Let’s begin by thinking carefully about numbers. Numbers are part of the mathematical alphabet, just like letters are used in English to form words. We use numbers for counting and representing quantities. When we think of the number *one*, we have in mind a picture:

Similarly, the number *two* describes a different quantity:

We could use a picture with dots to describe the number two. For instance, we could draw:

We call this way of thinking of numbers the “set model.” There are, however, other ways of representing numbers.
Another way to represent numbers is to describe locations with the **number line model**, which is visually similar to a thermometer. To construct a number line, begin by drawing a straight line and picking some point on the line. We call this point the **origin**. Label the origin with the number 0. We can think of 0 as the address of a certain location on the number line. Notice that the line continues in both directions without ending. We show this with arrows at the ends of the line.

Next, mark off some distance to the right of the origin and label the second point with the number 1.

Continue marking off points the same distance apart as above and label these points with the numbers 2, 3, 4, and so on.

The points you have constructed on your number line lead us to our first definition.

<table>
<thead>
<tr>
<th><strong>DEFINITION 1.1: COUNTING NUMBERS (POSITIVE INTEGERS)</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>The <strong>counting numbers</strong> are the numbers in the following never-ending sequence:</td>
</tr>
<tr>
<td>1, 2, 3, 4, 5, 6, 7, …</td>
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<tr>
<td>We can also write this as</td>
</tr>
<tr>
<td>+1, +2, +3, +4, +5, +6, +7, …</td>
</tr>
<tr>
<td>These numbers are also called the <strong>positive integers</strong> or <strong>natural numbers</strong>.</td>
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</table>
One interesting property of the natural numbers is that there are “infinitely many” of them; that is, if we write down a list of natural numbers, there is always some natural number that is not on the list.

When we include the number 0, we have a different collection of numbers that we call the \textbf{whole numbers}.

\begin{definition}
\textbf{Whole Numbers (Non-Negative Integers)}
The \textbf{whole numbers} are the numbers in the following never-ending sequence:
\[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\]
These numbers are also called the \textbf{non-negative integers}.
\end{definition}

In order to label points to the left of the origin, we use \textbf{negative integers}: 
\[-1, -2, -3, -4, \ldots\] The sign in front of the number tells us on which side of zero the number is located. Positive numbers are to the right of zero; negative numbers are to the left of zero. \textit{Zero is not considered to be either positive or negative.}

So we have seen that numbers can be used in different ways. They can help us to describe the quantity of objects using the set model or to denote a location using the number line model. Notice that the number representing a location also can tell us the distance the number is from the origin if we ignore the sign.

\begin{definition}
\textbf{Integers}
The collection of \textbf{integers} is composed of the negative integers, zero, and the positive integers:
\[\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\]
\end{definition}
Definition 1.3 leads to the **trichotomy** property, which states that there are exactly three possibilities for an integer: positive, negative, or zero.

Such collections of numbers are often called sets of numbers. For example, assume $S$ is the set $\{1, 2, 3, 4, 5\}$ and $T$ is the set $\{2, 3, 4\}$. Notice that every number in $T$ is also an element of the set $S$. That means that $T$ is a subset of $S$. This relationship is represented by the following diagram:

![Diagram](https://via.placeholder.com/150)

**EXAMPLE 1**

Call the set of positive integers $P$, the set of whole numbers $W$, and the set of integers $Z$. Draw a diagram of the relationship of these three sets.

**SOLUTION**

$P = \{1, 2, 3, \ldots\}$

$W = \{0, 1, 2, \ldots\}$

$Z = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$
**EXPLORATION: CONSTRUCTING A NUMBER LINE**

1. Draw a straight line.
2. Pick a point on the line and call this point the origin. Label the origin with the number 0.
3. Locate the numbers 1, 2, 3, …, 10, and –1, –2, –3, …, –10.
4. Where would 20, 30, 50 be located? 100? 1000?
5. Find the negative numbers corresponding to the numbers in question 4.

**EXERCISES**

1. The post office is located at the origin of Main Street. We label its address as 0. The laboratory has address 6 and the zoo has address 9. Going in the other direction from the origin, we find a candy shop with address –4 and a space observatory with address –7. Draw a number line representing Main Street. Label each of the above locations on the number line. Watch your spacing.

