - 2$, $y + 1$, $y + 4$, $y - 1$. 

$y$ 0 1 $x$
Chapter 1  Variables, Expressions and Equations

It isn’t possible to compute the exact value of $x$ or $y$ in the exploration. But it is possible to estimate the value of the variables $x$ and $y$. We just need to compare their positions to where 0 and 1 are.

Recovering the Location of a Variable on a Number Line

In this exploration, you will discover how to find a variable on a number line given the position of an algebraic expression. This is the opposite of the process that used in the previous exploration.

EXPLORATION 5

In each of the problems below, locate the point that represents $x$. Explain how you find your answer. Estimate the value of $x$ from its location. In each of the following explorations, 0 and 1 are marked.

1. \[ \begin{array}{c}
0 & 1 & x + 2 \\
\hline
\end{array} \]

2. \[ \begin{array}{c}
0 & 1 & x - 2 \\
\hline
\end{array} \]

3. \[ \begin{array}{c}
x + 3 & 0 & 1 \\
\hline
\end{array} \]

4. \[ \begin{array}{c}
x - 3 & 0 & 1 \\
\hline
\end{array} \]

In part 1, begin at point $x + 2$ and move 2 units to the left to find the location of the point $x$. This has the effect of adding $-2$ to the number $x + 2$. This is the same as subtracting 2 from $x + 2$:

$$x + 2 + (-2) = x + 0 = x.$$ 

The result of this move on the number line and the calculation above is the number $x$. 

16
In part 2, begin at $x - 2$ and move 2 units to the right to find the location of the point $x$. This has the effect of adding 2 to the number $x - 2$:

$$x - 2 + 2 = x + 0 = x.$$ 

Again, the result of this process is to locate the position of $x$.

**PROBLEM 4**

For each number line below, locate the point that represents $y$. Estimate the value of $y$ from its location.

1. 

2. 

3. 

4. 

We now explore another type of problem using the number line. Here we are given the location of an algebraic expression and its numerical value. For example, in the next problem, a point is labeled as both $2x + 1$ and 9.
PROBLEM 5

In each of the following problems, assume that this line segment gives the length 1 unit:

\[1\]

The given expression equals a numerical value. Locate the points that represent the numbers \(x\) and 0. Estimate the value of \(x\) by its location. Substitute the value of \(x\) in the expression to check your work.

1. \(2x + 1\) is the same as 9:

2. \(2x - 1\) is equal to 5:
EXERCISES

1. Plot a point that represents each expression and label it:

\[ 2y + 2, \quad 2y - 1, \quad 2y + 5, \quad \frac{y}{2}. \]

2. Plot a point that represents each expression and label it:

\[ a-1, \quad a+2, \quad 2a, \quad 2a-2, \quad 2(a-1), \quad 2a+4, \quad 2(a+2). \]

Are any of the expressions located at the same point? Explain why.

3. Locate the point that represents \( x \) for each of the following:
   a. \[ 2x + 3 \]

4. In each of the sections below, locate the point that represents \( x \). Estimate the value of \( x \) by its location.
   a. \[ x + 2 \]
   b. \[ 2x + 1 \]
   c. \[ 2x - 1 \]
5. In each of the sections below, locate the point that represents the variable. Estimate the value of the variable by its location.
   a. 
   \[ y - 6 \]
   \[ 0 \quad 1 \]
   b. 
   \[ 2y + 1 \]
   \[ 0 \quad 1 \]
   c. 
   \[ 3y - 1 \]
   \[ 0 \quad 1 \]
   d. 
   \[ 2y - 9 \]
   \[ 0 \quad 1 \]

6. Plot a point that represents each expression and label it:
   \[ a - b, \quad a + b, \quad b - a. \]
   \[ 0 \quad b \quad a \]

7. Plot a point that represents each expression and label it:
   \[ x - y, \quad x + y, \quad y - x. \]
   \[ y \quad 0 \quad x \]

8. Plot a point that represents each expression and label it:
   \[ 1 - x, \quad 2 - x, \quad 3 - x, \quad 1 - y, \quad 2 - y. \]
   \[ y \quad 0 \quad 1 \quad x \]
9. The line segment below is 1 unit long. Complete the following steps for each part:
   • Locate the point that represents the solution \( x \).
   • Locate the point that represents the number 0.
   • Estimate the value of \( x \) by its location.
   • Substitute the value of \( x \) in the expression to check your work.

   \[ \frac{1}{1} \]

   a. \( x - 3 \) is the same as \(-5\)

   \[ \begin{array}{c}
   x - 3 \\
   \hline
   -5
   \end{array} \]

   b. \( 2x + 3 \) is equal to \(-6\)

   \[ \begin{array}{c}
   2x + 3 \\
   \hline
   -6
   \end{array} \]

10. For each of the following expressions and equivalent numbers, draw a number line to determine and locate the value of the variable. It is up to you to choose the length of the unit and the location of the given expression.
   a. \( 4x + 7 \) is 19
   b. \( 3y + 1 \) is 10
   c. \( 2a - 3 \) is 7
   d. \( 5z - 9 \) is \(-24\)

11. Given the location of the two expressions, find the location of \( x \) and 0:

   \[ \begin{array}{c}
   3x \\
   \hline
   5x
   \end{array} \]

12. Given the location of the two expressions, find the location of \( x \) and 0:

   \[ \begin{array}{c}
   4x \\
   \hline
   2x
   \end{array} \]
13. **Investigation:**
Place the number $a$ on the number line. Can we find $2a$ without locating 0 first? Can we find $a - 1$ without finding 0 and 1 first? If the answer to these questions is yes, explain how to do so. If the answer is no, explain why not.

14. **Investigation:**
Locate the point that represents the number $x$ on the number line. Also, locate the point that represents the number 0.

\[ \begin{array}{c}
1 \\
\hline
\end{array} \]

a. $2x + 1$ is the same point as $3x - 2$:

\[ \text{2x + 1} \quad \text{3x - 2} \]

b. $2x + 1$ is the same point as $5x + 7$:

\[ \text{2x + 1} \quad \text{5x + 7} \]

15. **Ingenuity:**
Consider the following sequence:

\[ 4, 7, 10, 13, 16, 19, 22, \ldots \]

If the pattern continues:

a. What is the 20th member of the sequence?

b. What is the 100th member of the sequence?

c. The number $n$ is a positive integer. What is the $n$th member of the sequence?
In Section 1.2 variables represented numbers on the number line whose value was unknown. In this section, we see how to use variables to create algebraic expressions that describe a problem mathematically.

Variables and Expressions in Context

Here is an example that illustrates what variables are and how to use them.

EXAMPLE 1

Use the variable \( J \) to represent John’s age in years. Though you don’t know his age, you do know that John is 5 years older than Sue. Use the variable \( S \) to represent Sue’s age and write an equation that expresses the relationship between \( J \) and \( S \).

Solution  Define 2 variables in the following way:

Let \( J \) = John’s age in years.

Let \( S \) = Sue’s age in years.

Now translate the sentence “John is 5 years older than Sue” into an equation. Substituting variables for words, “\( J \) is 5 more than \( S \)”, or \( J = S + 5 \). “Is” translates to “\( = \)”. “5 more than” means add 5. Expressions involving algebraic symbols are called algebraic expressions. In this case, \( S \), \( S+5 \) and \( J \) are all algebraic expressions.
Once we have an algebraic equation, it is easy to make a table of some possible values of $J$ and $S$ that satisfy the conditions of this problem. For example, if Sue were 2 years old, then $S = 2$ and $J = S + 5 = 2 + 5 = 7$. So John would be 7 years old. Fill in the table below with other possible values.

<table>
<thead>
<tr>
<th>$S$</th>
<th>$J = S + 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

**EXPLORATION 1**

We look at a family of 4 people and their ages. Today, Adam’s age is 4 more than his sister Bonnie’s age. Their mother Carmen is twice as old as Bonnie and their father Daniel’s age is one year more than twice Adam’s age. Let the variable $A$ represent Adam’s age right now.

1. Write an expression that represents Bonnie’s age in terms of Adam’s age.
2. Using what we learned about Bonnie’s age, write Carmen’s age in terms of Adam’s age.
3. Write Daniel’s age in terms of Adam’s age.
4. Let the variable $B$ represent Bonnie’s age at this time. Write an expression that represents Adam, Carmen and Daniel’s ages in terms of $B$. Explain in words and in symbols how you arrived at your expressions.

Some of these relationships and expressions take careful examination and thinking. Expressions give a simple way of looking at relationships between quantities. However, do not forget that there is meaning behind the symbols. Do not lose sight of what that meaning is while working with the expressions.
EXAMPLE 2

Mary takes a bike ride everyday at an average rate of 12 miles per hour. Use the variable $t$ to represent the length of time for Mary’s bike ride, measured in hours. Notice that $t$ represents how long Mary rides her bike on the day in question. The reason $t$ is a variable is that it can assume different values depending on other information that is given. In spite of the fact that $t$ can vary, it can still give us interesting information about Mary’s rides.

Find the distance that Mary travels if she rides for

1. 2 hours,
2. 5 hours,
3. $t$ hours.

Solution

1. The distance Mary travels in one hour is 12 miles, so the distance Mary travels in 2 hours should be twice as far as 12 miles, that is, 24 miles. Her speed is $12 \text{ miles/hour}$, so we can write this correspondence between time and distance as

$$12 \text{ miles/hour} \cdot 2 \text{ hours} = 24 \text{ miles}.$$

2. When Mary rides for 5 hours, she travels

$$12 \text{ miles/hour} \cdot 5 \text{ hours} = 60 \text{ miles}$$

all together.
3. Since variables are just like numbers,: 

\[
\text{12 miles per hour} \cdot t \text{ hours} = 12 \cdot t \text{ miles.}
\]

In \( t \) hours, Mary will average 12 miles for each hour, so she rides \( 12 \cdot t \) miles in total. The expression \( 12t \) represents the total number of miles Mary rides, depending on the value of \( t \). This expression is very useful. For example, if \( t = 2 \), Mary rides \( 12(2) = 24 \) miles, but if \( t = 0.5 \), Mary travels \( 12(0.5) = 6 \) miles. When Mary rides \( t \) hours, the distance she travels is \( 12t \) miles. Note that \( 12t \) is again an algebraic expression and it represents the distance Mary travels in \( t \) hours when she rides an average rate of 12 miles per hour.

