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## TEACHER VERSION

# Math Explorations Part 2 <br> 2013 Teacher Edition 

Terry McCabe<br>Hiroko Warshauer<br>Max Warshauer

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Welcome to a book we hope will bridge the gap between arithmetic and algebra for you and your students. A team of professors, math camp instructors, teachers and university students have worked together to produce a book that we hope will introduce all young students to the excitement and joy of mathematics.

Thank you for implementing this curriculum. Please let us know any way we can enrich this experience for you and your students. Math Explorations, as the title suggests, is filled with explorations, examples and problems that invite students to discover the patterns that lead to algebra. The philosophy of the curriculum is that students best learn mathematics by doing it, and not by simply being told rules without real understanding. Thus, students explore examples, make conjectures, and are guided to a deep understanding of basic principles. At the same time, they are learning algebraic thinking, and laying the foundation for future studies in math and science.

The exercises at the end of each lesson are rich and non-standard. They have been carefully sequenced to provide practice and extend ideas from the section. The ingenuities are meant to challenge even the most adventuresome students. The investigations lead to the concepts that will be developed in the next lesson. Math Explorations is filled with diagrams, number lines, graphs and figures that are directly relevant to the material being discussed.

The student textbook is a durable hardcover book. It has no colored pictures or distracting visual trivia.

The accompanying supplementary notebooks are to be used in the classroom to engage students in doing the explorations and problems while also writing and reflecting on the work. Melinda Kniseley and Barbara Marques help develop the part 1 workbook based on the Math Explorations Part 1 text. While Amy Warshauer and Alexandra Eusebi developed the Part 2 workbook based on the Math Explorations Part 2 text.

We invite your comments, criticisms, and suggestions, which are critical to improving this curriculum so that it will best help all students.

The teacher's edition is a tabbed, three-ring binder that has every student page on the right and accompanying teacher notes on the left. All the exercises are answered and the more challenging ones are carefully explained. The accompanying CD contains activities for every section, assessments for every chapter, copy masters for the many number lines, figures and charts, test banks for every chapter, a sample mid-term and a semester exam, and some additional exercises in the form of challenge problems and science extensions. The form of the teacher's edition invites you to add copies of our material or your own activities where you wish them to be.

Each section in the teacher's edition starts with an overview of the big ideas, objectives, materials needed, activity and any helpful initial hints. Materials needed for the activities are listed in the activity sheet. A file of warm-up problems appropriate for each section is included in the accompanying CD. At least one of the problems is formatted as a multiple choice question. Each section has a Launch problem that leads into the topic of the section.

You are encouraged to take advantage of the many teaching benefits of graphing utilities throughout the text, though if graphing calculators or computer applications might not be available for you, all of the activities can be accomplished by hand

We trust you will enjoy using the book as much as all of us enjoyed producing it, and look forward to your comments and suggestion.

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# MATH EXPLORATIONS Guiding Principles 

Based on available research, our experience in mathematics education and after long hours of discussion and trial and error, the Mathworks team have developed a set of guiding principles that we feel will help foster a learning environment in which all children are challenged, engaged and have an opportunity to learn.

1. Math is about making sense of things, thinking deeply of fundamental concepts.Students need to:

- Make connections using mathematics.
- Investigate challenging, well-sequenced problems.
- Build on prior knowledge.
- Explore problems to make sense of mathematical ideas for themselves.
- Focus on the math problems, not the answers.
- Reflect on what they have learned.

2. Teachers need to establish a classroom culture where students are not afraid of failure. The keys to establishing this culture are to:

- Encourage students to take risks without fear of failure.
- Value curiosity. Make the problems fun, interesting and relevant.
- Promote productive struggle.
- Allow sufficient time for learning ideas deeply.
- Consider all attempts seriously.

3. Persistence is critical to success. To develop students with the confidence to not give up easily, students need to:

- Develop a "growth mindset."


# MATH EXPLORATIONS Guiding Principles 

- Take ownership of their own learning.
- Understand and believe that ability can be developed with hard work.
- Be challenged by problems with high expectations.

4. Communication between students and teachers is critical for learning. To facilitate this, teachers should:

- Expect students to present work and defend reasoning using precise mathematical language.
- Take student ideas seriously, and examine both right and wrong approaches.
- Expect students to articulate the big ideas (justify reasoning).
- Balance individual and group work; both can be appropriate depending on the problem. Group work needs careful management.

5. Dispositions and external factors are powerful and need to be taken into account and dealt with. To do this, the teacher must:

- Identify and deal with extraneous issues that can affect learning.
- Build a nurturing classroom environment that inspires a desire to learn and where students take charge of their own learning.


## MATH EXPLORATIONS Guiding Principles

## Section 1.1 - Building the Number Line

## Big Idea

Constructing the number line and modeling integers

## Key Objectives

- Use the linear model to order numbers: number lines.
- Realize importance of order and spacing when constructing number lines.
- Identify types of numbers (counting, natural, whole, and integers).
- Compare and order integers within the context of temperature.


## Materials

- Thermometer
- Objects to count (bananas might be fun, but counters are more useful)
- Tape
- Index cards


## Pedagogical/Orchestration

- Several of the homework questions require students to think about and draw a thermometer. You might want to have one handy to show or pass around. The big ones for outside patios and gardens would be easy to read.
- Within launch, include background of Roman Numbers being adopted from letters-does not have a zero, starts with counting numbers.


## Activity

"United We Stand" at the end of this section and on the CD.

## Exercises

\#13 Ingenuity foreshadows operations with integers.
\#14 Investigation is a good problem for a discussion on proper scaling of number lines.
Exercises \#1-11.

## Vocabulary

set model, number line model, origin, counting numbers, positive integers, natural numbers, whole numbers, nonnegative integers, integers, negative integers, trichotomy, degrees, Celsius, Fahrenheit

## TEKS

6.1(C);
6.12(A);
7.1(A);
7.13(A)
7.14(A); 7.15(A,B);
8.14(A,D);
8.15(A,B); 8.16(A,B)

## WARM-UPS for Section 1.1

1. For temperatures measured in degrees Celsius, the freezing point of water is $0^{\circ} \mathrm{C}$. If the temperature in a small town in Kansas is $6^{\circ} \mathrm{C}$ and the temperature falls 10 degrees, which of the following is the new temperature? Explain how you get your answer.
a. $16^{\circ} \mathrm{C}$
b. $-6^{\circ} \mathrm{C}$
c. $4^{\circ} \mathrm{C}$
d. $-4^{\circ} \mathrm{C}$
Answer: d
2. The game of golf is a sport in which negative scores are better than positive scores. Can you think of other examples in which negative numbers are used? Can you find any examples where negative numbers are better than positive ones?
3. Max was sitting in his math class and his mind began to wander. His teacher, Ms. Polly Hedron, was talking about number lines and Max began thinking of places outside of his classroom where he might see examples of number lines. List as many examples as you can to help Max out and explain why each example reminds you of a number line. Answer: Street addresses, thermometers, football fields, classroom numbers in hall way, clocks, rulers, etc.

## Launch for Section 1.1

Lead the class through the beginning of the section by modeling the idea of the set model using bananas or other counters. Ask the class, "Why is this called the 'set model'?" A possible response might be that numbers are thought of as sets of objects (e.g. bananas). Another possibility is that a number can describe how many objects are in a set, or a collection, of objects. Ask students, "In the set model, what does adding objects to a collection do to the total set?" Students will respond that it makes the set larger. "What about subtraction?" The response should be that it makes the total set smaller. The set model mostly describes "How many?" as in, "How many bananas do I have?" Tell the class that there are other models for numbers, and we are going to learn one that answers "How far and where?" Ask the students if they can think of examples in the real world where people use negative numbers. There are some examples below:

- For a sports example, use yards gained or lost in attempting a first down.
- The old, but good, standby of very cold temperatures and warm temperatures.

Tell students they will be expanding their knowledge of number lines in today's lesson.

## Alternative to Exploration: Constructing a Number Line

In this class activity, you might want the students to draw their line on the board, or use tape to draw the line on the floor. Some important issues to discuss include: Are the numbers evenly spaced? How big is big? i.e. where are the large numbers located? How do you decide which numbers are greater and which numbers are less? Does the origin have to be placed in the center of the line? For an alternative to this exploration, see the end of this section.

## EXPLORING <br> INTEGERS ON THE NUMBER LINE

## 1

## SECTION 1.1 BUILDING NUMBER LINES

Let's begin by thinking carefully about numbers. Numbers are part of the mathematical alphabet, just like letters are used in English to form words. We use numbers for counting and representing quantities. When we think of the number one, we have in mind a picture:


Similarly, the number two describes a different quantity:


We could use a picture with dots to describe the number two. For instance, we could draw:


We call this way of thinking of numbers the "set model." There are, however, other ways of representing numbers.

Introduce the number line to students. It is likely many will have already used one before, but there might be aspects of the number line they don't know well or don't fully understand. For example, point out that although we may draw only part of a number line, it does continue past what we can draw. It "goes on forever", and we use arrows to indicate this. As you draw a number line, have the students draw one as well in their notebooks. Go through the vocabulary (counting numbers, whole numbers, integers, etc.) as you add to your number line.

Point out that zero is very important. It is the origin, the starting place, and we use it as a reference point to describe positive and negative numbers; those to the right are positive and those to the left are negative. Here we see how the number line helps us think of numbers as locations. Where you are on the number line relative to zero is important.

You might consider having students either highlight vocabulary on their own or write their own glossary. Suggest using index cards that are hole-punched (to be placed in a 3-ring binder) to write new words and their meaning.

Throughout this book your students will be exposed to different forms of mathematical notation. The "..." known as an ellipsis, in math means that the series, in this case, continues in the same way it starts. For instance, in Definition 1.1 , students can assume that the number after " 7 " is " 8 " in the first sequence and that the number after " +7 " is $"+8 "$ in the second sequence. Another inference with the ellipsis is that the series continues without end.

Another way to represent numbers is to describe locations with the number line model, which is visually similar to a thermometer. To construct a number line, begin by drawing a straight line and picking some point on the line. We call this point the origin. Label the origin with the number 0 . We can think of 0 as the address of a certain location on the number line. Notice that the line continues in both directions without ending. We show this with arrows at the ends of the line.


Next, mark off some distance to the right of the origin and label the second point with the number 1 .


Continue marking off points the same distance apart as above and label these points with the numbers $2,3,4$, and so on.


The points you have constructed on your number line lead us to our first definitions.
DEFINITION 1.1: COUNTING NUMBERS (POSITIVE INTEGERS)
The counting numbers are the numbers in the following neverending sequence:

$$
1,2,3,4,5,6,7, \ldots
$$

We can also write this as

$$
+1,+2,+3,+4,+5,+6,+7, \ldots
$$

These numbers are also called the positive integers or natural numbers.

Emphasize regularly that left is negative and right is positive on the number line. Right now this directionality only involves identifying integers, but it is also essential in computation with integers.

What is the relationship between the set of integers and the sets of whole numbers and natural, or counting numbers?

One interesting property of the natural numbers is that there are "infinitely many" of them; that is, if we write down a list of natural numbers, there is always some natural number that is not on the list.

When we include the number 0 , we have a different collection of numbers that we call the whole numbers.

## DEFINITION 1.2: WHOLE NUMBERS (NON-NEGATIVE INTEGERS)

The whole numbers are the numbers in the following neverending sequence:

$$
0,1,2,3,4,5,6,7,8,9,10,11, \ldots
$$

These numbers are also called the non-negative integers.

In order to label points to the left of the origin, we use negative integers: $-1,-2,-3,-4, \ldots$ The sign in front of the number tells us on which side of zero the number is located. Positive numbers are to the right of zero; negative numbers are to the left of zero. Zero is not considered to be either positive or negative.


So we have seen that numbers can be used in different ways. They can help us to describe the quantity of objects using the set model or to denote a location using the number line model. Notice that the number representing a location also can tell us the distance the number is from the origin if we ignore the sign.

## DEFINITION 1.3: INTEGERS

The collection of integers is composed of the negative integers, zero, and the positive integers:
$\ldots,-4,-3,-2,-1,0,1,2,3,4, \ldots$

When discussing trichotomy, you might discuss it in contrast to parity (evenness or oddness). An integer can be either even or odd -2 possibilities. Ask them what 0 is and why. Some acceptable explanations include:

1. 0 is a number that is divisible by 2 , and any number divisible by 2 is even.
2. Every other number is either even or odd, and if 1 is odd, then 0 must be even.

Then say that when it comes to position on the number line, there are three possibilities: 1) positive, to the right of zero; 2) negative, to the left of zero; and 3) neither positive nor negative, but exactly zero. Don't worry about trichotomy too much; it's mostly a useful word for students who are confused about the difference between positive, negative, and zero.

## Exploration: Constructing a Number Line

For this activity, have the students turn their papers horizontally. Parts 1,2 , and 3 require that the students draw and label -10 and 10 on their papers. Some students may go all the way to the edge of their papers. As you get to part 4, the students may observe that they needed more room in order to plot the larger numbers. This is okay, and the student would not need to redraw the number line since part 4 only requires you to IDENTIFY approximately where these larger numbers belong. For example: at the end of the desk, in the next classroom, outside the building, in the next room, etc. This can bring out a good discussion foreshadowing how you decide what numbers to use on your number line (intervals and spacing).

Definition 1.3 leads to the trichotomy property, which states that there are exactly three possibilities for an integer: positive, negative, or zero.

Such collections of numbers are often called sets of numbers. For example, assume $S$ is the set $\{1,2,3,4,5\}$ and $T$ is the set $\{2,3,4\}$. Notice that every number in $T$ is also an element of the set $S$. That means that $T$ is a subset of $S$. This relationship is represent by the following diagram:


## EXAMPLE 1

Call the set of positive integers $P$, the set of whole numbers $W$, and the set of integers $Z$. Draw a diagram of the relationship of these three sets.


11 (4)

## EXPLORATION

4. All to the right +10 in order from least to greatest.

## EXERCISES

Teacher Tip: Have students create a number line from about-20 to 20 on a half sentence strip, punch holes in it so they can keep it in their binders. Students can have a horizontal number line and a vertical number line. They can also write "clue" words that identify negative direction and positive direction. You may suggest using a "folding" technique to ensure that the number line is evenly spaced.
1.

2. Be careful labeling. You might not have enough room. Students should be able to label every number, so the student number line should be larger than ours.

2 (b).


Ask students, "Why won't the origin 0 be on these number lines?"
Be sure to remind students that 771-77 does not need to be in the middle of their number line.

## EXPLORATION: CONSTRUCTING A NUMBER LINE

1. Draw a straight line.
2. Pick a point on the line and call this point the origin. Label the origin with the number 0.
3. Locate the numbers $1,2,3, \ldots, 10$, and $-1,-2,-3, \ldots,-10$.
4. Where would $20,30,50$ be located? 100? 1000? See TE.
5. Find the negative numbers corresponding to the numbers in question 4.

## EXERCISES

1. The post office is located at the origin of Main Street. We label its address as 0 . The laboratory has address 6 and the zoo has address 9 . Going in the other direction from the origin, we find a candy shop with address -4 and a space observatory with address -7 . Draw a number line representing Main Street. Label each of the above locations on the number line. Watch your spacing. See TE.
2. a. Copy the line below to mark off and label the integers from 0 to 10 and from 0 to -10. Use a pencil to experiment because you might need to erase. Watch your spacing.

b. Make a number line from -20 to 20 . See TE.
3. Draw a section of the number line containing the number 77. Mark the number 77 on your line. Now label your number line with the first few integers after 77 and the last few before 77, at least three each way. See TE.
4. Do the same thing you did in the previous exercise, but this time start with the number -77 instead of 77 . See TE.

5 (a). 50, 50. Yes
5 (b). 10, 10, 10. Yes. They should check their spacing between these points in the number line.
5 (c). Answers may vary.

5 (d).

6. Plotting a point is an exact placement whereas sketching is an estimation.

7.

8. Answer: The number to the right is greater.

This problem foreshadows the next section and will give students practice with creating their own mathematical language.
9.

Jan, Dec, Feb, Nov, Mar, Oct, Apr, Sept, May, Aug, June, July

5. Use a line like the one below to mark off the numbers with equal distances by tens from 0 to 100 and from 0 to -100 . Use a pencil to experiment.

a. What is the distance from 0 to 50 and from 50 to 100? Are they the same? See TE.
b. What is the distance from 10 to 20,30 to 40 , and 70 to 80 ? Are they the same? See TE.
c. Explain whether you need to rework your markings on the number line.

See TE.
d. Estimate the location of the following numbers and label each on your number line: See TE.

$$
15,25,55,-15,-25,-75,34,31,-34,-31,87
$$

6. Draw a number line so that the number -1000 is at the left end and 1000 is at the right end. Estimate the locations of the following integers: See TE.
500, -500, 250,-100, -800, 10, -990, 342, -781, 203, -407
7. Draw a number line. Find all the integers on your number line that are greater than 15 and less than 20. Color each of these locations blue.
8. If you find two numbers on the number line, how do you decide which number is greater?

Notice that we can move the number line from the horizontal position to a vertical position. We would then have a number line that looks like a thermometer. Draw a thermometer (vertical number line) on the side of your paper to help you answer questions 9 through 12.
9. The chart below shows the monthly average temperatures for the city of Oslo, Norway. Based on the data, put the twelve months in order from coldest to warmest. See TE.
10. $\cdot 7^{\circ} \mathrm{C}$ is closer by $2^{\circ} \mathrm{C}$.
$11.4^{\circ} \mathrm{C}$ is closer by $6^{\circ} \mathrm{C}$.
12. $-20^{\circ} \mathrm{F}$ is colder by $5^{\circ}$.

Ingestitigation


## MONTHLY AVERAGE TEMPERATURE: OSLO, NORWAY


10. Chris visits Edmonton, Canada where it is $-7^{\circ} \mathrm{C}$. Carmen visits Winnipeg, Canada where it is $9^{\circ} \mathrm{C}$. Which temperature is closer to the freezing point? Explain. Remember, when we measure temperature in degrees Celsius $\left({ }^{\circ} \mathrm{C}\right)$, $0^{\circ} \mathrm{C}$ is the freezing point of water. See TE.
11. The temperature in Toronto, Canada, one cold day, is $-10^{\circ} \mathrm{C}$. The next day the temperature is $4^{\circ} \mathrm{C}$. Which temperature is closer to the freezing point? See TE.
12. The temperature in Fargo, North Dakota is $-15^{\circ} \mathrm{F}$ while it is $-20^{\circ} \mathrm{F}$ in St. Paul, Minnesota. Which temperature is colder? How much colder? See TE.

## 13. Ingenuity:

On a cold winter day in lowa, the temperature at 6:00 P.M. is $10^{\circ} \mathrm{F}$. If the temperature decreases an average of $4{ }^{\circ} \mathrm{F}$ for each of the next five hours, what will the temperature be at 11:00 P.M.? See TE.

## 14. Investigation:

Use the number line below as a thermometer with the Celsius scale above the line and the Fahrenheit scale below the line to discover how the two scales are related.


## Summary

Ask the students to describe how the number line model differs from the set model. Answer: Direction and location are important; can visually see negative numbers. Ask them for examples of number lines in daily life. Answer: Thermometers, streets with addresses, rulers, classroom door numbers in hallways. For these examples ask which ones include a 0 point and where it is located.

- Streets might not include a 0 . Although streets often have an intersection representing 0 , they do not necessarily have a building at 0 . In these cases, the 0 intersection often marks the difference between addresses known as 119 South Guadalupe vs. 119 North Guadalupe OR 13 West Congress vs. 13 East Congress. Ask the class whether North or South would correspond to negative or positive? East or West?
- Rulers are number lines with a zero, sometimes unmarked, but always the left edge of the ruler. Ask why rulers don't have negatives.
a. At what temperature does water boil on each scale? See TE.
b. At what temperature does water freeze on each scale? See TE.
c. What is the Fahrenheit reading for $50^{\circ} \mathrm{C}$ ? See TE.
d. Is the Celsius reading for $25^{\circ} \mathrm{F}$ a positive or negative number? See TE.
e. A nice day is $77^{\circ} \mathrm{F}$. What is this temperature in Celsius? See TE.
f. A hot day is $100^{\circ}$. Estimate this temperature in Celsius. See TE.


## United We Stand

Objective: The students will work corroboratively to physically position themselves on a number line.

## Materials:

1) Index Cards, numbered with random integers. You will need one card per student.
(Be sure to include positives, negatives, and zero when you are numbering your cards.)
2) Colored Tape

## Activity Instructions:

Before the students get to class, use the colored tape to make an unmarked number line along the floor in the front of your room. Do not mark intervals on the number line.

After shuffling the cards to make sure they are in random order, pass one card to each student.
Once all of the cards have been handed out, the students will come up one at a time and position themselves on the number line that you made on the floor. Allow the important issue of spacing to arise naturally as the students arrange themselves.

When all students are standing in the correct order at the front of the room, take this opportunity to ask them some extension questions to check for understanding of the lesson. Some examples of questions you could ask are:
-Who is standing in the middle of the line, and why?
-If we had a card with the number one million, where would it go?
-If we had a card with the number____ it would be between which two students?
-Who represents the biggest number on our line?
-Who represents the smallest number on our line?
-Is the spacing between integers important?

## Section 1.2 - Less Than and Greater Than

## Big Idea

Comparing and Ordering Integers

## Key Objectives

- Construct different types of number lines.
- Develop the linear model of numbers.
- Use Inequality Notation: < and > .
- Define $a<b$ : $a$ is less than $b$, and $a$ is to the left of $b$.


## Materials

- Tape
- Index cards


## Pedagogical/Orchestration

- Prerequisites of this section are creating a number line: spacing/placing zero, positive, \& negative integers.
- Have students work out problems to compare which number is bigger than another. Example: Compare which is bigger. 7 or 2 ? 28 or 40 ? 13 or 25 ? Raise the question: "What do you mean by a number being bigger? What about 2 or -2? Both are 2's?" Have students re-evaluate the word "bigger" \& change to "greater than." Help students tie "greater than or less than" to a relative position on the number line. A number to the left is always less than a number to the right on a number line, and a number to the right is always greater than a number to the left on a number line.


## Activity

"Integer War" found at the end of this section and on the CD.

## Exercises

Be sure to assign Exercises 2, 3, and 4 so that patterns can be verbalized. Also make sure you assign Exercise 13 so that you can use this information to begin the next lesson, Section 1.3: Applications of the Number Line.

## Vocabulary

less than, greater than, numerical expression

## TEKS

$6.1(\mathrm{~A}, \mathrm{C}) ; \quad 6.11(\mathrm{~B}) ; \quad 7.1(\mathrm{~A}) ; \quad 7.13(\mathrm{~A}, \mathrm{D}) ; \quad 7.14(\mathrm{~A})$

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Teacher Edition
Section 1.2 Less Than and Greater Than
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## WARM-UPS for Section 1.2

## Launch for Section 1.2:

To launch Section 1.2, students will be reviewing what they learned yesterday about number lines and building on that concept. Ask students, "What are the important characteristics of a number line?" Put students' responses on the chalkboard or chart paper. Essential characteristics should include: numbers should be in order, numbers should be equally spaced, greater numbers are to the right of lesser numbers, there are arrows on each end of the line to show that the numbers extend infinitely in both directions, etc. It is important to locate the zero and identify 1 unit for scaling purposes. Assign students to groups, and have each group make a number line on the floor using tape for the line and index cards for the numbers. The numbers should extend from -10 to 10 and be spaced at least one foot apart. Have groups compare the number lines from other groups to make sure they satisfy the essential criteria of the number line as outlined by the students at the beginning of class. Also check to see that the markings are symmetrical, for instance that 5 and -5 are the same distance from 0 . If you see a number line that is not spaced properly, measure and cut a piece of string so that it is the length of the distance between two numbers like 3 to 5. Use the string length to compare equivalent distances on the same number line such as 7 to 9 or -3 to -1 . This has a great visual impact and will encourage the students to correct their spacing. Tell students, "Today we will be using these number lines to model the concepts of greater than and less than." Give the students an example such as $-1<x<5$, and have them figure out where $x$ would go on the number line. What integers would it be in between? Would -1 and 5 be included?

Have one student stand at 5 and another at 2 . Ask the class which is bigger (further along to the right)? As they answer, write " 5 is greater than 2 " and " $5>2$ " on the board. Discuss this notation if necessary. Ask how else we might represent the relationship between 2 and 5 . " 2 is less than 5 " and " $2<5$." Write these on the board and again go over the notation.

Next, introduce the definition in box 1.4 and take as much time as needed to ensure the students understand $x$ and $y$, less than and greater than, and the notation.

If your students are having trouble figuring out which symbol to use, explain to them that the < and > symbols match the symbols or arrows at the ends of the number line. If they are asked to compare 2 and 5 , they can just place these numbers on the number line and choose the symbol that is closest to 2 because 2 came first in the comparison, proving that $2<5$. Similarly, if they are asked to compare 5 and 2 , they would choose the symbol or arrow that is closest to 5 . Because 5 came first in the comparison, 5 must be greater than 2 . Sometimes it helps the students understand the concept of these symbols if they have a visual like this.

## SECTION 1.2 LESS THAN AND GREATER THAN

We say that 2 is less than 5 because 2 is to the left of 5 on the number line. "Less than" means "to the left of" when comparing numbers on the number line. We use the symbol "<" to mean "less than." We write "2 is less than 5" as " $2<5$." Some people like the "less than" symbol because it keeps the numbers in the same order as they appear on the number line.

We also say that 5 is greater than 2 because 5 is to the right of 2 on the number line. "Greater than" means "to the right of" when comparing numbers on the number line. We use the symbol ">" to mean greater than, so we write " 5 is greater than 2" as " $5>2$."

## DEFINITION 1.4: LESS THAN AND GREATER THAN

Suppose that $x$ and $y$ are integers. We say that $x$ is less than $y, \boldsymbol{x}<\boldsymbol{y}$, if $x$ is to the left of $y$ on the number line. We say that $x$ is greater than $y, \boldsymbol{x}>\boldsymbol{y}$, if $x$ is to the right of $y$ on the number line.

Here $x$ and $y$ are called variables, which we will formally introduce in Chapter 3 . A variable is a letter or symbol that represents an unknown quantity. Variables give us a convenient way to describe properties and ideas because variables can represent many values. Variables give us a simple way to describe math objects and concepts. In this case, $x$ and $y$ represent two integers, and the way that we tell which is greater is to compare their positions on the number line. The two number lines below demonstrate the cases $x<y$ and $x>y$. Can you tell which is which?


Be sure to address the fact that "less than or equal to" means either less than or equal to, but not both. Also, remind your students, if you need to, that the equals sign can be translated as "is" in English, short for "is equal to."

## EXAMPLE 1

When ready to begin Example 1, ask students to represent (a) 3 and 7, and ask the class which is greater and which is less. Have students write these symbolically on the board and in their notes. Discuss how they know. How does the number line help? Continue with parts (b), (c), and (d).
a. $3<7$
b. $-2<9$
c. $-1>-5$
d. $4>-4$

Teachers, what do your students know about the relationship between any positive number and any negative number? Do they recognize that positive numbers are always greater than negative numbers?

What about the relationship between zero and the positives and negatives? Zero is always less than any positive number and greater than any negative number.

What about the order of negative numbers? Ask students to think about -10 and -100. Which is greater and why?

## ACTIVITY

Play the Integer War activity (from the end of this section) for more practice on this important concept.

## EXAMPLE 2

For Example 2, ask the class to determine the order from least to greatest of the numbers given. Give them a few minutes to think and record a response. Have students come forward to represent each number. When in position, have the student who is left most of zero (-5) identify that spot as least, and continue to the student at position 7. Ask students how they know -5 is least and 7 is greatest of those two numbers. Emphasize (if needed) that when ordered along a number line you can easily tell which integers are least, greatest, and everything in between.

Also, if $a<b$ and $b<c$, then $a<c$, and if $a>b$ and $b>c$, then $a>c$.
For the number line in the solution for Example 2, notice that we didn't label all of the integers on the number line this time. It is a common practice not to write all of the numbers on the diagram if you don't need them and if doing so would make the picture look cluttered.

Talk to students about determining how to label the numbers on a number line. Talk about the multiples of five or ten as possibilities.

Mathematicians use the symbols " $\leq$ " and " $\geq$ " to mean "less than or equal to" and "greater than or equal to." We can write " $2 \leq 5$ " because 2 is less than 5 , but we can also write " $5 \leq 5$ " because 5 is equal to 5 .

## EXAMPLE 1

For each pair of integers below, determine which one is greater and which one is less. Express your answer as an inequality of the form $x<y$ or $x>y$, where $x$ and $y$ are the given integers.

| a. | 3 |
| :--- | ---: |
| b. | -2 |

$\begin{array}{lrll}\text { c. } & -1 & \square & -5 \\ \text { d. } & 4 & \square & -4\end{array}$

## SOLUTION

a. Begin by drawing a number line from -10 to 10 . Using this number line, we see that 3 is to the left of 7 , so $3<7$.
b. We observe that -2 is to the left of 9 on the number line, so $-2<9$. We can also see this in a different way: We know that -2 is to the left of 0 , because -2 is negative, and 0 is to the left of 9 , because 9 is positive. Thus -2 must be to the left of 9 , and we have $-2<9$.
c. We notice that -5 is to the left of -1 , so $-5<-1$ or $-1>-5$.
d. Because -4 is to the left of 0 , and 0 is to the left of 4 , we have $-4<4$ or alternatively $4>-4$.

## EXAMPLE 2

Put the following integers in order from least to greatest:

$$
2,-2,7,-1,-4,-5,4,6,3
$$

## SOLUTION

Again, we can use the number line to help us put the integers in order:


27 (10)

## Summary:

Before they begin the exercises, ask the students to explain again how they know if numbers are less than, greater than, or equal to one another.

A final activity might be to ask each student in order around the room to volunteer an integer, you might have students write their integers on their own personal white board or an index card so they can order them more easily. Record these as you go. Then have the students order these from least to greatest. Encourage students to volunteer negative numbers if they are not doing so. If no one offers 0 , then you can provide it as your example integer. If students volunteer really big numbers or small numbers ( 1,000 or $-1,000$ ) ask them to estimate where those might be if the class number line were to continue past the classroom.

## EXERCISES

2. a. $3<4$
d. $-2>-3$
$4>3$
$-3<-2$
b. $0<1$
e. $-8<9$
$1>0$
$9>-8$
c. $-2<0$
$0>-2$
f. $6>-5$
$-5<6$
3. a. $5>3$
$-5<-3$
b. $-10<-4$
$10>4$
c. $-12<0$
$12>0$
d. $-15<-13$
$15>13$
e. $\quad-9<-7$
$9>7$
f. $-21<21$
$21>-21$
4. In Exercise 2, the students should observe that when the order in which two numbers are written changes, then the inequality changes from greater than to less than or vice versa. The relationship between the two numbers, however, has not changed; that is " 3 is less than 4 " is merely a restatement of " 4 is greater than 3 " and 3 is still to the left of 4 .
The pattern in Exercise 3 is to observe that when one examines the relationship between the inequality between a pair of numbers $a$ and $b$ to its negatives, $-a$ and $-b$, that the inequality changes signs. For example, in the first relationship between 7 and 3 , we have 7 is greater than 3 . However, when we look at -7 and -3 , we see that -7 is less than -3 . The students should observe this on the number line and the relative position of the paired numbers. We do not expect the students to conclude that when one multiplies an inequality, $a<b$ by a negative number such as -1 , that $-a>-b$. This exercise foreshadows this property, but we do not expect to formalize this relationship here.

We can locate our nine given integers on the number line. You might try doing this by copying the diagram above and labeling the given numbers on your diagram. After comparing the nine numbers given, we get the following order:

$$
-5,-4,-2,-1,2,3,4,6,7
$$

We can write this list using the < symbol as:

$$
-5<-4<-2<-1<2<3<4<6<7
$$

## EXERCISES

1. Rewrite each of the following as a statement using $<,>, \leq$, or $\geq$. Compare your statements to the relative locations of the two numbers on the number line. Example: -3 is less than $8 . \quad-3<8$.
a. 9 is greater than $6 . \quad 9>6$
b. 4 is less than 7. $4<7$
c. -3 is greater than $-5 . \quad-3>-5$
d. -4 is less than $1 . \quad-4<1$
e. 8 is greater than or equal to $0.8 \geq 0$
f. -6 is less than or equal to $2 .-6 \leq 2$
2. Compare the numbers below and decide which symbol, < or >, to use between the numbers. Make a number line to show the relationship
a. $3 \square 4$
4 $\square$3
b. $0 \square 1$

3. Compare the numbers below and decide which symbol, $<$ or $>$, to use. Use a number line to show the relationship of these numbers.
a.

| 5 | $\square$ | $\square$ |
| ---: | :--- | ---: |
| -5 | $\square-3$ |  |
| -10 | $\square-4$ |  |
| 10 | $\square$ | $\square$ |
| -12 | $\square$ | $\square$ |
| 12 | $\square$ | $\square$ |

d. $-15 \square-13$
$15 \square 13$
e. $-9 \quad \square \quad-7$
$9 \begin{array}{r}\square \\ \hline\end{array}$
21

4. Describe any patterns you see in Exercises 2 and 3. See TE.
5. a. $6>2$
$2<6$
b. $13>-11$
$-13<11$
c. $\quad-9<3$
$-3<9$
d. $\quad 0<14$
$-14<0$
e. $12<28$
$-12>-28$
$-12<28$
$12>-28$
7. Answer: $4,5,6$.

8. Answer: (a)True. (b) False. An integer to the right of 5 is also to the right of -5 , this can be seen by drawing a number line. But, if a number is less than 5 , it is not necessarily less than -5 as there are numbers to the left of 5 that are not to the left of -5 . Take 3 for example. The number 3 is less than 5 but is greater than -5 because it is to the left of 5 but to the right of -5 .

9. The temperature fell $3^{\circ} \mathrm{C}$ or changed $-3^{\circ} \mathrm{C}$.
10. Albert is 9 more steps away from the ground level because $87-78=9$. Thus 87 is further from ground level than -78 by 9 steps. In other words, from 0 to 87 is 87 steps. From -78 to 0 is 78 steps. The difference between 87 steps and 78 steps is 9 . Therefore, Albert is 9 steps farther from ground level.
11. The answer is all integers greater than or equal to 2 .
12. Suppose that $x$ and $y$ are integers. We say that $x$ is less than or equal to $y(x \leq y)$ if $x$ is either to the left of $y$ on the number line or $x$ is $y$. We say that $x$ is greater than or equal to $y(x \geq y)$ if $x$ is either to the right of $y$ on the number line or $x$ is $y$.