2. a. Copy the line below to mark off and label the integers from 0 to 10 and from 0 to –10. Use a pencil to experiment because you might need to erase. Watch your spacing.

   ![Number Line Diagram]

   -10    0    10

b. Make a number line from –20 to 20.

3. Draw a section of the number line containing the number 77. Mark the number 77 on your line. Now label your number line with the first few integers to the right of 77 and the last few to the left of 77, at least three each way.

4. Do the same thing you did in the previous exercise, but this time start with the number –77 instead of 77.
5. Use a line like the one below to mark off the numbers with equal distances by tens from 0 to 100 and from 0 to -100. Use a pencil to experiment.

![Number Line]

-100 0 100

a. What is the distance from 0 to 50 and from 50 to 100? Are they the same?
b. What is the distance from 10 to 20, 30 to 40, and 70 to 80? Are they the same?
c. Explain whether you need to rework your markings on the number line.
d. Estimate the location of the following numbers and label each on your number line:
   - 15, 25, 55, -15, -25, -75, 34, 31, -34, -31, 87

6. Draw a number line so that the number -1000 is at the left end and 1000 is at the right end. Estimate the locations of the following integers:
   - 500, -500, 250, -100, -800, 10, -990, 342, -781, 203, -407

7. Draw a number line. Find all the integers on your number line that are greater than 15 and less than 20. Color each of these locations blue.

8. Given two numbers on the number line, how do you decide which number is greater?

   Notice that we can move the number line from the horizontal position to a vertical position. We would then have a number line that looks like a thermometer. Draw a thermometer (vertical number line) on the side of your paper to help you answer questions 9 through 12.

9. The chart below shows the monthly average temperatures for the city of Oslo, Norway. Based on the data, put the twelve months in order from coldest to warmest.
10. Chris visits Edmonton, Canada where it is \(-7\, ^\circ\text{C}\). Carmen visits Winnipeg, Canada where it is \(9\, ^\circ\text{C}\). Which temperature is closer to the freezing point? Explain. Remember, when we measure temperature in degrees Celsius (°C), \(0\, ^\circ\text{C}\) is the freezing point of water.

11. The temperature in Toronto, Canada, one cold day, is \(-10\, ^\circ\text{C}\). The next day the temperature is \(4\, ^\circ\text{C}\). Which temperature is closer to the freezing point?

12. The temperature in Fargo, North Dakota is \(-15\, ^\circ\text{F}\) while it is \(-20\, ^\circ\text{F}\) in St. Paul, Minnesota. Which temperature is colder? How much colder?

13. **Ingenuity:**
   
   On a cold winter day in Iowa, the temperature at 6:00 P.M. is \(10\, ^\circ\text{F}\). If the temperature decreases an average of \(4\, ^\circ\text{F}\) for each of the next five hours, what will the temperature be at 11:00 P.M.?
14. **Investigation:**

Use the number line below as a thermometer with the Celsius scale above the line and the Fahrenheit scale below the line to discover how the two scales are related. Write the temperature above and below the number line.

![Number line diagram with Celsius and Fahrenheit scales]

a. At what temperature does water boil on each scale?
b. At what temperature does water freeze on each scale?
c. What is the Fahrenheit reading for 50 °C?
d. Is the Celsius reading for 25 °F a positive or negative number?
e. A nice day is 77 °F. What is this temperature in Celsius?
f. A hot day is 100 °F. Estimate this temperature in Celsius.
We say that 2 is less than 5 because 2 is to the left of 5 on the number line. “Less than” means “to the left of” when comparing numbers on the number line. We use the symbol “<” to mean “less than.” We write “2 is less than 5” as “2 < 5.” Some people like the “less than” symbol because it keeps the numbers in the same order as they appear on the number line.

We also say that 5 is greater than 2 because 5 is to the right of 2 on the number line. “Greater than” means “to the right of” when comparing numbers on the number line. We use the symbol “>” to mean greater than, so we write “5 is greater than 2” as “5 > 2.”

