Key HW 9_2

1. Work problem 9.5 in the text. Note that the previous key I posted for this problem was only correct up to amplitude relations following from the boundary conditions. This key is correct all the way through.

Let us denote the total wave by $f(z,t)$. We have

$$f(z,t) = \begin{cases} g_I(z - v_1 t) + h_R(z + v_1 t) \ , & z < 0 \ , \\ g_T(z - v_2 t) \ , & z > 0 \ . \end{cases}$$

The boundary condition $[f] = 0$ at $z = 0$ (where the square brackets denote the jump in the quantity evaluated across $z = 0$) becomes

$$g_I(-v_1 t) + h_R(v_1 t) = g_T(-v_2 t) \tag{1}$$

while the boundary condition $[\partial f/\partial z] = 0$ gives

$$g_I'(-v_1 t) + h_R'(v_1 t) = g_T'(-v_2 t) \ , \tag{2}$$

where a prime denotes differentiation with respect to the argument, i.e. $g'(u) = dg/du$. Now, we use a trick, noting that

$$g_I'(-v_1 t) = -\frac{1}{v_1}\frac{d}{dt} g_I(-v_1 t)$$

and similarly for $h_R'(v_1 t)$ and $g_T'(-v_2 t)$. (This follows for $u = -v_1 t$ because $du = -v_1 dt$.) Equation (2) therefore can be written

$$-\frac{1}{v_1}\frac{d}{dt} g_I(-v_1 t) + \frac{1}{v_1}\frac{d}{dt} h_R(v_1 t) = -\frac{1}{v_2}\frac{d}{dt} g_T(-v_2 t) \ . \tag{3}$$

This equation can be integrated to give

$$-\frac{1}{v_1} g_I(-v_1 t) + \frac{1}{v_1} h_R(v_1 t) = -\frac{1}{v_2} g_T(-v_2 t) + C \ , \tag{4}$$

where C is an arbitrary constant. Combining equations (1) and (4), we obtain

$$h_R(v_1 t) = \frac{v_2 - v_1}{v_1 + v_2} g_I(-v_1 t) + C' \ , \quad g_T(-v_2 t) = \frac{2v_2}{v_1 + v_2} g_I(-v_1 t) + C' \ , \tag{5}$$

where $C' = C v_1 v_2/(v_1 + v_2)$.

Consider now the result for $h_R(v_1 t)$. Let $t = t_0 + (z/v_1)$. From equation (5) it follows

$$h_R(z + v_1 t_0) = \frac{v_2 - v_1}{v_1 + v_2} g_I(-z - v_1 t_0) + C' \ .$$

To evaluate g_T, let $t = t_0 - (z/v_2)$. Substituting into the second of equations (5), we get

$$g_T(z - v_2 t_0) = \frac{2v_2}{v_1 + v_2} g_I\left(\frac{v_1}{v_2}(z - v_2 t_0)\right) + C' \ .$$

Thus, up to an additive constant (which must be determined by initial conditions),

$$h_R(u) = \frac{v_2 - v_1}{v_1 + v_2} g_I(-u) \ , \quad g_T(u) = \frac{2v_2}{v_1 + v_2} g_I(u v_1/v_2) \ .$$

This result can also be obtained using equations (9.25) and (9.30) of Griffiths by representing an arbitrary function $g_I(u)$ by a Fourier integral (linear superposition of waves).

Note that the substitution in the latter part of the key can also be understood from the following reasoning:

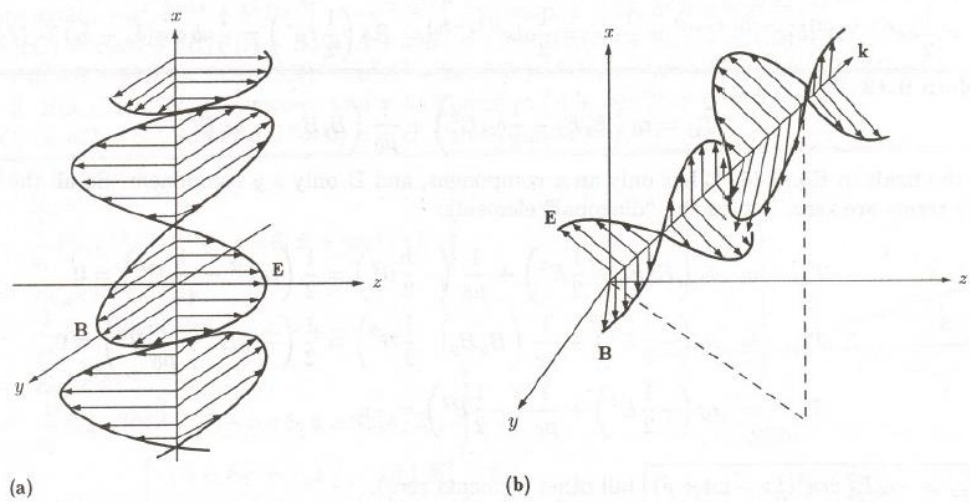
Instead of looking to the displacement at z=0 of the transmitted wave, I could look at the displacement at z=a where a is a positive value in the propagation direction of the transmitted wave. This value should correspond to the displacement of the transmitted wave at z=0 at an earlier moment in time. How much earlier, i.e. z/v_2. So I will have to find the displacement of the incident wave at $t-z/v_2$, so replace t in $g_I(-v_1t)$ with $t-z/v_2$ which results in $g_I(v_1(t-z/v_2))=g_I(-v_1t+zv_1/v_2)= g_I(zv_1/v_2-v_1t)$