## Ingenuity

13. Teachers should encourage students to read the question carefully and completely before starting their work. If they read all instructions for this exercise and write down all the information they know about the number they are looking for before they start their work then they know that: (a) the tens digit of the whole number is greater than the ones digit, (b) the number is between 80 and 90 , (c) the tens digit is less than twice the ones digit, (d) the integer is even. Using (b) and (d) they can narrow down their search to $80,82,84,86,88$, and 90. Then they can check for (b) and (c) on each number or make a chart to organize their information. Also be sure to point out that "whole number" and "integer" are synonymous in this case.
Teachers, remind your students that the digits 0-9 are the symbols that our numeral system uses to make all numbers. The tens digit is the symbol in the tens place of a number.
14. Compare the numbers below and decide which symbol, $<$ or $>$, to use. Use your rules from Exercise 4 to help you.
a.

| a. | 6 | $\square$ |
| :--- | ---: | :--- |
|  | 2 | $\square$ |
| b. | 13 | $\square$ |
|  | -11 |  |
|  | -13 | $\square$ |
| c. | -9 | $\square$ |
| c. |  | $\square$ |
|  | -3 | $\square$ |

d. $0 \square 14$
-14 0
e.

| 28 |  |
| ---: | ---: |
| -12 | $\square-28$ |
| -12 | $\square$ |
|  | $\square$ |
| 12 | $\square$ |

6. List the following integers in order using a number line. $83,26,-59,-62,-75$

$$
-62,-75,26,83,-59
$$

7. What are the possible values for an integer that is greater than 3 and less than 7? Mark these possible values on a number line. See TE.
8. Determine whether each of the following statements is true or false. Explain your answers.
a. If an integer is greater than 5 , then it is greater than -5 . See TE.
b. If an integer is less than 5 , then it is less than -5 . See TE.
9. In Madison, Wisconsin, the morning temperature is $-2^{\circ} \mathrm{C}$. In the evening the temperature reads $-6^{\circ} \mathrm{C}$. Did the temperature rise or fall? How much did it change? See TE.
10. Albert is on a flight of stairs 87 steps above the ground floor. Elaine has gone into the sub-basement 78 steps down from ground level (let's call it the $-78^{\text {th }}$ step). Who is farther from ground level? Why? See TE.
11. What are the possible values for an integer that is closer to 5 than it is to -2 ? Mark these possible values on the number line. See TE
12. Earlier we introduced "greater than or equal to" and "less than or equal to." Write a formal definition, using definition 1.4 as a model, for these two concepts. See TE.

## 13. Ingenuity:

Suppose that the tens digit of a whole number between 80 and 90 is greater than the ones digit, but less than twice the ones digit. If the integer is even, what is its value? 86
14.

This is a good activity to use as an interdisciplinary lesson.
Students are to choose 5 other periods between 2000 BC and 2100 AD to place on the timeline.


## 14. Investigation:

Make a large timeline from the year 2000 BC to the year 2100 AD. Research the years that mark the following periods on this timeline.
a. The life of William Shakespeare
b. The U.S. Civil War
c. The Mayan civilization
d. The Roman Empire
e. People driving cars
f. The United States has been a country
g. Texas was a state
h. Texas was a country

## Integer WAR

Objective: The students will work in pairs or small groups to practice comparing the values of certain integers.

## Materials:

One complete deck of cards (including Jokers) for each group

## Activity Instructions:

This card game is played like "War". The students will shuffle the deck and pass out all of the cards so that all players start with the same number of cards in their pile. Each student in the group will then take the top card from their pile and turn it face up. The students will decide which card has the highest value, and the winner will pick up all of the cards played and put them face down in the bottom of their pile. The game is called "Integer WAR" because if there is ever a tie, the players with the tied cards must announce "WAR". These two players will then count out 6 cards (the first 5 will be face down, the 6 th and last will be turned face up). As they count out these six cards, they will recite the words "I Declare War On You". (There are six syllables in this sentence, and they lay down one card per syllable). The player with the highest value on the card that is face up wins the whole pile. If they are still tied, the students will declare war again and follow the same procedure until there is a winner for this round. The game continues until either 1 player has collected all of the cards, or until you designate when they will stop playing. If you stop the game before 1 player has collected all of the cards, the player with the most cards wins.

The values of the cards are as follows:
All red cards are negative.
All black cards are positive.
All aces are 1.
All jacks are 10.
All queens are 50.
All kings are 100.
All jokers are 0 .

## Section 1.3 - Distance Between Points

## Big Idea:

Understanding and using absolute value

## Key Objectives:

- Define absolute value as distance from origin (magnitude).
- Find numeric distances between integers by counting from one to the other.
- Compare absolute values.


## Materials:

- Rulers
- Number Line Handouts at the end of the section or the CD
- White boards for students (optional)


## Pedagogical/Orchestration:

- Absolute Value prerequisite is the importance of spacing from Section 1.1.
- Be aware that some students confuse absolute value with opposite of a number.
- Before the students start the exercises, it might be a good idea to go through a few examples together. It is really important to ask the class "What is the absolute value of zero?" and make sure they know the correct answer and more importantly know the reason why their answer is correct.


## Internet Resources:

Battleship Game: Integers Review- http://www.quia.com/ba/164425.htm|
Jeopardy Game: Integers Review- http://www.quia.com/cb/64603.html
Rags to Riches: Integers Review- http://www.quia.com/rr/41496.html

## Activity:

"Absolute Value Bingo" Activity at the end of the section and on the CD

## Exercises:

Exercises 7-9 foreshadow subtraction.
Exercises 10-12 foreshadow addition, and would be nice problems to refer to when introducing Section 2.1 on addition of integers.

## Vocabulary:

absolute value, magnitude

## TEKS:

6.13(B); 7.1(A); 7.13(A,C); 7.14(A); 7.15(A)

## WARM-UPS for Section 1.3

1. Determine which of the following is not true. Explain your choice.
a. $-9<2$
b. $-6>3$
c. $-1>-5$
d. $-4<-2$

Answer: b
2. Which of the following sets of numbers are written in order from least to greatest? Correct the other ones to make them true.
a. $-2,-5,-7,0,4,8$
b. $-9,-6,0,-1,3,5$
c. $-13,-10,-2,1,6,9$
d. $0,1,-2,-3,4,-5,9$
3. Find the distance between each pair of numbers. What do you notice?
a. 23 and 32
b. 4 and -5
c. -46 and -55
d. What do you notice? Answer: They are the same distance apart.

## Launch for Section 1.3

Ask the students to describe how they know whether 7 or -7 is greater. They should respond that 7 is greater because it is further to the right of zero. Tell students, "Today we are going to be talking about the size or magnitude of a number rather than its direction from zero." At this point, lead the students through the measuring activity using the Number Line Handouts.

Pass out the Number Line Handout. Assign students pairs of numbers such as $1,-1 ; 2,-2 ; 3,-3$; etc. Have students use rulers to measure the distance from 0 to each of their numbers. For example, 5 is 5 units from 0 and -5 is also 5 units from 0 . Ask them to share their results and generalize. They should discover that a number and its negative are the same distance from 0 . Also, this distance is non-negative as are all distances. Point out that yesterday the class was concerned with relative position (left/right of 0). Today we are working on distance from 0 and less concerned
with left/right. After the students have tried out some number pairs and discovered that both a number and its negative (i.e. -11 and 11) are the same distance from 0 , then tell them that this distance is called the absolute value. Also show them the notation.

Ask the students to share an integer (try to encourage a mix of negatives and positives and zero). Record their integers. Choose a few and have the class determine the absolute value of that integer. Reinforce the notation by writing it.

Teacher Tip: Use personal white boards so that students can work through the process of the notation, maybe show a number line to prove the distance.

Next choose a few pairs and have the class determine which absolute value is greater. For example, if you have |-14| and $|7|$, the absolute value of -14 is 14 and the absolute value of 7 is 7 . Therefore $|-14|>|7|$. This might be difficult for some students to grasp because $-14<7$. Point out again if needed that absolute value measures the distance from 0 . If you draw -14 and 7 on a number line then it is visually apparent that -14 is further from 0 than 7 is and so -14 has the greater absolute value.

Finally, still using the student examples of integers, choose a few pairs to determine the distance between them on the number line. For example, if we look at 8 and -9 , and we plot them on a number line, how many units are they apart from one another? Although there are rules you can generalize that will make solving this type of problem more efficient, try not to lead the conversation towards these generalizations just yet. Some students may need more experience just counting "tick" marks before they are ready to learn a rule. In fact, Exercise \#14 will ask students to generalize this process so be sure to assign it. This can be a great way to start tomorrow's lesson (see review problems section).

Students seem to understand the rope analogy for the absolute value more readily than other explanations
Many students have difficulty with the concept of absolute value. Thinking of a length of rope tethered at the origin helps many of these students. If you stretch the rope 5 units to the left of the origin, it will touch -5 . If you stretch the same rope 5 units to the right of the origin, it will touch 5 . The rope's length represents the distance from 0 or the absolute value of a number. In that case, $|-5|=|5|$.

Students will count from one point to the other. This is OK because they have not learned how to subtract integers yet. They might because subtraction for two positive integers, i.e. 2 and 8 gives us $8-2=6$.

Don't force this on them yet.

## SECTION 1.3 DISTANCE BETWEEN POINTS

We locate the numbers 10 and -10 on the number line.


Notice that 10 and -10 are each 10 units from 0 . We have a special name for the distance of a number from 0 : the absolute value of the number.

In mathematics, we have a special symbol to represent absolute value. For example, we write |10| and read it as "absolute value of 10 ." We write $|-10|$ and read it as "absolute value of -10 ." Because 10 and -10 are both 10 units from 0 we have the following:

The absolute value of 10 equals 10 or $|10|=10$.
The absolute value of -10 equals 10 or $|-10|=10$.
The absolute value gives us a measure of the size or the magnitude of a number, or its distance from the origin. The positive or negative sign tells us the direction of the number relative to 0 . Because 10 and -10 are the same distance from 0 , they have the same absolute value. In other words, $-10<10$ but $|-10|=|10|$. Note: The absolute value symbol '||' should not be confused with parentheses '()'.

## EXPLORATION

Using the number line that you have constructed, find the distance between each pair of numbers:
a. 2 and 8
b. -4 and -1
c. -4 and 1
d. 5 and -3
e. 0 and 6
f. 0 and -6
g. 9 and 0
h. 0 and 9
i. 0 and -9

In addition to the previous examples, you may also see -|10| which is read as "the negative absolute value of 10 " or $-|-10|$ which is read as "the negative absolute value of -10 ." Since $|10|=10$ and $|-10|=10$, then we have

$$
-|10|=-10 \text { and }-|-10|=-10 \text {. }
$$

## Summary

Before concluding class or allowing the students to work on the exercises, ask them to summarize what the absolute value of a number means. What is it? The distance from 0 , therefore always positive. What is it not? The relative position to the left or right of zero.

Use Absolute Value Bingo at the end of this section as a review.
Although the absolute value concept is often dealt with at a later time, conceptually, it is essential in the understanding of the four basic operations with integers.
1.

| a. | 7 | d. | 10 | g. | 21 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| b. | 8 | e. | 10 | h. | 42 |
| c. | 8 | f. | 19 | i. | 33 |

2. a. 2
b. 15
c. 0
d. -17
e. -34
3. The absolute value of -44 is 44 . The absolute value of 44 is 44 . The answer, therefore, is 44 .
4. a. 9
b. 0
c. 9
5. a. $|-7|>|5|$
f. $|-6|>-8$
b. $|7|>|-5|$
g. $|-6|<8$
c. $|9|>-9$
h. $6<|-8|$
d. $|9|=|-9|$
i. $25<|28|$
e. $|32|<|-47|$
j. $|-28|>25$
6. The difference between 5 and 8 is so automatic and students have known it so long that thinking about it in two new ways might be hard. Direct them to find the two numbers on the number line and to think of their distances from the origin.
7. This exercise foreshadows using subtraction and absolute value to find the distance between any two numbers on the number line. Students may either draw a number line and use counting or they can use what they know about the number line to find the distance.
a. 1
b.
5
c.
8
8
42

## EXERCISES

1. Find the absolute values of the following numbers.
a. -7
b. 8
c. -8
d. 10
e. -10
f. 19
g. -21
h. -42
i. 33
2. Calculate the following:
a. $|-23|$
b. (15)
c. $|0|$
d. $-|17|$
e. $-|-34|$
3. What is the absolute value of the absolute value of -44 ?
4. Find the distance between each number and zero:
a. 9
b. 0
C. -9
5. For each pair of numbers below, place the correct symbol $\langle$,$\rangle , or =$.
a. $|-7|$

$\begin{array}{ccc}\text { f. } & |-6| & \square \\ \text { g. } & |-6| & \square \\ \text { h. } & 6 & \square \\ \text { i. } & 25 & \square \\ \text { j. } & |-28| & \square\end{array}$
b. $\quad|7|$

|-5|
c. $\quad|9|$
d. $\quad|9|$

e. |32|
|5|
8
6. Find the distance between 5 and 8 . Did you use the number line? Can you use absolute values? 3. See TE.
7. For each pair of integers given below, find the distance between the two integers on the number line. See TE.
a. 2 and 3
b. 4 and 9
c. $\quad 25$ and 17
d. -12 and 12
2 and -3
-4 and 9
-2 and 3
-4 and -9
17 and 25
12 and 12
-2 and -3
4 and -9
-17 and 25

| 8. a. 30 | b. | 10 | c. | 40 | d. | *62 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 5 |  | 3 |  | 2 |  |  |
| *35 |  | *13 |  | *38 |  |  |

*They might use subtraction on these pairs. This is foreshadowing subtraction of integers.
9. No, because there are two such numbers: 8 and 18 .

Teachers, note that students may have some difficulty distinguishing the two given numbers: 5 and 13. You may wish to talk with the students to see if they are able to clearly distinguish that 5 is a distance of 5 units and that 13 is a location on the number line.
11. a. $-3,7$

b. $-7,1$
c. $-11,-5$

These questions are more algebraic in nature and ask the students to write an expression in terms of x . Part a is foreshadowing of part b and part c. The students should be encouraged to write the process by which they get the response to part a: $2+5$ and $2-5$. Then the students may see that in the case of x rather than 2 , the two numbers a distance of 5 units from 2 can be written as $x+5$ and $x-5$. Don't force the issue if no one comes up with these answers.
d. $x-5, x+5$
e. $2-x, 2+x$

13. This number line will be used in the launch for section 2.1.

## Investigation

15. The students may generalize from their investigation that if they subtract the smaller of the two numbers from the larger, I - s , whether the numbers are positive, negative, or zero, the difference will always be positive and give the distance. The students may, however, look at cases of positives and negatives also, but still reach the same conclusion. Some students may decide to take the difference without regard to order and then take the absolute value of the difference to find the distance. These results will become more apparent in Section 2.4 when we discuss subtraction of integers.
16. For each pair of integers given below, find the distance between the two integers on the number line. See TE.
a. 40 and 70
b. 30 and 20
c. -80 and -40
d. -126 and -64
3 and 8
-2 and -5
-7 and -9
43 and 78
28 and 15
-87 and -49
17. A number is a distance of 5 from 13 . Is it possible to determine a single value for the number? Explain using a number line. See TE.
18. A negative number is a distance of 105 from 52. Is it possible to determine the value of the number? Explain. Yes; -53 is the only such number.
19. a. Find numbers that are a distance 5 from 2. See TE.
b. Find numbers that are a distance 4 from -3 .
c. Find numbers that are a distance 3 from -8 .
d. If $x$ represents some integer, what numbers are distance 5 from $x$ ?
e. What numbers are distance $x$ from 2?
20. If $x$ is the same distance from 6 as it is from 2 , what is $x$ ? 4
21. Create a number line that extends from -15 to 15 .
22. Ingenuity:

The distance between two cities on a highway is 118 miles. If all the exits between these two cities are at least 5 miles apart, what is the largest possible number of exits between these two cities? 23

## 15. Investigation:

Write a process for finding the distance between two numbers. Remember to address all possible cases: two positive numbers, two negative numbers, one of each, and at least one number equal to zero.

## Absolute Value Bingo

Objective: To help students practice their absolute value skills.

## Materials:

Copy of blank "Absolute Value Bingo" grids (one per person)
Two color counter chips (or any small item to be placed on the bingo cards)
Overhead transparency of bingo cards, cut them out to make 27 individual cards
"Absolute Value Bingo Answers" transparency

## Activity Instructions:

Pass out one blank "Absolute Value Bingo" grid to each student. Lay the "Absolute Value Bingo Answers" transparency on the overhead and have students copy 15 of the answers, in any random order, onto their grids. There are 16 spaces on the grids, the students can write FREE SPACE in the 16th box. (You might want to have them fill in their grids in pen or marker to avoid any temptation of cheating, and so the cards can be reused later). Pass out 10-15 chips to each student, and now the game is ready.

One at a time, place a bingo card on the overhead and have the students figure out the answer. To keep track of the correct answers, you may want to make a mark on your answer transparency of each problem you have presented to the class. The students will continue to figure out the absolute value cards until someone in the class has shouted "Bingo." You can decide whether to play a row, column, or diagonal at a time, or if you want to play blackout, around the world, or any other method.

After each round, it might be a good idea to have the kids exchange cards before you play again. Another good idea is to give the students more than one grid each.


Students: Select 16 numbers from the Absolute Value Bingo Answers sheet. There should be ONE number per box

## BINGO CARDS

| FIND THE DISTANCE <br> BETWEEN: <br> -10 AND 10 | FIND THE DISTANCE <br> BETWEEN: <br> 0 AND -12 | WHICH IS SMALLER? <br> 4 OR $\|-3\|$ |
| :---: | :---: | :---: |
| WHICH IS SMALLER? <br> \|-2| OR -2 | WHICH IS SMALLER? <br> 15 OR \|-10| | WHICH IS SMALLER? <br> 50 OR \|-100| |
| $>,<$, OR $=?$ <br> $\|15\| \square\|-15\|$ | $>,<$, OR $=?$ <br> $20 \square\|25\|$ | $>,<$, OR $=?$ <br> $-1 \square-\|-5\|$ |

## BINGO CARDS



## BINGO CARDS

| WHICH IS GREATER? $-\|10\| \text { or }\|-10\|$ | WHICH IS GREATER? $\|-8\| 0 R\|4\|$ | WHICH IS GREATER? $\|7\| \text { OR \|-13\| }$ |
| :---: | :---: | :---: |
| WHICH IS GREATER? \|-28| OR 25 | WHICH IS GREATER? \|13| OR 17 | FIND THE DISTANCE BETWEEN $\text { -10 AND } 0$ |
| FIND THE DISTANCE BETWEEN <br> 4 AND 9 | FIND THE DISTANCE BETWEEN <br> 3 AND -1 | FIND THE DISTANCE BETWEEN -25 AND 5 |



| 14 | 3 | 2 |
| :---: | :---: | :---: |
| 7 | 0 | 41 |
| 22 | -5 | -12 |
| -10 | $\|-8\|$ | $\|-13\|$ |
| $\|-28\|$ | 17 | 5 |
| 4 | 30 | 20 |
| $\|-3\|$ | -2 | $\|-10\|$ |
| 50 | $=$ | $<$ |
| $>$ | 12 | 10 |

## Number Lines for 1.3 Launch



1. a

b


C

d

2. The temperature rose $13^{\circ} \mathrm{C}$.
3. The temperature rose by $20^{\circ} \mathrm{F}$.
4. $1,2,3$ 5. The lliad was written 40 years before the Odyssey. 6. -5 7. 28

## Review Tips

Possible review activities might include running a game-style activity with the exercises, or have students work in groups or individually on problems before discussing as a class.

You might also try a card activity. There are many ways you can vary this, but here are a few ideas.
Take a deck of cards and tell the class that red cards are negative, black cards are positive, and jokers are zero. Aces are 1 or -1 depending on their color. You can choose to remove the face cards if you want, or you can assign each face card a value (i.e. Jacks are 11/-11, Queens are 12/-12, Kings are 13/-13).

Variation A: Split the class into two teams and have individuals representing each team come forward to the front of the room. Have the deck split roughly into three stacks, one in front of each team player, and one in front of you or another student serving as the moderator. The moderator (or you) draws a card and displays it to the class and the two teams. They then each draw from their pile and have to declare whether their number is greater than or less than the community card drawn. They can both be wrong and get no points, both be correct and get a point each for their team, or one can be correct (get a point) and the other not correct (no point).

Variation B: You can also play the same game, but use absolute values instead of greater than/less than. That is they compare their absolute value to the community card.

Variation C: The basic game is the same, but the students now decide the distance from their card to the community card.

You can play several rounds in one variation before switching to another variation.

## REVIEW PROBLEMS

1. For each part below, draw a number line with the three given integers marked. See TE.
a. $3,-4,6$
b. $20,-45,55$
c. $-8,0,-3$
d. $-1214,-1589,-1370$
2. At 7:00 A.M., Chicago's $0^{\prime}$ Hare Airport has a temperature of $-9^{\circ} \mathrm{C}$. At 11:00 A.m. that same day, the temperature reads $4^{\circ} \mathrm{C}$. Did the temperature rise or fall? Why? Can you determine by how many degrees? See TE.
3. At 4 P.M. in London, the temperature was $77^{\circ} \mathrm{F}$. At 6 A.M. the same day, the temperature was $57^{\circ}$. Did the temperature rise or fall? By how much? See TE.
4. Maggie has locked all of her money inside a safe and forgotten the combination. Luckily, Maggie left this note for herself: "The combination to open this safe is three positive integers. These positive integers are represented, in order, by the variables $a, b$, and $c$, where $c$ is three, $b<c$, and $a<b$." What is the combination to open the safe? See TE.
5. Homer was a Greek poet who produced several well-known epics during his lifetime. He wrote The Odyssey around 680 B.C.E. and The lliad around 720 B.C.E. Which of these two works was written first? How many years passed between these two dates? See TE.
6. a. Which of the following numbers is farthest from $0:-2,3$, or -5 ? See TE.
b. Which of the following numbers is closest to $14: 28,-1$, or -5 ? See TE.
7. Compare the pairs of numbers below and place the appropriate symbols between them. Use < or > .
a. 457
81
b. -23
$-32$
c. 191 $\square$-3
d. $|-9|$|5|
e. $|-17|$|-22| f. |156| $\square$|204|

## Studying the Planets:

Ask the students for a scientific explanation of the relationship they find between temperature and distance from the sun.

## 9. b. Mercury; Neptune

c. In general, the closer a planet is to the sun, the hotter the average temperature. But students may also find relationships between the length of days, time of revolution around the sun, and many other factors.

## Summary

Ask students to share the major ideas of the chapter, any issues that still concern them, and any questions they have. You might want them to write a short paragraph summarizing their learning in this chapter.
8. Find the distance between the following pairs of numbers.

Example: The distance between -2 and 9 is 11
a. 0 and -1 1
b. -8 and $0 \quad 8$
c. 0 and $-14 \quad 14$
d. 14 and $0 \quad 14$
e. -12 and $16 \quad 28$
f. -4 and $24 \quad 28$
g. -20 and 1030
h. -20 and $-40 \quad 20$
i. 20 and $-20 \quad 40$
j. 25 and -1540
k. -25 and 2550
l. 17 and $17 \quad 0$
9. Answer the following questions using the table of information for each planet in the solar system. Justify your answers.

| Studying the Planets |  |  |
| :---: | :---: | :---: |
| Planet | Distance from Sun <br> (million km) | Average temperature <br> (C) $)$ |
| Mercury | 57.9 | 427 |
| Venus | 108.2 | 464 |
| Earth | 149.6 | 15 |
| Mars | 227.9 | -63 |
| Jupiter | 778.3 | -144 |
| Saturn | 1427 | -176 |
| Uranus | 2869.6 | -215 |
| Neptune | 4496.6 | -215 |

a. Which planet is the hottest? Which is the coldest? Venus; Uranus \& Neptune
b. Which planet is closest to the sun? Which is the farthest from the sun?
c. Is there a relationship between a planet's distance from the sun and its average temperature? Explain. See TE.

## Section 1.2:

Answer: 84
Solution:
Consider letting $b$ have vales from 2 through 8 . When $b$ is 2 , there is one possible value for $a$ and seven possible values for c . Thus there are $1 \cdot 7=7$ solutions. When b is 3 there are two possible values for a and six possible values for c so there are $2 \cdot 6=12$ solutions. Continuing in this fashion we have $1 \cdot 7+2 \cdot 6+3 \cdot 5+4 \cdot 4+5 \cdot 3+6$ $\cdot 2+7 \cdot 1=84$

## Section 1.3:

Answer: 6

## Solution:

We certainly need at least 5 integers, so that there are 10 distances to measure. Without loss of generality, our collection includes 0 and 10 to give us the distance 10. Now we need to get a distance of 9 , so we can assume that 1 is in our collection, which gives us $10-1=9$ and 1-0 $=1$. We need a distance of 8 , which we can get by adding 2,8 , or 9 . 2 and 9 would give us duplicate distances to the previous numbers, though, so we have to use 8 , which gives us $8-0=8,8-1=7$, and $10-8=2$. Now we have the numbers $0,1,8$, and 10 , and we still need distances of $3,4,5$, and 6 . There is no number that has those four distances from the previous numbers, but we can get them by including 5 and 6 , for a total of 6 integers. Thus $\{0,1,5,6,9,10\}$ is a solution. Others are $\{0,1,3,4,8,10\},\{0$, $2,3,4,9,10\}$ and $\{0,2,5,6,9,10\}$.

## CHALLENGE PROBLEMS

## Section 1.2:

If $a, b$, and $c$ are positive integers such that $a<b<c<10$, how many possible values are there for the three numbers?

## Section 1.3:

A collection of integers has the property that every integer from 1 to 10 inclusive is the distance between some two numbers in the collection. What is the least possible number of integers in the collection?

## Section 2.1 - Addition of Integers

## Big Idea:

Adding integers using a number line
Addition is going forward on the number line.

## Key Objective:

Observe that the direction to face depends on the sign of the integer.

## Pedagogical/Orchestration:

Using the activities, begin to stress key words for positive and negative, such as forward, backwards, ascend, descend, deposit, withdrawal, up, down, rise, drop, below sea level, etc. Maybe as an exercise, have kids use their own number lines from Chapter One to add the key words to the number lines created in this section. Teachers could keep a running list of words that mean positive and negative.

## Internet Resource:

Matching Game: Adding Integers- http://www.quia.com/cm/112736.html

## Materials:

- Counters, Cars or cutouts of cars from the back of the section or the CD
- Sentence strips for making number lines
- Adding Machine Tape
- Number line created in Exercise 12 from Section 1.3
- Deck of playing cards


## Activity:

Car Activity at the end of the section and on the CD
Double Sided Chip Addition Activity at the end of the section and on the CD

## Exercises:

Students work on number lines with their cars to solve the addition problems and start the reflection process of noticing patterns.

## Vocabulary:

Net yardage, (For the teacher) nested parentheses
TEKS:
6.13(A); 7.2(C); 7.2(E)(F); 7.9(A); 7.14(A); 7.15(A) 8.1(A); 8.2(B); 8.15(A); 8.16(A);

## Note to Teacher:

Challenge 2.1 needs to be done before Challenge 2.2

## WARM-UPS for Section 2.1

1. Which of the following fractions is between $\frac{2}{3}$ and $\frac{3}{4}$ ? Explain why.
a. $\frac{3}{5}$
b. $\frac{4}{5}$
C. $\frac{7}{10}$
d. $\frac{1}{2}$
2. Polly Nomial's science class wants to make a thermometer, but they have learned from experience that it is better to make a sketch before they begin construction. The class wants to include the following temperatures on their thermometer: $\quad-23^{\circ} \mathrm{F}, 7^{\circ} \mathrm{F}, 38^{\circ} \mathrm{F}$, and $-9^{\circ} \mathrm{F}$ (it's a thermometer for a cold place). Sketch a design of a thermometer that is practical for these temperatures. Do not label every degree. Ans: (Possible answer) Range from $-60^{\circ} \mathrm{F}$ to $60^{\circ} \mathrm{F}$ and label every $5^{\circ} \mathrm{F}$.
3. Look at the table below and answer the questions:

| Element Name | Approximate melting point <br> in degrees Celcius |
| :---: | :---: |
| Hydrogen | -259 |
| Fluorine | -220 |
| Radon | -71 |
| Mercury | -39 |
| Bromine | -7 |
| Francium | 27 |
| Rubidium | 40 |
| Sodium | 98 |
| Tin | 232 |
| Neon | 248 |
| Silver | 961 |
| Tungsten | 3422 |

a. How much hotter is the melting point of Francium than Bromine?
b. How much hotter is the melting point of Rubidium than Mercury?
c. How much hotter is the melting point of Sodium than Radon?
d. How much hotter is the melting point of Tin than Fluorine?
e. How much hotter is the melting point of Silver than Neon?
f. How much hotter is the melting point of Tungsten than Hydrogen?

1. Kayla McNutt has a family of squirrels living in a tree in her yard. She observes them for a few days and records her findings.

| Day of the Week | Pecans Collected | Pecans Eaten (by the <br> squirrels, of course) |
| :---: | :---: | :---: |
| Monday | 30 | 12 |
| Tuesday | 23 | 9 |
| Wednesday | 25 | 11 |

If the squirrel family has 321 nuts before Monday, how many do they have Thursday morning?
a. 370
b. 367
c. 365
d. 362

Answer: (b) because $321+30-12+23-9+25-11=367$ pecans

## Launch for Section 2.1:

Demonstrate how to think of the set model of addition. Have a handful of counters in each hand and demonstrate putting them together and totaling their sum. This is a model most students are familiar with from previous grades. Tell your students that today they will learn a new model for adding integers. Have the students take out their number lines that they created in Exercise 12 from Section 1.3 that extends from -15 to 15 . This is a good size so that the little cars can be used later on for "driving" on the number line. Make sure that students are putting some thought into equally spacing the numbers; however, allow them to come up with their own strategies: paper folding, use of ruler, etc. Discuss the different strategies that students use. Tape the number lines to the students' desks, or hole punch them so they can be put in their binders and used as needed. Tell students, "Today we will use these number lines to develop the concept of adding integers."

Start a brainstorm list on the board about patterns that the students notice throughout Sections 1 and 2, including exercises. This will continue through Section 2.2.

Walk your students through the CLASS EXPLORATION so they know how to use their cars. Model this at the board with the two board cars you created from the pdfs. Be sure that they understand the meaning of pointing the car right (positive) or pointing the car left (negative). Because we are adding, the car moves forward in the direction it is pointing.

# ADDING AND <br> SUBTRACTING ON THE NUMBER LINE 



## SECTION 2.1 ADDITION OF INTEGERS

Addition is a mathematical operation for combining integers. Pictorially, using the "set model," when we add two integers we are combining the sets. To add 4 and 3 we draw the picture below:


We can also use our number line model to describe addition.

## CLASS EXPLORATION: DRIVING ON THE NUMBER LINE WITH ADDITION

We can visualize adding two numbers using a car driving on the number line. The final location gives the sum. Let us practice how this works on a small scale. Use the number line you constructed in Chapter 1 from -15 to 15 as your highway. You will also need a model car or something that can represent this model car.


Step 1: Place your car at the origin, 0 , on the number line.
Step 2: If the first of the two numbers that you wish to add is positive, the car faces right, the positive direction. If the first of the two numbers is negative, the car faces left, the negative direction. Drive to the location given by the first number. Park the car.

Step 3: Next examine the second of the two numbers. If this number is positive, point the car to the right, the positive direction. If the second number is negative, point the car to the left, the negative direction.

Remind your students that the work and check, or more often called the guess and check, method is one of the strongest in learning mathematics.

Go through the three examples together. Notice the differences.

- $\quad$ The first is a positive + a positive.
- $\quad$ The second is a negative + a positive.
- $\quad$ The third is a negative + a negative.

In Section 2.4, we will discuss the commutative property, which will help students see that adding a negative and a positive is just like adding a positive and a negative.

After completing Example 1, have students work with a partner to demonstrate additional problems using the car model. This should be done after each example for this section.

Step 4: Because you are adding, move the car forward, in the direction that it is facing, the distance equal to the absolute value of the second number.

Use your car and the four-step process to compute each of the following examples. Attempt the process on your own first, and then compare your answer with the provided solution.

## EXAMPLE 1

Find the sum $3+4$, and describe how you obtain your answer using the number line model.

## SOLUTION

The two numbers we are adding are 3 and 4 (which we also know as +3 and +4 ).
Step 1: Begin with your car at 0 .
Step 2: Because the first number is positive, the car faces to the right. Drive to the location +3 . Park the car.

Step 3: Point the car to the right because the second number, +4 , is positive.
Step 4: Move the car 4 units to the right. Park the car.
You are now at location 7.


## EXAMPLE 2

Find the sum $-3+4$. How do we start the process? In which direction does your car move first and how far? Explain how you reached your solution using a car on your number line.

Don't worry about discussing the rules of addition now. That is, do not feel you have to tell your students that two positives results in a positive; two negatives results in a negative; or when adding two different signs, subtract and take the sign of the larger absolute value. The first problem in the exercises is designed to get students thinking about this pattern. Let them discover it.

## Summary

Pair students and distribute one playing card to each student. Red cards represent negative numbers and black cards represent positive numbers. Ignore the face cards if you want. Jokers equal 0 and Aces are 1 or -1 .

Have each student pair create an illustration of a number line and the car process that models their two cards. The first illustration should be Student A's card + Student B's card. The next illustration should be Student B's card + Student A's card.

Possible dialog: "If in your group Student A's card + Student B's card was the same as Student B's card + Student A's, let me know. Why do you think this happened?"

It is because of the commutative property, covered later in Section 2.4.

When ready, ask the class to summarize this process of using the number line for addition.

After the exercises, make a list of the four example problems you did together as a class and the four correct solutions. See if the class can come up with any theories about adding integers. Why are some of the answers positive and some negative? Why are some of the answers the sum of the two numbers, while others are the difference? Please don't give the students the "rule" for adding integers, but see if you can lead them into figuring it out for themselves.

## SOLUTION

Step 1: Begin at 0 .
Step 2: Because -3 is a negative number, point the car to the left and drive 3 units. Park the car.