**Variables and Expressions in Geometry: the Area Model**

**EXAMPLE 3**

Sketch a rectangle that has length \( L \) cm and width 5 cm. Express the area and perimeter of the rectangle in terms of \( L \).

**Solution**

```

```

Make a table with the area and perimeter of various rectangles that satisfy the given condition. That is, the width must be 5 cm.
Section 1.3  VARIABLES AND EXPRESSIONS

<table>
<thead>
<tr>
<th>Length in cm</th>
<th>Width in cm</th>
<th>Area in cm²</th>
<th>Perimeter in cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>5 · 1 = 5</td>
<td>5 + 5 + 1 + 1 = 12</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>5 · 2 = 10</td>
<td>5 + 5 + 2 + 2 = 14</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>5 · 3 = 15</td>
<td>5 + 5 + 3 + 3 = 16</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>L</td>
<td>5</td>
<td>_____ =</td>
<td>_____ =</td>
</tr>
</tbody>
</table>

We can now write algebraic expressions that represent the patterns we see in the table. Let \( L \) be the length in cm. The area is \( 5L \). The perimeter is \( 5 + 5 + L + L = 10 + 2L \) cm.

**PROBLEM 1**

Sketch a rectangle whose length is twice its width. Let \( L \) represent the length and \( W \) the width. Write an algebraic expression for:

1. the area in terms of \( W \).
2. the area in terms of \( L \).
3. the perimeter in terms of \( W \).

**PROBLEM 2**

Sketch a rectangle whose length is 3 inches greater than its width. Let \( L \) represent the length and \( W \) the width. Write an algebraic expression for

1. the area in terms of \( W \).
2. the area in terms of \( L \).
3. the perimeter in terms of \( W \).

**Variables to Describe Sets: Set Notation**

In Section 1.1, we constructed the number line and discussed many different sets of numbers. Sets are so important that we have many ways of describing them. We can describe sets using just words. For example, the whole numbers or the even numbers are 2 different
sets of numbers. Sometimes we can graph sets on a number line. Also, variables can be used to describe elements in a set using set notation. We demonstrate all 3 methods in the following example.

EXAMPLE 4

Write the set of all numbers greater than or equal to 1 in set notation. Graph this set on the number line.

Solution  We want to represent this set using a variable and set notation. We use \( x \) to describe the numbers in the set, then translate the condition "\( x \) is greater than or equal to 1" to the language of algebra: \( x \geq 1 \).

Set Notation: \( \{ x | x \geq 1 \} \)

The brackets \( \{ \} \) denote a set. The variable \( x \) denotes any element in the set. The vertical bar \( | \) means "such that" and is followed by a condition to be satisfied by every element \( x \) in the set. So we read this notation as:

Words: “the set of all numbers \( x \) such that \( x \) is greater than or equal to 1”.

Notice that in this case the variable \( x \) is used to describe an infinite set of numbers!

We would also like to graph the set on a number line. First we locate 1, then indicate all the numbers greater than or equal to 1 by shading 1 as well as all the numbers to the right of 1 on the number line. Note, we can’t really shade in the point 1 by itself, so we draw a small shaded circle around it to signify that it is included in the set.

Graph:
EXAMPLE 5

Write the set of all numbers greater than 2 in set notation. Indicate this set on a number line.

Solution  We can use $x$ again to describe the numbers in this set. The condition “$x$ is greater than 2” with algebraic symbols is written as $x > 2$. The set can be written

**Set Notation:** \( \{x | x > 2\} \),

which reads as “the set of all numbers $x$ such that $x$ is greater than 2”

On the number line, we shade in every number greater than 2, but not 2 itself. Again, 2 is only a single point, so it is not possible to shade it in such a way that can be seen clearly. To denote that it is excluded from the set, we draw an unshaded circle around it and only shade numbers to the right of the circle.

Every number greater than 2 is greater than or equal to 1. Therefore the second set is a subset of the first. On the other hand, 1.5 is in the first set but not in the second. So the first set is not a subset of the second.

PROBLEM 3

1. Express the statement that "$x$ is greater than or equal to 2 and smaller than 7" in the language of algebra.
2. Write the set of numbers greater than or equal to 2 and smaller than 7 using set notation.
3. On the number line below show the set of numbers that are greater than or equal to 2 and smaller than 7.
4. Find 5 numbers $x$ that satisfy this condition.

**Summary**

When working through problems in algebra, use variables and symbols to state things mathematically, just as you write sentences by combining letters and words. It is often possible to translate sentences into math using variables and expressions to form equations. Once we have translated the sentences into equations, we can often solve these problems more easily than we could have when simply looking at the text of the problem. For many of the problems in this section, you may be able to find the answer without formally making equations. However, the approach of using equations becomes important as the problems become more complicated. Writing equations will organize our work and provide a clearer process for obtaining a solution.

Variables and algebra are part of the language of mathematics, and learning this language allows you to make mathematical statements using variables that you choose. When using variables, the important thing is to describe what the variable represents (in the examples above, variables represented units of time, lengths and numbers on the number line). Then translate real world problems into algebraic expressions, and operate on those expressions to solve the problems. In this way, algebra becomes a tool for modeling and solving problems.

**EXERCISES**

1. Draw a rectangle and call the lengths of the sides of the rectangle $x$ units and $y$ units. What is the perimeter of the rectangle? What is the area of the rectangle?

2. A rectangle has width $W$ and a length that is twice as long as the width.
   a. Write an expression for the perimeter of this rectangle.
   b. Write an expression for the area of this rectangle.
Section 1.3  VARIABLES AND EXPRESSIONS

3. A rectangle has length $L$ inches.
   a. The width is 6 inches longer than the length. Write an expression for the perimeter and an expression for the area of this rectangle.
   b. The width is 6 inches shorter than the length. Write an expression for the perimeter and an expression for the area of this rectangle.
   c. How long can the rectangle in part 3a be? How long can the rectangle in part 3b be?

4. Express each of the statements algebraically:
   a. $x$ is two units less than $y$.
   b. $x$ is three units more than $y$.
   c. $x$ is three times the number $y$.
   d. $x$ is one half the number $y$.
   e. $x$ is $y$ units less than $z$.

5. Express each of the statements algebraically:
   a. $x$ is less than the number that is the sum of $y$ and $z$.
   b. $z$ is greater than 3.
   c. $y$ is less than twice $x$.

6. Marisa is $M$ years old, and Jenny is 5 years older than Marisa. Write an expression that gives Jenny’s age.

7. Jessica is 10 years old today. How old will she be in 3 years? In 5 years? In $x$ years?

8. Jacob is $J$ years old today. How old will he be in $y$ years?

9. Joe travels at a rate of 20 miles per hour for $t$ hours. How far does he go?

10. Jeremy travels at a rate of $J$ miles per hour for 3 hours. How far does he go?

11. Sam has marbles.
   a. He has 10 marbles and gives away $x$ marbles. How many marbles does he have left?
   b. He has $y$ marbles and gives away 10 marbles. How many marbles does he have left?
   c. He has $u$ marbles, then Joe gives him 7 marbles and after that he gives half of the marbles he has to Matt. How many marbles does he have left?
   d. He has $v$ marbles, gives half of them to Matt, then Joe
gives him 7 marbles. How many marbles does he have?
e. In each of the questions above, what are the possible values of the variable?

12. Juan is 5 years younger than Maria and twice as old as Pedro.
a. Pedro’s age is \( P \). Write an equation that relates Maria’s and Juan’s ages.
b. Call Maria’s age \( M \) and write an equation that relates Pedro’s and Juan’s ages.

13. Translate each mathematical statement into an equation or expression:
a. \( x \) is two greater than \( y \).
b. the number that is two greater than \( x \).
c. the number that is two less than \( x \).
d. \( x \) is less than \( y \).
e. \( x \) is five smaller than a number that is twice as large as \( y \).

14. Express in words what each set represents.
a. \( \{ x : x < 10 \} \) all numbers less than 10.
b. \( \{ x : x > 5 \} \) all numbers greater than 5.
c. \( \{ x : 2 < x \leq 8 \} \) all numbers greater than 2 but less than or equal to 8.
d. \( \{ x : -5 \leq x \} \)

15. If \( x \) were 2 more, it would be 10 less than \( z \). Express the relation between \( x \) and \( z \).

16. A triangle has a base \( x \) cm long.
a. When the height is 6 cm, write an expression for the area of the triangle.
b. When the height is 3 times the base, write an expression for the area of the triangle.
c. When the height is one half the base, write an expression for the area of this triangle.

17. Describe the set of numbers that are larger than 2 and less than 5 using set notation. Represent this set on the number line.
18. The low temperature near the Exit Glacier in Alaska on January 5 was $-8 \, ^\circ F$, and the high temperature was $22 \, ^\circ F$.
   a. Describe in words the set of all the temperatures $T \, ^\circ F$ at this location on this day.
   b. Write this set using set notation.
   c. Represent this set on the number line.

19. Using set notation, describe 3 subsets of whole numbers that are each infinite. Then describe another infinite subset of whole numbers that is a subset of one of your first 3 subsets.

20. We can write the set of even natural numbers as $\{2, 4, 6, 8, \ldots \}$. It can also be described in set notation using a variable as $\{2n | n \text{ is a natural number}\}$. Write each of the following sets, using a variable in set notation.
   a. $\{1, 3, 5, 7, 9, \ldots \}$
   b. the set of positive multiples of 5
   c. the set of multiples of 10
   d. $\{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots \}$

21. What is the distance from $-8$ to $14$? From $-12$ to $-5$?
SECTION 1.4 SOLVING LINEAR EQUATIONS

Solving real-world problems using mathematics is often a 3-step process. First, you must translate the words in the problem to the language of mathematics. The second step is to use the tools that mathematics gives you to solve these problems. Finally, you check the answer. We will review different approaches to the second step.

EXAMPLE 1

Sue sells sandwiches at some price we don’t know and candy bars for $2 each. Mark buys only one candy bar and 4 sandwiches for a total cost of $14. How much do the sandwiches cost?