Here \( x \) and \( y \) are called variables, which we will formally introduce in Chapter 3. A variable is a letter or symbol that represents an unknown quantity. Variables give us a convenient way to describe properties and ideas because variables can represent many values. Variables give us a simple way to describe math objects and concepts. In this case, \( x \) and \( y \) represent two integers, and the way that we tell which is greater is to compare their positions on the number line. The two number lines below demonstrate the cases \( x < y \) and \( x > y \). Can you tell which is which?

Mathematicians use the symbols “\( \leq \)” and “\( \geq \)” to mean “less than or equal to” and “greater than or equal to.” We can write “2 ≤ 5” because 2 is less than 5, but we can also write “5 ≤ 5” because 5 is equal to 5.
EXAMPLE 1

For each pair of integers below, determine which one is greater and which one is less. Express your answer as an inequality of the form $x < y$ or $x > y$, where $x$ and $y$ are the given integers.

a. $3$ ______ $7$
   
   b. $-2$ ______ $9$

SOLUTION

a. Begin by drawing a number line from $-10$ to $10$. Using this number line, we see that $3$ is to the left of $7$, so $3 < 7$.

b. We observe that $-2$ is to the left of $9$ on the number line, so $-2 < 9$. We can also see this in a different way: We know that $-2$ is to the left of $0$, because $-2$ is negative, and $0$ is to the left of $9$, because $9$ is positive. Thus $-2$ must be to the left of $9$, and we have $-2 < 9$.

c. We notice that $-5$ is to the left of $-1$, so $-5 < -1$ or $-1 > -5$.

d. Because $-4$ is to the left of $0$, and $0$ is to the left of $4$, we have $-4 < 4$ or alternatively $4 > -4$.

EXAMPLE 2

Put the following integers in order from least to greatest:

$2, -2, 7, -1, -4, -5, 4, 6, 3$

SOLUTION

Again, we can use the number line to help us put the integers in order:

![Number Line Diagram]
We can locate our nine given integers on the number line. You might try doing this by copying the diagram above and labeling the given numbers on your diagram. After comparing the nine numbers given, we get the following order:

\[-5, -4, -2, -1, 2, 3, 4, 6, 7\]

We can write this list using the < symbol as:

\[-5 < -4 < -2 < -1 < 2 < 3 < 4 < 6 < 7\]

**EXERCISES**

1. Rewrite each of the following as a statement using <, >, ≤, or ≥. Compare your statements to the relative locations of the two numbers on the number line.

   Example: -3 is less than 8. \(-3 < 8\).

   a. 9 is greater than 6.  
   b. 4 is less than 7.  
   c. -3 is greater than -5.  
   d. -4 is less than 1.  
   e. 8 is greater than or equal to 0.  
   f. -6 is less than or equal to 2.

2. Compare the numbers below and decide which symbol, < or >, to use between the numbers. Make a number line to show the relationship.

   a. 3 □ 4  
   d. -2 □ -3  
   b. 0 □ 1  
   e. -8 □ 9  
   c. -2 □ 0  
   f. 6 □ -5  

3. Compare the numbers below and decide which symbol, < or >, to use. Use a number line to show the relationship of these numbers.

   a. 5 □ 3  
   d. -15 □ -13  
   b. -10 □ -4  
   e. -9 □ -7  
   c. -12 □ 0  
   f. -21 □ 21

4. Describe any patterns you see in Exercises 2 and 3.
5. Compare the numbers below and decide which symbol, < or >, to use. Use your rules from Exercise 4 to help you.
   a. 6 2
d. 0 14
   b. 13 -11
e. 12 28
   c. -9 3

6. List the following integers in order using a number line.
   -62, -75, 26, 83, -59

7. What are the possible values for an integer that is greater than 3 and less than 7? Mark these possible values on a number line.

8. Determine whether each of the following statements is true or false. Explain your answers.
   a. If an integer is greater than 5, then it is greater than -5.
   b. If an integer is less than 5, then it is less than -5.

9. In Madison, Wisconsin, the morning temperature is -2 °C. In the evening the temperature reads -6 °C. Did the temperature rise or fall? How much did it change?

