Abstract:

Let $G$ be a finite group. A commutator of $G$ is an element of the form $a^{-1}b^{-1}ab$ where $a, b \in G$. The commutator subgroup of $G$, denoted $G'$, is the subgroup generated by the set of all commutators. It is known that each element of the commutator subgroup need not be a commutator, but only a product of commutators. We define $\lambda(G)$ to be the minimal integer such that each element of $G'$ may be expressed as a product of $\lambda(G)$ commutators. Modifying a technique of P.X. Gallagher, we show,

$$|G'| \geq (\lambda(G) + 1)! (\lambda(G) - 1)!.$$ 

This improves the earlier bound of Gallagher (1965),

$$|G'| \geq \frac{1}{2} (\lambda(G) + 1)! (\lambda(G) - 1)! + 1.$$ 

Further, in the most recent edition of the Kourovka Notebook of Unsolved Problems in Group Theory, V.G. Bardakov conjectured that for any finite group $G$,

$$\frac{\lambda(G)}{|G|} \leq \frac{1}{6},$$

with the bound attained only at the symmetric group on three letters, $S_3$. We verify his conjecture, and using our improvement of Gallagher’s result, we show that if $|G| \geq 1000$,

$$\frac{\lambda(G)}{|G|} \leq \frac{1}{250}.$$ 

Explicit values of $\lambda(G)$ have been determined for all groups of smaller order. We will also give a brief introduction to some of the key ideas that we use from the character theory of finite groups.