Solution

First, notice that the total cost of $14 is the sum of the cost of the 4 sandwiches and the cost of the candy bar. So:

\[
\text{cost of 4 sandwiches} + \text{cost of one candy bar} = \text{total cost}.
\]

How do we translate this into a mathematical sentence? Begin with the information given in the problem.

\[
\begin{align*}
$14 &= \text{the total cost} \\
$2 &= \text{cost of the one candy bar Mark buys} \\
4 &= \text{number of sandwiches Mark buys}
\end{align*}
\]

We define the variable as the quantity that we want to compute:

\[
s = \text{the price of a single sandwich in dollars}.
\]

Now combine the information to form an equation.

\[
\begin{align*}
\text{cost of 4 sandwiches} + \text{cost of one candy bar} &= \text{total cost} \\
4s + 2 &= 14
\end{align*}
\]
To demonstrate different ways we can think about solving equations, we will use three different approaches to solve this single problem. What do you notice about the three methods? How are they similar?

Solve $4s + 2 = 14$

**Balance Model**

**Algebraic Method**

$4s + 2 = 14$

Subtract 2 from both sides

$4s + 2 - 2 = 14 - 2$

$4s = 12$

Divide both sides by 4

$\frac{4s}{4} = \frac{12}{4}$

$s = 3$

**Number Line Model**

Now we check our answer.

$4 \cdot 3 + 2 = 14$

$14 = 14$
EXPLORATION 1

In the previous example, we discussed how to write equivalent equations by performing the same operations to both sides of the equation: addition, subtraction, multiplication or division. Discuss which of these steps you should perform on each of the following equations. Some equations require two steps to solve for \( x \). Does it matter in what order you perform these steps? Explain.

1. \( x - 4 = 10 \)
2. \( x + 4 = 10 \)
3. \( 3x = 15 \)
4. \( \frac{x}{4} = 3 \)
5. \( 15 = 2x + 7 \)
6. \( 4x + 3 = 17 \)
7. \( 2x - 4 = 10 \)

EXPLORATION 2

Think about the problems above. What steps were involved in translating the problem in Example 1 to the language of mathematics? What steps did you use to solve the equations in Exploration 1? How do you decide the order of the steps?

PROBLEM 1

Susan asks Fred to make some cookies for a party. Fred decides to make twice as many as Susan requested, and also 3 extra cookies for his neighbor. Fred makes 57 cookies. How many cookies did Susan request?

Now we can formally define the concepts and properties we have discussed so far.
A solution to an equation with a variable $x$ is a number that when substituted for $x$, makes the 2 sides of the equation equal. If the equation has more than one solution, then the collection of solutions is called the solution set.

Two equations are equivalent if they have the same solution or solution set.

The process of solving an equation for $x$ is to find simpler equations that are equivalent. In each step, the equations are equivalent and will have the same solution. There are 4 basic ways to produce equivalent equations:

**Addition Property of Equality**

Starting with any equation, if you add the same amount to both sides of the equation you obtain an equivalent equation.

For example, the equations $x - 5 = 7$, $x - 5 + 5 = 7 + 5$ and $x = 12$ are equivalent.

**Subtraction Property of Equality**

Starting with any equation, if you subtract the same amount from both sides of the equation you obtain an equivalent equation.

For example, the equations $x + 5 = 7$, $x + 5 - 5 = 7 - 5$ and $x = 2$ are equivalent.
MULTIPLICATION PROPERTY OF EQUALITY

Starting with any equation, if you multiply both sides of the equation by the same non-zero number you obtain an equivalent equation.

For example, the equations $\frac{x}{5} = 7$, $\frac{x}{5} \cdot 5 = 7 \cdot 5$ and $x = 35$ are equivalent.

DIVISION PROPERTY OF EQUALITY

Starting with any equation, if you divide both sides of the equation by the same non-zero number you obtain an equivalent equation.

For example, the equations $5x = 35$, $\frac{5x}{5} = \frac{35}{5}$ and $x = 7$ are equivalent.

In the next 3 examples, we use the properties of equality to solve equations. To solve an equation for $x$, we must find an equivalent equation giving us the value of $x$. In order to achieve this, we need to transform the original equation through a chain of equivalent equations into an equation that has $x$ by itself on one side and a number on the other.

EXAMPLE 2

Solve the equation:

$$x - 3 = 20.$$

Solution

To solve for $x$, we need to get rid of the $-3$. By the addition property of equality, the original equation is equivalent to:

$$x - 3 + 3 = 20 + 3.$$

Evaluating the sums leads to the solution $x = 23$. 

38
EXAMPLE 3

Solve the equation:

\[ 4x = -8. \]

**Solution** By the division property of equality, our original equation is equivalent to:

\[ \frac{4x}{4} = \frac{-8}{4}. \]

Evaluating gives:

\[ x = -2 \]

as the solution.

EXAMPLE 4

Solve the equation:

\[ 3r + 5 = 17. \]

**Solution** In order to solve this equation, we need to isolate \( r \). However, in this case, we are unable to isolate \( r \) by performing a single operation. We observe that, in the expression on the left, \( r \) has been multiplied by 3, and a 5 has been added. This suggests that we should divide both sides by 3, and subtract 5 from each side. But in what order should we perform these operations? Let's think about how the expression on the left is constructed:

\[ 3r + 5 = \text{“Five more than three times } r\text{”} \]

The order of operations tells us that when we evaluate an expression like this, we do the multiplication first (\textbf{three times } \( r \)), and then the addition (\textbf{five more than three times } \( r \)). In Exploration 1 you discovered that to isolate the variable, we can reverse these operations, beginning with the last operation we did. Using the
substraction property of equality, let’s start by subtracting 5 from each side in order to reverse the addition:

\[ 3r + 5 - 5 = 17 - 5 \]
\[ 3r = 12 \]

Using the division property, solve for \( r \) by dividing both sides by 3:

\[ \frac{3r}{3} = \frac{12}{3} \]
\[ r = 4. \]

**EXERCISES**

1. For each equation below, solve for the unknown by using the Addition Property of Equality or the Subtraction Property of Equality. Name which one you use. Refer to Example 2.
   a. \( x - 5 = 15 \)
   b. \( y - 10 = -14 \)
   c. \( z + 17 = 35 \)
   d. \( a + 20 = 5 \)
   e. \( b + 2 = -4 \)

2. For each equation below, solve for the unknown by using the Multiplication Property of Equality or the Division Property of Equality. Name which one you use. Refer to Example 3.
   a. \( 2a = 30 \)
   b. \( 3c = -12 \)
   c. \( \frac{S}{3} = 2 \)
   d. \( -4d = 2 \)
   e. \( \frac{1}{8}d = -2 \)
   f. \( -\frac{x}{5} = -7 \)
3. Combine the previous 2 methods to solve each equation using multiple steps. Name which properties of equivalence you use.
   a. $2n + 5 = 37$
   b. $2E - 8 = 20$
   c. $3g + 4 = 40$
   d. $\frac{x+3}{4} = 1$
   e. $-4a + 3 = 15$
   f. $5(C - 2) = 25$
   g. $-\frac{d}{3} + 1 = 5$
   h. $\frac{1}{6}x - 4 = 1$

4. For each equation below:
   • solve the equation for the unknown variable by using the Properties of Equality.
   • name which property or properties you have used.
   • check that the solution satisfies the original equation.
   a. $2z = 7$
   b. $\frac{b}{3} = 5$
   c. $3T = 5$
   d. $2x + 3 = 12$
   e. $3S = -8$
   f. $4l - 5 = 7$
   g. $2X + 5 = 16$
   h. $2a - 8 = 7$
   i. $3y + 4 = 20$

5. For each situation described below:
   • define a variable that represents the unknown quantity.
   • write a mathematical equation.
   • solve the equation to answer the question.
   a. Sam went on a bike ride averaging 10 miles per hour. He biked a total of 25 miles. How long did it take him?
   b. A square garden has a fence around it. The total length of the fence is 92 feet. What is the length of each side?
   c. A rectangular playing field is twice as long as it is wide. The perimeter of the field is 246 meters. What is its width?
6. A rectangular pool is 3 meters longer than it is wide. The perimeter is 38 meters.
   a. Define the width as \( W \) meters, then write an equation using \( W \) and solve it for \( W \).
   b. Define the length as \( L \) meters, then write an equation using \( L \) and solve it for \( L \).
   c. How do the answers to parts 6a and 6b compare?

7. Think about consecutive whole numbers. For example, 2 and 3 are consecutive whole numbers.
   a. Write three pairs of consecutive whole numbers.
   b. Let \( n \) represent a whole number. Write an expression for the next larger whole number.
   c. Write an expression for the sum of two consecutive whole numbers.
   d. The sum of two consecutive whole numbers is 37. What are the two numbers?

8. Sara has 82 inches of string candy. She wants to divide it equally and share with Sandra. She had already promised to give a 10-inch piece to Juanita. If Sara gives half to Sandra first and then gives Juanita 10 inches of candy from her part, how much does Sara have left? To find out, let \( S \) be the amount of candy Sara gets. Write an equation using \( S \) and solve for \( S \).

9. Joe has 82 inches of string candy. He wants to divide it equally and share with Max. But he had promised to give a 10-inch piece to Jeremy. If Joe gives 10 inches of candy to Jeremy first, then divides the rest into equal pieces for himself and Max, how much does he get? To find out, let \( J \) denote the amount of candy Joe gets. Write up an equation using \( J \) and solve.

10. Do you get the same or different result in the previous two questions? Explain.

11. **Investigation:**
    Return to the sandwich stand of Example 1. Suppose Isabel has $20 and she wants to spend all her money (no change) to buy sandwiches and candy bars for her friends. How many of each could she buy?
12. **Ingenuity:**

   Consider the equation:

   \[ 3x + 5y = 33. \]

   What would it mean to solve this equation for \( x \)?

13. Explain the difference between an expression and an equation. Give an example of each.
EXPLORATION 1

Consider the square grid below, representing an \(8 \times 8\) swimming pool with a shaded border of width 1. How many squares are shaded in? Answer this without talking, without counting one by one, and without writing.
PROBLEM 1
Consider the grid below, representing a $12 \times 12$ pool with a shaded border of width 1. Calculate the number of shaded tiles without counting one by one.

EXPLORATION 2
Now look at a $n \times n$ square swimming pool with a border of width one. Determine the number of squares in the border using two of the methods your classmates described in Exploration 1. For each of the methods write out in words what the method is doing. Write an algebraic expression that explains each method you used. Make sure to say what the variable in your expression represents.

Equivalent Expressions

EXAMPLE 1
Write the following relationship mathematically:

"Twice the amount of money that Jessica has in the bank now"

in two different ways.
Solution  Let \( J \) = amount of money Jessica has in the bank now. One way we can write "twice this amount" is \( 2J \). Another way to write twice the amount is \( J + J \).