Step 3: Because 4, the number added to -3 , is positive, turn the car to face right.
Step 4: Move 4 units to the right, ending up at location 1. Park the car.
The result $-3+4=1$ is demonstrated below:


## EXAMPLE 3

Find the sum $-3+(-4)$, or simply $-3+-4$, using the same process.

## SOLUTION

Point the car to the left and move forward 3 units. Leave the car pointing to the left because the next number is negative. Move the car forward 4 units to the location -7 . We have: $-3+(-4)=-7$.


Remember, when you are adding, the car always moves forward, in the direction that it is facing. The signs of the integers tell us whether we face right, if positive, or left, if negative, before moving.

## EXERCISES

1. a. 3
C. -5
b. 3
d. -2
2. a. 0
c. 0
b. 0
d. 0
Foreshadowing additive inverse.
3. а. 7
f. 7
b. -9
g. -9
c. 4
h. 4
d. -2
i. -2
e. -7
j. -7
4. Your students should observe that adding two positives yields the positive sum of the numbers. Adding two negative yields the negative sum. Adding a positive and a negative yields the difference between the two numbers. The sign in this case is the sign of the number with the largest absolute value.
5. 

a. 6, 6
d. 0,0
b. 1, 1
e. $-5,-5$
c. 7,7

## EXERCISES

For exercises 1-4, find each sum. You may use your car and number line.

1. a. $3+0$
c. $0+-5$
b. $0+3$
d. $-2+0$
2. a. $-1+1$
c. $8+-8$
b. $2+-2$
d. $0+0$
3. a. $3+4$
f. $4+3$
b. $-2+-7$
g. $-7+-2$
c. $-1+5$
h. $5+-1$
d. $6+-8$
i. $-8+6$
e. $0+-7$
j. $-7+0$
4. Write rules to describe any patterns you see in problems $1-3$. See TE.

- Do you see a pattern when adding two positives?
- adding two negatives?
- adding a positive and a negative?
- adding a negative and a positive?

Explain how your rules work using a number line.
5. For this exercise, let's pay careful attention to the order in which we add. We use parentheses to specify order. For example, $(1+2)+3$ means first add 1 to 2 , and then add the result and $3 ; 4+(5+6)$ means first add 5 and 6 , and then add 4 and the result. Calculate the following sums:
a. $(1+2)+3$
d. $-3+(0+3)$
$1+(2+3)$
$(-3+0)+3$
b. $(2+4)+-5$
e. $(-7+8)+(-2+-4)$
$2+(4+-5)$
$-7+((8+-2)+-4)$
c. $8+(-7+6)$
$(8+-7)+6$
6. a. 17
e. -1
b. -12
f. 6
c. -11
g. 3
d. -12
h. 0

Teacher Tip for Exercise 7: You might demonstrate how to do this with vertical lines and arrows so students can see that the number line does not have to be horizontal. Later you might remind them about the $y$-coordinate.

Teacher Tip: Throughout these exercises, we keep emphasizing working through the algorithm physically. We feel this is very important for students learning a new skill like adding integers.
7. (c) and (d) are good activities for a class floor number line. You might find that you have some students who only respond if they can act out the addition of integers themselves.
11. Ask your students if they know what overdraft protection is. If not, explain to them how they can have a negative balance with overdraft protection and not get into trouble.
6. Predict the sign of the answer. Then find the sums. Use the number line if you need to.
a. $8+9$
h. $4+-5$
b. $-7+-5$
i. $-4+5$
c. $-7+5$
j. $-3+9$
d. $-4+-5$
k. $7+-4$
e. $-9+-2$
|. $-7+4$
f. $9+-2$
m. $-6+-6$
g. $-6+6$

For exercises 7-14, write each problem as an addition problem and use positive and negative numbers where appropriate. Show your work on a number line.
7. a. Jacob observes that the temperature is $-3^{\circ} \mathrm{C}$. If it rises $7^{\circ} \mathrm{C}$ in the next two hours, what will the new temperature be? $-3+7=4^{\circ} \mathrm{C}$
b. Marissa observes that the temperature is $2^{\circ} \mathrm{C}$. During the night, it falls $9^{\circ} \mathrm{C}$. What was the low temperature that night? $2+-9=-7^{\circ} \mathrm{C}$
c. Adrian takes 8 steps forward then takes 6 steps back. How far is Adrian from where he started? 2 steps forward, $8+-6=2$
d. David takes nine steps backward and then three steps forward. At what location does he end? 6 steps backwards, $-9+3=-6$
8. Jennifer checks the temperature and it is $-9^{\circ} \mathrm{C}$. If the temperature warms up $8^{\circ} \mathrm{C}$, what is the new temperature? $-9^{\circ} \mathrm{C}+8^{\circ} \mathrm{C}=-1{ }^{\circ} \mathrm{C}$
9. It was $-5^{\circ} \mathrm{C}$ in the morning. The temperature rose $8^{\circ} \mathrm{C}$. What is the temperature now? $-5+8=3^{\circ} \mathrm{C}$
10. Juan checks the temperature and it is $5^{\circ} \mathrm{C}$ at 5 P.M. By 10 P.M., the temperature has dropped $8^{\circ} \mathrm{C}$. What is the new temperature? $5+-8=-3^{\circ} \mathrm{C}$
11. Chris has $\$ 17$ in his bank account. If he withdraws $\$ 20$, what will his balance be? $\quad 17+-20=-\$ 3$
12. The temperature in St. Paul, Minnesota is $-4^{\circ} \mathrm{F}$ on a cold winter day. If the temperature falls another $7^{\circ} \mathrm{F}$, what will the new temperature be? $-4+-7$ $=-11^{\circ} \mathrm{F}$
13. Some students may not be familiar with football. Some introduction to the basics of football may be in order. The main idea needed in this problem is for the students to be aware that a player moves the ball up or down the field, and the opposing team tries to prevent the player from progress towards his team's goal. Forward progress measured in yards is a gain, and we designate this as a positive number. We will designate loss in yards as a negative number.

## Investigation

15. At this point we are not concerned with a correct answer. We are concerned with students looking for patterns, making conjectures, and testing them. You might want to expand this into a class exploration in anticipation of the next section.
16. If a football player loses 6 yards in one play, loses 2 yards in another play, and then gains 4 yards in the final play, what is the net gain or loss? $-6+-2+4$ $=-4$ yards

## 14. Ingenuity:

Assume that, in the diagrams below, each of the small squares has sides of length one inch. Find the perimeter of each of the figures below. What surprising result do you notice?


40 inches, 40 inches, 40 inches, all of the perimeters are the same!
15. Investigation:

With our car model, the car moves forward when we add. What do you think we should do when we want to subtract a number from another number? Write your best guess about how to subtract two numbers. See TE.

## CAR ACTIVITY



Objective: This activity is an extension to the example in the math book that will help the students visualize the motion of the car as it moves on the number line.

## Materials:

An evenly spaced number line on the board marked from -15 to 15
Both car cut-outs (left and right)
Magnetic Tape

## Activity Instructions:

1) Make your number line on the board.
2) Copy both car cut-outs, and color them however you see fit. Attach the magnetic tape to the back of the cut-outs so the cars will stick to the board.
3) Use the number line and the cars to demonstrate the motion of several examples of integer addition and subtraction problems. Let the students come to the board and move the cars if time allows.


Diagram for Ingenuity 2.1


## Double-Sided Chip Addition

Objective: Students will increase their ability to add integers using double-sided chips

## Materials:

Double-sided chips (coins or checker pieces can be used as long as the two sides are different)

## Activity Instructions:

1) We establish one side of the chip to be positive (+) and the other side to be negative (-).
2) We also consider any pair of (+) and (-) to be equivalent to zero and we'll say: $(+)(-)=0$
3) The Chip Addition is based on a set union concept. For example: $3+2=5$ is demonstrated with $(+)(+)(+) \cup(+)(+)$ gives us $(+)(+)(+)(+)(+)$ all together.
$3+-2=1$ is demonstrated with $(+)(+)(+) \bigcup(-)(-)$ giving us $(+)(-)(+)(-)(+)$ or $(+)$
*Note that subtraction exercises have not been pre-written for this activity.

## Section 2.2 - Subtraction of Integers

## Big Idea:

Subtraction is moving backward.

## Key Objectives:

- Observe that subtraction is like adding a negative.
- Observe that subtracting a negative is like adding its positive.


## Pedagogical/Orchestration:

- Students continue the reflection process of noticing patterns.
- Teacher can start a list on the board or chart paper of patterns noticed.
- Students continue using cars and number lines to work through examples and exercises.


## Materials:

- Sentence strip number lines created by students
- Cars
- Number line handout from CD to help with exercises
- Playing cards


## Activity:

Continue the Car Activity from Section 2.1. Use examples in book with students physically (using cars) and demonstrating (acting out) the given subtraction problems.

Lily Pad Race Activity from the end of this section and on the CD

## Exercises:

After Exercise 1, have class discussion and ask students what patterns they see. They should discover the equivalence of these two computations: $3-4$ means 3 subtract a positive $4 ; 3+(-4)$ means 3 plus the opposite of positive 4 . Make sure you use groups of related problems to help students see patterns, such as: $-4-3 ;-4+-3 ;-4--3$; $-4+3$.

## Vocabulary:

backwards, subtraction

## TEKS:

6.13(A); 7.2(C); 7.13(A); 7.14(A); 7.15(A) 8.1(A); 8.2(B); 8.15(A); 8.16(A);

## WARM-UPS for Section 2.2 (Subtraction of Integers)

1. Anna has $\$ 20$, saved $\$ 32$, spent $\$ 17$, earned $\$ 5$, and lost $\$ 15$. How much money does she have?

Answer: 20 + $32-17+5-15=\$ 25$
2. Ben is designing a new chip for a computer. The length of the chip needs to be between 7.45 and 7.75 centimeters. Which of the following lengths are within these specifications? Explain your answers.
a. 7.6 cm
b. $\quad 7.4 \mathrm{~cm}$
c. $\quad 7.38 \mathrm{~cm}$
d. 7.8 cm

## Answer: a

3. (Extra) Draw a model, using a number line to illustrate the following. What do you notice?
a. $6+-5$
b. $-5+6$
c. $-2+-5$
d. $-5+-2$

## Launch for Section 2.2:

Have a number line with integer values spaced about a foot apart on the wall. Place a rolling chair at the zero and ask for a student volunteer. Have the student sit in the chair, and ask the class the following question: "How can we model $2+3$ with this car model?" Follow the students' directions by pushing the student in the chair. Now ask the students, "What do you think the car should do when we want to subtract?" Allow students to answer. If no one suggests moving backwards, then move the student backwards in the chair and ask students what that would represent. Then tell students, "Today we will be making sense of subtracting numbers on a number line." Proceed with Example 1.

It might help some of your students if you remind them that going backward is like driving with the car in reverse.

It is very important for kids to write these steps in their own words so that they are clear about what they are doing. This is helpful to English Language Learners so that they are "learning" the language as they write each step.

Have students explore Examples 2 and 3 at the same time. Have class share their attempts.

## EXAMPLE 1

Your students already know that $5-2$ equals 3 . This example will simply show proof that the answer is indeed 3. Before moving on to the next example, be sure that the students understand the movement procedure that produces the correct answer. Students might also use a sentence strip number line for this problem.

Have students work with a partner to demonstrate subtraction problems using the car model. This should be done after each Example in this section.

Model the three examples with the class, having your students draw and act out the movements on their number lines. Summarize the process with your students.

Proceed thoughtfully and carefully through this to make sure each student is understanding the model. Subtraction can be tricky.

A student might reason in the following way: Because 5 is a positive number, the car faces to the right. The second number is also positive, so the car remains facing to the right. The subtraction sign just means we are putting the car in reverse 2 units. The car will end up at 3 .

## SECTION 2.2 SUBTRACTION OF INTEGERS

With our car model, addition involves moving the car forward in the direction indicated by the signs of the numbers we are adding. What do you think we should do when we want to subtract one number from another number? One way to model subtraction is to use our number line and move the car backward, the opposite of forward.

Try each example first, and then check your answer by comparing it to the solution given. Describe each of the steps you are using in words.

## EXAMPLE 1

Compute the difference $5-2$, and show how to model this with a number line. Use a model car and a number line to simulate your solution. What is your final location?

## SOLUTION

Step 1: Place your car at the origin. Because the first number is positive, face the car to the right.

Step 2: Move the car forward 5 units to the location given by the first number. Park the car.

Step 3: Point your car to the right because the number being subtracted is positive.

Step 4: Now instead of moving forward 2 spaces, move backward 2 spaces, ending up at location 3 . Remember, we move backward because we are subtracting. We can write this movement as $5-2=3$.


## EXAMPLE 2

At this point, ask your students to consider why this problem produced a negative solution.
Compare to set model: You have 2 apples. Take every 5 apples. You have -3 apples left. (Laughter)

## EXAMPLE 2

Compute the difference $2-5$. Use a number line to show how you solved the problem.

## SOLUTION

Step 1: Place your car at the origin. Because the first number is positive, face the car to the right.
Step 2: Move the car forward 2 units to the location given by the first number. Park the car.

Step 3: Point your car to the right because the number being subtracted is positive.

Step 4: Now since we are subtracting, move backward 5 spaces, ending up at location -3 . We can write this movement as $2-5=-3$.


## EXAMPLE 3

Compute the difference $-7-3$. Use a number line to show how you solved the problem.

## SOLUTION

Step 1: Place your car at the origin. Because the first number is negative, the car faces left.

Step 2: Move the car forward 7 units to the location given by the first number, -7 . Park the car.

Step 3: Point your car to the right because the number being subtracted is positive.

EXAMPLE 4 Complete this example as a class.

Teacher Tip: Before moving on to the exercises, make sure your students have a really good understanding about why the car changed directions in this particular problem. You might need to do a few more examples together with the class before they are ready to work alone.

The first number is positive, so the car faces right and moves two spaces to the right. The second number is negative, so the car faces left. This time, the car needs to move opposite the direction it is facing, so we end up at 7 .

## Summary

Have a class discussion of how subtraction is different from addition. Students should understand that to subtract, you follow the same procedures as when adding, but go backwards as a final step.

Using the deck of cards from the Integer War, deal the cards giving one to each student, and pair students up to create examples of their subtraction problems. Remember, reds are negative, blacks are positive, and jokers are zero. Have each group subtract Student A's card - Student B's card followed by Student B's card - Student A's card.

When finished, have the class share its observations.
When you subtract a negative, be aware that students will ask how to write a subtraction of a negative. For instance, $4--3$ or $4-(-3)$.

Step 4: Now move backward 3 spaces, ending up at location -10. Be careful! This time your car was pointing to the right, so when you back up you will move backwards to the left. We can write this movement as $-7-3=-10$.


## EXAMPLE 4

Compute the difference $2-(-5)$. Use a number line to show how you solved the problem.

## SOLUTION

Step 1: Place your car at the origin. Because the first number is positive, face the car to the right.
Step 2: Move the car forward 2 units to the location given by the first number. Park the car.

Step 3: Point your car to the left because the number being subtracted is negative.
Step 4: Now move backward 5 spaces, ending up at location 7. Be careful! This time your car was pointing to the left, so when you back up you will move to the right. We can write this movement as $2-(-5)=7$.


## SUMMARY

In order to compute $(x-y)$, we proceed as follows:

## EXERCISES

1. a. $4-3=1$
$4+-3=1$
f. $\quad 7--3=10$
$7+3=10$
b. $3-5=-2$
$3+-5=-2$
c. $-2-4=-6$
$-2+-4=-6$
g. $1--8=9$
$1+8=9$
h. $-2--4=2$
$-2+4=2$
d. $0-5=-5$
$0+-5=-5$
i. $0--9=9$
$0+9=9$
e. $-3-2=-5$
$-3+-2=-5$
j. $\quad-7-4=-11$
$-7+-4=-11$

Reflect on patterns noticed. Start a brainstorm list on the board or wall. Add observations to the list after each example and exercise.
2. a. $4-2=2$
$2-4=-2$
e. $\quad 6--7=13$
$-7-6=-13$
b. $2-3=-1$
f. $5--2=7$
$3-2=1$
$-2-5=-7$
3. For this exercise $e$ use the cumulative class list of of patterns $\ddagger=0$ have the class write rules for adding and subtracting integers.
Subtraction is the inverse, or opposite, of addition. ${ }^{-1} \overline{\text { Subbtracting }}-7$ a positive is like adding a negative. Students

 $(-y)$." Another observation students might make is "Adding is subtracting the inverse, or $x+y=x-(-y) . "-x-y$ and y - x are opposites.

Step 1: Place the car at 0 , the origin. Then face the car in the direction of the sign of the first number x .

Step 2: Move the car $|\mathrm{x}|$ units forward in the direction the car faces. Park the car.
Step 3: Next, face the car in the direction of the sign of the second number, $y$.
Step 4: Move the car |y| units backward, the opposite direction from what the car faces. The car is positioned on the difference $(x-y)$.

## EXERCISES

Use the car model with your number line to calculate each of the following exercises. Drive carefully.

1. a. $4-3$
$4+-3$
b. $3-5$
$3+-5$
c. $-2-4$
$-2+-4$
d. $0-5$
$0+-5$
e. $-3-2$
$-3+-2$
2. a. $4-2$

2-4
b. $2-3$

3-2
C. $-4-8$

8--4
d. $0-5$

5-0
f. $7--3$
$7+3$
g. $1--8$
$1+8$
h. $-2--4$
$-2+4$
i. $0--9$
$0+9$
j. $\quad-7-4$
$-7+-4$
e. 6--7
-7-6
f. $5--2$
-2 - 5
g. $-1--8$
-8--1
h. $-3--2$
$-2--3$
3. What patterns do you see in Exercises 1 and 2? In your own words, write a rule for each pattern that you observe. See TE.
4. a. $8+2=10$
d. $-5+-4=-9$
$-5-4=-9$
b. $2+6=8$
$2--6=8$
c. $-3--4=1$
$-3+4=1$
e. $-7+3=-4$
$-7--3=-4$
f. $-6--6=0$
$-6+6=0$
5. a. $3-9=-6$
b. $8-5=3$
c. $-2-5=-7$
d. $-1-(-5)=4$
$3+-9=-6$
$8-(-5)=13$
$-2+-5=-7$
$-1+5=4$
$3-(-9)=12$
$8+-5=3$
$-2-(-5)=3$
$-1+-5=-6$
4. Solve the following exercises using the car model and your rules from the previous exercise.
a. $8+2$
$8--2$
d. $-5+-4$ -5 - 4
8-2 $-5+4$
b. $2+6$
e. $-7+3$
2--6
$-7--3$
2-6
c. $-3--4$
$-3+4$
-3-4
f. $-6--6$
$-6+6$
5. Calculate the following sums and differences. Use the car model as needed.
a. 3-9
$3+-9$
3 - (-9)
b. 8-5
$8-(-5)$
$8+-5$
c. $-2-5$
$-2+-5$
$-2-(-5)$
d. $-1-(-5)$
$-1+5$
$-1+-5$

For Exercises 6 and 7, write a subtraction expression and compute.
6. It was $7^{\circ} \mathrm{C}$ at 8 A.M. The temperature dropped $8^{\circ} \mathrm{C}$ over the next four hours. What was the temperature at noon? $7{ }^{\circ} \mathrm{C}-8^{\circ} \mathrm{C}=-1{ }^{\circ} \mathrm{C}$
7. Nick opens a savings account at the bank in September. He deposits $\$ 20$ in September, $\$ 35$ in October, and $\$ 25$ in November. He needs to withdraw $\$ 50$ in December for holiday presents he wants to buy for his family. What is his balance after he makes his December withdrawal? (\$20 + \$35 + $\$ 25)$ - $\$ 50=\$ 30$ balance
8. Start at the beginning. Maria takes 2 steps forward and one back, ending at the first step. For every step's progress, Maria takes 3 steps. Check the number of steps Maria takes to reach step 2 on her journey. She starts on step 1 , goes to step 3, then back to step 2. Maria's progress is from step 1 to step 2, but to do that, she has taken three more steps for a total of six steps to end up at step 2. So for every step's progress, she must take 3 steps. After step 28, Maria takes 2 steps forward and reaches the mailbox before she has a chance to step back. So (3)(28) $+2=86$ steps. The best way to really understand this is to do it physically, counting steps.

Investigation: Skip Counting and Scaling
The investigation is foreshadowing the operation of multiplication.
9. For (a) and (b) it would be useful to have a sheet with number lines ready for students so they do not spend too much time drawing number lines.

Use 9 (c), (d), and (e) to help your students realize that they are not dealing with another mathematical universe when they are skip counting left, using the negative numbers. Also remind them what skip counting led to in the third grade. Let them remember how skip counting is to multiplying as crawling is to walking. When you crawl, you can get to most of the places you can when you walk. It's just a longer and harder way to get there.

The purpose of this investigation is to let students discover that the product they get when they multiply a positive by a negative is logical when they consider the number line.

## 8. Ingenuity:

Maria has an unusual morning ritual that she performs when she goes outside to get her mail. The distance from her front door to her mailbox is 30 steps. She steps outside the front door, takes two steps forward, and then takes one step back. She then takes another two steps forward and one step back. She continues doing this until she reaches her mailbox. In all, how many steps does Maria have to take before she gets to her mailbox? 86 steps
Hint: Working this out for a 30 -step trip can be quite difficult. You might want to start by seeing what happens if the mailbox is closer to the front door, perhaps 5 steps rather than 30 .
9. Investigation: Skip Counting and Scaling

When we initially built our number line, we used each mark to indicate one unit. The number line then corresponded to the integers $1,2,3, \ldots$. For larger numbers, we let the marks represent bigger lengths. So if each mark represents 5 units, then the marks correspond to the multiples of 5 , and we have $5,10,15,20, \ldots$ using "skip counting" by 5 's.
a. Build a number line where each mark represents 3 units, skip counting by $3^{\prime}$ 's. What is the $10^{\text {th }}$ number to the right of 0 ? What is the $10^{\text {th }}$ number to the left of 0 ? $30-30$
b. Build a number line where each mark represents 10 units, skip counting by 10 's from -40 to 40.
c. Make a table of the numbers you get when skip counting by 2's to 20 , skip counting by 3 's to 30 , skip counting by 4's to $40, \ldots$ and skip counting by 10 's to 100 .
d. Now skip count by $2^{\prime} s, 3$ 's, 4 's, ... up to $10^{\prime}$ 's in the opposite direction. Make a table of the numbers you get when skip counting by 2's to -20, by 3 's to -30 , by 4 's to -40 , etc.
e. Do you notice any patterns when you skip count? For example, the $5^{\text {th }}$ number when skip counting by 3 's is 15 . This is the same as the 3 rd number when you skip count by 5's. Does this kind of symmetry always hold? See TE.

## Lily Pad Race

Objective: Adding / Subtracting Integers

## Materials:

Integer cards from the CD (copy on card stock and cut out)
Lily Pad board game from the CD (one per group)
Coins or some other small items to use as game pieces
Paper / pencil (if necessary to help with solving integer problems)

## Activity Instructions:

Students should be divided into small groups and given the following set of instructions for playing the game.

1) Pick a card and solve problem.
2) Move the number of spaces as your answer.
3) If your answer is:
a. Positive = move forward
b. Negative = move backwards
4) The player who reaches a Lily Pad first on either end of the board game wins the game.

## Section 2.3 - Adding and Subtracting Large Integers

## Big Ideas:

Transition from concrete to abstract models of adding/subtracting large integers.

## Key Objectives:

- Develop rules for adding/subtracting integers.
- Use rough number lines to add and subtract larger integers.


## Pedagogical/Orchestration:

- Encourage students to use rough number lines to visualize adding and subtracting larger integers.
- Make sure students are rewriting subtraction equations as addition equations. Subtracting is like adding the opposite.
- A major goal of this section is to encourage students to create their own rules for adding integers. These include rules for adding two positives, two negatives, and one positive/one negative. The textbook purposely does not state the rules. Although students could count marks along the number line when adding and subtracting small integers, that technique becomes much more cumbersome with large numbers. Developing a more efficient strategy for larger numbers is important as long as the students still understand the reasons these strategies work.
- Have students create posters of the rules for adding integers created from the discussion of Class Explorations. Individual students might want to write the rules down and keep them in a math notebook.
- Students need to work through the example problems without using their books, as the solutions are given below each example.


## Materials:

- Number lines
- Poster paper or chart paper for students to create posters
- Marker
- Deck of Cards for playing Twenty-Five


## Activity:

Twenty-Five Activity at the end of the section and on the CD
Addition Board Game Activity at the end of the section and on the CD

## Exercises:

Exercises 1-12 appropriate for 6th grade.

## Vocabulary:

rules, inverse operations

## TEKS:

6.11(A)(C);
7.2(C);
7.13(A);
7.14(A); 7.15(A)
8.1(A); 8.2(B);
8.15(A);
8.16(A);

## WARM-UPS for Section 2.3

1. The national debt at the beginning of July 1,2010 was approximately $\$ 13,000,000,000,000$. If the debt increases by 4 billion a day, how much will the debt be at the end of July 12,2010 ?
a. $13,480,000,000,000$
b. $13,004,800,000,000$
c. 13,048,000,000,000
d. $13,000,480,000,000$

Answer: c
2. The high and low temperatures for a village in Alaska for a four-day period are given in the chart below. What was the average high temperature for these four days?

| Day | 1 | 2 | 3 | 4 |
| :---: | :---: | :---: | :---: | :---: |
| High | $24^{\circ} \mathrm{F}$ | $16^{\circ} \mathrm{F}$ | $30^{\circ} \mathrm{F}$ | $22^{\circ} \mathrm{F}$ |
| Low | $8^{\circ} \mathrm{F}$ | $-8^{\circ} \mathrm{F}$ | $16^{\circ} \mathrm{F}$ | $-4^{\circ} \mathrm{F}$ |

## Answer: Average high temperature was $23^{\circ} \mathrm{F}$ and average low was $3^{\circ} \mathrm{F}$

## Launch for Section 2.3:

Ask the class the following three questions, allowing just 15 to 30 seconds for responses. (1) "Who can come up with an example where a positive integer plus a negative integer is negative?" (2) "Will a positive plus a negative always equal a negative value?" (3) "Give an example where a positive integer plus a negative integer is positive."
"So, in your groups, discuss why sometimes the sum of a positive and a negative is greater than zero, and sometimes it is less than zero. Also discuss when the sum of a positive and negative integer can equal zero."

After groups discuss the launch questions for a few minutes, tell students they will be looking for patterns to create their own rules for adding integers. At this point, the whole class can proceed with the Exploration.

Ask your students what is similar about parts (a) and (c) and about parts (b) and (d). They should observe that (a) and (c) are the sum of two positive numbers, whereas (b) and (d) are both negative. Ask them to also look for patterns in the negative or positive signs for each pair, (a) and (c) and (b) and (d). If they suggest that positive addition results in a positive sum and negative addition results in a negative sum, that is sufficient. If, on the other hand, only a few members understand, do not rush to confirm the insight. Ask for different numbers that fit the pattern in the CLASS EXPLORATION and work through the new examples. After this, again ask students to give a rule or rules for adding integers that are both positive or both negative. When the class has developed a rule, record it in front of the class in language they are comfortable with. Then lead them through part 2 to test their rule. When ready, share the rules below to see if their rules agree. If not, help them alter their rules, unless the only difference is in choice of vocabulary or sentence structure.

Repeat a similar process for the explorations in parts 3 and 4, in which one positive and one negative integer are added. Then choose more examples to help develop a rule. Finish by comparing the class rules to the ones below.

Rule 1: If you are finding the sum of two positive numbers, add their absolute values and the answer is positive.
Rule 2: If you are finding the sum of two negative numbers, add their absolute values and the answer is negative.
Rule 3: To add two numbers with opposite signs we use the following steps.
Step 1: $\quad$ Take the absolute value of each of the numbers.
Step 2: Subtract the smaller absolute value from the larger absolute value.
Step 3: The answer is the difference found in step 2 with the sign of the number with the larger absolute value.

Integer Song (to the tune of "Row Your Boat"):
Same sign add and keep
Different sign subtract
Take the sign of the farthest number
Then you'll be exact

Suggestion: Create posters for rules of addition.

## SECTION 2.3 ADDING AND SUBTRACTING LARGER NUMBERS

We continue to explore addition and subtraction with larger numbers using the patterns we observed with smaller numbers. Let's begin with some problems using two digit numbers.

## CLASS EXPLORATION: WORKING WITH LARGE NUMBERS

1. Find the following sums. To be done as a class.
a. $12+17$
c. $\quad 19+28$
b. $-12+(-17) \quad-29$
d. $-19+(-28) \quad-47$

What do you observe? Is there a simple way of combining two integers that have the same sign, both positive or both negative? Write a rule that explains the process. Use your rule for the following problems:
2. Find the following sums. To be done individually.
a. $13+19$
32
c. $16+13$
29
b. $-13+(-19) \quad-32$
d. $-16+(-13) \quad-29$

Now let's explore some problems where the numbers we are adding have opposite signs and try to modify our rules to solve these.
3. Find the following sums.
a. $-13+19$
c. $26+(-33)$
b. $13+(-19) \quad-6$
b. $13+(-19) \quad-6$
d. $-26+33 \quad 7$
-7

Do you see a pattern? Try writing a rule for these problems. How can you use absolute values to describe what you have done? Sketch a number line to show your rule is correct.
4. Find the following sums using what you discovered about adding a positive and a negative integer.
a. $28+(-33) \quad-5$
b. $-28+335$
c. $-45+32-13$
d. $45+(-32) \quad 13$

Any time you take ensuring all of your students understand how to use the number line with integers is not wasted. The number line is an essential building block in mathematics, a building block towards coordinate planes and graphs.

Teacher Tip: You might ask students to break into small groups. They can each explain one part of a problem to all the other group members or make presentations to the class. Encourage your students to use visual aids in their presentations.

In general, we hope that the teachers will use the examples to generate exploration and approaches to the problems by students without reading on to the solutions. The students should be encouraged to look at the solutions only after attempting the examples and problems for themselves. It is always a good idea to compare their answers with the approach the book used and generate discussions about different strategies.

Below is a picture for the sum $-26+33$. Does your rule give you the answer of 7 ? Explain why your rule works.


## EXAMPLE 1

Find the sum: $-103+94$. Explain how you obtained your answer.

## SOLUTION

Step 1: We first find the absolute values $|-103|=103$ and $|94|=94$.
Step 2: Because the numbers have opposite signs, we compute the difference of the absolute values $103-94=9$.

Step 3: The answer is -9 because the number with the larger absolute value is -103 , a negative number.

We can also find the answer by driving along the number line. The first number says to drive 103 units to the left, and the second number says to drive 94 units to the right. The result is that you will have driven 9 more units to the left than to the right, so you are still left of 0 and your final position will be -9 .


## EXAMPLE 2

Net yardage is the total number of yards gained or lost at the end of the series of plays. This is similar to displacement in science and net income in finance. Net refers to the end result of the total changes from the original position, not the individual steps from A to B. In this case, for part (a) we add 13 yards and 22 yards to find his position in relation to where he started.

## Summary

Take your students through the examples and solutions from the book. You might also find it helpful to use a deck of cards to create some more examples, but card numbers are smaller than the ones in this section. You might distribute two cards per student, taking care that the colors of the suits are the same. Assume the first card is the tens place and the second, the units place. For another activity, ask students to create two addition story problems with numbers with absolute value greater than 50 and at least one negative number. Have the students share problems with other students to solve. Before leaving or starting on the exercises, have the class go over their rules from the board again.

Note that this number line model is just a sketch, not to scale..

## EXAMPLE 2

a. During a football game, David gains 13 yards on one play and gains 22 yards on the next play. What is his net yardage? Net yardage is the total number of yards gained or lost at the end of a series of plays.
b. On the next series of downs, he gains 16 yards on the first play and loses 9 yards the second play. What is his net yardage this time?

## SOLUTION

a. His total or net yardage is calculated by addition: $13+22=35$ yards.
b. His net gain can be calculated by $16-9=7$ yards. We can also think of this as addition by writing the sum $16+(-9)=7$. In the first play he gained 16 yards and in the second play he "gained" -9 yards. Thinking of the problem in this way will help in working the following exercises. You can draw pictures or diagrams to help visualize what is going on.

## EXPLORATION 1

Draw a number line from -15 to 15 . Find the distance between each of the following pair of numbers. Explain how you can compute these distances without using the number line.
a. 8 and 3
b. 8 and -3
c. -8 and 3
d. -8 and -3

## PROBLEM 1

For each pair of numbers below, sketch a number line to illustrate the distance between the two numbers. Then, show how to use subtraction to compute the distance. For example, to illustrate the distance between 22 and 8 , we would draw the following number line.

a. 22 and 8
b. -24 and 12
c. -14 and -26
d. 15 and -23
e. -25 and 5
f. -8 and -24
g. 0 and -29

## EXERCISES

For exercises 1-12, students can write either an addition or subtraction problem.
a. 27
b. -27
c. -137
d. -35
e. 35
2. $-17+25=8$; or $25-17=8 ; 8$ yard gain
3. $9+-17=-8$; or $9-17=-8 ; 8$ yard loss
4. $-52+-21=-73^{\circ} \mathrm{F}$; or $-52-21=-73^{\circ} \mathrm{F}$
5. $-23+16=-7^{\circ} \mathrm{F}$
6. $-45+-132=-177 ;-45-132=-177 ; \$ 177$ debt to the bank
7. a. $18+15=33$ pounds lost.
b. $-18+-15=-33.33$ pounds lost.
8. A liger is a cross between a lion and a tiger. $-18+13=-5 ; \quad 5$ pounds lost
9. $4+-7=-3 ; 4-7=-3$. 3 feet under its original position

## EXERCISES

For exercises 2-13, write and compute an expression. Label your answers.