10. Albert is on a flight of stairs 87 steps above the ground floor. Elaine has gone into the sub-basement 78 steps down from ground level (let’s call it the -78th step). Who is farther from ground level? Why?

11. What are the possible values for an integer that is closer to 5 than it is to -2? Mark these possible values on the number line.

12. Earlier we introduced “greater than or equal to” and “less than or equal to.” Write a formal definition, using definition 1.4 as a model, for these two concepts.

13. **Ingenuity:**
   Suppose that the tens digit of a whole number between 80 and 90 is greater than the ones digit, but less than twice the ones digit. If the integer is even, what is its value?
14. **Investigation:**

Make a large timeline from the year 2000 BC to the year 2100 AD. Research the years that mark the following periods on this timeline.

a. The life of William Shakespeare
b. The U.S. Civil War
c. The Mayan civilization
d. The Roman Empire
e. People driving cars
f. The United States has been a country
g. Texas was a state
h. Texas was a country
SECTION 1.3 DISTANCE BETWEEN POINTS

We locate the numbers 10 and -10 on the number line.

<table>
<thead>
<tr>
<th>Ten Units Left of Zero</th>
<th>Ten Units Right of Zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>0</td>
</tr>
</tbody>
</table>

Notice that 10 and -10 are each 10 units from 0. We have a special name for the distance of a number from 0: the absolute value of the number.

In mathematics, we have a special symbol to represent absolute value. For example, we write |10| and read it as “absolute value of 10.” We write |-10| and read it as “absolute value of -10.” Because 10 and -10 are both 10 units from 0 we have the following:

The absolute value of 10 equals 10 or |10| = 10.
The absolute value of -10 equals 10 or |-10| = 10.

The absolute value gives us a measure of the size or the magnitude of a number, or its distance from the origin. The positive or negative sign tells us the direction of the number relative to 0. Because 10 and -10 are the same distance from 0, they have the same absolute value. In other words, -10 < 10 but |-10| = |10|.

Note: The absolute value symbol ‘| |’ should not be confused with parentheses ‘()’.

EXPLORATION

Using the number line that you have constructed, find the distance between each pair of numbers:

a. 2 and 8  d. 5 and -3  g. 9 and 0
b. -4 and -1  e. 0 and 6  h. 0 and 9
c. -4 and 1  f. 0 and -6  i. 0 and -9

In addition to the previous examples, you may also see -|10| which is read as “the negative absolute value of 10” or -|-10| which is read as “the negative absolute value of -10.” Since |10| = 10 and |-10| = 10, then we have

-|10| = -10 and -|-10| = -10.
EXERCISES

1. Find the absolute values of the following numbers.
   a. -7  d. 10  g. -21
   b. 8  e. -10  h. -42
   c. -8  f. 19  i. 33

2. Calculate the following:
   a. |-23|  b. (15)  c. |0|  d. |-17|  e. |-34|

3. What is the absolute value of the absolute value of -44?

4. Find the distance between each number and zero:
   a. 9  b. 0  c. -9

5. For each pair of numbers below, place the correct symbol <, >, or =.
   a. -7 |5|  f. |-6| -8
   b. 7 |-5|  g. |-6| 8
   c. 9 -9  h. 6 |-8|
   d. 9 |-9|  i. 25 |28|
   e. 32 |-47|  j. |-28| 25

6. Find the distance between 5 and 8. Did you use the number line? Can you use absolute values?

7. For each pair of integers given below, find the distance between the two integers on the number line.
   a. 2 and 3  b. 4 and 9  c. 25 and 17  d. -12 and 12
   2 and -3  -4 and 9  17 and 25  12 and 12
   -2 and 3  -4 and -9  -17 and 25
   -2 and -3  4 and -9
Section 1.3 Distance Between Points,

8. For each pair of integers given below, find the distance between the two integers on the number line.
   a. 40 and 70  b. 30 and 20  c. -80 and -40  d. -126 and -64
   3 and 8  -2 and -5  -7 and -9
   43 and 78  28 and 15  -87 and -49

9. A number is a distance of 5 from 13. Is it possible to determine a single value for the number? Explain using a number line.