The expressions \( J + J \) and \( 2J \) are called equivalent expressions and we write \( J + J = 2J \). This means that the two sides of the equation are equal no matter what value we choose for \( J \).

PROBLEM 2

A rectangle has length \( L \) and width \( W \). What is the perimeter of the rectangle? Write as many expressions for the perimeter as you can and explain how you arrived at the expressions. It might be helpful to draw a sketch of the rectangle.

Properties that Establish Equivalence

EXPLORATION 3

Determine which of the following number sentences are true. Try to justify your answer without calculating each side.

1. \( 5 + 7 = 7 + 5 \)
2. \( 56 + 89 = 89 + 56 \)
3. \( 457 + 684 = 684 + 457 \)
4. \( 578943 + 674321 = 674321 + 578943 \)
5. \( 7 - 5 = 5 - 7 \)
6. \( 56 - 89 = 89 - 56 \)
7. \( 457 - 684 = 684 - 457 \)
8. \( 578943 - 674321 = 674321 - 578943 \)
9. \( 1 + 2 - 3 = 2 + 1 - 3 \)
10. \( 4 + 5 - 6 = 5 - 6 + 4 \)
11. \( 11 - 14 + 21 = 11 - 21 + 14 \)
12. \( 11 + (-14) + 21 = 11 + 21 + (-14) \)
Two equivalent expressions for the perimeter of the rectangle in problem 2 are \( L + W + L + W \) and \( W + L + W + L \). The order in which the sides of the rectangle are added does not affect the perimeter. This fact is an important property in mathematics. The property is called the *commutative property of addition*. It means that for any two numbers \( n \) and \( m \), \( n + m = m + n \). "Commute" here means to change the order of the terms.

<table>
<thead>
<tr>
<th>COMMUTATIVE PROPERTY OF ADDITION</th>
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<tr>
<td>For any two numbers ( x ) and ( y ),</td>
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<tr>
<td>( x + y = y + x ).</td>
</tr>
<tr>
<td>For example, ( 24 + 6 = 6 + 24 ) and ( 8 + (-5) = (-5) + 8 ).</td>
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</tbody>
</table>

**PROBLEM 3**

The number line is a good way to visualize properties of addition. Use a number line to illustrate that each of the following mathematical sentences is true.

1. \( 24 + 6 = 6 + 24 \)
2. \( 8 + (-5) = (-5) + 8 \)
3. \( n + m = m + n \) where each of \( m \) and \( n \) are numbers

**PROBLEM 4**

Is it possible to change the order with other operations and maintain equivalence? Check to see if this works with subtraction, multiplication and division. For each operation, if it is commutative, make a rule. If not, give an example of how it fails to be commutative.

The *commutative property of multiplication* is also very useful. The expression \( 50t \) is equivalent to \( t50 \). It is a convention in algebra to use the form \( 50x \) rather than \( x50 \). This is intended to make
expressions with lots of variables and numbers simpler to read.

<table>
<thead>
<tr>
<th>COMMUTATIVE PROPERTY OF MULTIPLICATION</th>
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<tbody>
<tr>
<td>For any two numbers ( x ) and ( y ),</td>
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<tr>
<td>( xy = yx ).</td>
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</table>

For example, \( 5 \cdot 6 = 6 \cdot 5 = 30 \) and \( 3 \cdot -2 = -2 \cdot 3 = -6 \).

Let’s return to equivalent expressions for the perimeter. Another equivalent expression that you might have found is \( 2L + 2W \). Consider \( L + W + L + W \) and use the commutative property to write \( L + L + W + W \), then group the first two and last two terms to get \( 2L + 2W \). Grouping is called the associative property of addition, because the numbers are changing their group or association. In general, for each number \( a, b \) and \( c \), \( (a + b) + c = a + (b + c) \). Just as the commutative property of addition means that adding 2 numbers when placed in different orders does not change the sum, the associative property of addition means that the order of evaluating the addition of 3 or more numbers does not change the sum.
ASSOCIATIVE PROPERTY OF ADDITION

For any numbers \( x, y, \) and \( z, \)

\[
(x + y) + z = x + (y + z).
\]

For example, \( 3 + 4 + 5 \) can be viewed as both:

\[
(3 + 4) + 5 = 7 + 5 = 12
\]

or as:

\[
3 + (4 + 5) = 3 + 9 = 12.
\]

PROBLEM 5

1. Does the same property work for subtraction? Is \( (6 - 5) - 3 = 6 - (5 - 3) \)?
2. Does multiplication have an associative property? Is \( (8 \cdot 4) \cdot 2 \) equal to \( 8 \cdot (4 \cdot 2) \)? Is \( (ab)c = a(bc) \)?
3. Does division have an associative property? Check to see if \( (8 \div 4) \div 2 \) and \( 8 \div (4 \div 2) \) are equivalent.
4. Discuss why when carrying out any operation it is important to be particularly mindful of parentheses when subtracting and dividing are involved.

Let’s explore another property that is useful when multiplying two expressions.
EXPLORATION 4

Compute the area of each of the large rectangles below in at least 2 ways:

1. 
   
   \[
   \begin{array}{c}
   5 \\
   6 \\
   9
   \end{array}
   \]

2. 
   
   \[
   \begin{array}{c}
   x \\
   4 \\
   8
   \end{array}
   \]

3. 
   
   \[
   \begin{array}{c}
   x \\
   y \\
   z
   \end{array}
   \]

You have just discovered the **distributive property**! For any numbers \(a, b\) and \(c\), \(a(b + c) = ab + ac\). This is true even for negative numbers or zero. This property allows us to find one more equivalent expression for the perimeter, \(2L + 2W = 2(L + W)\), and also to simplify or transform other expressions.
**Section 1.5  EQUIVALENT EXPRESSIONS**

**DISTRIBUTIVE PROPERTY**

For any numbers $x$, $y$, and $z$,

$$x(y + z) = xy + xz.$$  

For example, $3(2 + 4) = 3 \cdot 2 + 3 \cdot 4 = 6 + 12 = 18$.

**PROBLEM 6**

Use the area model to show that $3(x + 4) = 3x + 3 \cdot 4$.

Let's explore equivalent expressions involving fractions.

**EXAMPLE 2**

Use the three rectangles below to explain why the expressions $\frac{2x}{3}$, $\frac{2}{3}x$, and $\frac{1}{3}(2x)$ are equivalent.

**Solution**  To relate the rectangle to an algebraic expression, we need to imagine how the rectangle was drawn. On the left, we have shaded one third of the 2 by $x$ rectangle which represents $\frac{2x}{3}$. On the top right, we have shaded two thirds of the 1 by $x$ rectangle which represents $\frac{2}{3}(1 \cdot x) = \frac{2}{3}x$. And on the bottom right, the shaded rectangle is $\frac{1}{3}$ by $2x$: $\frac{1}{3}(2x)$. 

51
By cutting each shaded region into two equal pieces we can see the areas are all equal. So \( \frac{2x}{3}, \frac{4x}{3}, \) and \( \frac{1}{3}(2x) \) are equivalent.

So we see that \( \frac{2}{3}x = \frac{2x}{3} = \frac{1}{3}(2x) \).

To understand and manipulate fractions algebraically, we need to study the multiplicative inverse (reciprocal) of a number:

**MULTIPLICATIVE INVERSE**

For any non-zero number \( n \),

\[
    n \cdot \frac{1}{n} = 1.
\]

The expression \( \frac{1}{n} \) is called the *multiplicative inverse or reciprocal* of \( n \).

For example, \( 5 \cdot \frac{1}{5} = 1 \). So \( \frac{1}{5} \) is the multiplicative inverse of 5.
EXAMPLE 3

Using the properties of numbers, show that the following expressions are equivalent:

\[ \frac{4y}{5} \quad \text{and} \quad \frac{4}{5y}. \]

**Solution** It is easy to test for equivalence for a particular value of \( y \). For example, if \( y = 10 \), then \( \frac{4y}{5} = \frac{40}{5} = 8 \), and \( \frac{4}{5}y = \frac{4}{5} \cdot 10 = 4 \cdot 2 = 8 \). But how do we know this is true for every value of \( y \)? There are infinitely many choices for \( y \)!

We will use the properties from this section. But first, what exactly is meant by \( \frac{a}{b} \)? We say that it is \( a \) divided by \( b \). But what does this mean precisely? The number \( \frac{a}{b} \) is the product \( a \cdot \frac{1}{b} = \frac{1}{b} \cdot a \). Now we can use the properties of multiplication.

Now, return to the original question. Using the above notation for fractions and the associative property of multiplication,

\[ \frac{4y}{5} = \frac{1}{5} \cdot (4 \cdot y) = \left( \frac{1}{5} \cdot 4 \right) \cdot y = \frac{4}{5}y. \]

Therefore, \( \frac{4y}{5} = \frac{4}{5}y \) for every value of \( y \). So \( \frac{4y}{5} \) and \( \frac{4}{5}y \) are equivalent.

PROBLEM 7

Show that these two expressions are also equivalent to \( 4 \left( \frac{y}{5} \right) \).

EXAMPLE 4

Show that \( \frac{4x+6}{2} \) and \( 2x + 3 \) are equivalent.

**Solution** Using the distributive property and fraction notation, we can write

\[ \frac{4x + 6}{2} = \frac{1}{2} \cdot (4x + 6) = \frac{1}{2} \cdot 4x + \frac{1}{2} \cdot 6 = 2x + 3. \]
So \( \frac{4x+6}{2} = 2x + 3 \) for all values of \( x \). Fortunately it is not necessary to write all the steps when we work with algebraic expressions. In the future we can just show the distributive step:

\[
\frac{4x + 6}{2} = \frac{4x}{2} + \frac{6}{2} = 2x + 3.
\]

**PROBLEM 8**

Kate believes that \( \frac{2x+4}{2} = x + 2 \). Jamie says, "No. It should be \( \frac{2x+4}{2} = x + 4 \)." Who is right? Draw an area model to support your answer.

**Combining Like Terms**

The algebraic expression \( 4y + 3x + 2x + 4 + 7 \) contains five terms, two terms that involve the same variable \( x \), two that involve only numbers and the term \( 4y \). In this case, \( 3x \) and \( 2x \) are called like terms because they each have the same variable \( x \). The two numbers \( 4 \) and \( 7 \) are also like terms. However, there are no terms that are “like” \( 4y \). It is often helpful to write an expression in a simpler form by combining like terms.