1. Compute the following sums and differences:
a. $-55+82$
b. $55+-82$
c. -55-82
d. 118-153
e. $-118+153$
2. During a football game, Francisco lost 17 yards on one play. On the next play, he gained 25 yards. Find Francisco's net yardage for these two plays.
3. During a football game, Ramon gained 9 yards on the first play but lost 17 yards on the next play. What was his net yardage for these two plays?
4. In Fairbanks, Alaska, the temperature on February $2^{\text {nd }}$ was $-52^{\circ}$. The next day a cold front moved in, and the temperature dropped $21^{\circ} \mathrm{F}$. How cold was it then?
5. In Juneau, Alaska, the temperature on January $10^{\text {th }}$ was $-23^{\circ}$. The next day the temperature rose $16^{\circ} \mathrm{F}$. What was the temperature on January 11th? $-7^{\circ} \mathrm{F}$
6. David's current balance is $-\$ 45$. He needs to withdraw $\$ 132$ more. What will David's balance be after he withdraws the $\$ 132$ ? $-\$ 177$
7. a. Alex had a sick pig. During one week the pig lost 18 pounds. The next week, the pig lost 15 pounds. How many pounds did the pig lose during these two weeks? 33 lbs .
b. Suppose we consider weight gained by a positive number and weight lost by a negative number. Do part a again, using this idea. In other words, he gained -18 pounds the first week. -33 lbs.
8. Our family's pet liger Jordan lost 18 pounds during May and gained 13 pounds during June. What was the net gain during these two months?
9. The water level at Falcon Lake rose 4 feet and then dropped 7 feet from its original level. What is the change in the water level from its original position?
10. $10-(-3)=13 ; 13$ meter downward change. Total change is the difference from an original position. Hence, $10-(-3)=13 \mathrm{~m}$. However, the solution can be written as -13 to represent a downward change. $8-(-4)=12 ; 12$ meter downward change.
11. $-1582+-798=-2380 ; 1582-798=2380$ feet deep

With problems like 10, we usually say that the mine is $2,380 \mathrm{ft}$ deep. If some of your students choose to use the positive amount, discuss the difference between talking about a situation mathematically and casually. In everyday conversation we say a mine is $2,380 \mathrm{ft}$ deep, but we know that means the bottom of the mine is $-2,380 \mathrm{ft}$ from the surface.
12. $-800+1079=-279^{\circ} \mathrm{F} ; 800-1079=-279^{\circ} \mathrm{F} . \quad 279{ }^{\circ} \mathrm{F}$
13. a. $15800-2200=13,600 \mathrm{ft} ; 15800+(-2200)=13600 \mathrm{ft}$.
b. $15800+1900=17,700 \mathrm{ft}$
c. $15800+2200=18,000 \mathrm{ft}$
d. $15800-1900=13800 \mathrm{ft} ; 15800+-1900=13,800 \mathrm{ft}$

## Ingenuity

14. a. 5 , because the middle number in the sequence from 1 to 9 will be the middle number in the magic square.
b. One magic square looks like this: $6 \quad 1 \quad 8$
$7 \quad 5 \quad 3$
$\begin{array}{lll}2 & 9 & 4\end{array}$
All other correct magic squares will be reflections, horizontally, vertically, and diagonally of this one. All rows, columns and diagonals sum to 15 .
c. The digit 1 cannot be in any corner cell. Any number in a corner cell must have three ways it can sum to 15. For instance, $6+5+4=15,6+7+2=15$ and $6+1+8=15$, but the only two ways that 1 sums to 15 are $1+5+9=15$ and $1+6+8=15$.
15. Marcus dives off a 10 m diving board and goes 3 m below the surface. What is his total change in position? What is his change in position when he dives off an 8 m platform and goes 4 m below the surface?
16. After a coal miner descends $1,582 \mathrm{ft}$, he is 798 ft from the bottom of the mine. How deep is the mine?
17. The temperature on the surface of Mercury drops $1,079{ }^{\circ} \mathrm{F}$ in a day's revolution. If the temperature is $800^{\circ} \mathrm{F}$ during the hottest part of the day, what is the coolest temperature at night?
18. A plane is flying at an altitude of $15,782 \mathrm{ft}$. For the following situations, estimate the plane's altitude to the hundred's place. See TE.
a. the plane descends $2,200 \mathrm{ft}$
c. the plane ascends $2,200 \mathrm{ft}$
b. the plane ascends $1,933 \mathrm{ft}$
d. the plane descends $1,933 \mathrm{ft}$
19. Ingenuity: See TE.
a. A magic square is a three-by-three grid containing each of the whole numbers from 1 to 9 exactly once with the interesting property that the sum of the numbers on each row, the sum of the numbers on each column, and the sum of the numbers on each diagonal are all the same. What number must be in the middle of a magic square?
b. Find a magic square with the number 1 in the upper center cell.
c. Find any cells of a magic square that cannot contain a 1 .

## Investigation

1. Let students locate, plot, and label -x and -y for each case. "What do you notice?"
2. Students should be encouraged to use substitution, guess and check with this problem. It often is the case that students will gravitate towards the model demonstrated in the last number line but if students experiment with different numbers in place of $x$ and $y$ they may be surprised at the results.

Also, this is a good opportunity to remind students that x and y are variables that can represent any number, but their placement in relation to zero tells us whether or not they are positive and negative.

On the first number line we note that $x$ and $y$ are both negative numbers where $x<y$. Students may choose numbers such as -4 and -3 to represent $x$ and $y$. Then, after plotting $-(-4)=4$ and $-(-3)=3$, students may note that $3<4$ or, symbolically, $-\mathrm{y}<-\mathrm{x}$.

Remind students that $-x$ and -y can also be stated to be "the opposite of x " and "the opposite of y ," respectively.

$-x>-y$ in all cases

## 15. Investigation:

Let us explore one of the patterns you discovered in Chapter 1 more generally using the number line. Suppose and are two numbers such that $x<y$. Remember, this just means that is some number to the left of some other number on the number line.
Let us also think about where the two numbers are relative to zero. Remember, by the trichotomy property in Section 1.1, is negative, zero, or positive. If is negative, or to the left of zero, then must also be to the left of zero, or negative. Do you see why? If is zero, then must be to the left of zero, or negative. If is positive, we just know that is to the left of ; it can be negative, zero, or positive.
There are a total of five different cases depicted below. Remember, in every case has to be to the right of , since < . For each case, copy the number line, plot - and - , and decide if - is greater than, less than, or equal to ${ }^{-}$. Does the pattern you see match the pattern from Chapter 1? If not, try to write a new rule to describe the pattern.



Objective: The students will practice the skills of adding and subtracting integers.

## Materials:

Deck of cards (one per group)
Activity Instructions:Arrange students into groups of two or more. Have students deal out as many cards as possible from a deck of cards so that each student has an equal number of cards. Put aside any extra cards. Explain to students that every black card in their pile represents a positive number. Every red card represents a negative number. For example, a black seven is worth +7 (seven); a red three is worth -3 (negative 3 ).
(Note: If this game is new to students, you might want to discard the face cards prior to dealing. If students are familiar with the game, or if you want to provide an extra challenge, leave the aces and face cards in the deck. In that case, explain to students that aces have a value of 1 , jacks have a value of 11, queens have a value of 12, and kings have a value of 13.)

At the start of the game, have each player place his or her cards in a stack, face down. Then ask the player to the right of the dealer to turn up one card and say the number on the card.

For example, if the player turns up a black eight, he or she says "8."
Continue from one player to the next in a clockwise direction. The second player turns up a card, adds it to the first card, and says the sum of the two cards aloud.

For example, if the card is a red 9, which has a value of -9 , the player says, " $8+(-9)=(-1)$ "
The next player takes the top card from his or her pile, adds it to the first two cards, and says the sum.
For example, if the card is a black 2 , which has a value of +2 , the player says, " $(-1)+2=1$."
The game continues until someone shows a card that, when added to the stack, results in a sum of exactly 25 .

## Extra Challenging Version <br> To add another dimension to the game, you might have students always use subtraction. Doing that will reinforce the skill of subtracting negative integers.

Magic Squares for Ingenuity 2.3







## Number Lines for 2.3 Investigation



## Addition Board Game

Objective: Students will increase their ability to add integers using a board game.

## Materials:

Blank game board
Bag of cards with an addition problem
Bag of cards with number of spaces to move

## Activity Instructions:

1) The student draws a card and states the answer.
2) If the answer is correct, the student draws a number card and moves the pawn the specified number of spaces. If the answer is incorrect, the student does not move the pawn.
3) Either place the correct answer on the back of the card or have an answer sheet for the other student to check.

Bank of Addition Problems for Index Cards: (answers are included for teacher's use only)

$$
\begin{aligned}
& -34+17=-17 \\
& 56+-4=52 \\
& 78+59=137 \\
& 30+23=53 \\
& -94+17=-77 \\
& 26+-14=12 \\
& -88+19=-69 \\
& 40+-83=-43 \\
& -24+-7=-31 \\
& 66+-34=32
\end{aligned}
$$

$$
\begin{aligned}
& -8+59=51 \\
& -70+-23=-93 \\
& -64+-57=-121 \\
& 56+-24=32 \\
& -98+-9=-107 \\
& 50+23=73 \\
& -72+17=-55 \\
& -56+-74=-130 \\
& 71+-52=19 \\
& -20+23=3
\end{aligned}
$$

## Section 2.4 - Integer Properties and Terminology

## Big Idea:

Formalizing patterns to define properties of integers.

## Key Objectives:

- Use Double Opposite theorem (negative of a negative).
- Learn formal definition of absolute value.
- Discover distance as absolute value of difference.
- Find $-x$ on number line when given $x$.
- Formalize the rule: $x-y=x+(-y)$


## Pedagogical/Orchestration:

- Have students close book before doing examples as answers are in book.
- Brainstorm properties of integers. Give names to what they already know.
- This lesson involves learning the definitions of most of the properties involved with adding integers, many of them quite formal. Have places ready on the walls or bulletin boards to write down properties as your students learn them. You'll want all of them displayed simultaneously by the end of the lesson, with examples.


## Materials:

- Large number line for demonstration purposes, Posters, Markers, Plenty of student workspace


## Activity:

Property Posters from the end of the section and on the CD.

## Exercises:

Exercise 10 foreshadows multiplication of integers.

## Vocabulary:

additive identity, additive inverse, absolute value, double opposite, distance, commutative, associative

## TEKS:

7.2(C); 7.13(A); 7.14(A); 7.15(A) 8.1(A); 8.2(B); 8.15(A); 8.16(A);

## WARM-UPS for Section 2.4

1. During a 4 day period on the stock market, an energy company's rating dropped 24 points on day 1 , dropped 18 points day 2 , rose 34 points day 3 and dropped 10 points day 4 . What was the net change in this company's rating for these 4 days?
a. -16 points
b. -18 points
c. -20 points
d. -22 points

Ans: $-24+-18+34+-10=(b)-18$ points
2. Sarah needs to get a job because she owes her sister $\$ 8$, owes her mom twice as much, owes her brother half of what she owes her sister, and owes her dad half of what she owes her brother. How much money does she owe her family in all?
Ans: $8+16+4+22=\$ 30$

## Launch for Section 2.4:

Start by telling the class they will be learning some of the most important rules and patterns involving integers. Ask them to share any they know. If students have trouble coming up with a rule, remind them of how they have been changing a subtraction expression to an equivalent addition expression. Can they write a rule for this? ( $x-y=x$ $+-y$ ). Students can even give this rule a name. Give each group a couple minutes to come up with a name for the rule, and then the class can vote for which name is the most descriptive and helpful. Then tell the class they will be learning about some other important rules for integers, and proceed with the section.

Additive Identity Property: If x is any number, then $\mathrm{x}+0=\mathrm{x}$.
This is a property the students probably know already. You might go through a few examples using positives, negatives, and zero. You might also ask them why the property is called additive identity. Responses might include additive because of the addition $(x+0)$ and identity because the identity of $x$ did not change: $x+0$ remains $x$. The value of the number $x$ did not change. Write the property and an example or two in some prominent place as you conclude a discussion on each property.

## SECTION 2.4 INTEGER PROPERTIES AND TERMINOLOGY

In the previous sections' exercises, you have been writing rules to describe patterns you observed about the integers. In this section, we will review some of these patterns and ways to write rules for them. We will use the number line to help demonstrate the reason the rules work.

We use variables like $x$ to generalize the rules. Remember that these variables just represent a general number; if you were to replace every x in this section with a 4, the statements would be just as true, but less general.

Think about what happens when you add 0 to a number x . You first drive x units to the location x . Then you drive 0 units, remaining exactly where you were before:


In other words, adding 0 to a number doesn't change its value. Because 0 has this property, we call 0 the additive identity.

## PROPERTY 2.1: ADDITIVE IDENTITY

For any number x

$$
x+0=x
$$

## EXAMPLE 1

a. $2+0=2$
b. $\quad-5+0=-5$
c. $\quad 0+0=0$

Additive Inverse Property: If $x$ is an integer, then there exists a number $-x$, called the additive inverse of $x$, such that $x+-x=0$.

Despite being so wordy, this one is also pretty easy for students to grasp. Before formally stating it, ask, "Can you find a pair of numbers on the number line that are the same distance from zero?" Begin to help your students associate the pairs, like 5 and -5 , as additive inverses, numbers that added together equal 0 , or the additive identity. For example, $4+-4=0$, so 4 and -4 are additive inverses, or opposites. Finally, ask your students the additive inverse of 7 , of -11 , of 2 billion.

Emphasize that these are additive inverses because they are both the same distance from 0 and that when added, equal 0 .

Ask your students how they would find the additive inverse of any positive number; of any negative number; of zero. They might suggest that you either "take off" or "put on" a negative to find the additive inverse, or that you use 0 in the case of 0 . This question leads us to the next property, but first look more closely at this one:

If x is an integer, then there exists a number -x , called the additive inverse of x , such that $\mathrm{x}+-\mathrm{x}=0$.
If the integer, x , is 17 , then -x or -17 is fairly straightforward. Students "put on" a negative, so to speak.
What if the integer, $x$, is -17 ? According to the property, we must find a number, $-x$, and in this case $-(-17)$. Although you might already know that this is 17 , this can be very confusing for students.

This brings us to the double opposite property: the Double Opposite Property states that If x is an integer, then $-(-x)=x$.

Before sharing this, ask the class what they think $-(-x)$ is. Take responses then ask again with a numeric example like $-(-5)$. Take responses. Use the number line. The integer -5 is inside the parentheses, so place a marker at -5 . The - before the $(-5)$ is signaling the opposite or the additive inverse of -5 . What is that? What is the opposite of -5 ? What is the same distance from 0 as -5 ? The integer 5 , of course!

How can we find a pair of numbers on the number line that are the same distance from zero? To do this, we need to go the same distance from 0 but in opposite directions. For example, the numbers 1 and -1 are both 1 unit from 0 . Similarly, 5 and -5 are the same distance from 0 . We call pairs of numbers like 5 and -5 "opposites," or additive inverses.

What happens when you add a number to its additive inverse? Beginning at the origin, you first move a certain distance in one direction, and then move exactly the same distance in the opposite direction:


Your final position is back at the origin, 0 . Therefore, the sum of any number and its opposite is 0 . We call this the additive inverse property.

## PROPERTY 2.2: ADDITIVE INVERSE PROPERTY

For any number $x$, there exists a number $-x$, called the additive inverse of $x$, such that

$$
x+-x=0
$$

## EXAMPLE 2

a. $\quad 4+-4=0$
b. $\quad 15+-15=0$
c. $\quad 7+-7=0$

What number is $-(-\mathrm{x})$ ? Because $-(-\mathrm{x})$ is the opposite of -x ,

$$
-(-x)+-x=0
$$

On the other hand, because $-x$ is the opposite of $x$,

$$
x+-x=0
$$

Return to $-(-x)$. Ask the class to describe how $-(-x)=x$ just as $-(-5)=5$. Use the book for help if needed.

- From the additive inverse property, we know that if x is an integer, then there exists a number -x , called the additive inverse of $x$, such that $x+-x=0$.
- So if $-(-x)$ is the initial integer, then $-x$ must be its additive inverse and together they must sum to 0 . That is, $\quad-(-x)+-x=0$.
- Now look at $-x$ itself. What is its additive inverse? The additive inverse of $-x$ is $x$.
- Return to $-(-x)+-x=0$. If $x+-x=0$ then $-(-x)$ must also be $x$.

Do a few more numeric examples, if needed, asking what $-(-11)$ is and $-(-21)$ and so on.

The absolute value definition in this chapter can be tricky to grasp without examples, so again take your time with the students.

Absolute Value Definition: Let x represent an integer. The absolute value of x , written $|\mathrm{x}|$, is defined as follows:
$|x|=x$ if $x$ is positive or zero (greater than or equal to 0 )
$|x|=-x$ if $x$ is negative (less than 0 )

Try this with an example like 8 . Ask the kids to find |8|. They should say 8 . Do it again with 0 . The absolute value of a positive or 0 integer is that integer itself. No changes.

But try $|-7|$. The absolute value is 7 . If the integer, $x$, is negative, then the absolute value of that integer is $-x$. This is using the double opposite property. That is, $-(-7)=7$ or $-(-x)=x$.

One of the best ways of thinking about the absolute value of a number is as the distance from the number to zero on the number line. Visualize a rope 7 units long, tethered at zero. The absolute value of 7 is the same as the absolute value of -7 with the rope. Both numbers are 7 units away from zero.

Comparing these two equations shows us that $-(-x)$ must equal $x$. In short, the opposite of the opposite of $x,-(-x)$, is the number $x$ itself.

## THEOREM 2.1: DOUBLE OPPOSITE THEOREM

For any number x ,

$$
-(-x)=x
$$

We can see this more easily on a picture:


For example, if $x$ is 6 , the picture shows us that the opposite of the number 6 is -6 , and the opposite of -6 is $-(-6)$, which is the same as 6 . If x is -9 , the opposite of x is 9 , and the opposite of the opposite of x is -9 .

## EXAMPLE 3

a. $-(-12)=12$
b. $-(-3)=3$
c. $-(-56)=56$

Every negative number is the opposite of a positive number. If we think about negative numbers this way, the double opposite property gives us a nice way to express absolute values. If a number $x$ is positive, then $|x|=x$. For example, $|5|=5$. If a number $x$ is negative, the opposite of a positive number, the absolute value of $x$ will be the opposite of $x$. So $|-5|=-(-5)=5$.

## DEFINITION 2.1: ABSOLUTE VALUE

For any number $x$, the absolute value of $x$, written $|x|$, is defined as follows:

$$
\begin{aligned}
& |x|=x, \text { if } x \geq 0 \\
& |x|=-x, \text { if } x<0
\end{aligned}
$$

Compare this rule to Definition 1.5 (in Section 1.3) which says that the absolute value of a number is its distance from zero.

## PROBLEM 1

The distance between:
a. $\quad 10$ and 7 is 3
7 and 10 is 3
c. $\quad 14$ and -3 is 17
-3 and 14 is 17
b. $\quad 14$ and 7 is 7
7 and 14 is 7

The distance between:
d. 10 and 8 is 2

8 and 10 is 2
f. -6 and -4 is 2
-4 and -6 is 2
e. 10 and -8 is 18
-8 and 10 is 18

## PROBLEM 1

Let's try to figure out a formula for finding the distance between two points. Find the distance between each pair of points below. Subtraction
a. $\quad 10$ and 7
c. $\quad 14$ and -3
7 and 10 -3 and 14
b. $\quad 14$ and 7
7 and 14

What patterns do you notice in these examples? Try to write a rule for finding the distance between two numbers. What operation can you use to find the distance?

Try using your rule to help you find the distance between each pair of points below. Check your answers on a number line.
d. 10 and 8
f. -6 and -4
8 and 10
-4 and -6
e. 10 and -8
-8 and 10
Notice that the distance between the two numbers is the same no matter which one comes first. The distance is always nonnegative and equal in magnitude of the difference between the two numbers. This can be written as follows:

## DEFINITION 2.2: DISTANCE

For any two numbers x and y , the distance between x and y is the absolute value of their difference; that is,

Distance $=|x-y|$

From Section 2.2, Exercise 3, you probably noticed that $4-3$ and $4+-3$ are equivalent. This property can be summarized as follows:

## PROPERTY 2.3: SUBTRACTION PROPERTY

For any two numbers $x$ and $y$ :

$$
x-y=x+-y
$$

## Commutative Property of Addition

If $x$ and $y$ are integers, then $x+y=y+x$.
This property basically says numbers can be added in any order; e.g. $3+7=7+3$, and the sum is the same. To lead students to this property you might choose to do a series of examples on the number line and have them look for patterns. Work through some positives ( $5+7$ and $7+5$ ), some with $0(0+3,3+0)$, some with negatives $(-7$ +-1 and $-1+-7)$, and some mixed ( $-4+6$ and $6+-4$ ). Show that the result is the same regardless of the order. Be careful to select only addition problems because for subtraction this is not true (5-3 is NOT 3-5), but don't confuse them with this now. It will be covered in Exercise \#3.

Finally, we get to the last property: the Associative Property of Addition.
If $x, y$, and $z$ are integers, then $(x+y)+z=x+(y+z)$.
Have your students evaluate $(2+3)+4$ to get $5+4$ is 9 . Then have them add $2+(3+4)$ to get $2+7$ is 9 .
How numbers are grouped in addition does not affect the sum. This is what the associative property states.

## Summary

There are a number of properties to discuss. Now that each one is on the board, you might ask students to share any they consider confusing or simple, and tell why. Give a little extra attention to the ones they identify as confusing.

Suppose that $x$ and $y$ represent integers. Remember, to find the sum $x+y$, we started at the point 0 on the number line, moved $|x|$ units in the direction of $x$, and then moved |y| units in the direction of y .


To find the sum $y+x$, we started at 0 and performed these two steps in the reverse order.


Reversing the order of these steps does not change the final outcome. In either case, we end up in the same place. We call this the commutative property of addition.

## PROPERTY 2.4: COMMUTATIVE PROPERTY OF ADDITION

For any numbers $x$ and $y$,

$$
x+y=y+x
$$

## PROBLEM 2

Compute each side of the equalities to show that addition is commutative.
a. $2+3=3+2$
b. $-5+-10=-10+(-5)$
c. $-6+9=9+(-6)$

## EXERCISES

1. a. -6
d. $-x$
b. 7
e. $-y$
c. 0
f. $z$
2.. a. Commutative
b. Double Opposite
c. Associative
d. Additive Inverse
e. Commutative
f. Additive Identity
2. It is easy to show by counter example that there is no commutative property of subtraction. For instance, $2-3=-1$ but $3-2=1$.
3. This exercise can help students see that the absolute value of $|x|-|y| \leq|x-y|$ and $|x|+|y| \geq|x+y|$. A revisit after order of operations in Chapter 4 may help answer the question: Where does absolute value fall within the order of operations?

Now, consider the expressions $(x+y)+z$ and $x+(y+z)$, where $x, y$, and $z$ are integers. To calculate the first, we first add $x$ and $y$ and then add $z$; to calculate the second, we first add $y$ and $z$, and then add $x$. Draw a picture using the car model for each of the expressions. Just as before, the order does not matter in determining the final value. This is called the associative property of addition.

PROPERTY 2.5: ASSOCIATIVE PROPERTY OF ADDITION
For any numbers $x, y$, and $z$,

$$
(x+y)+z=x+(y+z)
$$

## PROBLEM 3

Verify the equality.
a. $(3+6)+1=3+(6+1)$
b. $(-7+4)+8=-7+(4+8)$

## EXERCISES

1. Find the opposites of the following. Remember, the opposite of $x$ is $-x$. See TE.
a. 6
b. -7
c. 0
d. $x$
e. $y$
f. $-z$
2. Name the property associated with each of the following:
a. $3+2=2+3$
b. $-(-7)=7$
c. $(-5+4)+6=-5+(4+6)$
d. $6+-6=0$
e. $-37+4=4+-37$
f. $879+0=879$
3. Is there a commutative property of subtraction? In other words, is it true that for any numbers $x$ and $y, x-y=y-x$ ? Explain. See TE.
4. Evaluate the following expressions:
a. i. $|1|+|-1|$
2 ii. $|1+-1|$
b. i. $|-14+-8|$
22 ii. $|-14|+|-8| \quad 22$
c. i. $|6-(-2)|$
8 ii. $|6|-|-2| \quad 4$
d. i. $|15|-|27|$
-12 ii. $|15-27| 12$

0

Exercise 5 leads students to work with the distance definition:
Let $x$ and $y$ represent integers (possibly the same). The distance between $x$ and $y$ is the absolute value of the difference between $x$ and $y$; that is, Distance $=|x-y|$.
6. It is not possible to tell which of the four integers is closest to zero because any or all of them could be negative. We do not know where zero is in relation to the variables.
8. Answers will vary based on perceived values of $-x$ and $-y$.


## Ingenuity

10. a. Fill the 3 -liter jug. Fill the 2 -liter jug from it. You will have 1 liter left in the 3 -liter jug.
b. Fill the 3 -liter jug and pour it into the 5 -liter jug. Fill the 3 -liter jug again and fill the 5 -liter jug. There will be 1 liter of water in the 3 -liter jug. Empty the 5 -liter jug. Pour the 1 liter of water from the 3 -liter jug into the 5 -liter jug. Fill the 3 -liter jug. Pour the 3 liters into the 5 -liter jug. The 5 -liter jug will have four liters of water in it.
An alternate solution to the proposed problem:

| Action | Liters in the 5-liter jug | Liters in the 3-liter jug |
| :--- | :--- | :--- |
| Fill the 5-liter jug | 5 | 0 |
| Fill the 3 from the 5 | 2 | 3 |
| Empty the 3-liter | 2 | 0 |
| Fill the 3 from the 5 | 0 | 2 |
| Fill the 5-liter | 5 | 2 |
| Fill the 3 from the 5 | 4 | 3 |

Now you have measured exactly 4 liters in the 5 -liter jug.

## Investigation

11. Do not tell your students how to do this. It is just a way of introducing what will be covered in Chapter 3. You can validate their thinking if they are on the right track, but let them discover this for themselves.
When we consider skip counting on the number line, we think of the first number in multiplication as the number of skips. A negative number of skips might represent past action. We think of the second number in multiplication model as the length of each skip. A negative length might represent skips to the left of zero on the number line.
Look ahead in the next section for the model.
12. Using the definition for Distance in this section, find the distances between the following points:
a. $\quad-8$ and $4 \quad 12$
b. 9 and -514
c. $\quad-9$ and $-5 \quad 4$
d. 23 and $102 \quad 79$
d. -23 and 4972
e. 34 and 232
f. -68 and 2593
g. 64 and -670
h. -104 and -51 53
13. Suppose that $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}$ are integers and $\mathrm{a}<\mathrm{b}<\mathrm{c}<\mathrm{d}$. Is it possible to determine which of these four integers is closest to zero? Explain. See TE.
14. Estimate to the tens place: See TE.
a. $|987-342| 650$
b. $-|653+47|-700$
c. $|64-734| 670$
d. $-|823-297|-520$
e. $|43-362| 320$
f. $\quad-|35-874|-830$
15. Locate the following integers on the number line: $x+y,-x_{1}-y_{1}-x+-y_{\text {, }}$ and $-(x+y)$.

16. For each of the following pictures, illustrate the distance from x to 5 .
a.

b.

17. Ingenuity:
a. Given a 3-liter jug, a 2-liter jug, and an unlimited supply of water, how can you obtain exactly one liter of water? See TE.
b. Given a 5 -liter jug, a 3 -liter jug, and an unlimited supply of water, how can you obtain exactly four liters of water? See TE.
18. Investigation:

Think about how you can model multiplication on the number line. How would you multiply two positive numbers? A negative by a positive? Think about what the first number and the second number mean using skip counting. See TE.


Objective: This activity will be used with section 2.4 to show that the students understand each of the properties listed in this section.

Materials:
Poster Paper
Colored Pencils or Map Colors

## Activity Instructions:

The students will break up into groups of two. Each group is responsible for making one poster that includes all six properties listed in this section: Additive Identity, Additive Inverse, Double Opposite, Absolute Value, Commutative, and Associative Properties.

Before distributing the poster paper, have each group work on a rough draft. (This will help eliminate poster paper waste!) Each poster should include a written explanation of the property, an algebraic example, a real example, and some sort of visual. Encourage the students to be creative. If they can relate the properties to any real world examples, that would be great.

Once the students have finished their rough drafts, explain to them that all six properties must be neatly displayed on one piece of poster paper. They will need to divide the poster into six equal sections to neatly display their properties.

Finished posters should be neat, bold, and colorful. Finished posters can be displayed in the classroom or in the hallway.

## Section 2.5 - The Chip Model

## Big Idea:

Using Chips to model addition/subtraction. (Will be useful for equations in Chapter 3).

## Key Objectives:

- Observe that subtracting is like adding the opposite.
- Learn to use zero pairs with a chip model.


## Pedagogical/Orchestration:

- Warning for teachers: Not a day to "wing it." During examples, do not allow students to open their books. The answers are there, and they may get distracted with what's on the pages.
- For Examples 5 \& 6: May be helpful to label each side of a chip as a positive (+) or negative (-).


## Materials:

- Chips or 2-color counters (red and yellow recommended), Transparent colored chips if using overhead projector


## Activity:

Ch. $1 \& 2$ Jeopardy at the end of the section and on the CD
Counting Chips at the end of the section and on the CD
Double Sided Chip Subtraction at the end of the section and on the CD

## Exercises:

Exercise 11 Ingenuity (Foreshadows Probability)

## Vocabulary:

zero pair, opposites

## TEKS:

$7.2(C) ; \quad 7.13(A) ; \quad 7.14(A) ; \quad 7.15(A) \quad 8.1(A) ; \quad 8.2(B) ; 8.15(A) ; 8.16(A)$;

## WARM-UPS for Section 2.5

1. We can write $5=7-2$ or $5=7+(-2)$. Find four ways you can write 10 as the sum of two positive numbers or as the sum of one positive and one negative number. Can you write 0 as the sum of two positive numbers? Can you write 0 as the sum of a positive and a negative number?
2. A teacher is mixing a punch drink for a school dance. The recipe calls for 1 cup of juice and she wants to make 100 times the original recipe. How many gallons of juice does she need?
a. between 5 and 6 gallons
b. between 6 and 7 gallons
c. between 7 and 8 gallons
d. between 8 and 9 gallons

$$
\text { Answer: b because the exact amount is } 6 \frac{1}{4} \text { gallons. }
$$

## Launch for Section 2.5:

Tell students they will be learning a new model for adding and subtracting integers called the chip model. After explaining that the yellow chips are positive and the red chips are negative, place 5 yellow chips on the overhead. Ask students, "What number does this represent?" Do the same with a number of red chips. Then place a yellow chip and a red chip on the overhead and ask, "What number does this represent?" After students make their responses, ask students to explain why it equals zero. Then introduce the term "zero pair" and start Example 1, modeling the examples using the chips.

Next, drop 5 chips onto the floor. What are the possible total values of the chips?
Have the students fill in this table of possibilities:

| Positive up | Negative up | Total value |
| :---: | :---: | :---: |
| 5 | 0 | 5 |
| 4 | 1 | 3 |
| 3 | 2 | 1 |
| 2 | 3 | -1 |
| 1 | 4 | -3 |
| 0 | 5 | -5 |

Notice that we can get all of the odd integers from -5 to 5 , but none of the even integers! Why do you think that happens?

## Introduction

If you are unfamiliar with the chip model, run through the examples several times before teaching this lesson. Also, try new examples of your own and compare the chip model result with the correct answer. For several examples like 3,4 and 5 , the book offers two methods for evaluating the expression. Become familiar with each of the methods. One method shows subtracting as adding a negative, or adding a negative as subtracting a positive. The other method involves adding a strategic zero and is discussed below.

A big idea developed in this section and expanded in Chapter 3: Modeling Problems Algebraically is adding a strategic zero. For example, consider the equation $7=x-5$. You already know that $x=12$, but back up and pretend this is a more difficult problem you are solving in steps. Logically to solve, add a 5 to each side. This gives $7+5=x-5+5$. Simplify to $12=x$ because the -5 and +5 summed to zero. You knew that adding a 5 to each side would not change the equation, and it would find the solution. Think about this, though. We could have added a 4 to each side to get: $7+4=x-5+4$, which simplifies to $11=x-1$. Although the solution to $x$ remains 12 , adding 4 is not very strategic or helpful.

Similarly, the chip model will sometimes require us to add a strategic zero. For example, compute $3-5$ with the chip model. With a collection of 3 chips and 5 chips, it is impossible to take 5 from 3 .

You need to add more to your stack of three so that you do not change the fact that your stack only has three in it. You need to add a strategic zero. One way to add zero is to add one negative and one positive chip to your stack of three.

## Body of Lesson

Begin by posing the various problems to your students. Let them try to model the problems as you observe their work or answer questions. When ready, ask a student to draw the solution, or show the solution based on a student's work.

Use the same method through the examples. When you get to those with multiple methods, encourage the students to try other ways or model the solutions for each other.

## SECTION 2.5 THE CHIP MODEL

Another way to think of positive and negative numbers is with positive and negative chips. Each positive chip counts as positive one. Each negative chip is a negative one. If you have 2 negative chips, then you "owe" the bank $\$ 2$. If you have 2 positive chips then the bank "owes" you $\$ 2$.

The chip model is based on the set model and the additive inverse property: $1+(-1)=0$. Many times, we call $1+(-1)$ a zero pair because the sum is zero. We can picture this as

$$
\oplus+\ominus=\bigoplus_{\ominus}^{-\ominus}=0
$$

## EXAMPLE 1

Model each equation with positive and negative chips:
a. $3+2=5$
b. $3-2=1$
c. $3+-2=1$

## SOLUTION

a.

b.



## EXAMPLE 2

Matching three of the positives with three of the negatives, you see that there are three zero pairs and then two negative chips remaining. The sum of -5 and 3 is -2 .

Notice that $3-2$ produces the same result as $3+(-2)$ because taking away two positive chips is like adding two negative chips. Each negative chip cancels out the positive chip so that you end up with one positive chip. Subtraction is like adding the opposite. We can think of adding $n$ opposites as "cancelling" or taking away $n$ chips.

## EXAMPLE 2

Represent the expression $-5+3$ with chips.

## SOLUTION



## EXAMPLE 3

Use chips to represent $-5-(-2)$.

## SOLUTION

This is the same as beginning with 5 negative chips and taking away 2 negative chips. You end up with 3 negative chips. Thus $-5-(-2)=-3$.


## EXAMPLE 4

In order to model subtraction, we have assumed that we have chips in our pile to take away. But what do we do if we begin with 5 positive chips and need to take away 8 positive chips? How do we model $5-8$ ?

## SOLUTION

Let's first take our 5 positive chips and add in three zero pairs. Because adding zero does not change the number, we have not changed the problem, but we now have enough positive chips.


We can now subtract 8 positive chips from our set of 8 positive chips and 3 negative chips. We are left with 3 negative chips.


$$
8+-3-8=-3
$$

## EXAMPLE 5

Use chips to model -3-2.