2. Work problem 9.9 in the text.

Problem 9.9

(a) $\boxed{\mathbf{k} = -\frac{\omega}{c}\hat{\mathbf{x}};\ \hat{\mathbf{n}} = \hat{\mathbf{z}}.}$ $\mathbf{k}\cdot\mathbf{r} = \left(-\frac{\omega}{c}\hat{\mathbf{x}}\right)\cdot(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) = -\frac{\omega}{c}x;\ \mathbf{k}\times\hat{\mathbf{n}} = -\hat{\mathbf{x}}\times\hat{\mathbf{z}} = \hat{\mathbf{y}}.$

$$\boxed{\mathbf{E}(x,t) = E_0\cos\left(\frac{\omega}{c}x + \omega t\right)\hat{\mathbf{z}};\quad \mathbf{B}(x,t) = \frac{E_0}{c}\cos\left(\frac{\omega}{c}x + \omega t\right)\hat{\mathbf{y}}.}$$



(b) $\boxed{\mathbf{k} = \frac{\omega}{c}\left(\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}}{\sqrt{3}}\right);\ \hat{\mathbf{n}} = \frac{\hat{\mathbf{x}}-\hat{\mathbf{z}}}{\sqrt{2}}.}$ (Since $\hat{\mathbf{n}}$ is parallel to the $x\,z$ plane, it must have the form $\alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{z}}$; since $\hat{\mathbf{n}}\cdot\mathbf{k} = 0, \beta = -\alpha$; and since it is a unit vector, $\alpha = 1/\sqrt{2}$.)

$$\mathbf{k}\cdot\mathbf{r} = \frac{\omega}{\sqrt{3}c}(\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}})\cdot(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}) = \frac{\omega}{\sqrt{3}c}(x+y+z);\ \hat{\mathbf{k}}\times\hat{\mathbf{n}} = \frac{1}{\sqrt{6}}\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 1 & 1 & 1\\ 1 & 0 & -1\end{vmatrix} = \frac{1}{\sqrt{6}}(-\hat{\mathbf{x}}+2\hat{\mathbf{y}}-\hat{\mathbf{z}}).$$

$$\boxed{\begin{aligned}\mathbf{E}(x,y,z,t) &= E_0\cos\left[\frac{\omega}{\sqrt{3}c}(x+y+z) - \omega t\right]\left(\frac{\hat{\mathbf{x}}-\hat{\mathbf{z}}}{\sqrt{2}}\right);\\ \mathbf{B}(x,y,z,t) &= \frac{E_0}{c}\cos\left[\frac{\omega}{\sqrt{3}c}(x+y+z) - \omega t\right]\left(\frac{-\hat{\mathbf{x}}+2\hat{\mathbf{y}}-\hat{\mathbf{z}}}{\sqrt{6}}\right).\end{aligned}}$$

3. Work problem 9.10 in the text.

Problem 9.10

$P = \frac{I}{c} = \frac{1.3 \times 10^3}{3.0 \times 10^8} = \boxed{4.3 \times 10^{-6}\ \mathrm{N/m^2}.}$ For a perfect reflector the pressure is twice as great: $\boxed{8.6 \times 10^{-6}\ \mathrm{N/m^2}.}$ Atmospheric pressure is $1.03 \times 10^5\ \mathrm{N/m^2}$, so the pressure of light on a reflector is $(8.6 \times 10^{-6})/(1.03 \times 10^5) = \boxed{8.3 \times 10^{-11}\ \text{atmospheres.}}$

4. Work problem 9.11 in the text.
   a. Use Newton's 2nd law to determine the acceleration from the applied force, i.e.

$$a = \frac{F}{m} = \frac{qE}{m} = \frac{qE_o \cos(kz - \omega t)}{m}$$

Now I use the relation between acceleration and velocity, i.e.

$$a = \frac{dv}{dt} \Leftrightarrow dv = adt \Leftrightarrow v = \int adt = \int \frac{qE_o}{m}\cos(kz - \omega t)dt = -\frac{qE_o}{\omega m}\sin(kz - \omega t)$$

   b. I calculate the Lorentz force from:

$$F_{lor} = qvB = -\frac{q^2 E_o}{\omega m}\sin(kz - \omega t)B$$

Since this is an EM plane wave in vacuum there is a relation between E and B given by equation 9.47. So:

$$F_{lor} = -\frac{q^2 E_o}{\omega m}\sin(kz - \omega t)\frac{E}{c} = -\frac{q^2 E_o^2}{\omega mc}\sin(kz - \omega t)\cos(kz - \omega t) = -\frac{q^2 E_o^2}{2\omega mc}\sin(2kz - 2\omega t)$$

c. Integrating over a period gives:

$$\langle F_{lor} \rangle = \frac{1}{T}\int_0^T -\frac{q^2 E_o^2}{2\omega mc}\sin\left(2kz - 2\frac{2\pi}{T}t\right)dt$$

And without the math I can say that this function is periodic with period T/2 and it is a sine so I believe that this is zero.

d. We skip d.