**EXPLORATION 5**

1. The number \( a \) is marked on the number line below. Locate \( 2a \), \( 3a \) and \( 5a \) on the number line. Explain why \( 2a + 3a = 5a \).

   ![Number Line with Points](image)

2. The numbers \( 2a \) and \( 3b \) are marked on the number line. Locate \( a, b \) and \( 2a + 3b \) on the number line. In part 1, we "combined" \( 2a \) and \( 3a \) into \( 5a \). Do you think it is possible to write \( 2a + 3b \) as one term? Explain.
Using the distributive property and combining like terms can often simplify a complicated expression.

EXAMPLE 5

Find an expression that is equivalent to $3(x+4) - 2(2x - 3) + 8x - 1$ by:

1. Using the distributive property to remove the need for parentheses,
2. Combining like terms.

Solution  Use the distributive property and then combine like terms:

\[
3(x + 4) - 2(2x - 3) + 8x - 1 \\
= 3x + 12 - 4x + 6 + 8x - 1 \quad [\text{Dist. Prop.}] \\
= 3x - 4x + 8x + 12 + 6 - 1 \quad [\text{Assoc. and Comm. Props.}] \\
= x + 17 \quad [\text{Combining Like Terms}]
\]

Note that the resulting expression $x + 17$ is much simpler than the original expression $3(x + 4) - 2(2x - 3) + 8x - 1$. However, in some cases a more complicated expression is preferred, because it is easier to interpret in the context of a problem.

PROBLEM 9

Match each expression on the left to an equivalent expression on the right. Explain.
Chapter 1  Variables, Expressions and Equations

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<table>
<thead>
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<tbody>
<tr>
<td>1. $4(2x - 3) + 3x - 2$</td>
<td>a. $-(x + 2)$</td>
</tr>
<tr>
<td>2. $2(3 - x) + 3(x + 5) - 1$</td>
<td>b. $11x - 14$</td>
</tr>
<tr>
<td>3. $5(2x - 3) - 2(3 - 2x)$</td>
<td>c. $8(x + 3)$</td>
</tr>
<tr>
<td>4. $7(x + 3) + 3(x + 3) - 2x - 6$</td>
<td>d. $x + 20$</td>
</tr>
<tr>
<td>5. $4x - 5(x - 2) + 8$</td>
<td>e. $14x - 21$</td>
</tr>
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**Number Sense, Mental Math and Equivalent Expressions**

Number sense is an understanding of numbers, their relationships, and how they are affected by operations. You have been developing your number sense since you began to count. Having a strong number sense gives you the ability to solve problems in many ways. Often schools host "number sense" competitions. In these competitions, students solve a long list of arithmetic problems in a short time without using paper and pencil. The winners usually have a bag of mathematical tricks that let them solve certain types of problems very quickly in their head. These mathematical tricks are just a set of useful equivalent expressions. Let's explore one of these tricks.

**EXPLORATION 6**

1. Compute each of the following.
   a. $54 \div 10$
   b. $54 \div 100$
   c. $54 \div 1000$

2. A fourth grader is learning how to divide and is given the problems above. How would you explain the "easy way" to find the answer?

3. Compute each of the following. What do you notice?
   a. $100 \div 50$
   b. $10 \div 50$
   c. $27 \div 50$
   d. $132 \div 50$
4. Use the properties in this chapter to show that \( x \div 50 = 2x \div 100 \).

5. Discuss with your neighbor how \( x \div 50 = 2x \div 100 \) can be used to compute \( 1234 \div 50 \). Do you think this is easier than using long division?

6. Naveen is traveling in India to visit his grandparents. The exchange rate is 50.5 Indian Rupees to the dollar. He sees a shirt he wants to buy. The price is 332 rupees. Naveen wants to know if this is a good price. Use the trick you explored here to estimate how much the shirt costs in dollars.
EXERCISES

1. A rectangle is twice as long as it is wide.
   • Write expressions for the perimeter in as many equivalent ways as you can.
   • Write expressions for the area in as many equivalent ways as you can.

2. Use the distributive property as the first step in solving the following equations. Then solve the equation without using the distributive property.
   a. \(2(x - 3) = 4\)
   b. \(3(4 - y) = 6\)
   c. \(5(2z + 1) = 9\)
   d. \(2(5 - 3S) = 4\)

3. Solve each of these equations by transforming them into equivalent, simpler equations using the properties of equality developed in the previous section. Name the properties used.
   a. \(2A = 100\)
   b. \(b - 7 = 9\)
   c. \(6 + 5x = 21\)

4. Determine which of the following expressions are equivalent. Explain.
   a. \(4a - 12, 4(a - 3), 4(a - 8)\)
   b. \(5 \cdot \left(\frac{x}{5}\right), \frac{5x}{5}, \frac{5x}{5}\)
   c. \(8 \cdot \left(\frac{x}{8}\right), \frac{4x}{2}, 2x\)
   d. \(\frac{6a + 4}{2}, 3a + 2, 6a + 2\)
   e. \(-24b + 12, -4(6b - 3), -4(6b + 3)\)
5. Match each expression on the left to an equivalent expression on the right.

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<tr>
<td>1. ((2x + 3) + x + 7)</td>
<td>a. (x - 4)</td>
</tr>
<tr>
<td>2. ((2x + 3) - (x + 7))</td>
<td>b. (4x + 44)</td>
</tr>
<tr>
<td>3. (5(2x + 4) + 3(8 - 2x))</td>
<td>c. (3x - 3)</td>
</tr>
<tr>
<td>4. (\frac{1}{2}(2x + 6) + 3(3x + 4))</td>
<td>d. (10x + 15)</td>
</tr>
<tr>
<td>5. (\frac{2x+6}{2} + 2(x - 3))</td>
<td>e. (3x + 10)</td>
</tr>
</tbody>
</table>

6. Jack took a group of students on a river boat ride. He was charged the same amount for each student and $8 for himself. Let \(x\) be the number of students. If each student was charged $5 and the total bill was $73, how many students were there? Set up an equation in terms of \(x\) and solve.

7. Jill took a group of students on a tour of a museum. She was charged the same amount for each student and $6 for herself. Let \(c\) be the cost per student. If she took 22 students and the total bill was $61, what is cost per student?

8. Jay and Sally are baking cakes using the same recipe. The recipe calls for \(x\) cups of flour per regular size cake. Jay triples the recipe, drops the bowl and spills half of the batter. He then bakes the remaining batter in a big pan. Sally mixes half the ingredients from the recipe and bakes a smaller cake. She does this 3 times. Which cook uses the most flour in the cakes they bake? Explain your answer.

9. **Investigation:**

Suppose \(a\) and \(b\) are numbers with \(a < b\) as shown on the number line:

```
  a   b
```

Notice the position of 0 and 1 on the number are both unknown.

a. Write an expression for a number that is greater than \(b\).
b. Write an expression for a number greater than \( b \) that you can locate on the number line without knowing the length of the unit.

c. Find a number less than \( a \) that you can locate precisely on the number line without knowing the length of the unit.

d. Find a number between \( a \) and \( b \) that you can locate precisely on the number line without knowing the length of the unit or the distance between \( a \) and \( b \).

e. Find two more numbers between \( a \) and \( b \) that you can locate precisely.

10. Kiran is traveling in India. The exchange rate is 50.5 Rupees to the dollar. She wants to buy a painting for her house. The price is 42123 rupees. Use the method from 6 to estimate the price in dollars.

\[
\frac{2 \times 42123}{100} = \$842.46
\]

11. The Double-Half method is a number sense mathematics "trick". When multiplying two numbers, it is sometimes easier to first multiply the first number by 2 and divide the second by 2 and then find the product of the resulting numbers. For example, if you wanted to multiply 35 \( \times \) 16, you could find:

- \( 35 \times 2 = 70 \)
- \( 16 \div 2 = 8 \)
- \( 35 \times 16 = 70 \times 8 \)

a. Explain why this method makes it easier to compute the product in your head.

b. For what pairs of numbers do you think this method would work best?

c. Express this "trick" using variables in an equation.

12. Explore on the internet for other number sense mathematics tricks. Find two tricks that make sense to you. For each trick:

- Give an example of how it is used.
- Explain for what kinds of numbers the trick seems the most useful.
- Express the trick using variables in an equation.

13. Ingenuity:

Andrew uses 1 \( \times \) 1 \( \times \) 1 blocks to make larger cubes. For example, he uses 4 of the small blocks to make a 2 \( \times \) 2 \( \times \) 2 cube. As he makes larger and larger cubes, some of the small blocks are
hidden on the inside of the cube and some he can see on the surface of the cube.

a. How many of the small blocks does Andrew need to make a $3 \times 3 \times 3$ cube? How many of the blocks are hidden in the inside and how many are on the surface?

b. How many of the small blocks does Andrew need to make a $4 \times 4 \times 4$ cube? How many of the blocks are hidden in the inside and how many are on the surface?

c. How many of the small blocks does Andrew need to make a $n \times n \times n$ cube? How many of the blocks are hidden in the inside and how many are on the surface?

14. Ingenuity:
Joe invests $P$ dollars in a fund. At the end of the year, his investment is worth one dollar more than twice his original investment. This pattern is repeated for the next 3 years. How much does he have in his investment at the end of the fourth year? Write an expression for how much the account will be worth at the end of 7 years.
In Section 1.4 we used the properties of equality to create equivalent equations and to find the solution of an equation. Recall that a solution is a value of the variable that makes both sides of the equation equal. In Section 1.5 we used the properties of arithmetic to manipulate expressions to find equivalent expressions.

The phrases equivalent expression and equivalent equation sound so similar it’s easy to think they describe the exact same thing. But this is not true. Expressions are equivalent if they are equal for all possible values of the variable. Equations are equivalent if the set of solutions is the same.

EXPLORATION 1

1. Write two different expressions which are equivalent. Explain why they are equivalent.
2. Write two different equations which are equivalent. Explain why they are equivalent.

Now, we will combine the two ideas of equivalence to solve more complicated problems.