## SOLUTION

We begin with 3 negative chips. We now need to take away 2 positive chips. Again, we add 2 "zero pairs."

## EXERCISES

1.a.

b.
$\stackrel{\oplus \oplus}{\oplus} \oplus \stackrel{\oplus}{\oplus} \oplus+\stackrel{\oplus}{\oplus} \oplus \oplus \oplus \oplus \oplus$

c.


3. Add three zeros pairs to the 6 positive chips in the form of 3 positive chips linked to 3 negative chips, so you can cancel 3 negative chips from both numbers, leaving $6+3$ positive chips. You are adding 3 to 6 to get 9 .
4. Sam owes Sarah $\$ 2$. His debt is represented as $-\$ 2$.
5. Carlos will owe Claudia $\$ 8$. Carlos' debt is represented as $-\$ 8$.
6. Alejandro has 20 positive chips and 15 negative chips, or $20-15=5$ positive chips. Alma has 32 positive chips and 29 negative chips, or $32-29=3$ chips. Therefore, Alejandro has 2 more positive chips than Alma.

$$
\begin{aligned}
& -3=2+-2+-3=-5+2
\end{aligned}
$$

Now we can take away 2 positive chips, and are left with 5 negative chips:

$$
\begin{gathered}
\ominus \ominus \ominus \ominus \ominus \\
\ominus
\end{gathered}
$$

## EXERCISES

1. Find the following sums using the chip model. Check your answers using the number line model. Do your answers agree? Which model do you prefer? Why?
a. $5+38$
b. $5+(-3) \quad 2$
c. $-3+5 \quad 2$
d. $-3+-5 \quad-8$
2. Compute the following sums and differences using both the chip model and the number line model.
a. 5-3 2
c. $-7-(-9) \quad 2$
b. $5-8-3$
d. $-7+(-9)-16$
3. When using the chip model to compute the difference $6-(-3)$, how do you subtract the negative? Explain what you are doing in words. See TE.
4. Sam owes Sarah $\$ 5$. In the chip model, we represent a debt with negative chips. If he pays her \$3, how much does he still owe her? See TE.
5. Carlos owes Claudia $\$ 5$. If Claudia loans Carlos another $\$ 3$, how much will he then owe? See TE.
6. Alejandro has 20 positive chips and 15 negative chips. Alma has 32 positive chips and 29 negative chips. Whose chips represent the greater sum? See TE.
7. Some of the solutions include the following: 11 positive chips and 1 negative chip; 12 positive chips and 2 negative chips; 13 positive chips and 3 negative chips; 14 positive chips and 4 negative chips; 15 positive chips and 5 negative chips. However, other ways are possible.
8. Only two ways: 5 positive chips, or 6 positive chips and 1 negative chip.
9. There are an unlimited number of ways to represent the number 5. Look at Exercise 7. You can see that it could extend indefinitely.
10. One possible solution is $-1-(-5)$.

Investigation
12. Add -1 for each pair that ends in an even number, so divide by 2 and make the answer negative.

$$
\begin{array}{cl}
1-2+3-4+\ldots+25-26 & -26 / 2=-13 \\
1-2+3-4+\ldots+25-26+27 & -13+27=14
\end{array}
$$

7. Find 5 ways to represent the number 10 using positive and negative chips. See $T E$.
8. How many ways are there to represent the number 5 using positive and negative chips if you only have 6 positive chips and 6 negative chips to use? See TE.
9. How many ways are there to represent the number 5 using positive and negative chips if you have an unlimited supply? Explain. See TE.
10. Use the chip model and addition or subtraction to show how two negative numbers can result in the answer 4. Answers will vary.
11. Ingenuity:

Elena and Becky play a game in which a coin is tossed repeatedly. If the coin comes up heads twice in a row, Elena wins. If the coin comes up tails twice in a row, Becky wins. If Elena wins after five tosses, then which side of the coin came up on each toss? The toss winners were T, H, T, H, H.
12. Investigation:

Using the chip model, evaluate the following sums:
a. 1-2
b. $1-2+3-4$
c. $1-2+3-4+5-6 \quad-3$
d. $1-2+3-4+5-6+7-8 \quad-4$
e. Do you see a pattern? Can you find a quick way to evaluate each expression below?

$$
1-2+3-4+\cdots+25-26
$$

$$
1-2+3-4+\cdots+25-26+27
$$

# Ch 1 \& 2 Review Integer Jeopardy 

Objective: Students will compare numbers \& absolute value, add and subtract integers, and solve equations.

## Materials:

Game Template on CD (file name Ch 1 \& 2 Integer Jeopardy.ppt)
Pocket holders (3"x 4") (if computer access is not available)
Cards with questions written on them corresponding to the four categories (if computer access is not available)
Timer/stopwatch

## Activity Instructions:

1) Divide the class into two groups.
2) Have index cards prepared with problems to solve for each category or use the Jeopardy game on the CD.
3) The questions should range from easy for 10 points to hard for 50 points. If the student gets the question correct, the team gets the number, or point value, of the question. The students can work together to come up with the answer. If the team says the wrong answer the other team has a chance to answer the question.
4) Continue to play until all the cards are used. The team with the most points wins. Teacher uses a timer to have the students answer the question in that amount of time: 30 seconds to 1 minute.

How to use the jeopardy game on the CD (note that the following file can be edited by teacher, if desired):

Open file Ch 1 \& 2 Integer Jeopardy.ppt.

## Counting Chips

Objective: The students will practice counting integers using the chip model.

## Materials:

Reproduce and cut out the circles from the next sheet, coloring smiley faces yellow and sad faces red. You will need 20 circle cut-outs ( 10 red and 10 yellow) per group
** (You could just use some of your old integer chips and draw your smiley's and sad faces with a permanent marker. If you do this, be sure to only draw on one side of the chips!!)
Brown paper bags (one per group)
Die (one per group)
Students will need paper and pencil to record their totals
Activity Instructions:

1) Divide your students into groups of either $2,3,4$, or 5 . Each group will need one die, a brown paper bag with 20 circle cut-outs (or integer chips) inside (10 red and 10 yellow), and paper and pencil to record their totals per player, per turn.
2) Explain to your students that they will roll the die to see who gets to go first. The first player will then roll the die. If the student rolls a 6 on the die, then they will reach into the bag (without looking) and pull out that many circles/chips. The student will total the chips and record this number on their sheet of paper. The circles are then returned to the bag.
3) The game will continue in a clock-wise direction until all students have had a set number of turns, or until they have played for a set amount of time.
4) The winner of the game will be the student with the highest total from all of their turns. Below is a sample scoresheet.

|  | Susan | Mark | Jake |
| :--- | :--- | :--- | :--- |
| Turn \#1 | -2 | 5 | 1 |
| Turn \#2 | 4 | -4 | 4 |
| Turn \#3 | 0 | 3 | 3 |
| Turn \#4 | -1 | 0 | -5 |
| Turn \#5 | 6 | -5 | -2 |
| Turn \#6 | 1 | 1 | 0 |
| Turn \#7 | 2 | 6 | 3 |
| Total | 10 | 6 | 4 |
|  | **Winner** |  |  |

## Circle Cut-Outs

(Color sad faces red and smiley faces yellow)

# JJJJ 

## JJJJJ

## LLLLL



## Double-Sided Chip Subtraction

Objective: Students will increase their ability to subtract integers using double-sided chips.

## Materials:

Double-sided chips (coins or checker pieces can be used as long as the two sides are different).

## Activity Instructions:

1) We use the Take Away Model for subtraction. For example: 3-2 is demonstrated with $(+)(+)(+)$ take away $(+)(+)$ leaves ( + ).
2) $-3-2=-1$ is demonstrated with $(-)(-)(-)$ minus $(-)(-)$ leaves $(-)$.
3) $3--2=5$ is demonstrated with $(+)(+)(+)$, but unable to take away $(-)(-)$, we introduce into our holdings two zeros sums so that we have $(+)(+)(+)(+)(-)(+)(-)$. When we take away, we are left with $(+)(+)(+)(+)(+)$.
4) $-3-2=-5$ is demonstrated with (-)(-)(-), but unable to take away (+)(+), we introduce into our holdings two zero sums so that we have $(-)(-)(-)(+)(-)(+)(-)$. When we take away $(+)(+)$ we are left with $(-)(-)(-)(-)(-)$.
*Note that exercises for chip subtraction have not been pre-written for this activity.

## REVIEW PROBLEMS

1. Write a problem and its solution that represents the model: $6+-10=-4$

2. Marci is snorkeling in the San Marcos River 5 feet below the surface. She dives 3 feet deeper. How many feet below the surface is Marci now? 8 ft . or -8 ft.
3. Isaac has $-\$ 20$ in his account. He needs to borrow $\$ 10$ more. What will his balance be now? $-\$ 30$. He will be in debt $\$ 30$.
4. Solve each problem below, and model the operation with a number line and a chip model:
a. $9+(-2)$
d. $-8+7+5$
4
b. $-8+3$
-5
e. $4-6 \quad-2$
c. $-4+(-6)$
-10
f. $-3-(-7)$
4
5. A football team is on its 25 -yard line. In four consecutive plays the offense rushes for $+8,-4,-6$ and +16 yards. What yard line will this team be on after these four plays? The team will be on the 39 -yard line.
6. Find the following sums or differences. Show all your steps and use the number line if necessary.
a. 54-63
-9
d. $-26+21 \quad-5$
b. $-31-24$
-55
e. $-47+-75 \quad-122$
c. $35--2257$
f. $-36--44$ 8
7. The temperature in Anchorage, Alaska, is $-15^{\circ}$. The temperature rises $12^{\circ} \mathrm{F}$ during the day. What is the new temperature? $-3{ }^{\circ} \mathrm{F}$
8. Lisa's bank account reads $-\$ 56$. If Lisa withdraws $\$ 23$ from her account, what will her balance be? -\$79
9. a. additive inverse d. associative
b. double negative e. commutative
c. zero identity
f. subtraction property
10. The temperature in Billings, Montana is $13^{\circ}$. The temperature then dips $15^{\circ} \mathrm{F}$.

What is the temperature now? $-2^{\circ} \mathrm{F}$
10. Name the property associated with the following:
a. $72+(-72)=0$
b. $-(-12)=12$
c. $39+0=39$
d. $(-15+3)+(-8)=-15+(3+-8)$
e. $6+-4=-4+6$
f. $8+-10=8-10$

## Section 2.1: 23

Solution: Note that 23 is not legal since subtracting $0,7,14$, or 21 does not leave a multiple of 5 . Note that 0,7 , 14,21 , and 28 (multiples of 7 ) leave all possible remainders when divided by 5 , so subtracting one of those from any number will leave a multiple of 5 , thus any amount over $\$ 27$ is legal. , , , and , so 23 is the largest illegal amount. In general, if a and b are relatively prime positive integers, then is the largest number that cannot be written in the form where x and y are nonnegative integers.

Section 2.2: 11
Solution: Well, we know we can pay any amount over $\$ 23$ in exact change, and now $23=30 \cdot 7$ is ok. We just work our way down from there: $22=7+5 \cdot 3,21=7 \cdot 3,20=5 \cdot 4,19=7 \cdot 2+5,18=25-7,17=7+5$ $\cdot 2,16=7 \cdot 3-5,15=5 \cdot 3,14=7 \cdot 2,13=5 \cdot 4-7$, and $12=75$. Since neither 16 nor 18 is a multiple of 5 or 7 it is not possible to make a purchase of $\$ 11$.

## Section 2.3: 10

Solution: There is a pattern to the numbers $1,1+2,1+2+3$, etc. We can just keep computing them until we get the first sum over 200, or discover and use the formula $S=\frac{n(n+1)}{2}$ which first passes 200 for $n=20$, giving 210. Thus the book had 20 pages but page 10 was missed. Note that with 21 pages we'd have a sum of 231, which is 31 bigger than 200, but there is no page number 31 . In general, every positive integer has a unique representation of the form $\frac{n(n+1)}{2}-k$ where $k \leq n$.

## Section 2.4: 53

Solution: The largest possible total is 5 , the smallest is -53 and all possible totals are odd integers. Changing the sign of 2 increases or decreases the total by four, changing the sign of 3 increases or decreases the total by six, etc. Since we cannot change the sign of 1 , we can not change 55 or -53 by two. Thus totals of 53 and -51 are not possible. There are 55 odd numbers in the range from -53 to 55 inclusive and two values we can not obtain, thus $55-2=53$. .
Section 2.5: 1
Solution: Anne wins $1+2+4+8+16+32+64+128+256+512=1023$ pennies, and bill wins 1024 , or 1 more. In general, since $1+2+4+\cdots=2^{n-1}=2^{n}-1$, the answer would be 1 no matter how many rounds Anne wins before Bill's first win.

## CHALLENGE PROBLEMS

## Section 2.1:

The country of Fakestan prints all of its currency in 5-and 7-dollar bills. By law, all purchases must be made with exact change and it is illegal to charge an amount that is impossible to pay. What is the largest whole dollar amount that cannot be charged?

## Section 2.2:

After much public discussion, the government of Fakestan, whose currency only comes in 5-and 7-dollar bills, decided to allow purchases to be made either with exact change or with one bill in change. What is the largest whole dollar amount that cannot be charged now?

## Section 2.3:

Edward tried to add the page numbers of a book, but he left one number out of the sum and got the incorrect total of 200 . What number did he miss?

## Section 2.4:

If each $\pm$ represents addition or subtraction, how many different values are possible for $1 \pm 2 \pm 3 \pm 4 \pm 5 \pm 6 \pm 7 \pm 8 \pm 9 \pm 10$ ?

## Section 2.5:

Anne and Bill play a game where Bill tries to guess what number Anne is thinking of. If Bill is correct, then he wins, otherwise Anne wins. The first round is worth one penny to the winner, and each later round is worth twice as much as the one before (the pennies come from an endless supply). Anne wins the first 9 rounds, but Bill wins the 10th round. How many more pennies does Bill have than Anne?

## Section 3.1 - Variables and Expressions

## Big Idea:

Understanding the meaning of a variable within an expression.

## Key Objectives:

- Adding, subtracting, and comparing values (multiplying and dividing not included here).
- Translating numerical expressions into words.
- Translating words into numerical expressions.
- Using the symbols for "greater than" and "less than."


## Pedagogical/Orchestration

- A variable is a letter/symbol that represents a number.
- Emphasize that $\mathrm{a}<\mathrm{b}$ means the number a is to the left of the number b on the number line.


## Materials:

Chart Paper

## Activity:

Use the Jeopardy Game (from the end of Section 3.2) at any time, but may be best after completing Section 3.2.

## Exercises:

1-6 are good practice for this section.
7-10 foreshadow Section 3.2.
11 Brain Teaser
12 This problem is used to launch Section 3.2, so exposure to it beforehand is beneficial.

## Vocabulary:

variable, expression

## TEKS:

6.1(C); 6.12(A); 7.5(A); 7.13(A); 7.14(A) 8.5(A); 8.14(A); 8.15(A);

## WARM-UPS for Section 3.1

1. For each of the following sums or differences, determine whether the answer is positive or negative. Do not compute the answer.
a. $-58+78 \quad$ Ans: positive Students should sketch a number line to explain.
b. $124+-265$

Ans: negative
c. $-158+-234$

Ans: negative
d. $-58-78$

Ans: negative
e. $72--58$

Ans: positive
f. $-135--280$

Ans: positive
2. For each of the number lines below, write the sum or difference that picture is modeling:
a.

b.

c.


Ans: $-8+5=-3$

Ans: $8+-5=3$

Ans: $8+5=13$
3. A school has 480 students. If 3 out of every 5 students have black hair, how many students have black hair?
a. 280 students
b. 288 students
c. 290 students
d. 300 students

Answer: b because $\frac{3}{5}=\frac{x}{480} \rightarrow \frac{3}{5} \cdot 480=x \rightarrow x=288$

## Launch for Section 3.1:

Tell your students, "I'm curious to find out how many languages each of you know." Ask students to raise their hands if they speak one language, two, three, etc. Have just a short discussion on what languages they know. Then tell them that this year, they will be learning a new language called "algebra," and this language is used for problem solving. Tell students, "Every language has words and phrases. What is the difference between a word and a phrase?" Allow for responses. "Today we will be discussing how words and phrases are used in algebra. Pay close attention to the lesson, and see if you can pick up on what a "phrase" is called in algebra, and what the "words" of algebra would be. Write "Word" and "Phrase" on the board leaving room to add additional examples throughout the lesson.

## Instructions:

Brainstorm a list of phrases that describe the following expressions using these two expressions.

| Addition <br> $\mathbf{n}+\mathbf{3}$ | Subtraction <br> $\mathbf{n}-\mathbf{3}$ |
| :---: | :---: |
| 3 more than n | n minus 3 |
| the sum of n and 3 |  |
| 3 added to n |  |
| n increased by 3 | 3 less than n |
| the difference of n and 3 |  |
| 3 subtracted from n |  |

You might remember the term "unknown" instead of "variable" for $x$. To use "unknown" suggests both that $x$ is a specific number and that its identity is the reason for algebra. Algebra today emphasizes the role of function of a variable x that can take many forms and be many possible numbers.

# MODELING <br> PROBLEMS <br> ALGEBRAICALLY 

Cosers)

## SECTION 3.1 VARIABLES AND EXPRESSIONS

Numbers give us a way to describe different quantities. The number 5 might be 5 marbles, or it might be 5 units on a number line. When we write 5 , we have in mind a definite amount or quantity. Often, however, we will want to represent an unknown quantity. To do this, we use variables. For example, Amy has some marbles, but we don't know how many she has. We could then write

$$
M=\text { the number of marbles Amy has. }
$$

A variable is a letter or symbol that represents an unknown quantity. Therefore, M is a variable that represents the number of marbles Amy has, even though we do not know what that number is. Variables give us a convenient way to describe properties and ideas because variables can represent many different things.

We use numbers, variables, and mathematical operations to form expressions. For example, Lisa has one more marble than Amy. We could then write " $M+1$ " to describe how many marbles Lisa has. Expressions are mathematical phrases, like " $\mathrm{M}+1$," that we use to describe quantities mathematically. Expressions can be numbers, variables, or a combination using math operations. Some other examples of expressions are $2 x+3,5 \cdot 2,2 x+y-3$, and 15 .

The following table summarizes the different ways a mathematical expression can be translated into a word phrase.

After you have finished looking at the examples, Jeopardy is an excellent way for your students to reinforce their control of algebraic translation and for you to assess their mastery of the variable and expression translations.

Emphasize that although the answers are the same in the "more than" expression in Example 1, it is still important to write the expression correctly, as we see in the next example. Remind your students about the fact that position in addition doesn't change the sum. Ask them to recall the Commutative Property for Addition. As a last thought, you will always have students who call the property the "Commutative." Remind them that numbers don't talk, or communicate. They can, however, switch positions, or commute. Talk about commuters moving from their home to work and back.

## EXAMPLE 2

Emphasize the importance of order. In fact, the order is changed in expressions like "less than" or "greater than." In Example 2, what is the difference between " $2-5$ " and " $5-2$ "?. Talk about the fact that the two answers are opposites of each other.

The Language of Algebraic Expressions

| Addition $x+5$ <br> - The sum of $x$ and 5 <br> - x plus 5 <br> - $x$ increased by 5 <br> - 5 added to $x$ <br> - 5 greater than x <br> - 5 more than $x$ <br> - The addition of 5 and $x$ | Subtraction $n-10$ <br> - The difference of n and 10 <br> - 10 less than n <br> - 10 subtracted from n <br> - n minus 10 <br> - $n$ decreased by 10 <br> - take away 10 from n |
| :---: | :---: |
| Multiplication 12d <br> - 12 times d <br> - 12 multiplied by d <br> - The product of 12 and | Division $\frac{y}{4}$ <br> - The quotient of $y$ and 4 - y divided by 4 |

## EXAMPLE 1

Translate "five more than two" into a mathematical expression.

## SOLUTION

"Five more than two" translates as $2+5$.
Notice that we would get the same answer if we calculated $5+2$, but that would really represent the expression "two more than five."

## EXAMPLE 2

Translate "five less than two" into a mathematical expression.

## SOLUTION

"Five less than two" translates as $2-5$.
In this case, this is not the same as 5-2. Do you see the difference?

Teacher Note: Continually emphasize that x can be any number, depending on the situation, and that any letter can be used for the variable. You might want to encourage the use of meaningful variables like "A" for "area" or "r" for "radius."

Rather than talking about crocodiles or other mnemonics that do not reinforce mathematical concepts, remind students that "<" points to the negative numbers, or the numbers that are "less than" every positive number. Conversely, " >" points to the positive numbers, or the numbers that are "greater than" every negative number.

## EXAMPLE 4

" 12 less than x " or " x minus 12 "

## EXAMPLE 3

Translate "five more than $x$ " into a mathematical expression. Illustrate this on the number line with an arbitrary point with coordinate $x$.

## SOLUTION

"Five more than $x$ " translates as $x+5$. Notice that you get this answer no matter what the variable x represents.


Remember that the symbol < is the "less than" symbol, and the symbol > is the "greater than" symbol. So, the inequality $x<2$ says that " $x$ is less than 2 ." Although we do not know what the variable $x$ is, this inequality says that $x$ is some number that is less than 2 . On the other hand, the expression "x less than 2 " is written mathematically as $2-x$.

## EXAMPLE 4

Translate the expression, " $x-3$ " into a word phrase. Locate $x-3$ on the number line in Example 3.

## SOLUTION

" $x-3$ " translates as " 3 subtracted from x." What are other ways that this expression can be written as a word phrase? In this case, why is it wrong to say, " $x$ less than 3 "? What is the difference?

## EXAMPLE 5

Translate the expression " $-4+6$ " into a word phrase.

## EXERCISES

1. a. $5+2=7$
$5-2=3$
b. $6-3=3$
$3-6=-3$
2. a. $-14+26=12$
d. $8+12=20$
b. $-38-20=-58$
e. $9+13=22$
c. $9+14=23$
f. $7-11=-4$
3. a. five less than eight
b. five more than negative four
c. two more than four
d. two less than negative three
e. two more than $x$
f. five less than $R$
g. $m$ is less than 4.
h. $m$ is greater than zero.

## SOLUTION

" $-4+6$ " translates to "the sum of -4 and 6 ." What other ways can this be written as a word phrase?

## EXAMPLE 6

Using the symbol >, write an inequality that says " 5 is greater than $x . "$

## SOLUTION

$5>x$. Caution: " 5 is greater than $x$ " is not the same as " 5 greater than $x$." Do you see the difference?

## EXERCISES

1. Translate and compute each of the following into a mathematical expression. See TE.
a. 2 more than 5
b. 3 less than 6
2 less than 5
6 less than 3
2. Translate and compute each of the following: See TE.
a. 26 added to -14
d. 8 increased by 12
b. the difference of -38 and 20
e. 13 greater than 9
c. the sum of 9 and 14
f. 11 subtracted from 7
3. Translate the following expressions or inequalities into word phrases or sentences. Try to use words other than plus or minus. See TE.

Example: -2-7 7 less than -2
a. $8-5$
b. $-4+5$
c. $4+2$
d. $-3-2$
e. $x+2$
f. $R-5$
g. $m<4$
h. $m>0$
4. a. " 2 is less than 5 " is a relational statement in which 2 is to the left of 5 . " 2 less than 5 " is an operational statement that says to decrease 5 by 2 so that the statement is equivalent to 5 minus 2 .
b. $2<5$ and $5-2$
5. a. $8-5$
d. $A>B$
b. $5<8$
e. $A<B$
c. $7>3$
f. $B+A$
6. $\{-1,0,1,2,3,4\}$
$7 \& 8$. Foreshadowing Section 3.2.
$9 \& 10$. Note that these two problems are more abstract and foreshadow the types of equations the students will write in Section 3.2.
9. $3=P+7$
$P=-4$
10. $x+(x+6)+(x+6)+6+[(x+6)+6]+6=4 x+6 \cdot 6=52$
$4 x=52-36=16$
$x=4$
$x+6+6+6=4+18=22$ units long

## Investigation

11. c. The answer to part (a) is the result of addition. The answer to part (b) is a process of working with an equation with an unknown.
The difference between an expression and an equation is that an equation relates expressions with an equal sign.
An expression involves the verbs "evaluate" or "simplify."
An equation involves the verb "is."
12. a. Using words, not mathematical symbols, explain the difference between the statement " 2 is less than 5 " and the expression " 2 less than 5." See TE.
b. Translate part a into mathematical expressions or inequalities. See TE.
13. Translate each of the following into a mathematical expression or inequality. See TE.
a. 5 less than 8
d. $A$ is greater than $B$
b. 5 is less than 8
e. $A$ is less than $B$
c. 7 is greater than 3
f. A more than B
14. If $A$ is less than 5 and greater than -2 , what integers could $A$ represent? See TE.
15. Translate the equation $\mathrm{N}=3+2$ into a sentence. Answers may vary.
16. Translate the sentence "A is four more than 2 " into a mathematical statement. $A=2+4$
17. Three is seven more than $P$. What integer does $P$ represent? See TE.
18. Ingenuity:

Four sticks are placed end to end to form a line 52 units long. If each of the last three sticks is six units longer than the one before it, how long is the longest stick?
11. Investigation:
a. Evaluate: $-5+7.2$
b. For what value of $x$ is $5+x=7$ ? $\quad x=2$
c. What is different about the answers to $\mathbf{a}$ and $\mathbf{b}$ ? What is the difference between an expression and an equation? What verb is associated with the answer to an expression? What verb is associated with the answer to an equation? See TE.

The Language of Algebraic Expressions

| Addition $x+5$ | Subtraction $n-10$ |
| :--- | :--- |
| Multiplication 12d |  |

## Section 3.2 - Equations

## Big Idea:

Translating a word problem by writing an equation.

## Key Objectives:

Setting up equations and informally solving them using a number line or picture, though at this point, we are not isolating variables.

## Pedagogical/Orchestration:

- Teacher may remind students that an equation is just like setting up problems that they have seen before. For example: $3+$ $\qquad$ $=10$ and instead of "box" we call it $N$ or variable.
- The teacher needs to be aware that a numerical expression is not an equation. The definition of equation needs to be made clear and reinforced at this point.
- Teacher may lead students to avoid setting up an equation incorrectly; for example: $x$ less than 10 equals 8 should be written as: $10-x=8$. 10 less than a number equals 6 should be written as: $x-10=6$.


## Internet Resource:

Battleship Game: Solving one step equations- http://www.quia.com/ba/36544.htm|

## Materials:

No extra materials needed.

## Activity:

"Ch 3 Jeopardy" from the end of the section and on the CD

## Exercises:

Exercises 9 \& 10 foreshadow solving equations.
Exercises 4, 5, 6 \& 7 may be set up by role playing them in front of class.

## Vocabulary:

equation, translate(change), twice, evaluate, algebraic expression

## TEKS:

$6.1(C) ; \quad 6.5 ; \quad 6.11(B) ; \quad 6.12(A) ; \quad 7.5(A)(B) ; \quad 7.13(A) ; \quad 7.14(A) \quad 8.5(A) ; \quad 8.14(A) ; \quad 8.15(A) ;$

## WARM-UPS for Section 3.2

1. A salesman gave George a $35 \%$ discount on a new pair of shoes. Which of the following is NOT equivalent to $35 \%$ ?
a. $\frac{7}{20}$
b. 0.35
c. $\frac{35}{100}$
d. $\frac{3}{5} \quad$ Answer: d
2. What is the greatest common factor of each of the following pairs of numbers?
a. 15 and 16
Ans: 1
b. 32 and 33
Ans: 1
C. 28 and 27
Ans: 1
d. What do you notice? Do you see a pattern?

## Ans: Two consecutive whole numbers can only have 1 as a common factor.

## Launch for Section 3.2

Investigation 12 from Section 3.1 is a good springboard into today's lesson. Write the following words on the board leaving room to write in student responses:

$$
\begin{array}{lll}
\text { Words } & \text { Phrases } & \text { Sentences }
\end{array}
$$

Ask students for words and phrases found in algebra, and write down their responses. The "Words" column should include things like numbers, variables, and operators. The "Phrases" column should include the word "expressions" with examples such as $x+5$ and others. Tell students, "Today we will expand our algebra language to include complete sentences. You have heard quite a few algebra sentences already. Who can give me an example of one?" Accept responses such as $x+2=5$ or $x>5$. Tell students any equations or inequalities are considered a sentence in algebra. Ask students, "What is the difference between a phrase and a sentence?" Listen to responses and make sure it comes out that a sentence has a verb and is a complete thought, whereas a phrase is a sentence fragment, or part of a sentence. Sentences are made up of phrases just as equations and inequalities are made up of two expressions that are being compared. An equation always has an equal sign which is the verb of the sentence. Tell your students, "Today we will be translating phrases and sentences into the language of algebra." Throughout the lesson continue to add examples of algebra words, phrases, and sentences to the board.

Lead your students through Examples 1 through 3 so they can consider and discuss them. Also offer the example from the Exploration, but don't tell your students to open their books to that page yet. Instead, let them consider the steps they might use to translate the Exploration scenario and solve for $x$. Then have them compare their steps to those in the book.

PROBLEM 1
a. $10-x=8$
b. $x-10=8$

## SECTION 3.2 EQUATIONS

Let's begin with the sentence "A number is 3 more than 7." We could figure out what this number is with relative ease, but how can we write this mathematically? We can use numbers, variables, and operations to form expressions. Then we can combine these expressions to form a mathematical sentence called an equation. An equation is a math sentence with an equality sign, $=$, between two expressions. As long as a math sentence contains an = sign, it is called an equation. So you could have equations like $2+3=5$ which is simply a numerical equation, or $x+2=10$ which is an equation with one variable $x$, or even an equation like $A=L \bullet W$ which is a formula that can be used to find the area of a rectangle. Now, explain verbally the difference between an equation and an expression.

## EXAMPLE 1

Translate the sentence "A number is 3 more than 7" into an equation.

## SOLUTION

Step 1: We give the unknown number a name, $N$, and write " $N=$ the number." $N$ is a variable. It represents the number we are trying to find.

Step 2: We translate the sentence into an equation.

$$
\begin{gathered}
\text { A number } \text { is } 3 \text { more than } 7 \\
N=7+3
\end{gathered}
$$

So, the equation form of the sentence is $N=7+3$.
Since $7+3=10$, we can conclude that $N=10$. $N$ now represents a known quantity, 10 , instead of an unknown quantity. Therefore, we say that we have solved the equation for $N$.

## PROBLEM 1

Translate the sentences below where $x$ is a number.

PROBLEM 2
a. $4 x=64 ; x=16$ because $4(16)=64$
b. $7 x=49 ; x=7$ because $7(7)=49$
a. $x$ less than 10 equals 8 .
b. 10 less than $x$ equals 8 .

## EXAMPLE 2

What number is twice as large as six?

## SOLUTION

Step 1: We define a variable to represent our number. Let $T$ be a number that is twice as large as six.
Step 2: The statement "A number is twice as large as six" translates as $T=2 \times 6=2 \cdot 6=(2)(6)$.

When we put the symbols 2 and 6 next to each other with parentheses around each, it is understood that we mean to multiply them. The small dot is also a symbol for multiplication. So (2) (6) $=2 \cdot 6=2 \times 6$. We usually do not use the symbol $\times$ for multiplication because it could be confused with a variable $x$.

Step 3: We know $2 \cdot 6=12$, so $T=12$.
Step 4: Check. Is 12 a number that is twice as large as 6? Yes.
Using variables to model problems is the beginning of learning algebra. Algebra enables us to translate problems into mathematical expressions and equations. We then use mathematics to solve for the unknowns, which provides solutions to our problems.

## PROBLEM 2

Translate each number sentence below. Can you determine the value of the variable?
a. The product of four and a number is 64 .
b. A number times seven is forty-nine.

Note: When we are able to determine the value of the variable in an equation, we say that we have solved the equation. The value of the variable is called a solution of the equation.

PROBLEM 3
a. $3 x+4=19 ; x=5$
b. $(36)-6=x=12$

Teacher Tip: Lead your students to create a poster, or post the steps of an equation-solving procedure for students to use as a guide until they have demonstrated a solid understanding of the process.

## EXAMPLE 3

Translate the sentence "A number is 2 less than four times 10 " into an equation and solve for the unknown variable. Does your answer make sense?

## SOLUTION

Step 1: Let's call our unknown number $N$.
Step 2: $N=(4)(10)-2$ is our equation for the statement above.
Step 3: The right side is equal to 38 . So, $N=38$ is the solution. We have solved the equation! Now let's check our answer.
Step 4: Is 38 equal to 2 less than 4 times 10 ? Four times 10 equals 40 , and 2 less than 40 is 38 , so our answer is correct.

## PROBLEM 3

Translate and solve each number sentence below.
a. Four greater than three times a number is 19 .
b. Half of 36 minus six is a number.

## EXAMPLE 4

We have seen how we can use numbers and variables to translate problems into equations. Consider the problem, "Jeremy is 9 years old. In how many years will Jeremy be 15 years old?"

How might you begin this problem? Did you define a variable? If so, how did you use this variable?

Here is a step-by-step approach. Do your steps resemble the following?

## Step 1: Define your variable

$Y=$ the number of years it takes for Jeremy to reach 15 .

## Step 2: Translate the problem into an equation

Let $m=$ amount of money Jacob needs. $73+m=98$. Since $73+25=98$, then $m=25$ is a solution.

We know that 15 is $Y$ more than 9 , so we write $15=9+Y$, an equation with one variable, $Y$.

## Step 3: Solve for the unknown variable

If you look on the number line, you'll notice you have to move right 6 units to go from 9 to 15. So $Y=6$.

## Step 4: Check your answer

Substitute $Y=6$ into the original equation to see that $15=9+6$.

## PROBLEM 4

Jacob has $\$ 73$. How much more does he need if he wants to have $\$ 98$ ?

Expressions can be evaluated by substituting in values for variables within the expression.

## EXAMPLE 5

The statement, "Alexandra has four more marbles than Denise" can be translated into the expression: $x+4$, where $x$ is the number of marbles Denise has. If Denise has 10 marbles, how many marbles does Alexandra have?

## Solution

Let $x=$ the number of marbles Denise has.
$x+4$ is the expression representing the number of marbles Alexandra has, relative to the number of marbles Denise has.