10. A negative number is a distance of 105 from 52. Is it possible to determine the value of the number? Explain.

11. a. Find numbers that are a distance of 5 from 2.
    b. Find numbers that are a distance of 4 from -3.
    c. Find numbers that are a distance of 3 from -8.
    d. If x represents some integer, what numbers are a distance of 5 from x?
    e. What numbers are a distance of x from 2?

12. If x is the same distance from 6 as it is from 2, what is x?

13. Create a number line that extends from -15 to 15.

14. Ingenuity:

   The distance between two cities on a highway is 118 miles. If all the exits between these two cities are at least 5 miles apart, what is the largest possible number of exits between these two cities?

15. Investigation:

   Write a process for finding the distance between two numbers. Remember to address all possible cases: two positive numbers, two negative numbers, one of each, and at least one number equal to zero.
REVIEW PROBLEMS

1. For each part below, draw a number line with the three given integers marked.
   a. 3, -4, 6  
   b. 20, -45, 55  
   c. -8, 0, -3  
   d. -1214, -1589, -1370

2. At 7:00 A.M., Chicago’s O’Hare Airport has a temperature of -9 °C. At 11:00 A.M. that same day, the temperature reads 4 °C. Did the temperature rise or fall? Why? Determine by how many degrees.

3. At 4 P.M. in London, the temperature was 77 °F. At 6 A.M. the same day, the temperature was 57 °F. Did the temperature rise or fall? By how much?

4. Maggie has locked all of her money inside a safe and forgotten the combination. Luckily, Maggie left this note for herself: “The combination to open this safe is three positive integers. These positive integers are represented, in order, by the variables a, b, and c, where c is three, b < c, and a < b.” What is the combination to open the safe?

5. Homer was a Greek poet who produced several well-known epics during his lifetime. He wrote The Odyssey around 680 B.C.E. and The Iliad around 720 B.C.E. Which of these two works was written first? How many years passed between these two dates?

6. a. Which of the following numbers is farthest from 0: -2, 3, or -5?  
   b. Which of the following numbers is closest to 14: 28, -1, or -5?

7. Compare the pairs of numbers below and place the appropriate symbols between them. Use < or >.
   a. 457 [ ] 81  
   b. -23 [ ] -32  
   c. 191 [ ] -3  
   d. |-9| [ ] |5|  
   e. |-17| [ ] |-22|  
   f. |156| [ ] |204|
8. Find the distance between the following pairs of numbers.

Example: The distance between -2 and 9 is 11

a. 0 and -1  
g. -20 and 10  
b. -8 and 0  
h. -20 and -40  
c. 0 and -14  
i. 20 and -20  
d. 14 and 0  
j. 25 and -15  
e. -12 and 16  
k. -25 and 25  
f. -4 and 24  
l. 17 and 17

9. Answer the following questions using the table of information for each planet in the solar system. Justify your answers.

<table>
<thead>
<tr>
<th>Studying the Planets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Planet</td>
</tr>
<tr>
<td>--------</td>
</tr>
<tr>
<td>Mercury</td>
</tr>
<tr>
<td>Venus</td>
</tr>
<tr>
<td>Earth</td>
</tr>
<tr>
<td>Mars</td>
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<tr>
<td>Jupiter</td>
</tr>
<tr>
<td>Saturn</td>
</tr>
<tr>
<td>Uranus</td>
</tr>
<tr>
<td>Neptune</td>
</tr>
</tbody>
</table>

a. Which planet is the hottest? Which is the coldest?
b. Which planet is closest to the sun? Which is the farthest from the sun?
c. Is there a relationship between a planet’s distance from the sun and its average temperature? Explain.


**CHALLENGE PROBLEMS**

**Section 1.2:**
If $a$, $b$, and $c$ are positive integers such that $a < b < c < 10$, how many possible values are there for the three numbers?

**Section 1.3:**
A collection of integers has the property that every integer from 1 to 10 inclusive is the distance between some two numbers in the collection. What is the least possible number of integers in the collection?