EXAMPLE 1

Solve the following equation, explaining each step and naming the property you use:

\[ 3x + 1 = x - 7. \]

**Solution** To solve for \( x \), we use the properties of equality and arithmetic to simplify things. The goal is to find an equivalent equation in which \( x \) is by itself on one side of the equation and the solution is on the other. Notice that the equation has \( x \) terms on both sides of the equal sign. So we begin by grouping the \( x \)
terms on one side of the equation. Use the subtraction property to subtract \(x\) first:

\[
3x + 1 = x - 7
\]

\[
3x + 1 - x = x - 7 - x
\]

\[
2x + 1 = -7.
\]

Now we have an equation of the form studied in Section 1.4. We still have number terms on both sides of the equal sign. Use the subtraction property to subtract 1:

\[
2x + 1 = -7
\]

\[
2x + 1 - 1 = -7 - 1
\]

\[
2x = -8.
\]

Finally, use the division property to divide by 2 to obtain the solution:

\[
2x = -8
\]

\[
\frac{2x}{2} = \frac{-8}{2}
\]

\[
x = -4.
\]

Note, we could have begun by grouping the number terms first and achieved the same result. Since we have 2 kinds of like terms (those with \(x\) and one with only numbers) and 2 sides of an equation, we can group the like terms separately on each side of the equation.

**PROBLEM 1**

Solve the following equations:

1. \(2(2x + 2) + 3(x + 2) = x - 2\).
2. \(3(x + 5) - (x - 3) = 2(x - 3)\).
3. \(-2(x - 3) + 4x = 5x - 7\).
In Section 1.4 we used the balance model to visualize solving the equation $4s + 2 = 14$. In the next example, we use this model on a more complicated equation.

**EXAMPLE 2**

Use the balance model to solve the equation $\frac{3}{2}x + 3 = \frac{1}{2}x + 4$.

**Solution**

**Balance Model**

- Subtract $3$ from both sides

**Algebraic Method**

\[
\frac{3}{2}x + 3 = \frac{1}{2}x + 4
\]

Subtract $3$ from both sides

\[
\frac{3}{2}x + 3 - 3 = \frac{1}{2}x + 4 - 3
\]

\[
\frac{3}{2}x = \frac{1}{2}x + 1
\]

Subtract $\frac{1}{2}x$ from both sides

\[
\frac{3}{2}x - \frac{1}{2}x = \frac{1}{2}x + \frac{1}{2}x + 1
\]

\[
x = 1
\]
Now we check our answer.

\[
\frac{3}{2} \cdot 1 + 3 = \frac{1}{2} \cdot 1 + 4 \\
\frac{3}{2} + 3 = \frac{1}{2} + 4 \\
\frac{3}{2} + \frac{6}{2} = \frac{1}{2} + \frac{8}{2} \\
\frac{9}{2} = \frac{9}{2}
\]

EXPLORATION 2

1. Consider the equation \(5(2x - 3) - 4x = 6x - 15\).
   a. Try to solve the equation. What happens?
   b. Substitute \(x = 1\) on both sides of the equation. What happens? Substitute \(x = 0\). Now what happens?
   c. Use the distributive property and combining like terms to show that \(5(2x - 3) - 4x\) is equivalent to \(6x - 15\). What does this say about which values of \(x\) make the equation true?

2. Consider the equation \(3(x - 2) + 2x = 5x + 4\).
   a. Try to solve the equation. What happens?
   b. Use the distributive property and combining like terms to show that \(3(x - 2) + 2x\) is equivalent to \(5x - 6\).
   c. Is there any value of \(x\) so that \(5x - 6\) is the same as \(5x + 4\)? Explain. What does this say about which values of \(x\) make the equation \(3(x - 2) + 2x = 5x + 4\) true?

In part 1 of the exploration above, you found that an equation can have more than one solution. In this case, we talk about the set of solutions or solution set. Since \(5(2x - 3) - 4x\) and \(6x - 15\) are equivalent, the solution set for \(5(2x - 3) - 4x = 6x - 15\) is the set of all numbers. In part, we saw that no value of \(x\) makes the equation \(3(x - 2) + 2x = 5x + 4\) true. So the equation has no solution and we say the solution set is empty.
EXAMPLE 3

Suppose $n$ is a number so that if you triple it and subtract 4 you obtain the same number as if you decrease it by 3 and then double the result. What is $n$?

**Solution** The problem describes two different procedures you can perform on the number $n$ that give the same result. First we need to write each procedure as an expression involving $n$.

Procedure One: Triple it and subtract 4: $3n - 4$

Procedure Two: Decrease it by 3 and double the result: $2(n - 3)$

Second, we set the the two expressions equal to one another. Then we solve for $n$.

\[
3n - 4 = 2(n - 3)
\]

\[
3n - 4 = 2n - 6
\]

\[
3n - 2n - 4 = 2n - 6 - 2n
\]

\[
n - 4 = -6
\]

\[
n - 4 + 4 = -6 + 4
\]

\[
n = -2
\]

It is a good idea to check our answer. $3 \cdot (-2) - 4 = -10$ and $2(-2 - 3) = 2(-5) = -10$. So yes it works.

PROBLEM 2

Montserrat has two job offers to deliver fliers around the neighborhood. The first offers to pay her $50 per week plus $0.10\frac{1}{2}$ cents per flier. The second will pay only $30 per week, but will give 20 cents per flier.

1. Set up an equation to find $x$ the number fliers she must deliver so that the two offers pay the same per week.
2. Solve for $x$. Which job would you take and why?
EXERCISES

1. In each of the following equations, the variable appears on both sides of the equal sign. Use the properties of equality to solve them for the unknown.
   a. \(4x = 2x + 3\)
   b. \(16z + 7 = 3z - 4\)
   c. \(2A - 1 = 9 - 3A\)
   d. \(y + 4 = 4y - 8\)
   e. \(3n - 4 = \frac{3}{11}n - 1\)

2. Use the distributive property as the first step in solving the following equations. Then solve the equation without using the distributive property.
   a. \(2(x - 3) = 4\)
   b. \(3(4 - y) = 6\)
   c. \(5(2z + 1) = 9\)
   d. \(2(5 - 3S) = 4\)

3. Solve the following linear equations:
   a. \(2(x - 3) = 4x + 7\)
   b. \(3x - 2(x - 4) = 5x - 13\)
   c. \(4(x + 3) = 2x + 2(x - 5)\)
   d. \(-3a + 7 = a - 5(3 - a)\)
   e. \(2a + 4.1 = 3(a - 1.3)\)
   f. \(a + \frac{1}{3} = 3a - \frac{1}{2}\)
   g. \(\frac{1}{3}(2c - 11) = c + 7\)
   h. \(9(x - \frac{3}{x}) = 5x\)
   i. \(3x - \frac{1}{2}(x - 1) = 2(x + 10)\)
   j. \(3x - 2(2x + 6) = 4(x + 1) - 5x - 16\)
   k. \(2.3(x + 7) = 4(2x - 3)\)
   l. \(17 = 3(x - 5) - 2(x + 1)\)
   m. \(11(x + 2) = 45\)
   n. \(9(x - 8) = 27\)
   o. \(\frac{2}{5}(2x + 3) = 7x - 1\)
   p. \(6x + 7 = 3(2x - 1)\)
   q. \(11 - 5(x + 6) = 3(x - 4)\)
   r. \(13x - [3(2x + 5)] = 6(x + 1)\)
   s. \(\frac{2}{3} = 5(x - 1)\)
   t. \(3(x + \frac{1}{2}) = 2(x - \frac{1}{2})\)
   u. \(5x - \frac{1}{2}(2x - 4) = 2x + 2(x + 1)\)
v. \(4.7(3x - 2) = 2.3 + 4(x + .6)\)
w. \(-6(1 - 3x) = 4(2x - 1)\)
x. \(2(-3a + 2) = (a + 2) - 4(a - 2)\)
y. \(33 - 10(c + \frac{3}{2}) = \frac{2}{3}(6c - 9) + 1\)
z. \(13x - 17 - (-2x + 1) = 37 - 3(x - 11)\)

4. Juan and Kate each plan to bake the same number of cookies. Juan tripled his planned number of cookies and then added 5 more. Kate decides to increase her planned number of cookies by 8 and then double this number of cookies. They then realized that they would still bake the same number of cookies. How many did they originally plan to bake?

5. Suppose \(n\) is a number so that if you take one third of it and add 12 you obtain the same number as if you double 1 and decrease it by 9. Write an equation and solve for \(n\).

6. **Investigation:**
   Recall two formulas for the circumference of a circle: \(C = 2\pi r\) and \(C = \pi d\).
   a. In your own words, write down the meaning of circumference.
   b. \(C\) represents the circumference. What do \(r\), \(\pi\) and \(d\) represent?
   c. In your house find 4 cylinder shaped objects. Good examples: cans of food or soda, rolls of toilet paper. For each object:
      i. Measure the diameter of the base (the circle on the bottom) in mm.
      ii. Measure the circumference. You can do this using a string or long sheet of toilet paper to wrap around the object. Or consider rolling the object on a piece of paper and marking when it completes a full circle.
      iii. Make a table of each pair of measurements.
7. Each cell phone text messaging plan charges a flat monthly fee for having text-messaging service plus an additional cost per text message.

<table>
<thead>
<tr>
<th>Plan</th>
<th>Service Cost per Month</th>
<th>Cost per Msg</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$15</td>
<td>$0.05</td>
</tr>
<tr>
<td>B</td>
<td>$10</td>
<td>$0.10</td>
</tr>
<tr>
<td>C</td>
<td>$25</td>
<td>unlimited texting</td>
</tr>
</tbody>
</table>

a. Set up an equation to determine how many text messages you must make so that plan A and plan B cost the same.

b. Set up an equation to determine how many text messages you must make so that plan A and plan C cost the same.

c. Solve the two equations.

d. Which plan would you choose and why?

8. Explain what it means for two equations to be equivalent. Give an example.

9. Explain what it means for two expressions to be equivalent. Give an example.
SECTION 1.7 FORMULAS AND LITERAL EQUATIONS

Mathematicians and scientists use formulas to describe relationships between different quantities. These formulas are often referred to as literal equations. Formulas usually involve two or more variables or letters and hence the term “literal” equation is used. If possible we choose the letter to remind us of the quantity we are describing. A familiar literal equation is the formula \( P = 2L + 2W \) that relates the perimeter of a rectangle to its length and width.

EXPLORATION 1

What formulas do you know? Which have you worked with in other grades? Which formulas have you used in other classes besides math? Remember to define what each variable in a formula means.