If $x=10$, substitute 10 in for $x$ and the expression gives us:
$(10)+4=14$, therefore, Alexandra has 14 marbles.

## Example 6

Ryan and Annie are downloading songs from the internet. Ryan purchased three fewer songs than Annie. If Annie bought 12 songs, how many songs did Ryan purchase?

## Solution

Let $s=$ the number of songs Annie purchased.
$s-3$ is the expression representing the number of songs Ryan purchased, which is less than the number of songs Annie purchased.

If $s=12$, substituting 12 into the expression for $s$, gives us:
(12) $-3=9$

Ryan purchased 9 songs

## Example 7

Eric and Alison are playing a board game. Alison is six spaces ahead of Eric. On his next turn, Eric draws a card that requires him to move back three spaces. How many spaces away from Alison is he?

## Solution

Let $\mathrm{n}=$ the number of spaces Eric moves.
$6-\mathrm{n}$ is the expression representing the number of spaces Eric is away from Alison.
If $\mathrm{n}=-3$, substituting -3 into the expression for n gives us:
$6-(-3)=9$
Eric is 9 spaces away from Alison.

1. a. $N=43+2, N=45$
b. $\quad N=15-4, N=11$
c. $\quad N=2(60), N=120$
d. $\quad N-4=15, N=19$
e. $\quad N=2(5)+3, N=13$
f. $\quad N-8=50, N=58$
g. $N=2(10)-6, N=14$
h. $N-7=13, N=20$
i. $\quad 2 N-6=2, N=4$

Remind students to "define a variable" and "translate the problem into an equation." The problems appear simple. Students will want to "skip" the process.
2. Let $y=$ number of years. $10+y=24$
3. Let $D=$ Daniel's present age. $\quad D+7=16$
4. Let $\mathrm{N}=$ the amount of money he needs. $\mathrm{N}+73=98 ; \mathrm{N}=25$
5. Let $L=$ money Samantha has left. $85-58=L ;$ or $58+L=85$

Note: In Exercise 5, the variable can be on the left or right side without affecting the solution.

## EXERCISES

Try the following exercises using the four-step process. When you "solve" your problem, you should not only find the answer, but also show the way you got your answer, which is just as important.

1. Write the following statements as either an expression or an equation.

Let $N=$ the number.
a. A number is 2 more than 43 .
b. Four less than a number.
c. A number is twice as large as 60 .
d. The difference of a number and 4 .
e. A number is the product of 5 and 2 increased by 3 .
f. 8 more than a number.
g. Ten less than fifteen is five.
h. 7 subtracted from a number is thirteen.
i. Six more than eight is fourteen.
2. Translate each of the expressions or equations below into a mathematical statement, and read your statement verbally.
a. $N-2$
b. $17-N$
c. $2+4=6$
d. $N+2=15$
e. $3 N-2=13$
f. $N-M$
g. $N+M$

Do steps 1 and 2 of our four-step process for Exercises 3-6. See TE.
3. Jack is 10 . In how many years will Jack be 24 ?
4. If Daniel will be 16 in 7 years, how old is Daniel now?
5. Jacob has $\$ 73$. How much more does he need if he wants to have $\$ 98$ ?
6. Samantha has $\$ 85$ and lends George $\$ 58$. How much does she have left?
8. The numbers in this problem are very large. The students may need some guidance on how to scale the number line to include such large numbers as -282 and 20, 320. This might be a good time to discuss with the students that though every number need not be included, they should be aware of the numbers that "fall through the cracks" due to scaling. Students should recognize that they must travel a distance of 282 feet to get to sea level or 0 , then continue to climb 20,320 more feet for a total of $282+20,320=20,502$ feet. What is important here is for the students to see the process of combining the lengths. This foreshadows the absolute values that we will use in the next two sections.
This exercise may be used as an informal assessment assigned as a class project.

In Exercises 9 and 10, make sure your students follow all four steps. At this point in algebra, the answer is the easiest part of the work. We are trying to establish good algebraic work habits.
9. Let $\mathrm{t}=$ the increase in temperature; $65+\mathrm{t}=87 ; \mathrm{t}=22^{\circ} \mathrm{F} ; 65+22=87$
10. Let $d=$ degrees needed to warm-up; $-8+d=5 ; d=13^{\circ} \mathrm{C} ;-8+13=5$
11. $3=P+7$
$P=4$
12. $M-3=1$
$M=4$
13. Let $B$ stand for how long it took Brad to complete the bicycle course.
$B+8=20 ; B=12$
14. Let $P$ stand for the number of points JD scored.
$120-15=P ; P=105$

Write a story problem for the equations in Exercises 6 and 7:
7. $X+8=25$ Answers will vary.
8. $64-X=12$ Answers will vary.
9. The lowest and highest points in North America are Death Valley in California and Mount McKinley in Alaska. Death Valley is below sea level. In fact, it is 282 feet below sea level! On a number line, we represent this elevation by -282. Mount McKinley is 20,320 feet above sea level. There is a big difference in the two elevations. Use $D$ to represent the height we must climb to go from the elevation of Death Valley to the elevation of Mount McKinley. The equation that models this situation is $-282+D=20,320$. Solve for $D .20,602 \mathrm{ft}$.

In exercises 9 and 10, define a variable, set up an equation, solve, and check. Remember to include your units of measurement, such as ${ }^{\circ} \mathrm{F}$ and ${ }^{\circ} \mathrm{C}$, in the definition and in the answer.
10. In the morning it was a cool $65^{\circ} \mathrm{F}$. By the afternoon the temperature had reached $87^{\circ} \mathrm{F}$. What was the increase in temperature from morning to afternoon? $22^{\circ} \mathrm{F}$
11. On a cold day in Canada, the temperature was $-8^{\circ} \mathrm{C}$ at $6: 00$ A.M. How many degrees must it warm up to reach $5^{\circ} \mathrm{C}$ ? $13^{\circ} \mathrm{C}$
12. Three is seven more than $P$. What integer does $P$ represent? See TE.
13. Emily eats three cookies and has 1 cookie left. Let $M=$ the number of cookies Emily had at the beginning. What integer does $M$ represent? See TE.
14. Brad was riding his bike with his father on a bike trail. Brad finished 8 minutes ahead of his father. If his father took 20 minutes to complete the course, how long did it take Brad? Write the expression for the number of minutes it took Brad to complete the course. Solve the problem.
15. Amy is bowling with her friend JD. Amy scored a total of 120 points. JD scored 15 points less than Amy. How many points did JD score? Write the expression for the number of points JD scored and solve the problem.
16. a. $\triangle=\Delta=2$
b. $\mathbb{Z}=6, \nabla=3$
c. $\quad=2, \square=6, ~=1,\rangle=6$

## 16. Ingenuity:

In a certain sequence of numbers, each term after the first is 4 greater than the previous term. We don't know the value of the first term; but we do know that the sixth term of the sequence is 47 . We want to find the value $x$, the second term. We can represent this information as

$$
\text { _, } x, \ldots, \ldots, \ldots, 47, \ldots
$$

What is the value of $x$ ? 31

## 17. Investigation:

A mobile is a type of hanging sculpture in which several objects are suspended in balanced equilibrium. In the mathematical mobiles below, each shape has an associated weight, and for each horizontal beam, the total weight hanging from one side is equal to the total weight hanging from the other (we assume that the wire has no weight). For example, in the mobile on the left, the two circles together weigh as much as the rectangle; the rectangle weighs twice as much as a circle. In the mobile on the right, the circle has the same weight as the hexagon, and the rectangle has twice the weight of the hexagon.


If the rectangle has weight 4 , the hexagon and circle each have weight 2 . We can write this symbolically as $\square=4, \square=\bigcirc=2$. Based on the weight given and the mobile balance property, deduce each shape's weight.



## Chapter 3

## Jeopardy!

Objective: This game is designed to help students review all sections in Chapter 3.

## Materials:

Computer (preferably with access to project the screen to the front of the room)
Scratch paper (or individual dry erase boards)

## Activity Instructions:

The class will need to be divided into 3 groups, not necessarily with the same number of members. You will, however, want to make sure that the groups are equitable as far as student ability levels. Each group will allow one person at a time to come up and compete for Jeopardy points. If a group has one more student than another, it will only change the rotation of the contestants competing.

After deciding on groups, position the students so that they are lined up in the order in which they will compete. The first members of all three groups will then come to the front of the room. (I find that it works best if you line up desks in 3 long rows. The students then take turns coming to the desks in the front row, and the front row represents the students competing in each round.)

The Jeopardy power point should be accessed and projected to the front of the room. The teacher will be the person operating the Jeopardy game (or the teacher can assign a responsible student to do this job). After choosing which team will go first, the student in the front row of that team will choose a category and a point level to begin the game.

The power point is set up so that when you click on the selected category and point level, the question will pop up on the screen. The first student to get the correct answer will get the points. When you click to the next screen, the answer to this question will pop up so the whole class can see the correct answer. The teacher (or a responsible student) should be in charge of calculating total points for each team. At this time, the students should rotate so the next group of three students are now in the front row, and the group that just won the last round will now pick the new category and point level. The game continues until you either run out of time or you run out of questions. The team with the most total points wins.

In the case of a tie, the teacher will need to decide upon a Final Jeopardy question. The game does not provide a Final Jeopardy question for you.
*If you do not have a way to project your computer screen to the whole class, you can modify this game by copying the questions onto index cards and displaying them in a pocket chart, or just taping them to the board in the front of the room. You could also print out the power point pages and cut and paste the problems to the index cards. (http://www.magicnet.net~itms/jeopardy/index.htm)

## Section 3.3 - Solving Equations with Subtraction

## Big Ideas:

- Using the balance model to solve simple algebraic equations.
- Establishing the 4 step method to solve equations.


## Key Objectives:

- Observing that equations maintain balance on each side of the equal sign like a scale.
- Using the Addition Property of Equality and Subtraction Property of Equality.


## Pedagogical/Orchestration:

- Students should have previous experience translating expressions.
- Use the Launch as the basis for discussion. Work through the examples without the book. Students should take notes and refer to the book as a resource after the lesson is presented.
- Emphasize use of the 4 step method.
- Teacher should read this section (Body of the Lesson) carefully before presenting. In the models, the gray tiles represent positive integers, and the white represent negative integers.
- Solving addition equations using subtraction.


## Materials:

- Balance scale from science department or commercial algebraic scale, Small printable balance scale models from the end of this section or the CD for doing the exercises, Chips or 2-color counters (red and yellow recommended)


## Exercises:

Exercises 1 - 9 appropriate for 6th grade.

## Vocabulary:

Subtraction Property of Equality, equivalent, balanced, Addition Property of Equality

## TEKS:

$6.4 ; \quad 6.5 ; \quad 6.12(A) ; \quad 7.5(A)(B) ; \quad 7.13(A)(B)(D) ; \quad 7.14(A)$

## WARM-UPS for Section 3.3

1. Suppose a rectangular garden is twice as long as it is wide. If we build a rectangular fence to enclose the garden and the garden is 15 feet wide, how much fencing will we need?
a. 80 feet
b. 90 feet
c. 450
d. 100 feet

Ans: $b$ because Perimeter $=2 L+2 W=2(2 W)+2 W=6 W=6(15)=90$ feet
2. Mary has M number of marbles and Jack has twice as many marbles as Mary. Cecelia has 5 more marbles than Jack. If Cecelia has 27 marbles, write an equation for how many marbles does Cecelia has in terms of M .
a. $2 \mathrm{M}+5=27$
b. $2 \mathrm{M}-5=27$
c. $2(M+5)=27$
d. $2 \mathrm{M}+5=27$

Ans: $d$ because Jack has $2 M$ marbles, Cecelia has $2 M+5$ marbles. So, $2 M+5=27$.

## Launch for Section 3.3:

Tell your students that today they will learn how to think about solving equations using a physical model. Draw and explain what a balance scale is, or have a physical balance scale in the room. Many students are familiar with them from science activities; some might know what one is without knowing the name, and others might know nothing about the balance scale at all. A helpful metaphor is a teeter-totter, or seesaw. If a person sits on one end of a seesaw, it will rise on the other end. If a friend gets on the other end, the seesaw will be more balanced. Ask the students who have been on a seesaw with an adult what happened. They will probably say that the adult weighed much more and therefore sank while the student rose into the air. Remind them that the seesaw will not balance if there are unequal weights at either end. Similarly, a balance scale will not balance unless both sides are equal.

Ask your students to remember the chip model of addition and subtraction from a few sections ago. Ask if they remember how to express zero using the chips. Also ask if they remember how to add zero pairs to evaluate a problem like $5--2$. Ask a student to explain or demonstrate the concept and process. Tell them just like balancing a scale or a teeter-totter, they will be using these concepts to balance equations.
*Note: Have students write equations on index cards. These will be used as part of the summary at the end of the section.

## SECTION 3.3 SOLVING EQUATIONS WITH SUBTRACTION

This is a balance scale:


When we put a weight on one side of the scale, we must place the same weight on the other side in order for the scale to be balanced. If a scale is balanced and equal weights are added or subtracted from both sides of the scale, the scale will remain balanced.

In much the same way, an equation is a statement that two expressions are equal. We can think of the expressions on each side of the equality sign as representing the weight placed on each side of a balanced scale. When we add or subtract the same amount from each side of the equation, the equation will remain balanced.

## EXAMPLE 1

If Jeremy was three years older, he would be the same age as his twelve-year-old sister. What is Jeremy's age?

## SOLUTION

We let J be Jeremy's age, and translate the sentence into an equation as follows:

| Jeremy's | three years | same age | twelve-year-old |
| :---: | :---: | :---: | :---: |
| age | older | as | sister |
| J | +3 | $=$ | 12 |

## Body of Lesson

Teachers should read carefully before presenting. In the models, gray tiles represent positive integers and the white represent negative integers.

Draw an empty balance scale. Ask the class what will happen to one end if you add 4 to the other end. (It will rise and the side with 4 will sink.) Model several extensions with illustrations. What would happen if you then added two to the other side? (The four side would rise a little bit and the 2 side would fall a little bit, but one is still heavier and so unbalanced.) How much would they need to add to the 2 side to make the scale balance? (2) Why? What would happen if you added 10 to both sides? (Each side would have 14, but the scale is still balanced.) What if you subtracted half from each? (Half of 14 equals 7, so subtracting 7 would result in 7 on each side, so the scale remains balanced).

Teachers, note that students often have trouble understanding the two sides of an equation are equal to each other. The left side is the same "size" as the right side. The equal symbol is supposed to help with this understanding, but many students do not appreciate the significance. Because the two sides often look visually different, some students do not see them as equal. Take time throughout the lesson to let students gain an understanding of algebraic equality.

Pose Example 1 for students to translate into a symbolic form. When the class has agreed on the equation, show how your students can use balance model to represent the equation, as modeled in the book. Ask the students to explain why you drew the balance scale diagram the way you did. Where are the variable, the equality, and the other numbers? Why is the left side the way it is? The right side? Ask them how they can use the fact that they want to maintain balance between the two sides when they find the value of J? Draw pictures to represent their next steps.

Continue through the examples from the text, but after Example 1 allow your students to draw, possibly at the board, their own pictures rather than passively watching them being drawn each time. The act of deciphering the text into symbols and diagrams is helpful for understanding.

Now we have the equation $\mathrm{J}+3=12$. The unknown is J , Jeremy's age. Pictorially, this sentence says that $J+3$ is the same as 12 , which we can show on a balance scale:


In order to solve this equation for J, we must find what balances J. To do this, we remove three "blocks" from each side of the balance scale:


This is what we have left:


We can express this algebraically as follows:

$$
\begin{array}{rlr}
\mathrm{J}+3 & =12 \\
(\mathrm{~J}+3)-3 & =12-3 & \\
\mathrm{~J}+(3-3) & =9 \quad \text { by associative property } \\
\mathrm{J}+0 & =9 \quad \text { by additive inverse property } \\
\mathrm{J} & =9 \quad \text { by additive identity property }
\end{array}
$$

Because we have now solved for J, we can go back and check the solution. Substituting $\mathrm{J}=9$ into the original equation $\mathrm{J}+3=12$ gives us $9+3=12$, which is correct. Jeremy's age is 9 .

## EXAMPLE 2

If Wesley finds 5 more marbles, he will have the same number of marbles as John. John has 11 marbles. How many marbles does Wesley have?

## SOLUTION

We can use a four-step method similar to the one in the previous section to solve the equation.

## Step 1: Define a variable.

Let $W=$ the number of marbles that Wesley has.

## Step 2: Translate the problem into an equation.

Wesley's marbles finds five more same number as John (11 marbles)

$$
\begin{gathered}
W+5 \\
W+5=11
\end{gathered}
$$

Step 3: Solve the equation.
Draw a balance scale which represents the problem. Solve the equation with the balance scale, and then solve it algebraically.



$$
\begin{array}{rlr}
W+5 & =11 & \\
(W+5)-5 & =11-5 \\
W+(5-5) & =6 \quad \text { by associative property } \\
W+0 & =6 \quad \text { by additive inverse property } \\
W & =6 \quad \text { by additive identity property }
\end{array}
$$

## Step 4: Check the solution.

Substituting the solution $W=6$ into the original equation $W+5=11$ gives $6+5=11$, which is correct. Wesley has six marbles.

## EXAMPLE 3

Jeffrey checks his bank balance and finds that he has been charged a $\$ 6$ fee. His balance is now $-\$ 10$. What was Jeffrey's balance before the fee was assessed?

## SOLUTION

## Step 1: Define a variable.

Let $\mathrm{X}=$ Jeffrey's original bank balance.

## Step 2: Translate the problem into an equation.

$$
\begin{gathered}
\text { Jeffrey's original balance with a } \$ 6 \text { fee becomes } \\
\hline \mathbf{~} \begin{array}{c}
-\$ 10 \\
+-6
\end{array} \\
x+-6=-10
\end{gathered}
$$

Step 3: Solve the equation.
We would like to use our balance scale model again, but how do we represent-6 and-10? In Section 2.5, "The Chip Model," we used positive chips for positive numbers and negative chips for negative numbers. In the balance scale model, a gray, or filled, block on the balance beam represents +1 , and a white, or empty, block represents the opposite, -1 .


## Step 4: Check the solution.

Substitute $X=-4$ into our original equation $X+-6=-10$ to get $-4+-6=-10$, which is true. Jeffrey's original balance was $-\$ 4$.

The rule we are using to solve these problems is called the subtraction property of equality because in each, we are subtracting the same number (removing the same number of blocks) from both sides of an equation. The new equation we obtain is said to be equivalent to the original equation because the two equations have the same solution: any value for a variable that makes one of the equations balance will make the other balance as well.

## PROPERTY 3.1: SUBTRACTION PROPERTY OF EQUALITY

$$
\text { If } \mathrm{A}=\mathrm{B} \text {, then } \mathrm{A}-\mathrm{C}=\mathrm{B}-\mathrm{C} \text {. }
$$

## Summary

You can go over the final student created equations from their index cards. Ask the students to draw balance scale pictures of the equations.

Before leaving, ask students to write in their notebooks how an equation is like a scale balance or a seesaw.

In Section 2.2 you learned that subtracting is the same as adding the opposite. The subtraction property of equality is the same as the addition property of equality for negative numbers.

## PROPERTY 3.2: ADDITION PROPERTY OF EQUALITY

$$
\text { If } A=B \text {, then } A+C=B+C \text {. }
$$

This property states that a balance scale stays balanced if we add the same number of blocks to both sides. The new equation this creates is equivalent to the original. Any value for a variable that balances the new equation will balance the original equation as well.

The property is easier to see in terms of the balance scale model:
If $A=B$,


Then $\mathrm{A}+\mathrm{C}=\mathrm{B}+\mathrm{C}$.


## EXERCISES

Exercises 1-9 are appropriate for 6th graders. Exercises 10-12 are appropriate for 7th graders.
1.

2.

3.

4.

11. Look for a pattern by starting with the differences between the first terms in each sum: $2-1=1$ and the differences between the second terms in each sum: 4-3=1, until students realize that for each pair of numbers the difference will be 1 . So for the 30 pairs, the answer is $1+1+\ldots$ thirty times, or 30 .
12. a. Add $x$ to both sides, then subtract 2 .
b. Subtract 2 from both sides.

## EXERCISES

Illustrate how the equation would appear on the balance scale. Solve the problem by adding or subtracting chips to both sides of the balance. Then solve each equation algebraically. Check that your answer is correct.

1. $B+3=8$
2. $7=2+\mathrm{A}$
3. $8=N+5$

$$
N=3
$$

4. $x+-2=-6 \quad x=-4$

Solve the following equations using the pattern seen in the problems above. Check that your answer is correct.
5. $x+10=8 \quad x=-2$
6. $x+-7=-13 \quad x=-6$
7. $-3+5=2+n \quad n=0$
8. $3+v-7=-6 \quad v=-2$
9. $b+c=d$ Solve for $c($ in terms of $b$ and $d) . \quad c=d-b$
10. $2 a+x=3 y \quad$ Solve for $x$ (in terms of $a$ and $y$ ). $x=3 y-2 a$
11. Ingenuity:

What is the difference between the sum of the first 30 even positive integers $(2,4,6,8, \ldots)$ and the sum of the first 30 odd positive integers $(1,3,5,7, \ldots)$ ? Try to answer this problem without calculating any large sums. See TE.
12. Investigation:

We have solved some algebraic equations using the four-step model. Consider the following two equations:
a. $5-x=2$
b. $5=2+x$

How can you use some of the ideas we have used previously to solve these equations? See TE.

## Balance Scale Models for Exercises



## Section 3.4 - Solving Equations with Addition

## Big Idea:

Solving Subtraction Equations by using the balance scale model.

## Key Objectives:

- Using the Chip Model and use of zero pairs.
- Adding negative variables, like (-m) in Example 4.


## Pedagogical/Orchestration:

- Use of additive inverses from Section 2.4.
- Chip Model from Section 2.5


## Materials:

- Balance Scale from science department or commercial algebraic scale, Small printable balance scale models from the end of this section or the CD for doing the exercises


## Activities:

Jeopardy Game from section 3.2

## Exercises:

Students should show proficiency using the balance scale model.

## Vocabulary:

No new vocabulary

## TEKS:

6.4; 6.5;
6.12(A)
7.5(A)(B);
7.13(A)(B)(D);
7.14(A)
8.5(A);
8.14(A);
8.15(A)

## WARM-UPS for Section 3.4

1. Juan has M dollars. Brandon has $\$ 13$ more than Juan. Joe has $\$ 20$ less than Brandon. Mandy has twice the amount of money as Joe. How much money does Mandy have in terms of M dollars.
a. $2(M-7)$
b. $M+13-40$
c. $2 \mathrm{M}-7$
d. $M-14$

Ans: a
2. If you knew that the difference of two numbers was 45 and the smaller number was 7 , how would you find the larger number? If you knew that the difference of two numbers was 4 and one of the numbers was 7 , could there be more than one possible value for the other number? Draw a number line to model this problem?
Ans: Part 1: $X-7=45$. So $X=45+7=52$. Part 2: Either, $X-7=4$ or $7-X=4$. So $X=11$ or $X$ $=3$.

## Launch for Section 3.4:

Let your students know that today they will learn how to solve equations that involve differences using a physical model. Review how a balance scale works with a physical balance scale or its equivalent. Remind the class that the scale will not balance if there are unequal quantities on either side. As a reminder of yesterday's lesson, demonstrate taking away equal weights from both sides to keep the scale balanced and to find the variable, x . Remind students that $\square$ represents 1 unit, $\square$ represents -1 , and $\square$ represents $x$.

Ask students how they would represent $x-1$. Discuss student responses. One representation that should be brought into the conversation is the x symbol with 1 "hole" in it like this:

Tell students that throughout the lesson they should be thinking about what they need to do to fill that hole so that x can be complete, yet the balance scale will still stay balanced.

## Body of Lesson

Draw a balance scale with weights on either side (this explanation will assume 12 weights on either side, adjust for your own choice of weights). Ask the class what will happen to one end if you subtract 4 from the other end. (It will fall and the side with 8 will rise.) Model several extensions with illustrations. What would happen if you then subtracted two from the other side? (The 8 side would fall a little bit and the 10 side would rise a little bit, but one is still lighter and so unbalanced.) How much would they need to add to the 8 side to make the scale balance? (2) Why? What would happen if you subtracted 5 from both sides? (Each side would have 3, and the scale is now balanced.) What if you added half of 14 to each? (Half of 14 equals 7 , so adding 7 would result in 10 on each side, so the scale remains balanced).

Teachers, remind your students who are having a hard time understanding the scale model that just because the two sides often look visually different, they can be equal. Take time throughout the lesson to let students gain an understanding of algebraic equality.

Pose Example 1 for students to translate into a symbolic form. When the class has agreed on the equation, ask one of your students to use the balance model to represent the equation, as modeled in the book. Ask the class to explain why the student drew the balance scale diagram the way he/she did. Where are the variable, the equality, and the other numbers? Why is the left side the way it is? The right side? Ask the class how they can use the fact that they want to maintain balance between the two sides to find the value of W . Have one of your students draw pictures to represent their next steps.

Continue through the examples from the text, allowing your students to draw, possibly at the board, their own pictures as you did yesterday. The active deciphering of the text into symbols and diagrams is helpful for understanding.

Teacher Tip: You can verify understanding of the model by drawing another scale and asking students, "What equation does this balance represent?" Students could create their own drawings and trade with partners for more variety and equation fun.

## SECTION 3.4 SOLVING EQUATIONS WITH ADDITION

Some equations use subtraction and would be difficult to solve using our balance scale model so far. The following examples show how to represent subtraction with the balance scale model. When solving equations that involve subtraction, the fact that subtracting is the same as adding the opposite is an important tool.

## EXAMPLE 1

Whitney was 10 years old 5 years ago. What is Whitney's age now?

## SOLUTION

## Step 1: Define a variable

Let $\mathrm{W}=$ Whitney's age now.

## Step 2: Translate the problem into an equation.

$W-5=10$
Step 3: Solve the equation.
Method 1:
We can use "holes" to represent subtraction from a variable:


The holes in the large variable block each represent a 1 that has been subtracted, so the block with five holes means $W-5$, and the entire balance scale represents our equation $\mathrm{W}-5=10$.

We now need to fill in the holes. To do this, we must add five blocks to both sides of the balance. So, we can use the addition property of equality to finish Whitney's problem:


We added five positive blocks to fill in the five holes, so the variable block is now complete and represents W .


We can rewrite these steps algebraically, justifying the equivalent expressions at each step using the integer properties from Section 2.4.

$$
\begin{aligned}
W-5 & =10 & & \\
(W-5)+5 & =10+5 & & \text { by additive property of equality } \\
(W+-5)+5 & =15 & & \text { by subtraction property } \\
W+(-5+5) & =15 & & \text { by associative property } \\
W+0 & =15 & & \text { by additive inverse } \\
W & =15 & & \text { by additive identity }
\end{aligned}
$$

## Step 4: Check the solution.

If Whitney is 15 , then Whitney's age five years ago was $15-5=10$.

## Method 2:

We could have solved this problem a slightly different way. Because $W-5=W+-5$, we could have changed the equation to $W+-5=10$. Remember that we model this on the balance scale by using empty blocks that each represent -1 each.


Now we want to remove the five negative blocks from the left. There are no negative blocks to remove on the right side, so instead we use the addition property of equality and add five positive blocks to both sides. As in the chip model, this creates five zero pairs (one positive block and one negative block each) on the left side of the balance.


We remove the five zero pairs from the left side, and we are left with the same solution as before, $W=15$.


We can rewrite these steps algebraically:

$$
\begin{aligned}
W-5 & =10 \\
W+-5 & =10 \\
(W+-5)+5 & =10+5
\end{aligned}
$$

$$
W=15
$$

## EXAMPLE 2

Amanda wrote a check for $\$ 9$. The balance on her checking account is now $-\$ 2$. How much money did Amanda originally have in her checking account?

## SOLUTION

## Step 1: Define a variable.

Let $\mathrm{Z}=$ Amanda's original balance.
Step 2: Translate the problem into an equation.
$z-9=-2$
Step 3: Solve the equation.
Again, we represent subtraction from the variable by putting holes in the variable block.


To fill in the nine holes, we add nine blocks.


The holes are filled in, and on the right side of the balance we have two zero pairs.


After cancelling the zero pairs, we are left with seven blocks on the right.


$$
\begin{aligned}
Z-9 & =-2 & & \\
(Z-9)+9 & =-2+9 & & \text { by additive property of equality } \\
(Z+-9)+9 & =7 & & \text { by subtraction property } \\
Z+(-9+9) & =7 & & \text { by associative property } \\
Z+0 & =7 & & \text { by additive inverse } \\
Z & =7 & & \text { by additive identity }
\end{aligned}
$$

## Step 4: Check the solution.

Our solution is correct because $7-9=-2$. Amanda originally had $\$ 7$ in her checking account.

## EXAMPLE 3

The temperature rises $4^{\circ} \mathrm{C}$ to $-7^{\circ} \mathrm{C}$ when the sun comes out. What was the original temperature?

## SOLUTION

## Step 1: Define a variable.

Let $\mathrm{Y}=$ the temperature, in Celsius, before the sun came out.
Step 2: Translate the problem into an equation.
$Y+4=-7$
Step 3: Solve the equation.


We add 4 negative blocks to each side in order to pair up four positive blocks to the four negative blocks on the left hand side. This will create a zero pair on the left hand side and leave $Y$ by itself.


$$
\begin{aligned}
Y+4 & =-7 \\
(Y+4)+-4 & =-7+-4 \\
Y & =-11
\end{aligned}
$$

Explain to students that we often rewrite equations such as $5-M=3$ as $5+-M=3$ because the balance scale works with sums. There may be instances in their mathematical careers where one way of writing equations makes more sense than another way. Through practice, students will develop flexibility in their thinking and adopt their own preferences.

## Step 4: Check the solution.

The solution is correct because $-11+4=-7$. The original temperature was $-11^{\circ} \mathrm{C}$.

## EXAMPLE 4

Tuesday morning Amanda had $\$ 5$. Later in the day, she checked and found she had only \$3 left. How much had she spent?

## SOLUTION

Step 1: Define a variable.
Let $\mathrm{M}=$ the amount of money, in dollars, Amanda spent Tuesday.

## Step 2: Translate the problem into an equation.

$5-M=3$

## Step 3: Solve the equation.

Because subtraction is the same as adding the opposite, we can rewrite the equation as $5+-\mathrm{M}=3$.


Since we want to solve for positive $M$, we use the addition property of equality to add M to both sides. This creates a pair of additive inverses on the left side of the balance; we can also think of our notation as shorthand for M zero pairs.


Make sure your students know that $2+3=5$ is the same thing as $5=2+3$.
The scale remains balanced even if we switch the quantities so that what was on the right is now on the left, and what was on the left is now on the right. This is an idea of symmetry.

## Summary

Make sure your students understand the concept of "holes" in representing negative numbers. Balancing the equation, then, is simply filling in the hole to represent subtracting zero.

Before leaving, ask students to write in their notebooks how they can use a scale balance to represent an equation that involves subtraction, or finding a difference.

Notice that after we remove these additive inverses, we arrive at a balance representing the addition problem $5=M+3$, which we know how to solve. Unlike before, however, the variable is on the right side of the balance.


After removing three blocks from each side of the balance, we are left with M on the right and two positive blocks on the left.


$$
\begin{aligned}
5-M & =3 \\
5+-M & =3 \\
(5+-M)+M & =3+M \\
5+(-M+M) & =M+3 \\
5+0 & =M+3 \\
5 & =M+3 \\
5-3 & =(M+3)-3 \\
2 & =M+(3-3) \\
2 & =M+0 \\
2 & =M
\end{aligned}
$$

## Step 4: Check the solution.

The solution is correct because $5-2=3$. Amanda spent $\$ 2$ Tuesday.

## EXERCISES

1. $x+9=7 \quad x=-2$
2. $-2=T+2 \quad T=-4$
3. $x+23=17 \quad x=-6$
4. $50=H+50 \quad H=0$
5. $-4=D-6 \quad D=2$
6. $-2=M+8 \quad M=-10$
7. $-10=9+x \quad x=-19$
8. $t-4=-3 \quad t=1$
9. Let $s=$ stolen coins by the third pirate; $s+5=23 ; s=18$ coins; $18+5=23$
10. Let $\mathrm{c}=$ number of cookies Jenny started with; $\mathrm{c}-12=32 ; \mathrm{c}=44$ cookies; $44-12=32$

## Ingenuity

11. Let $f=$ first test score; $4 f+96=70(4) ; f=46 ; 4(46)+96=70(4)$

This is a very good spot to lead your students to the realization that if the average of 4 tests is 70 , their sum is 280.

## EXERCISES

Solve the following equations. You may use the balance scale model first if you like, but you must show the steps algebraically. Label each step with the property that you are using. Check that your answer is correct.

1. $x+9=7$
2. $-2=T+2$
3. $x+23=17$
4. $50=H+50$
5. $-4=\mathrm{D}-6$
6. $-2=M+8$
7. $-10=9+x$
8. $t-4=-3$

In problems 9 through 11, define a variable, set up an equation, solve, and check.
9. One day three pirates found 23 gold coins. The next day there were only two pirates and 5 gold coins. How many gold coins did the third pirate steal?
10. After eating 12 cookies, Jenny only has 32 cookies left. How many cookies did Jenny start with? 44 cookies
11. Ingenuity:

Helen failed the first history test of the semester. On the second test, her score was 8 points higher. Helen's score on the third test was 28 points higher than her score on the second test. Her score on the fourth test of the semester was 16 points higher than her score on the third test. The average of her four test scores was 70. What was Helen's score on the first test? 46

## Investigation

12. Notice that we are skip counting by -4s. Instead of computing all the numbers up to the 100th number in the sequence, students can build a table to organize their information like the table below.

| Number | Term | Process |
| :--- | :--- | :--- |
| 1 | 11 | $11-4(0)$ |
| 2 | 7 | $11-4(1)$ |
| 3 | 3 | $11-4(2)$ |
| 4 | -1 | $11-4(3)$ |
| 100 | -385 | $11-4(99)$ |
| $n$ | $-4 n+15$ | $11-4(n-1)$ |

## 12. Investigation:

Suppose we have the following sequence of numbers: $11,7,3,-1,-5, \ldots$ What are the next 4 numbers in the sequence? What is the $15^{\text {th }}$ number in the sequence? What is the $100^{\text {th }}$ number in the sequence? $9,13,17,21$. The $15^{\text {th }}$ number in the sequence is 45 . The $100^{\text {th }}$ number in the sequence is 385.