When you work with a rectangle, you are often asked to find its area, \( A = L \cdot W \) where the variable \( A \) represents the area of a rectangle, \( L \) the length of the rectangle and \( W \) its width. The perimeter \( P \) of a rectangle with length \( L \) and width \( W \) is given by the formula \( P = 2L + 2W \) or \( P = 2(L + W) \). In both examples, \( A \) and \( P \) are expressed in terms of \( L \) and \( W \). The formulas written in this form are particularly useful when we are looking for the area and perimeter. However, suppose you must find the length or the width. Luckily, because you are working with equations, you can use properties to create equivalent equations and rearrange the variables to solve for \( L \) or \( W \).

EXPLORATION 2

Use \( P = 2L + 2W \) and find equations equivalent to it. One that has \( W \) by itself on one side of the equal sign and the other that has \( L \) by itself on one side of the equal sign. Explain each of your steps.
PROBLEM 1

The perimeter of rectangle $A$ is 45 meters and its width is 18 meters. Rectangle $B$ has perimeter 78 meters and width 10 meters. Determine the lengths of each of the two rectangles. Try using the information from your exploration.

EXAMPLE 1

The area of a rectangle is 54 square meters. Its length is 3 meters. Determine its width.

Solution  One way to approach this problem is to use the formula for the area of a rectangle is $A = LW$ and substitute the known values, $A = 54$ and $L = 3$. The equation then becomes $54 = 3W$. What times 3 equals 54? From the definition of division, this is another way of asking what is 54 divided by 3. The width $W = 18$. An alternate way to solve the equation $54 = 3W$ is to multiply both sides by $\frac{1}{3}$. This means $\frac{1}{3} \cdot 54 = \frac{1}{3} \cdot 3W$. The equivalent equation is $\frac{54}{3} = W$ or $18 = W$.

Another approach to solving literal equations is to work with the given formula, $A = LW$ and first solve for the specified quantity in the problem. The problem is asking you to solve for the width, $W$. Now, because $A = LW$ is an equation, you can solve for $W$ by dividing both sides by $L$. The equivalent equation is $W = \frac{A}{L}$. Now if we substitute the values for $A$ and $L$ then $W = \frac{54}{3}$ or $W = 18$. Using either approach, your answer is still $W = 18$. When might there be an advantage of one approach over the other?
PROBLEM 2

The area of Cody’s triangle is 20 square inches. The length of its base is 8 inches. The area of Althea’s triangle is 15 square inches. The length of its base is 5 inches. What is the height of each of these triangles?

EXPLORATION 3

A box has a square base of length $x$. The height of the box is 3 times the length of the base. Write an expression for each of the following:

1. Volume of the box.
2. Surface area of the box.

Try sketching the box.

Now let’s explore the data you collected in Exercise 6 from Section 1.6.

EXPLORATION 4

Recall the formula for the circumference $C$ of a circle in terms of its radius $r$, $C = 2\pi r$ and in terms of its diameter $d$, $C = \pi d$.

1. What is the relationship between $d$ and $r$? Write an equation to represent this relationship.
2. Solve the equation $C = \pi d$ for $\pi$ in terms of $C$ and $d$.
3. For Exercise 6 from Section 1.6 you measured the circumference and diameter of the base of some cylinders you found in your house. Discuss with your neighbors why you chose the objects you did, how your measured the circumference and any difficulties that arose.
4. As a group record the measurements of the objects from around your house in a table. Using a calculator compute $\frac{C}{d}$ for each object. What should this equal?
5. Use the results from your table to estimate the value of $\pi$.

6. Press the $\pi$ key on your calculator. How does this number compare to your estimate above?

The next example explores the relationship between different circles.

**EXAMPLE 2**

If we examine a second circle with a radius twice the radius of the first, what is the relationship between the circumference of the second circle to the circumference of the first?

![Diagram of two circles with radii r and 2r]

**Solution**  Let $r =$ the radius of the first circle. Then the radius of the second circle = $2r$. The circumference of the first circle is $C_1 = 2\pi r$. The circumference of the second circle, $C_2 = 2\pi(2r) = 4\pi r = 2(2\pi r) = 2C_1$. Therefore, we conclude that if we double the radius of a circle, then the circumference doubles.

**PROBLEM 3**

What happens to the circumference of a circle if the radius of a circle is tripled? What happens to the circumference of a circle if the radius of a circle is quadrupled? Do you see a pattern?
PROBLEM 4

Consider a box with length $L$, width $W$ and height $h$. The formula for the volume is $V = LWH$.

1. If the volume is 20 cubic centimeters, what is the formula for the width $W$?
2. If the surface area is 60 square centimeters what is the formula for the height $H$?

In many cases, the most difficult part of using a literal formula is figuring out what each variable represents. This can especially be true in geometry where different names can be used to describe the same figure. For example, in this book we have referred to the length and width of a rectangle. So we write the formula for the area as $A = LW$. However, you could also compute the area as $A = bh$ where $b$ is the base and $h$ is the height. These formulas are really the same but the names and letters used to describe the rectangle are different. The table at the end of this section shows more formulas from geometry. Use these formulas for the next problem.

PROBLEM 5

For each of the following situations:

• determine the formula to use
• specify what each variable means
• compute the area or volume

1. A model of a square pyramid has base edges of 10 inches and height of 12 inches. The slant height is 13 inches. What is its lateral area?
2. What is the volume of a sphere with a 6cm radius?
3. What is the volume of a cone with a height of 15 inches and a radius of 4 inches for its circular base?
4. What is the volume of a cylinder with a height of 15 inches and a radius of 4 inches for its circular base?
Section 1.7  FORMULAS AND LITERAL EQUATIONS

### Formulas from Geometry

<table>
<thead>
<tr>
<th>CIRCUMFERENCE</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Circle</td>
<td>( C = 2\pi r ) ( C = \pi d )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>AREA</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td>( A = \frac{1}{2}bh )</td>
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<tr>
<td>Rectangle or Parallelogram</td>
<td>( A = bh )</td>
</tr>
<tr>
<td>Trapezoid</td>
<td>( A = \frac{1}{2}(b_1 + b_2)h )</td>
</tr>
<tr>
<td>Circle</td>
<td>( A = \pi r^2 )</td>
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</table>

<table>
<thead>
<tr>
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<th>Total</th>
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<td>( S = Ph )</td>
<td>( S = Ph + 2B )</td>
</tr>
<tr>
<td>Pyramid</td>
<td>( S = \frac{1}{2}Pl )</td>
<td>( S = \frac{1}{2}Pl + B )</td>
</tr>
<tr>
<td>Cylinder</td>
<td>( S = 2\pi rh )</td>
<td>( S = 2\pi rh + 2\pi r^2 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>VOLUME</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Prism or cylinder</td>
<td>( V = Bh )</td>
</tr>
<tr>
<td>Pyramid or cone</td>
<td>( V = \frac{1}{3}Bh )</td>
</tr>
<tr>
<td>Sphere</td>
<td>( V = \frac{4}{3}\pi r^3 )</td>
</tr>
</tbody>
</table>
EXERCISES

1. The area of a triangle is one half of its base times its height: \( A = \frac{1}{2}bh. \)
   a. Solve for the base \( b \) in terms of the area \( A \) and height \( h \).
   b. Juan wants to make a triangle whose area is 1 cm² and height is 1 cm, how long should the base be?

2. The area of a rectangle is its length times its width: \( A = lw. \)
   a. Solve for the length \( l \) in terms of the area \( A \) and width \( w \).
   b. Julie wants to make a rectangular garden. Since the garden is along the side of her house, she knows the width will be 5 feet. How long should she make the garden if she wants to have an area of 40 square feet?

3. The area of a kite is one half of the product of its diagonals: \( A = \frac{1}{2}d_1d_2. \) See the figure below.

   ![](diagram.png)

   a. Solve for the diagonal \( d_2 \) in terms of the area \( A \) and the other diagonal \( d_1 \).
   b. Nate wants to build a kite whose area is 1 square meter. He already has one stick (diagonal) that is 80 cm long. How long should the other stick be?

4. The volume of a rectangular prism (in other words a box) is the area of its base, \( B \) times its height \( h \): \( V = Bh. \) Of course the base is a rectangle, so the area of the base is the length times the width \( B = lw \)
   a. Solve for the height \( h \) in terms of the volume, and the area of the base.
   b. Hiro wants to build a swimming pool. Its base will a
rectangle with length \( l = 15 \) meters, width \( w = 4 \) meters. She wants the volume equal to be 120 cubic meters. What should the height of the pool be?

c. Jian wants to have a pool that holds twice as much water as Hiro’s, but with the same base. What should the height of Jian’s pool be?

d. Both Hiro and Jian need to paint the inside of the pool, that is the 4 inside walls and the bottom. How many times more paint does Jian need than Hiro?

5. The average of two numbers is their sum divided by two. \( A = \frac{m+n}{2} \). Copy and fill in the blanks in the following table.

<table>
<thead>
<tr>
<th>( m )</th>
<th>( n )</th>
<th>( a )</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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<td></td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>( x )</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>( y )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6. The temperature in Fahrenheit is 1.8 times the temperature in Celsius plus 32: \( F = 1.8C + 32 \).

a. Find \( F \) if \( C = 0 \), \( C = 100 \). (Note: \( C = 0 \) is the freezing temperature for water, and \( C = 100 \) is the boiling point for water).

b. Solve for the temperature in Celsius \( C \) in terms of the temperature in Fahrenheit \( F \).

c. Find the high temperature in your town today in Fahrenheit and in Celsius. \( C = \frac{(F - 32)}{1.8} \)

d. Ricardo lives in Chile where they use Celsius for temperature. Ricardo calls his friend Pete in Texas. When Pete asks Ricardo about the weather, Ricardo says, “It’s crazy! Today’s high temperature was twice as high as the low temperature.” Ricardo is thinking in degrees Celsius, do you think his statement would still be true if he thought about the temperatures in Fahrenheit? Explain.
7. One idea in formulas or literal equations is to use letters for the variables that remind you what each variable represents. The following are formulas from mathematics or science you may not be familiar with. For each one, after looking at the word description and the formula, label what each variable means.

a. The density of an object is its mass divided by its volume: 
   \[ D = \frac{m}{V} \]
   \( D \) = density, \( m \) = mass, and \( V \) = volume.

b. The electric current through a wire is the amount of electric charge that goes through the wire divided by the amount of time over which this charge is measured.
   \[ I = \frac{Q}{t} \]
   \( I \) = current, \( Q \) = electric charge, and \( t \) = time.

c. The surface area of a sphere is 4 times \( \pi \) times its radius squared:
   \[ S = 4\pi r^2 \]

d. The area of a trapezoid is one half its height times the sum of its 2 bases.
   \[ A = \frac{1}{2}h(b_1 + b_2) \]

8. Rewrite each formula as indicated.
   a. Solve \( D = \frac{m}{V} \) for \( V \).
   b. Solve \( I = \frac{Q}{t} \) for \( Q \).
   c. Solve \( A = \frac{1}{2}h(b_1 + b_2) \) for \( h \).