## Balance Scale Models for Exercises



## Section 3.5 - Equations and Inequalities on Number Line

## Big Idea:

Develop the idea of solving equations using a number line.

## Key Objectives:

- Plot points on the number line to represent various expressions.
- Reflecting on the effects of adding/subtracting a positive integer to/from a variable.
- Visually connecting that $x+3=7$ is equivalent to $x+2=6$ and $x+1=5$ and $x=4$.


## Pedagogical/Orchestration:

- Use of additive inverses from Section 2.4.
- Chip Model from Section 2.5


## Vocabulary:

equation, inequality

## TEKS:

6.9 (A, B, C )

If $x<0$, then $-x>0$ and $2 x<x$. Students might object to $-x$ being a positive number. An extension question could be to plot points that represent $\frac{\mathrm{a}}{2}$ and $\frac{3 a}{2}$ in part 1 .

Students can make a little strip of paper with marks showing the distance of 1 and use it to measure 1 unit to the right or left of $x$. Students may try to estimate the value of $x+1$ or $x-2$. This is OK but it is not the purpose of this activity.

## SECTION 3.5 EQUATIONS \& INEQUALITIES ON NUMBER LINE

In this chapter we have used the balance scale to solve equations such as " $x+3=5$ " and " $x-4=3$ ". We can also explore equations using a number line. We begin by investigating how to visualize expressions on a number line.

## EXPLORATION 1

Suppose $a$ and $x$ are numbers located on the number line as seen below. Locate and label the points that represent the indicated numbers. Use string to act out how you determine your answer.

1. Plot points that represents each of the following: $2 a, 3 a,-a,-2 a,-3 a$

2. Plot points that represents each of the following: $2 x, 3 x,-x,-2 x,-3 x$

3. Compare the results from parts 1 and 2 . What do you notice?

## EXPLORATION 2

PART A: Suppose is a number that is located on the number line as seen below. Locate and label the points that represent the indicated expressions. The numbers 0 and 1 are also labeled. Plot a point that represents each of the following expressions:

$$
x+1, x+2, x-1, x-2, \frac{x}{2}, 1-x
$$

Reflect with students about any patterns that they notice. In particular, what is the effect of adding a positive integer to a variable? What is the effect of subtracting a positive integer from a variable?

Expect students to see that

$$
\begin{aligned}
& a+2=5 \text { is equivalent to } a+1=4 \text { and } a=3 \\
& b-2=-1 \text { is equivalent to } b-1=0 \text { and } b=1 \\
& c+5=-2 \text { is equivalent to } c+4=-3 \text { and so on until } c=-7 .
\end{aligned}
$$

Reflect with class that we are showing that if $a+2=5$, then $a=3$. This is a visual way of "solving the equation $a+2=5$. So, the equation $b-2=-1$ has the solution $b=1$ and the equation $c+5=-2$ has the solution $c=-7$.

## PROBLEM 1

a. $x=2$
b. $y=-3$
c. $z=6$


PART B: Suppose we know the location of each of the expressions as indicated on the number line below. Find the locations for $a, b$ and $c$. Explain how you locate each of these points on the number line.


In Part A in this exploration we used the location of a variable on the number line to locate expressions on the same number line. In Part B, we were given the location of an expression, such as $a+2=5$, and used it to find the location of the variable $a$ on the number line. We see that $a=3$. In other words, we solved the equation using the number line.

## PROBLEM 1

Use the number line to solve each of the following equations:
a. $x+3=5$
b. $y+5=2$
c. $z-4=2$
d. Discuss how solving these equations on the number line compares with the balance scale method.

Recall that an equation is a statement that two expressions are equivalent. A statement that one expression is always less than (or greater than) another is called an inequality.

## EXAMPLE 1

1. The number of apples, $x$, consumed is more than twice the number of bananas, $y$. Thus, $x>2 y$.
2. You can also say $0<B<35$

We say -3 is in $S$ because the statement that -3 is less than or equal to -3 is true because $-3=-3$.

If students ask if $x$ can be any number (not integer) it is a good conversation to have.
2. Bob's age, $B$, is less than 35 years. Thus, $B<35$.
3. The cost of three apples is less than $\$ 2.00$.

Write an inequality to represent the possible $\operatorname{cost} A$ of one apple.
Let $A=$ cost of an apple in cents. Then $3 A<200$
Sometimes an inequality is a statement of comparison between two quantities, such as, $4<7$. But we can also use an inequality to describe a condition that a variable satisfies, as in Bob's age, $B$, is less than 35 . So we say $B<35$.

We can use a number line to represent all the possible numbers that satisfy an inequality. For example, suppose $S$ is all numbers less than 3 . We say that $S$ is the set of numbers that are less than 4. Another way to describe this set is
" $S$ is the set of all numbers $x$ so that $x<4$."
We use the inequality " $x<4$ " to defined the set $S$. We can represent this set $S$ on the number line below.


Notice that the part of the number line to the left of 4 represents the set $S$. This means that each number to the left of 4 is in $S$ and every number of $S$ is located on the line to the left of 4 . Note that the point representing 4 is not filled in to indicate that 4 does not satisfy the condition that $x<4$.

## EXAMPLE 2

Draw a number line and represent the set $T$ of all numbers x such that $-3 \leq x$.


Notice that the point at -3 is filled in to indicate that -3 does satisfy the condition that $-3 \leq x$.

In solving an equation, such as $x+2=5$, we want to find all numbers x that satisfies this statement. Since the only solution is $x=3$, we say that the solution set is $\{3\}$.

Discuss how the inequality $x<3$ and $3>x$ are equivalent.

If we start with an inequality, such as $x+2<5$, we can ask:
For what numbers $x$ satisfy this inequality?
We can represent the inequality on the number line as shown below.


The shaded part of the number shows where the expression $x+2$ could be located on the number line. If $x+2<5$, then $x+1<4$ and $x<3$. We draw a new number line to represent where the variable $x$ can be. The resulting number line below is called the graphical solution to the inequality $x+2<5$.


Notice that adding or subtracting the same number to both sides of an inequality will result always preserve the inequality. An inequality is like a balance beam that is not balanced. If you add or subtract the same things from both sides of the balance, the balance beam will still be unbalanced in the same way. If $a<b$ then $a+c<b+c$, and if $a<b$ then $a-c<b-c$. Let's use this to solve an inequality algebraically, and then graph the solution.

## EXAMPLE 3

Draw a graphical solution to the inequality $x-4<8$ algebraically Then graph the solution.

## SOLUTION


4. a. $x=4$
b. $x=-4$
c. $x=12$
d. $x=-7$
e. $x=16$
f. $x=17$
g. $x=-14$

## EXERCISES

1. Plot a point that represents each expression:

$$
2 x, 2 x+1,2 x-1,3 x-1
$$


2. Plot a point that represents each expression:

$$
y+1, y-1,2 y, 2 y+1,2 y-1,2 y+8
$$


3. Solve the equation $x+5=2$ using the number line.


4 The cost of five apples is less than $\$ 8.00$.
Write an inequality to represent the possible cost A of one apple.

5 Use a number line to solve each of the following equations:
a. $x+3=7$
b. $x+7=3$
C. $x-5=7$
d. $x+9=2$
6. Solve each of these equations. Do you need to sketch a number line to help find the solution?
a. $x-10=6$
b. $x-24=-7$
c. $x+29=15$
7. Draw a number line and represent the set of all numbers $x$ such that $x<8$.
12. a. $x=3$
b. $x=4$
c. $x=-2$
8. Draw a number line and represent the set of all numbers $x$ such that $x<-5$.
9. Draw a number line and represent the set of all numbers $x$ such that $x>-4$.
10. Draw a number line and represent the set of all numbers $x$ such that $-6<x$.
11. Solve each of the following inequalities and graph their solution sets:
a. $x+3<2$
b. $x-3<2$
c. $x+5<6$
d. $x+5<2$
e. $x+3>6$
f. $x-4>2$
12. Graph the solution sets for each of the following inequalities:
a. $2<x+3<5$
b. $0<x-3<2$
c. $2<x+5<8$
d. $0<x+4<3$

## 13. Investigation:

Use a number line to solve each of the following equations:
a. $2 x-1=5$
b. $2 x+3=11$
c. $2 x+8=4$


## REVIEW PROBLEMS

1. Illustrate how the equation would appear on a balance scale. Then solve algebraically for the unknown variable. Check that your problem is correct.
a. $x+3=9$
b. $-7=n+-2$
c. $Z-8=5$

In problems 2-6, translate the sentence into an equation and solve for the unknown variable. Does your answer make sense?
2. A number is 4 less than 11 .

$$
x=11-4, x=7
$$

3. A number is 3 more than twice 15 .
4. A number is the difference of 28 and 4 .
$x=2(15)+3, x=33$
5. A number is 15 subtracted from 16 .
$x=28-4, x=24$
6. 8 more than a number is 15
$x=16-15, x=1$
7. 4 less than a number is -7
$x+8=15, x=7$
8. A number is the sum of 3 and another number that is half of 40 .

In problems 9-18, solve the equation.

$$
x=\frac{1}{2}(40)+3, x=23
$$

9. $3+t=2$
$t=-1$
10. $p-16=-3$
$p=13$
11. $y=20+-283$
12. $63+m=-18$
13. $28+d=32$
14. $9=a+8$
$y=-263$
15. $-8=m-7$
$m=81$
$d=4$
$a=1$
16. $x=32-50$
$m=-1$
$x=-18$
17. $54-a=25$
$a=29$
18. $17-b=40$
$b=-23$
19. $-34+\mathrm{A}=20$ so $\mathrm{A}=54$
20. The answer depends on the present year. Take the 21st century year and add 525. $x=P Y-1475$.
21. $m+27=83$ so $m=56$
22. $\mathrm{T}+22=8$ so $\mathrm{T}=-16$
23. $T-26=-15$ so $T=11$
24. 



We also know that there are 14 passengers after the 3rd bus stop, so that makes $i-17=14$, which means that there must have been 31 passengers initially.
19. Artemis was born in Greece in the year 34 BCE . Her grandson was born in the year 20 AD. How old was Artemis in the year when her grandson was born?
20. Rome is said to have been founded in 753 B.C.E. How many years passed before Christopher Columbus landed in North America in 1492? Represent 753 B.C.E. by -753 . Write an equation which models this problem, solve it and check your answer. $753+x=1492 ; \quad x=2245$
21. The great Renaissance artist Michelangelo was born in 1475. If Michelangelo were still alive, how old would he be on his birthday this year? Write a mathematical equation, solve and check. See TE.

For exercises 22-24, write an equation which models each problem. Solve it and check your answer.
22. Jack has a bag of marbles. Judy gives him 27 new marbles. If Jack now has 83 marbles, how many did he have in the beginning?
23. In an Alaskan village, the temperature rose $22^{\circ} \mathrm{F}$ from dawn until 10 a.m. What was the temperature at dawn if we know that it is $8^{\circ} \mathrm{F}$ at 10 a.m.?3
24. In a Canadian village, the temperature drops from $26^{\circ} \mathrm{C}$ from $4: 00$ p.m. until 7:00 p.m. If the temperature at 7:00 p.m. is $-15^{\circ} \mathrm{C}$, then what was the temperature at 4:00 p.m.?

In problems 25 through 27, define a variable, set up an equation, and solve.
25. Valerie bought 8 books. She now has 23 books. How many books did she start out with? $8+b=23 ; b=15$
26. In 24 years, Victor will be 42 years old. How old is Victor now? $V+24=$ 42; $V=18$
27. A city bus makes three stops and the following events take place:

- At the first stop, 5 passengers get off and 2 passengers get on.
- At the second stop, 8 passengers get off and 3 passengers get on.
- At the third stop, 9 passengers get off and none get on.

After the third stop there are 14 passengers on the bus. How many passengers were on the bus initially? $P-5+2-8+3-9=14 ; \quad P=31$
29. $L=$ Lynn's age

J = Joan's age
$B=$ Bobby's age
$\mathrm{L}=\mathrm{J}+7$
$B=L+2$
$B=(J+7)+2=J+9$
$B-J=9$ years
28. $2+a=6$ and $b=8+a$. Solve for $b$ and $a . \quad a=4$ and $b=12$
29. Ingenuity:

Lynn is seven years older than Joan but two years younger than Bobby. How much older than Joan is Bobby? 9 years

Section 3.1: 30
Solution: Call my age $x$, then $x+10=2(x-10)$. Thus $x+10=2 x-20$, so $x$ is 30 .
Alternate Solution: If a 20 year change in age doubles my age, I must have been 20 years old 10 years ago and now I'm 30.

## Section 3.3: \$15.00

## Solution:

Let $h$ be the price of a hamburger, $m$ for a milk shake, and $f$ for fries. Then $3 h+5 m=f=23.50$ and $5 h+9 m+f$ $=39.50$. The prices of the three items are not unique but we see that 4 times the first equation minus two times the second equation gives us $2 h+2 m+2 f=4(23.50)-2(23.50)-2(39.50)=39.50$.

Adding or subtracting multiples of two equations produces a "linear combination" of the equations which is satisfied by the same values of $h, m$, and $f$ that satisfy the original pair of equations.

## Bonus:

$(h, m, f)=(600,100,50),(500,150,100),(400,200,150)$ and $(300,250,200)$ with all the prices expressed in cents.

## Section 3.4: 75

## Solution:

Let $x$ be the total number of pieces of candy. If the first child takes $p$ pieces, then the first four take $p+2 p+4 p+$ $8 p=15 p$ pieces. The fifth child takes $x-15 p=\frac{15 p}{4}$ pieces, which is odd, so $p$ must be 4 times an odd number and $\frac{75 p}{4}$. Since $p$ is a power of 2 , it must be 4 , making $x=75$.

## CHALLENGE PROBLEMS

## Section 3.1:

I am older than I once was, and younger than I'll be. That's not unusual. Ten years from now I'll be twice as old as I was 10 years ago. How old am I now?

## Section 3.3:

The cost of 3 hamburgers, 5 milk shakes, and 1 order of fries at a certain fast food restaurant is $\$ 23.50$. At the same restaurant, the cost of 5 hamburgers, 9 milk shakes, and 1 order of fries is $\$ 39.50$. What is the cost of 2 hamburgers, 2 milk shakes and 2 orders of fries at this restaurant?

## Bonus:

If we restrict the price of a hamburger to a whole number of dollars and

$$
h>m>f>0,
$$

find four solutions for ( $h, m, f$ ). Do these values agree with your answer to the original question?

## Section 3.4:

Four children came across a pile of candy. Each one takes twice as many pieces as the one before. An odd number of pieces remain. A fifth child comes to take the remaining pieces, and is happy to see that he got as many as the average of the other four. If the number of pieces taken by the first child is a power of 2, how many pieces of candy were in the original pile?

## Section 4.1 - Multiplication of Integers

## Big Idea:

Using linear models to multiply integers

## Key Objectives:

- Connect multiplication to the number line.
- Discover multiplication rules for integers.


## Materials:

- Number Line handouts, Skip Counting Table handouts from the end of the section and on the CD, graph paper, graphing calculators, straight edges


## Pedagogical/Orchestration:

Teacher will guide students through Exploration 1 and have class discussion to connect multiplying and subtracting integers to solving linear functions.

## Activity:

"Leap Frog" from the end of the section and on the CD
"Skip Counting Race" Activity from the end of the section and on the CD

## Vocabulary:

factor, magnitude, multiplication, skip counting model, frog model

## TEKS:

6.11(C,D);
7.1(A); 7.2(A,B,C,G); 7.13(A,C);
7.14(A, B);
7.15(A,B) 8.2(B,C);
8.14(A,B);
8.15(A); 8.16(A)

## WARM-UPS for Section 4.1

1. In the list of numbers you get by skip counting by 6 , which of the following numbers does not appear in the list?
a. 192
b. 264
c. 506
d. 294
Ans: c
2. Mona builds a two room chicken house that has 272 square feet. The length of the house is 17 feet and the larger of the rooms is 9 wide. What is the area of each of the rooms?
Ans: The area of the larger room is 153 sq ft . So the area of the smaller room is 119 sq ft .

## Launch for Section 4.1:

Make a vertical list of skip counting by 5's. Use this to build a table as shown below. We want to talk about the different ways to express the product of two numbers.
$5=5 \times 1=(5)(1)=5 \cdot 1=5$ * 1
$10=5 \times 2=(5)(2)=5 \cdot 2=5 * 2$
$15=5 \times 3=(5)(3)=5 \cdot 3=5$ * 3

Reflect on the different notations for multiplication. Talk about the product of an integer and a variable.
Ex: 3x, 4A, -3y

Vocabulary Review for the word "factor": In the product of two numbers, each of them is called a factor. For example, in the product of $2 \cdot 3$, the numbers 2 and 3 are both factors.

# M U LTIPLICATION AND DIVISION 



SECTION 4.1 MULTIPLICATION OF INTEGERS

In Section 2.2, we introduced skip counting. We know that skip counting by 3's generates the list $3,6,9,12,15,18,21,24,27, \ldots$, which continues indefinitely. Skip counting provides a model for multiplication that we can represent on a number line.

When we added using the car model, we drove the distance that corresponded to each of the numbers we were adding. In order to multiply, we can think of a frog that jumps along the number line. For example, when you multiply 4.3

- the first factor indicates which direction the frog should face and the length of each jump;
- the second factor indicates the number of jumps.

The picture below models the multiplication $4 \cdot 3=12$. Notice the frog is facing in the positive direction because the first factor, 4 , is positive. The frog takes 3 jumps, and each jump is 4 units long.


At each step or jump in this process, the position of the frog changes by a constant amount, 4 units.

Hand out the tables from the end of this section. REMEMBER this is a section to do with the students' textbooks closed.

|  | Length of Jump | Number of Jumps | Frog's Location |
| :---: | :---: | :---: | :---: |
|  | 4 | 0 | 0 |
|  | 4 | 1 | 4 |
|  | 4 | 2 | 8 |
|  | 3 | 12 |  |
| 4 | 4 | 16 |  |
| 4 | 5 | 20 |  |
| 4 | 6 | 24 |  |
| 4 | 10 | 40 |  |
| 4 | 20 | 80 |  |
| 4 | $n$ | $4 n$ |  |

The students will be making "function tables" in Chapter 5.

## EXPLORATION 1: FROG JUMP MULTIPLICATION

Copy and fill Table 4.1a in which each jump is 4 units long.
Table 4.1a

| Length of Jump | Number of Jumps | Frog's Location |
| :---: | :---: | :---: |
| 4 | 0 | 0 |
| 4 | 1 | 4 |
| 4 | 2 | 8 |
| 4 | 3 |  |
| 4 | 4 |  |
| 4 | 5 |  |
| 4 | 6 |  |
| 4 | 10 |  |
| 4 | 20 |  |
| 4 | n |  |

What patterns do you notice? You might recognize these numbers from a multiplication table of 4 's where the pattern is $4 \cdot 1=4,4 \cdot 2=8,4 \cdot 3=12$, $4 \cdot 4=16$. You can think of (4)(3) as (4 units per jump) (3 jumps) $=12$ units.

You learned how to add positive and negative integers in the first chapter. Is there a way to think about multiplying a negative integer times a positive integer? You can use the frog model to multiply $-4 \cdot 3$, as shown below.


In groups, have students explain and show their thinking while filling in the table using the number line. What direction is the frog facing on each jump? How long is each jump?

Tables for students to fill out are at the end of this section and on the CD. As they fill in the tables, students may be able to use patterns from the table to make generalizations for the rules for multiplication of a negative by a positive. Start keeping track of these, you will add to this list throughout this chapter.

| Directed Length of Jump | Number of Jumps | Frog's Location |
| :---: | :---: | :---: |
| -4 | 0 | 0 |
| -4 | 1 | -4 |
| -4 | 2 | -8 |
| -4 | 3 | -12 |
| -4 | 4 | -16 |
| -4 | 5 | -20 |
| -4 | 6 | -24 |
| -4 | 10 | -40 |
| -4 | 20 | -80 |
| -4 | $n$ | $-4 n$ |

Have your students demonstrate this on the classroom number line.

It is very important for them to see that the product of $-4 \cdot 3$ is the same as the product of $-3 \cdot 4$.

Have students use the number lines that they previously created, as needed.
a. -18
C. -9
b. -15
d. -3

Copy and fill the skip counting Table 4.1b as you did in Table 6.1a, but this time use jumps of directed length -4 .

Table 4.1b

| Directed Length of Jump | Number of Jumps | Frog's Location |
| :---: | :---: | :---: |
| -4 | 0 | 0 |
| -4 | 1 | -4 |
| -4 | 2 |  |
| -4 | 3 |  |
| -4 | 4 |  |
| -4 | 5 |  |
| -4 | 6 |  |
| -4 | 10 |  |
| -4 | 20 |  |
| -4 | $n$ |  |

Using the pattern demonstrated in this table, compute the product $-3 \cdot 4$, or $-3 \cdot 4$.
The picture below models the product $-3 \cdot 4$. The first factor tells us which direction the frog should face and the length of each jump; the second factor tells us the number of jumps.


The frog is facing left because we are modeling a jump of -3 units per jump. Use the number line to compute the following products:
a. $(-3)(6)$
b. $(-3)(5)$
c. $(-3)(3)$
d. $(-3)(1)$

Our model for $n \cdot m$ is " $n$ is the length of the jump," and " $m$ is the number of jumps." This is the same as rate • time. For $2 \cdot-3$, the product will be where we were 3 jumps in the past.

|  | Directed Length of Jump | Number of Jumps | Frog's Location |
| :---: | :---: | :---: | :---: |
|  | 3 | -6 | -18 |
|  | 3 | -5 | -15 |
|  | 3 | -4 | -12 |
|  | -3 | -9 |  |
| 3 | -2 | -6 |  |
| 3 | -1 | -3 |  |
|  | 3 | 0 | 0 |
| 3 | 1 | 3 |  |
| 3 | 2 | 6 |  |
|  | 3 | 9 |  |

Once students solidify the use of a number line and have filled in the table, allow them to make conjectures for multiplying a positive by a negative. Add to your running list of conjectures.

How can we make sense of the product (3)(-4)? This is the first example where the second factor is negative.

The first number, 3 or +3 , gives the length of each jump, and the direction the frog is facing. Because the number is positive, the frog faces right.

The second factor gives the number of jumps. What do we mean by the number -4 as the number of jumps? If we think of the jumps taking place at equal time intervals, we can imagine the frog jumping along a line.

We pick one location, call it 0 and name the time as the " 0 jump." When the frog takes its first jump, jump 1, the frog lands at location 3. When the frog takes its second jump, jump 2, the frog lands at location 6.

Let's go back to the 0 , location and ask where the frog was on the jump before it arrived at 0 . We call this jump -1 . Because the frog jumps 3 units to the right every jump, the frog must have been at location -3 , which is 3 units to the left of 0 . Two jumps before reaching 0 , the frog was at location -6 . We can now copy and fill the table below.

Table 4.1c

| Directed Length of Jump | Number of Jumps | Frog's Location |
| :---: | :---: | :---: |
| 3 | -6 |  |
| 3 | -5 |  |
| 3 | -4 |  |
| 3 | -3 |  |
| 3 | -2 |  |
| 3 | -1 |  |
| 3 | 0 | 0 |
| 3 | 1 | 3 |
| 3 | 2 | 6 |
| 3 | 3 |  |

It is now possible to answer the earlier question. What do we mean by -4 jumps? This means we jump backward in time, or simply jump backward.

It's the teacher's discretion if students need to redraw a number line for each problem or if using one is sufficient.

|  | Directed Length of Jump | Number of Jumps | Frog's Location |
| :---: | :---: | :---: | :---: |
|  | -3 | -6 | 18 |
|  | -3 | -5 | 15 |
|  | -3 | -4 | 12 |
|  | -3 | 9 |  |
| -3 | -2 | 6 |  |
| -3 | -1 | 3 |  |
| -3 | 0 | 0 |  |
|  | -3 | 1 | -3 |
| -3 | 2 | -6 |  |
|  | -3 | -9 |  |

Use the number line to compute the following products. Verify that your answers agree with the table.
a. $(3)(-6)$
-18
c. $(3)(-3)$
-9
b. $(3)(-5)$
-15
d. $(3)(-1)$ -3

Let's summarize the frog model:

- The first factor tells us which direction the frog should face and the length of each jump.
- The second factor tells us the number of jumps and the direction of the jump. When the second factor is positive, the frog jumps forward; when the second factor is negative, the frog jumps backward.

Using the frog model, compute the product $(-3)(-4)$. The directed length of each jump is -3 . Determine what happens when the frog jumps backward in time.


Copy and fill the following table, starting at the bottom and working up.
Table 4.1d

| Directed Length of Jump | Number of Jumps | Frog's Location |
| :---: | :---: | :---: |
| -3 | -6 |  |
| -3 | -5 |  |
| -3 | -4 |  |
| -3 | -3 |  |
| -3 | -2 |  |
| -3 | -1 |  |
| -3 | 0 | 0 |
| -3 | 1 | -3 |
| -3 | 2 | -6 |
| -3 | 3 |  |

## EXERCISES

Provide your students with a number line.
2.

3. a. 42
b. -40
$-42$
40
-42
-40
42
40
4. The only difference is the signs. One, or any odd number, of negative factors gives a negative product. Two, or any even number, of negative factors gives a positive product.
5. The simplest rule is the product is negative if the signs of two factors are different (or if the number of negative factors is odd) and positive if the signs of two factors are the same (or if the number of negative factors is even).

Use this table to compute the following products:
a. $(-3)(-6) \quad 18$
c. $(-3)(-3)$
9
b. $(-3)(-5)$
15
d. $(-3)(-1)$
3

## EXERCISES

1. Use the frog model on the number line to compute the following products. As you multiply, visualize the process to verify the accuracy of the products.
a. $-3 \cdot 2$
-6
d. 8.-5 -40
b. $\quad-5 \cdot 3 \quad-15$
e. $11 \cdot-6 \quad-66$
c. $-12 \cdot 3 \quad-36$
f. $30 \cdot-2 \quad-60$
2. Use the number line to demonstrate $(-2)(-4)$. Do the same for $(-3)(-7)$. See TE.
3. Use the number line frog model to compute the following products: See TE.
a. (6)(7)
b. $(-5)(8)$
(6) (-7)
$(-5)(-8)$
$(-6)(7)$
(5) $(-8)$
$(-6)(-7)$
(5) (8)
4. Suppose $n$ and $m$ are positive integers. Using Exercise 3, how is the product $(n)(m)$ related to the product $(-n)(m) ?(n)(-m)$ ? $(-n)(-m)$ ? See TE.
5. We know from experience that when we multiply a positive number by another positive number we will always get a positive number. Find a rule for the product of a negative number and a positive number. Find a rule for the product of a positive number and a negative number. Find a rule for the product of two negative numbers. See TE.
6. Evaluate the following products:
a. 15(-13) -195
f. $(-118)(-3) \quad 354$
b. 22(-35) -770
g. $217(-12)-2604$
c. $(-8) 9 \quad-72$
h. $(-7) 8 \quad-56$
d. $8(-19) \quad-152$
i. $7(-8) \quad-56$
e. $(-12)(-11) 132$
7. Between 1000 and 2000 tater tots. Answers will vary in how students arrived at their conclusions.
8. -48 because $0+-6+-6+-6+-6+-6+-6+-6+-6=-48$ or $0+(-6)(8)=0+-48=-48$
9. $10+(-8)(3)=10+(-24)=-14$ dollars
10. $-4(7)=-28.50,70+-28=42^{\circ} \mathrm{F}$
11. $103^{\circ} \mathrm{F}$. Encourage your students to multiply by a negative integer to reach the solution.
12. If the temperature decreases an average of $4^{\circ} \mathrm{F}$ for each of the next four hours, the temperature at 10:00 p.M. is $-6^{\circ}$ F. Written as an addition problem: $10+-4+-4+-4+-4$. As a combination multiplication and subtraction problem: $10+(-4)(4)=10-4 \cdot 4$.
13. Have your students act out on a number line or explain using a number line; $18^{\prime}, 30^{\prime}, 18^{\prime}$.
14. Pedro has 17 bags of tater tots that have approximately 80 tots in each bag. Predict whether Pedro has fewer than 100 tots, between 100 and 500 tots, between 500 and 1000 tots, between 1000 and 2000 tots, or more than 2000 tots in all. Show how you arrived at your prediction. See TE.
15. Juan has a checking account with a balance of 0 dollars. He then makes 8 withdrawals of $\$ 6$ each. What is the new balance in Juan's checking account? Work this as an addition problem. See TE.
16. Andrew has an account with a balance of $\$ 10$. He then makes 3 withdrawals of $\$ 8$ each. What is the new balance in Andrew's checking account? See TE.
17. On a November day, a cold front blew into town. The temperature was $70^{\circ} \mathrm{F}$ before the temperature dropped an average of $4{ }^{\circ} \mathrm{F}$ an hour. What was the temperature after 7 hours? See TE.
18. One evening in San Antonio, the temperature drops for five hours, from 5:00 P.m. to 10:00 P.m. The temperature drops an average of $3^{\circ} \mathrm{F}$ per hour. The temperature at $10: 00$ P.M. is $88^{\circ} \mathrm{F}$. What was the temperature at 5:00 P.M.? See TE.
19. On a cold winter day in Roanoke, Virginia, the temperature at 6:00 P.M. is $10^{\circ} \mathrm{F}$. The temperature decreases an average of $4{ }^{\circ} \mathrm{F}$ for each of the next four hours. What is the temperature at 10:00 P.M.? Write this problem as an addition problem. Rewrite it as a combination multiplication and subtraction problem.
20. A bee flies by Tommy traveling east at 6 feet per second. Assuming the bee flies in a straight line, how far is the bee from Tommy after 3 seconds? After 5 seconds? How far west of Tommy was the bee at -3 seconds?

## 14. Ingenuity:

In Spring Branch, Texas, population 1,920, there are an average of 6 televisions for every 8 people. How many more televisions would there be if Spring Branch had 8 televisions for every 6 people? 1120

Although numbers that are multiplied need a sign or punctuation to distinguish (3)(2) from 32, in the case of multiplication with variables, the common notation has no sign or punctuation. When your students see $4 x$, they should understand that means " 4 times $x$ " or " $4 x^{\prime}$ s."
15. b.

| $x$ | $y$ |
| :---: | :---: |
| -2 | 9 |
| 0 | 6 |
| 2 | 3 |
| 4 | 0 |
| 6 | -3 |

## 15. Investigation:

Fernando the Frog is resting at the point 0 on the number line. He can make two kinds of jumps: "leaps" 6 units long and "hops" 4 units long. He can move in either a positive or negative direction.
a. Fernando wants to catch a fly that is buzzing around point 24 on the number line. His tongue inn't long enough to reach the fly unless he is at point 24 as well. What sequence of moves can Fernando make to end at point 24? Answers will vary.
b. Make a table of solutions. Do you notice any patterns? Find more solutions if possible. See TE.
c. Find one pair of integers $x$ and $y$ such that $6 x+4 y=24$. How does this help you answer the previous problem? Answers will vary.

Table 4.1a

| Directed Length of <br> Jump | Number of Jumps | Frog's Location |
| :---: | :---: | :---: |
| 4 | 0 | 0 |
| 4 | 1 | 4 |
| 4 | 2 | 8 |
| 4 | 3 |  |
| 4 | 4 |  |
| 4 | 5 |  |
| 4 | 6 |  |
| 4 | 20 |  |
| 4 | n |  |
| 4 |  |  |

Table 4.1b

| Directed Length of <br> Jump | Number of Jumps | Frog's Location |
| :---: | :---: | :---: |
| -4 | 0 | 0 |
| -4 | 1 | -4 |
| -4 | 2 |  |
| -4 | 3 |  |
| -4 | 4 |  |
| -4 | 5 |  |
| -4 | 6 |  |
| -4 | 10 |  |
| -4 | 20 |  |
| -4 | n |  |

Table 4.1c

| Directed Length of <br> Jump | Number of Jumps | Frog's Location |
| :---: | :---: | :---: |
| 3 | -6 |  |
| 3 | -5 |  |
| 3 | -4 |  |
| 3 | -3 |  |
| 3 | -2 |  |
| 3 | -1 |  |
| 3 | 0 | 0 |
| 3 | 1 | 3 |
| 3 | 2 | 6 |
| 3 | 3 |  |

Table 4.1 d

| Directed Length of <br> Jump | Number of Jumps | Frog's Location |
| :---: | :---: | :---: |
| -3 | -6 |  |
| -3 | -5 |  |
| -3 | -4 |  |
| -3 | -3 |  |
| -3 | -2 |  |
| -3 | -1 |  |
| -3 | 0 | 0 |
| -3 | 1 | -3 |
| -3 | 2 | -6 |
| -3 | 3 |  |

## LEAP FROG

Objective: This activity is an extension to the example in the book from Section 4.1 that will help the students visualize the motion of the frog as it leaps on the number line.

## Materials:

An evenly spaced number line on the board marked from -100 to 100
The frog cut-out below
Magnetic Tape

## Activity Instructions:

1) Make your number line on the board.
2) Cut out the frog below, and decorate him however you see fit.
3) Attach the magnetic tape to the back of the cut-out so the frog will stick to the board.
4) Use the number line and the frog to demonstrate the motion of several examples of integer multiplication problems. Let the students come to the board and move the frog if time allows.


## SKIP COUNTING RACE

Objective: Students will become more flexible with numbers and the number line. It will help reinforce their skip counting strategies using the number line.

## Materials:

Number line

## Activity Instructions:

The teacher will remind students that when they multiply $5 \cdot 7$, the first factor tells the direction of the jump and the length, the second factor tells the number of jumps. Since 5 is positive, the direction is positive and the length is 5 . Then, 7 says to jump that many jumps. Students record this on the number line written on paper; the answer is 35 .

Students will continue to try other combinations of numbers, using the number line and jumping frog strategies.
Variation of this activity could be for students to race each other on the board.

The teacher pre-writes the number lines on the board. Two or three pairs of students go up to the board. One student is the Coach and the other is the Player. The Coach prepares the number line numbered from any number to any number. The Coach will help only if needed and will not be allowed to write on the board. The Player will solve the problem.

Each Coach gives the Player a multiplication problem, after teacher counts off: " $1,2,3$." The pair who first answers the problem correctly wins.