9. Come up with 3 literal formulas that describe quantities that you deal with in your life.

10. Ms. Foss asked her students to come up with the literal formula for the area of a triangle. Most of her students gave the answer of \( \frac{1}{2}bh \), where \( b \) represents the base of the triangle and \( h \) represents the height of the triangle. However, a few students gave the answer of \( \frac{1}{2}xy \), where \( x \) represents the base of the triangle and \( y \) represents the height, and one student gave the answer of \( \frac{1}{2}bh \), where \( b \) represents the height of the triangle and \( h \) represents the base of the triangle. Which students are right?

11. Alfredo increased the sides of a square by 3cm and the area increased by 189 square cm. What is the area of the bigger square?

12. A cone with an open circular base at the top is filled with water to \( \frac{1}{2} \) of its full height. What fraction of its full volume is filled with water?
13. Two pizzas fit side by side on a large circular tray. See the figure below. If each pizza has a circumference of 36 inches, write an equation that could be used to find \( d \), the diameter of the tray.

\[ d = 2 \left( \frac{36}{\pi} \right) \]

14. **Investigation:**
   We have seen that doubling the radius of a circle also doubles the circumference.
   a. What happens to the perimeter of a rectangle when we double the length and width? When we triple the length and width?
   b. What happens to the perimeter of a triangle when we double the length of its sides? When we triple the length of its sides?
   c. What do you think the effect on the perimeter will be if we increase the lengths of the sides by a factor of \( r \)?

15. **Investigation:**
   In this investigation we will investigate the relationship between increasing the radius or sides of figures and the change in its area.
   a. What happens to the area of a circle when we double its radius? When we triple its radius?
   b. What happens to the area of a rectangle when we double the length of its sides? When we triple the lengths of its sides?
   c. What happens to the area of a triangle when we double the length of its sides? When we triple the lengths of its sides?
   d. What do you think the effect on the area will be if we increase the lengths of the sides of a polygon by a factor of \( r \)?
16. **Investigation:**
The number \( \pi \) is very important in mathematics. Mathematicians have been studying it for four thousand years. Use the internet to research the following questions about the history of \( \pi \).

a. When was \( \pi \) first discovered? What was their estimate of its value?

b. Archimedes is the first mathematician to write about a method to calculate \( \pi \). What was his estimate?

c. \( \pi \) is an irrational number. What does that mean?

d. What is the best estimate of \( \pi \) known today? Why isn’t included in the book?

17. A cylindrical flower vase is filled with 200\( cm^3 \) of water. If the height of the vase is 10 cm and the radius of the base is 5\( cm \), what percentage of the total volume of the vase is filled with water?

18. **Ingenuity:**
*The Rope Around the Earth Puzzle:* Imagine a rope tied around the Earth’s equator. If we increased the length of the rope by exactly one meter, a gap between the rope and the Earth’s surface will form all the way around. How large is the gap that is formed?

19. **Ingenuity:**
Show how to derive the area of a trapezoid: \( A = \frac{1}{2}h(b_1 + b_2) \). The figure with the shaded trapezoid may be helpful.
20. **Investigation:**

A company produces hollow 3-D shapes to use in schools. One packet of shapes has three shapes all with the same circular base with radius of 5 cm. The three shapes are:

- a cylinder with a height of 5 cm
- a cone with a height of 5 cm
- hemisphere (half of a sphere)

Imagine filling each of the objects with water.

a. Which of the objects has the largest volume? Which of the objects has the smallest volume?

b. If we fill the cone with water and then use this water to try to fill the cylinder, what fraction of the cylinder will be full?

c. If we fill the cone with water again, but now use this water to fill the hemisphere, what fraction of the hemisphere will be full?
## SECTION 1.8 CHAPTER REVIEW

### Key Terms
- algebraic expression
- radius
- algebraic method
- rational numbers
- circumference
- reciprocal
- constants
- rectangle
- equivalent equations
- set notation
- equivalent expressions
- sets
- integers
- solution set
- irrational numbers
- subset
- multiplicative inverse
- upper and lower bounds
- natural numbers
- variable
- perimeter
- whole numbers

### Properties and Theorems

<table>
<thead>
<tr>
<th>Additive Inverse Property:</th>
<th>Multiplicative Inverse:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n + (-n) = 0 )</td>
<td>( n \cdot \frac{1}{n} = 1 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Addition Property of Equality:</th>
<th>Subtraction Property of Equality:</th>
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<tr>
<td>If ( a = b ), ( a + c = b + c )</td>
<td>If ( a = b ), ( a - c = b - c )</td>
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</table>

<table>
<thead>
<tr>
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<th>Division Property of Equality:</th>
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<tbody>
<tr>
<td>If ( a = b ), ( ac = bc )</td>
<td>If ( a = b ), ( c \neq 0 ), ( \frac{a}{c} = \frac{b}{c} )</td>
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<table>
<thead>
<tr>
<th>Associative Property of Addition:</th>
<th>Commutative Property of Addition:</th>
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</thead>
<tbody>
<tr>
<td>( (x + y) + z = x + (y + z) )</td>
<td>( x + y = y + x )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Associative Property of Multiplication:</th>
<th>Commutative Prop. of Multiplication:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (xy)z = x(yz) )</td>
<td>( xy = yx )</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Distributive Property:</th>
<th>Double Opposite Theorem</th>
</tr>
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<tbody>
<tr>
<td>( x(y + z) = xy + yz )</td>
<td>( -(-n) = n )</td>
</tr>
</tbody>
</table>

### Formulas
- Distance = Rate \( \cdot \) Time
- Circumference = \( 2\pi r \)
Practice Problems

1. Compute:
   a. \(34 + (-6)\)  
   b. \(-21 + (-19)\)  
   c. \(\frac{2}{3} + \frac{1}{5}\)  
   d. \(5\frac{1}{4} - 2\frac{2}{3}\)  
   e. \(-11 \cdot (-6)\)  
   f. \(3\frac{2}{3} \cdot 4\frac{1}{6}\)  
   g. \(-7 \div \frac{3}{5}\)  
   h. \(-2\frac{1}{2} \div 1\frac{1}{3}\)

2. Evaluate:
   a. \(-6 + 3(4)\)  
   b. \(10 - (-3)(1)\)  
   c. \(3 - (-4)(-6)\)  
   d. \(-13 + (-6 - 3)\)

3. Compute the distance between each of the following pairs of numbers:
   a. \(-3\) and \(5\)  
   b. \(2\) and \(12\)  
   c. \(-\frac{1}{2}\) and \(7\)  
   d. \(5\) and \(x\)

4. Plot a point that represents each expression and label it:
   \(\frac{1}{2}(a + b),\ \frac{1}{2}a + \frac{1}{2}b.\)

5. Consider the number line below.

   \[\text{2x} \quad \text{6x} \quad 8x + 3 \quad 19\]

   a. Given the location of the two expressions find the locations of \(x\) and \(0\).
   b. Use the number line above to solve \(8x + 3 = 19\).

6. Write the following as algebraic expressions:
   a. \(x\) is 3 units more than \(y\).
   b. \(x\) is 4 times greater than \(y\).
   c. \(x\) is 6 units less than \(y\).
   d. \(x\) is three-fourths of \(y\).
   e. \(x\) is \(y\) units more than \(z\).

7. Max is \(M\) years old now. How old will he be in 10 years? Write an expression that models this.

8. A triangle has a base of length \(x\) cm.
   a. When the height of the triangle is 20 cm, write an expression for the area of the triangle.
Chapter 1  Variables, Expressions and Equations

b. When the height is 4 units longer than the base, write an expression for the area of the triangle.

9. The high temperature in San Antonio, Tx on July 9 was 107°F and the low was 88°F. Write the temperatures for July 9 in set notation and graph this set on the number line.

10. Solve on the number line.
   a. $3x = 24$
   b. $2x = 12$
   c. $8x = 32$
   d. $12x = 30$

11. Write an equation that models the problem and solve.
   a. A rectangular field has a length that is twice as long as its width. If the perimeter of the field is 326 m, what is the length of the field?
   b. The sum of three consecutive integers is 66. What are the integers?
   c. Eric has $30. He promised to give Xavier $10 and give one-fourth of the left over money to Anahi. How much money does Anahi get?

12. Solve each equation. Tell which property of equality you use in each step.
   a. $y + 4 = 6y - 8$
   b. $3x - 8 = x + 6$
   c. $3y - 2 = -y + 3$
   d. $4y + 1 = 5 - 3y$

13. Solve each equation. Tell which property of equality you use in each step.
   a. $18(x - \frac{2}{3}) = 5x$
   b. $13x - [3(4x - 2)] = 6(x - 1)$
   c. $33 - 10(c + \frac{2}{5}) = \frac{4}{3}(9c - 6)$

14. Suppose $n$ is a number such that if you triple it and subtract 1, you get the same as if you double the sum of $n$ and 2. What is $n$?

15. The volume of a rectangular pyramid is $V = \frac{1}{3}Bh$, where $B$ is the area of the base and $h$ is the height of the pyramid.
   a. Solve the equation for the height.
   b. If the volume of a rectangular pyramid is 24 $cm^3$ and the area of the base is 3 $cm^2$, what is the height?

16. The density of an object is its mass divided by its volume.
   a. Write an expression for the density of an object.
   b. If the objects density is 100 $g$ per cubic $cm$, and we have
50 g of the object, what is its volume?

17. A swimming pool has the shape of a rectangular prism. The volume of a rectangular prism is \( V = Bh \) where \( B \) is the area of the base.

   a. Suppose the length of the pool is 25 m and the width is 10 m. If we want the volume of the water in the pool to be 750 \( m^3 \), how deep should the pool be?

   b. Suppose we want to double the volume of the pool but leave the base alone. How tall should the walls of the pool be now?