## Section 4.2 - Area Model for Multiplication \& <br> The Distributive Property

## Big Idea:

Using area models to represent multiplication

## Key Objectives:

- Model one- and two-digit multiplication with areas.
- Discuss the use of place value in the Area Model.
- Use the distributive property to find partial products.
- Apply the area model to algebraic problems.


## Materials:

- Grid paper
- Rulers
- Algebra tiles could be used with area model (optional)


## Pedagogical/Orchestration:

Tie in the area model with the distributive property.

## Activities:

"Floor Plan Design" at the end of the section and on the CD
"Tile Floor Plans" at the end of the section and on the CD

## Vocabulary:

linear model; area; dimensions; area model; distributive property of multiplication; commutative property of multiplication; terms; combine like terms.

## TEKS:

$6.11(\mathrm{C}, \mathrm{D}) ; \quad 7.2(\mathrm{~A}, \mathrm{~B}, \mathrm{C}, \mathrm{G}) ; \quad 7.9(\mathrm{~A}) ; 7.13$ (A); 7.14(A,B); 7.15(B) 7.8(C); 8.2(B,C); 8.7(B); 8.10(A);
$8.14(\mathrm{~A}, \mathrm{~B}) ; \quad 8.15(\mathrm{~A}) ; \quad 8.16(\mathrm{~A})$

## WARM-UPS for Section 4.2

1. Sketch a rectangle with area greater than its perimeter.

Ans: 10x10 rectangle has area 100 and perimeter 40.
2. Sketch a rectangle with a perimeter greater than its area.

Ans: 50x2 rectangle has area 100 and perimeter 104.
3. Sketch a rectangle with a perimeter that is more than twice its area.

Ans: 100x1 rectangle has area 100 and perimeter 202.

## Launch for Section 4.2:

To introduce this section, ask your students to think about situations where the ability to multiply is necessary. Make sure area and volume are mentioned as the class is brainstorming. When the conversation reaches the subject of area, refer to the book and talk about the rectangular model of $4 \cdot 6$ at the beginning of this section. Proceed with the lesson informing your students, "Yesterday, we used a linear model, or number line, to model multiplication. Today, we will be using an area model to help us multiply."

Students may have encountered lengths and widths of rectangles with one being longer than the other or one being horizontal or vertical. In this text, we do not specify such conditions and allow one dimension to be length and the other to be width.

## SECTION 4.2 AREA MODEL FOR MULTIPLICATION \& THE DISTRIBUTIVE PROPERTY

In the previous section, we explored multiplication using the frog model of skip counting on the number line or repeated addition. This is also called the linear model for multiplication. In addition to the linear model, we can represent multiplication as area.

To multiply 4 by 6 , consider the picture of the rectangle below. The area of a rectangle is the number of units, or $1 \times 1$ squares, that it takes to cover the figure with no overlaps and no gaps. What is the area of the rectangle below assuming that each square in the grid has area 1 square unit?


Dimensions are measured in one direction: the length, width, or height of a figure. The dimensions of this rectangle, which is 4 units wide and 6 units long, are 4 units in width and 6 units in length. The area can be computed by summing the areas of the columns: $4+4+4+4+4+4=4 \cdot 6=24$. We can also think of this area as the sum of the area of the rows: $6+6+6+6=6 \cdot 4=24$. The rectangle is called a 4 by 6 , or a 6 by 4 , rectangle because the area is computed as the product $4 \cdot 6=6 \cdot 4=24$. Remember, in the frog model, it is the same as when a frog jumps 6 times on a number line, with each jump 4 units long. The visual representation of area above is the area model that describes multiplication.

## PROPERTY 4.1: COMMUTATIVE PROPERTY FOR MULTIPLICATION

$$
A \cdot B=B \cdot A
$$

PROPERTY 4.2: ASSOCIATIVE PROPERTY FOR MULTIPLICATION
For any numbers $x, y$ and $z$
$(x y) z=x(y z)$

## EXAMPLE 1

The Elliots are constructing a small building that is one room wide and two rooms long. Each room is 5 meters wide. The front room is 4 meters long, and the back room is 6 meters long. What is the floor space of each room? What is the floor space of the building? How are the areas of the two rooms related to the area of the building? The floor plan below shows the situation:

| 4 m | 6 m |
| :---: | :---: |
| 5 m |  |
|  |  |

## SOLUTION

The area of the room on the left is calculated by $(5 \mathrm{~m})(4 \mathrm{~m})=20$ square meters, or 20 sq . m . The area of the room on the right is $(5 \mathrm{~m})(6 \mathrm{~m})=30$ square meters. The total area is the sum of the areas of the two rooms:

$$
20 \text { square meters + } 30 \text { square meters }=50 \text { square meters. }
$$

Another way to compute the total area is to consider the larger rectangle and its width and length:
$(5$ meters $)(4$ meters +6 meters $)=(5$ meters $)(10$ meters $)=50$ square meters.
Notice that this is the same area.

## PROBLEM 1:

a. $8(4)+8(7)$
$32+56$
88 sq. units
b. $8(4+7)=8(11)=88$ sq. units.
(Do not give them the distributive property yet.)

## PROBLEM 1

a. Compute the area of the larger rectangle by computing the areas of the inner rectangles.

b. Compute the area of the outer rectangle. Compare to answer in part a.

## EXAMPLE 2

Now suppose the dimensions of the Elliots' building have not been decided yet. We need a formula for the areas. Call the width of the building $n$ feet and the lengths of rooms 1 and $2, k$ and $m$ feet respectively. Find the area of each room and the building's total area.

| $k \mathrm{ft}$ |  |  | $m \mathrm{ft}$ |
| :--- | :--- | :---: | :---: |
| ft |  |  |  |
|  |  |  |  |

Problem 2:
$a b+a c$

## SOLUTION

The area of room 1 is $(n \mathrm{ft})(k \mathrm{ft})=n \cdot k$ square ft .
The area of room 2 is $(n \mathrm{ft})(m \mathrm{ft})=n \cdot m$ square ft .
The area of the building is $n(k+m)$ square ft .
Remember, the area of the building can also be computed as the sum of the areas of the two rooms, $(n \cdot k+n \cdot m)$ square ft .

So, $n(k+m)=n \cdot k+n \cdot m$. We call this relationship the distributive property. This property tells us how addition and multiplication interact.

## PROBLEM 2

Use the distributive property to write the following product as a sum: $a(b+c)$.

## PROPERTY 4.1: DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION

For any numbers $k, m$, and $n$,

$$
n(k+m)=n \cdot k+n \cdot m .
$$

You have already learned to multiply two-digit and three-digit numbers. Now you can use the area model and the distributive property to explore this process carefully. Begin by modeling the product of a one-digit number and a two-digit number. To multiply $6 \cdot 37$, use place value to write the product $6 \cdot 37$ as $6(30+7)$. By the distributive property, $6 \cdot 37=6(30+7)=6 \cdot 30+6 \cdot 7=180+42=222$.


You can extend the same process to multiply 43 by 27 using the distributive property,

$$
\begin{aligned}
43 \cdot 27 & =(40+3)(20+7) \\
& =40(20+7)+3(20+7) \\
& =40 \cdot 20+40 \cdot 7+3 \cdot 20+3 \cdot 7 \\
& =800+280+60+21=1161
\end{aligned}
$$

Visualize the product as area with the picture below:


The total area is $800+60+280+21=1161$.

## EXAMPLE 3

Pat has a rectangle with a length that is 3 units more than $x$, a positive length. The width of the rectangle is 4 units. What is the area of the rectangle?

## SOLUTION

Begin by drawing a picture of the rectangle. How do you draw a side of length of $x$ ? Any of the following pictures could represent the rectangle.

or


## EXAMPLE 3

Note that x represents a length and $4 \cdot x=4 x$ represents an area.


Each of these visual representations helps us find the area. The area is the product of the length and width $4(x+3)$. Using the distributive property, the total area is equal to the sum of the areas of the two smaller rectangles: $4 \cdot x+4 \cdot 3=4 x+12$. Either way, the area is $4 x+12$. Note that this is an expression for the area of the outer rectangle.

## EXAMPLE 4

Luz sees an advertisement for \$3 off the regular price of DVDs in her favorite store. She buys 4 DVDs, all of which have the same regular price of more than \$3. Represent this situation with an algebraic expression and with an area model representation.

## SOLUTION

First of all, notice that you do not know the regular price of a DVD. Represent this unknown with the variable $x$, where $x=$ the regular price of a DVD. Then, $x-3$ represents the sale price, and $4(x-3)$ is the total cost of Luz's four DVDs.

The previous example shows us a way to visualize $x+3$ or 3 more than $x$. To represent 3 less than $x$ visually using the area model, we can rewrite the problem as adding the opposite rather than subtracting: $x-3=x+-3$. The rectangle then looks like this:

## EXPLORATION 1

In this context, $2 x$ and $3 x$ represent distances.
Do they see that $3 x+2 x=5 x$ ?


The outer rectangle has area 4 x , the original cost of 4 CDs. The $4 \times 3$ rectangle represents the discount. So the sale price is $4 x-12$, which is the same as $4(x-3)$. Therefore, $4(x-3)=4 x-12$.

In computing algebraic expressions, we often end up with several terms that look alike, such as $3 x+2 x$. In the following Exploration, we will model this sum in several ways.

## EXPLORATION 1

Use the number line below to locate and label $2 x, 3 x$, and $3 x+2 x$.


## EXPLORATION 2

Compute the area of each of the rectangles below. Explain how to use the distributive property. Show the connection between the area of rectangle A and $B$ to that of rectangle C.


## EXERCISES

1. a. $3(2)+3(7)=3(2+7)=3(9)=27$
b. $3 x+4 x=(3+4) x=7 x$
c. $2 a+5 a=(2+5) a=7 a$
d. $7 x-3 x=(7-3) x=4 x$
2. 


3.

| 2 units | $x$ units |
| :--- | :---: |

6. a. $2 x+4$
b. $3 x+12$
c. $2 x+10$
d. $2 x-10$
e. $6 x+8$
f. $6 x-8$
7. a. 45 cards
b. $\quad 120$ cards
c. $3(x+5)$ cards or $3 x+15$ cards

Be sure to explain the difference in $3 x+5$ and $3(x+5)$ to the students when they are working on part (c). Let them plug in values to discover which of these expressions is correct. This discussion will help the students with problems 7 through 12 as well.

## EXERCISES

1. Use the distributive property to simplify the following expressions. Compare the linear and area models for each.
a. $2(3)+7(3)$
27
b. $3 x+4 x$
$7 x$
c. $2 a+5 a$
$7 a$
d. $7 x-3 x$
$4 x$

This is also called combining like terms.
2. How do you apply the distributive property to the multiplication problem $3(x-2)$ ? If necessary, draw a picture in which the product is the area of a rectangle. It might be useful to rewrite the subtraction factor $(x-2)$ as the sum ( $x+-2$ ). See TE.
3. Draw a picture model that uses the product $2(x-4)$ as the area of the shaded interior of a rectangle. See TE.
4. Compute the area of shaded rectangle below. The dimensions of the large rectangle are 3 units wide and $x$ units long. $3(x-4)$ or $3 x-12$

5. Draw a rectangle model for the following problems and then compute the following products using the distributive property:
a. $2(y+5)$
$2 y+10$
b. $3(x+2) \quad 3 x+6$
6. Rewrite the following products using the distributive property. Draw a rectangle model for the first 2 problems, if necessary. See TE.
a. $2(x+2)$
b. $3(x+4)$
c. $2(x+5)$
d. $2(x-5)$
e. $2(3 x+4)$
f. $2(3 x-4)$
7. Mike has a certain number of baseball cards, and Jill has 5 more cards than Mike. Ramon has three times as many cards as Jill. How many cards does Ramon have if Mike has: See TE.
a. 10 cards?
b. 35 cards?
c. $x$ cards?
11. $2(J-3)$ or $2 J-6$ years old
12. b. The area of the large rectangular pen is equal to the sum of the areas of the smaller rectangular compartments. $2 x+4 x+3 x+5 x=14 x$
8. Eddie and Sam like to play marbles. Eddie has $x$ marbles. He buys 3 more marbles. Now Sam has twice as many marbles as Eddie. Express the number of marbles Sam has, in terms of $x . \quad 2(x+3)$ or $2 x+6$ marbles
9. Eddie's friend, llya, has $x$ marbles. Sam's sister, Natasha, has twice as many marbles as Ilya. Natasha buys three more marbles. How many marbles does Natasha have now? $2 x+3$
10. Sophia has two nephews, Juan and Ted. Ted is two years older than Juan. Sophia is three times older than Ted. If Juan is $J$ years old, what is Sophia's age? $3(J+2)$ or $3 J+6$ years old
11. Gloria has two nieces, Sara and Jane. Sara is three years younger than Jane. Gloria is twice as old as Sara. If Jane is $J$ years old, how old is Gloria?
12. Ronnie has a large rectangular pen that is fenced to create 4 smaller rectangular compartments. The plans below show the dimensions of the pen.

a. What is the area of the large pen? $14 x$
b. How is this area related to the areas of the smaller compartments? See TE.
13. Compute the area of the rectangle below. $a c+3 a+4 c+12=(a+4)(c+3)$

14. Find the areas of the smaller rectangles and add them together to compute the area of the larger rectangle. $a c+a d+b c+b d=(a+b)(c+d)$

16. a. $x^{2}+5 x+6$
b. $x^{2}+6 x+5$
c. $x^{2}+x-12$
14. Compute the area of the large rectangle below as well as the areas of the smaller rectangles. Explain your reasoning. Note that the horizontal length of the rectangle is $(c+d)$ units and the vertical width is $(a+b)$ units. Then compute the total area using the formula: Area $=($ width $)$ (length).
What is another way to compute the total area? See TE.

15. Draw a rectangle with area $\mathrm{ac}+3 \mathrm{a}+2 \mathrm{c}+6$.
16. Develop a formula for the area of each of the rectangles below. Write your answer in simplest terms. See TE.
a.

b. $x$

c.

17. The last two digits of the number must be divisible by 4 .

## Investigation

18. Although we will not formally discuss division until Section 4.4 , it is never too early to start teaching the close relationship between multiplication and division. Remind your students that division is the inverse operation of multiplication. Therefore, 183 divided by 14 could be restated as 14 times what equals 183 , or $14 q=183$, for $q$ a positive integer. This should stimulate them to begin their area model with one side length.

They will now begin building the other side of the rectangle until they reach the desired length that produces a product of 183. Of course, they won't be able to, which may lead to the discussion of what the quotient and remainder represent in this model.

The quotient is one of the dimensions, or length of one of the sides, and the remainder is outside the rectangle.
17. Ingenuity:

Numbers like 536 and 712 are divisible by 4, while 378 is not. Make a conjecture for a rule to determine whether a number is divisible by 4.
18. Investigation:

Use the area model to explain how to divide 183 by 14 . What do the quotient and the remainder represent? See TE.

## Floor Plan Design

Objective: The students will apply the area model of multiplication to design a floor plan for a room with specific area requirements.

## Materials:

Notebook paper or plain white paper
Rulers

## Activity Instructions:

1) The teacher will explain to the class that they are to design a floor plan (similar to the examples in the math book in section 5.2 ). The floor plan will have 4 rooms with a total area of 620 square feet. The four rooms must all be of different size.
2) The students will turn in a final product that shows their floor plan design with all dimensions labeled. The students will also provide a mathematical explanation of how they can prove that their floor plan design meets the 620 sq. feet requirement.

## Tile Floor Plans

Objective: Students will reinforce their flexibility with area vs. perimeter by figuring out floor plans with given area and perimeter, or given area and any perimeter, or any area and given perimeter, or any area and maximum perimeter.

## Materials:

Plastic or foam tiles
Grid paper

## Activity Instructions:

1) Teacher will hand out tiles to students to create different floor models so that area is equal to 28 square units and the perimeter is less than 60 . Once students try building each model possible, they can record their results on a grid sheet of paper.
2) Teacher will hand out tiles to students to create different floor models so that area has to be 30 tiles and the perimeter is the least possible. Once students try building each model possible, they can record their results on a grid sheet of paper.
3) Teacher will hand out tiles to students to create different floor models so that area has to be a set number of tiles and the perimeter is the maximum. Once students try building each model possible, they can record their results on a grid sheet of paper.

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## Section 4.3 - Applications of Multiplication

## Big Idea:

Finding area of rectangles and relationships between area and perimeter

## Key Objectives:

- Discover patterns when scaling up the rectangles or changing the widths and lengths of the rectangles.
- Given a set area, discover how changing dimensions affects perimeter of a rectangle.
- Given a set perimeter, discover how changing dimensions affects the area of a rectangle.


## Materials:

- Grid paper, straight edge, colored tape for Launch


## Pedagogical/Orchestration:

- This section sets the stage for a discussion of linear change (perimeter) versus quadratic, or square change, (area) as they relate to a change in the dimensions (also linear). This is also another opportunity for students to practice finding patterns and hypothesizing about them.
- Make sure questions like the following are brought up sometime during the lesson: Given a set perimeter, say 24 meters, what shaped rectangle would give you the largest area? In general, what shape will give you the largest area given a set perimeter? (Answers: For whole number dimensions only, the 6 m by 4 m rectangle will have the largest area for a rectangle with perimeter 24 meters. In general, the most squarelike rectangle will have the largest area for a rectangle with a given perimeter.)


## Internet Resource:

Rags to Riches: Perimeter and Area- http://www.quia.com/rr/91670.html

## Activities:

"Chicken Run" Activity, to be done when doing Exercise 4, at the end of the section and on the CD

## Exercises:

Exercise 7 would make a good assessment.

## Vocabulary:

perimeter, area, dimensions

## TEKS:

6.2(E); 6.8(B); $6.11(\mathrm{C}, \mathrm{D}) ; 7.2(\mathrm{C}, \mathrm{E}) ; \quad 7.4(\mathrm{~A}) ; 7.8(\mathrm{C}) ; 7.9(\mathrm{~A}) ; 7.13(\mathrm{~A}, \mathrm{C}) ; 7.14(\mathrm{~A}) ; 7.15(\mathrm{~A}, \mathrm{~B})$;
8.7(B); 8.4(A); 8.10(A); 8.14(A); 8.15(A); 8.16(A)

## WARM-UPS for Section 4.3

1. Compute each of the following products:
a. $(5+1)(5-1)$
b. $(6+1)(6-1)$
c. $(7+1)(7-1)$
d. What pattern do you notice? Speculate on a rule for the product: $(n+1)(n-1)$
2. Jack and Jane eat at a cafe and the bill is $\$ 19.00$. If the tax is $10 \%$, their final bill is within which of the following ranges:
a. between $\$ 19.00$ and $\$ 20.00$
b. between $\$ 20.00$ and $\$ 20.50$
c. between $\$ 20.50$ and $\$ 21.00$
d. between $\$ 21.00$ and $\$ 21.50$

Ans: c because the tax is $\$ 1.90$ so bill is $\$ 20.90$

## Launch for Section 4.3:

Today's Launch begins with the Exploration at the start of Section 4.3. Assign each student group one rectangle from rectangles $A$ through $D$ in the Exploration. Give each group some tape and have them outline the rectangles on the floor. If the floor has square tiles, they may use that as a guide, or they may have to use a ruler. Ask the students to visit each others' rectangles and determine whose rectangle is the biggest. If the students are only bringing up area in the discussion, ask students which group they think used the most tape to make their rectangle. Tell the students, "As you work through today's lesson, think about the different ways the size of the rectangle can be described." Finish the Exploration by bringing in the vocabulary of area and perimeter.

Answers may vary, depending on criteria. Encourage discussion and debate.

## SECTION 4.3 APPLICATIONS OF MULTIPLICATION

In the previous sections, you explored models for multiplication including the area model. You will now explore properties of rectangles.

## EXPLORATION

Of rectangles $A, B, C$, and $D$, which is the biggest? Explain your answer. See TE.


## DISCUSSION

There are several ways to think about what "biggest" means. One way to measure "biggest" is to find the area by counting the number of unit squares that are needed to cover each figure.

1. What are the areas of rectangles $A, B, C$, and $D$ ? $15,16,14$, and 10 sq. units
2. Which one has the largest area? $B$
3. Does this agree with the rectangle you chose? Answers will vary.

Another way to measure the size of a rectangle is to add the lengths of all the sides. This sum is called the perimeter. Its name comes from the Greek words peri, meaning "around," and metron, meaning "measure."
4. What are the perimeters of the 4 rectangles? $A: 16 ; B: 16 ; C: 18 ; D: 22$
5. Which one has the largest perimeter? D
6. Does this agree with the rectangle you chose? Answers will vary.

## EXERCISES

| Rectangle | Length | Width | Area | Perimeter |
| :---: | :---: | :---: | :---: | :---: |
| $E$ | 3 | 6 | 18 | 18 |
| $F$ | 3 | 4 | 12 | 14 |
| $G$ | 5 | 6 | 30 | 22 |
| $H$ | 7 | 8 | 56 | 30 |
| l | 4 | 11 | 44 | 30 |
| J | 2 | 19 | 38 | 42 |

Answers will vary. Possible relationships that hold true are that area $=$ length times width or that perimeter equals twice the length plus twice the width. Or, it appears at times, that as the length and width go up, the area and perimeter go up.

## EXERCISES

1. Calculate the area and perimeter of each rectangle on the following grid. Make a table like the one that follows showing the length, width, area, and perimeter of each rectangle. Assume the length is the horizontal distance.


| Rectangle | Length | Width | Area | Perimeter |
| :---: | :--- | :--- | :--- | :--- |
| $E$ |  |  |  |  |
| $F$ |  |  |  |  |
| $G$ |  |  |  |  |
| $H$ |  |  |  |  |
| $I$ |  |  |  |  |
| $J$ |  |  |  |  |

What patterns do you notice in the values in the table? Do you see any relationships between the four categories that you can state as a rule?
2. Students will most likely choose only integer sides. The chart to the right is one possibly correct chart. Emphasize the importance of organization to find all the cases and avoid repetition. Discuss the fact that, aside from keeping the organization, order doesn't matter.
If we don't assume that the sides are integers, we could have sides of 48 and $\frac{1}{2}$ or 4.8 by 5 . That opens many more possibilities.
Assume that there are no repeats, that is, a $6 \times 4$ rectangle is the same as a $4 \times 6$.
2. There are 4 different rectangles. Their sides are the factors of 24 . Avoid repeats.
3. Generally, if the two dimensions are closer, the area increases, and the perimeter decreases.
4. Explore the idea of what we mean by "best" just as the students explored the idea of "biggest" earlier in the Exploration. From the farmer's perspective, biggest may mean in area or in minimal cost for the fencing. For an athletic chicken practicing for a race, best may mean that it provides a long run.
The best dimensions could be 4 by 6 meters because it provides the smallest perimeter for a 24 square meter area with integer dimensions. The smallest perimeter is actually a square with side length square root of 24 . This problem would be a good problem to revisit after working with square roots in section 7.3.
5. It will be easier if students use grid paper to do this. The rectangles with integer sides are $1 \times 11,2 \times 10,3 \times 9$, $4 \times 8,5 \times 7$, and $6 \times 6$. Organizing the data might avoid missed possibilities and is a good mathematical habit. The students may also report some non-integer dimensions.
6. This exercise is a preview to the next section.

For a square with side length 2 , the area is 4 and the perimeter is 8 . If the side length is 3 , the area is 9 and the perimeter is 12 . If the side length is 4 , the area is 16 and the perimeter is 16 . If the side length is 5 , the area is 25 and the perimeter is 20 . If a square has side length $s$, the formula for the area of the square is $A$ $=s^{2}$, and the formula for the perimeter of the square is $P=4 s$. " $s^{2}$ " is called " $s$ squared" from the operation associated with the area of the square.

| Side Length | Area | Perimeter |
| :--- | :--- | :--- |
| 2 | 4 | 8 |
| 3 | 9 | 12 |
| 4 | 16 | 16 |
| 5 | 25 | 20 |
| $s$ | $s^{2}$ | $4 s$ |

2. Using a sheet of grid paper, make as many different rectangles of area 24 square units as possible. Make a chart like the one below to record information about these rectangles. Did you find more than four different rectangles?

Rectangles of area 24 square units

| Length | Width | Dimensions | Perimeter |
| :---: | :---: | :---: | :---: |
| 24 | 1 | $24 \times 1$ | 50 |
| 12 | 2 | $12 \times 2$ | 28 |
| 8 | 3 | $8 \times 3$ | 22 |
| 6 | 4 | $6 \times 4$ | 20 |
|  |  |  |  |
|  |  |  |  |

Looking at the information in the table, what do you notice? See TE.
3. Using another sheet of grid paper, make and label as many different rectangles of area 36 square units as possible. Make another chart to organize the information about these rectangles. How is the perimeter of a rectangle related to the area of the rectangle? See TE.
4. Jim wants to build a rectangular pen for his pet chicken. A friend says that chickens need 24 square meters to move around comfortably. What dimensions would be best for this pen? Why? See TE.
5. Draw as many different rectangles as you can that have perimeter 24 units. See TE.
6. What is the area and perimeter of a square with side length 2 ? What if it has side length 3, 4, or 5? Make a chart of these side lengths, areas, and perimeters, and look for patterns. If a square has side length $s$, what are formulas for the area and the perimeter of the square? Why is " $s^{2 "}$ called "s squared?" See TE.
7. a. The area should quadruple, and the perimeter doubles.
b. Answers will vary.
8. a. When the dimensions of a rectangle double, the area becomes 4 times as large as the original rectangle. The perimeter doubles.
b. When the dimensions of a rectangle triple, the area becomes 9 times as large, and the perimeter triples.
c. The area increases by the square of the multiple, and the perimeter increases by the multiple.
9. $2 L+2 W=42$

Using $L=2 W \longrightarrow 2(2 W)+2 W=42 \longrightarrow 6 W=42 \longrightarrow W=7$ and $L=2(7)=14$
10. $2 L+2 W=48$

Using $L=3 W \longrightarrow 2(3 W)+2 W=48 \longrightarrow 6 W+2 W=48 \longrightarrow 8 W=48 \longrightarrow W=6$ and $L=3(6)=18$
11. As in many of the concepts in this textbook, the pattern is developed inductively, with several examples leading to a pattern.
a. One square foot of carpet costs $\$ 2$; two square feet costs $\$ 4$; three square feet costs $\$ 6$; and $N$ square feet of carpet will cost $\$ 2 \mathrm{~N}$.
7. Draw two different rectangles, each with area 6 square units. Use the corner of a sheet of grid paper or a large grid. Document your work for this exercise on the grid. See TE.
a. Double the dimensions of the rectangles. Check to see if the rule or pattern you observed in exercise 6 still works. What happens to the area and the perimeter?
b. Triple the dimensions of the rectangles with area 6 square units. Does your rule predict the area and perimeter of these new rectangles correctly?
8. Explore the effect of changing length and width. See TE.
a. When the dimensions, the length and the width, of a rectangle double, what happens to the area? What happens to the perimeter?
b. What if the dimensions of a rectangle triple?
c. Write rules expressing any patterns you notice.
9. The perimeter of a rectangle is 42 cm , and the length is twice the width. What are the dimensions of the rectangle? The width is 7 cm and length 14 cm .
10. If the perimeter of a rectangle is 48 cm , and the length is three times the width, what are the dimensions of the rectangle? The width is 6 cm and length 18 cm .
11. Kenny wants to buy enough carpet to cover the floor in a rectangular room that is 12 feet by 15 feet. The carpet costs $\$ 2$ per square foot.
a. How much does one square foot of carpet cost? Two square feet? Three square feet? How much does it cost to buy $N$ square feet of carpet? See TE.
b. What is the total area of carpet that Kenny will need? 180 sq. ft.
c. How much will it cost for Kenny to carpet his room? \$360
d. Kenny has another rectangular room that is 15 feet long and has area 315 square feet. What is the width of this room? 21 ft

## Ingenuity

12. a. There are 8 passwords of length 3 that are comprised of only 1 's and 0 's. There are 16 passwords of length 4 comprised only of 1 's and 0 's.
c. The number of passwords is a power of 2 where the length of the password is the power. There are 64 6 -digit passwords. There are $2^{n} n$-digit passwords. To see why the number of passwords follows this pattern set up a tree diagram in which each branch has only two choices, 0 and 1 .

## Investigation

13. The areas of the 2 longer sidewalks are $8(3)=24 \mathrm{sq} \mathrm{ft}$ each. The areas of the 2 shorter sidewalks are $6(3)=18 \mathrm{sq} \mathrm{ft}$ each. The area of each of the 4 corner squares is $(3) 3=9 \mathrm{sq} \mathrm{ft}$. The area of the whole sidewalk is $2(24)+2(18)+4(9)=120 \mathrm{sq} \mathrm{ft}$, while the area of the rectangular garden is only $6(8)=48 \mathrm{sq} \mathrm{ft}$. The sidewalk has a greater area by 72 sq ft .

## 12. Ingenuity:

Joseph is a computer user with terrible security habits: every password he creates is a string of ones and zeros. There are only two passwords of length 1 that fit this description: 0 and 1. There are four passwords of length 2 that Joseph could use: 00, 01, 10, and 11. See TE and table for (a) - (c)
a. How many such passwords are there of length 3 ? length 4 ?
b. Complete the following table:

| Length of password | Number of passwords |
| :---: | :---: |
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 4 | 16 |
| 5 | 32 |

c. Did you notice a pattern in part b? How many such 6-digit passwords do you think there are? How many such $n$-digit passwords do you think there are? Why does the number of passwords follow this pattern?

## 13. Investigation:

Raylene wants to pour a sidewalk around her rectangular garden. The garden is 6 feet by 8 feet, and the sidewalk will be 3 feet wide. What will the area of this sidewalk be? Which has greater area, the garden or the sidewalk? How much greater? See TE.

## CHICKEN RUN



Objective: This activity is an extension on the chicken problem in Section 6.3. The students are asked to solve a problem involving the best pen for their pet chicken. This activity will provide the student with a worksheet in which to show all steps of the problem.

## Materials:

Chicken Run worksheet, one copy per student

## Activity Instructions:

The students will use the Chicken Run worksheet to solve problem \#4 in Section 6.3.
The teacher will explain that the purpose of the worksheet is to provide a work space for the problem that will allow the students to show the math processes used to come up with the solution.

Name: $\qquad$ Date: $\qquad$

## Chicken Run Worksheet

Problem: Suppose you want to build a rectangular pen for your pet chicken. A friend says that chickens need 24 square meters to move around comfortably. What dimensions would be the best for this pen? Why?

1. In the space below, please provide a diagram of your chicken pen. Your pen should be drawn to scale, and the dimensions should be provided. Feel free to be creative and personalize the space for your chicken.
2. In the space below, please provide an explanation for your decision. You will need to include mathematical proof that your dimensions fit the needs of your chicken. You should also address how you think the dimensions you chose would be cost effective considering the possible dimensions of other size pens.
$\qquad$
$\qquad$
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## Section 4.4 - The Linear Model for Division

## Big Idea:

Dividing using the linear model

## Key Objective:

- Understand and use the vocabulary of division.
- Understand what the missing factor looks like in the linear model.
- Find numeric distances between integers.


## Materials:

- Number line handout, graph paper, straight edge


## Pedagogical/Orchestration:

This section is a review of basic long division with the linear model. This method is for students who never "got" division, and will encourage them to look for patterns and develop an understanding of how division works.

## Internet Resource:

Rags to Riches: Linear Equations in one Variable- http://www.quia.com/rr/42586.html

## Activity:

"Division Decision" at the end of the section and on the CD

## Exercises:

15-factorials foreshadowed.
16-reflects back on solving equations and applying the skip-counting model.

## Vocabulary:

divisor, quotient, dividend, factor, factorial

## TEKS:

6.2(C)(D); 6.11(C); 7.2(C)(F)(G); 7.9(A); 7.13(A); 7.15(A)(B); 8.2(A,C); 8.14(A); 8.16(A)

## WARM-UPS for Section 4.4

1. A school is planning a field trip for 460 seventh and eighth grade students. If a school bus can seat 30 students, how many buses will the school need for the trip?
a. 15 buses
b. 16 buses
c. 17 buses
d. 18 buses

Ans: b because $460=30(15)+10$ so a sixteenth bus is needed for the 10 students left over.

## Launch for Section 4.4:

Tell your students, "Today we will be exploring different models for division. As we go through the examples, be thinking about patterns that you see, and how this model relates to long division."

Ask students to remember the chart in 3.1 about all the ways to describe $N+3$ and $N-3$. Build a chart and brainstorm with the class the phrases that describe the expression.

| Division | Multiplication <br> $12 \div 6=2$ |
| :---: | :--- |
|  | -product |
|  | -times |

In the past, you have learned about division. Thinking about what you already know about division, how do you write twelve divided by 3 ? What does twelve divided by 3 mean?

If you and two of your friends go out to eat at a restaurant, and the bill comes out to be $\$ 12.00$, and you share the bill, how much money will each of you pay?

Define and label the parts of the different representations of division (eg. dividend, divisor, quotient, and remainder). Now, lead your students through the "Division Decision" activity found at the end of the section.

## ACTIVITY

This is a class activity. Ask 1 or 2 students to make and implement a plan to divide the class into 3 equal groups for a game. The best number of students is a multiple of 3 . Notice that we are modeling a division problem; for instance, $21 \div 3$. If the number of students in the class is not a multiple of 3 , the class will have to decide what to do. For example, if the number of students is not a multiple of 3 , there will be a remainder of either 1 or 2 . If there is a remainder of 1 , the class might appoint the extra person to be a game director. If there is a remainder of 2 , the class might also have a scorekeeper.

Reflect on the models your students used. Ask if there are any other methods to divide, excepting long division with pencil and paper. Make sure they discover the first two methods below.

1. Partitive Model: Create piles or groups of objects by placing one object in each pile in each round, and then count the number of rounds it takes to use all the objects, as when a dealer deals the cards in a card game.
2. Repeated Subtraction: Remove 3 objects at a time, and count the number of removals it takes to remove the original objects.
3. Linear Model (See discussion below): Students skip count by 3 until the total is reached. This is an additive model. Be sure your students do not confuse the linear model with the skip counting method of multiplication. Ask your students to organize the piles of 3 objects into columns; create as many columns as possible, and line them up to form a rectangular shape. In doing so, they have discovered how to reverse the area model of multiplication to divide.
The divisor can be either the length of each jump or the number of jumps, depending on the problem. Remind your students factor is another word for divisor when the remainder is 0 , and, in that case, we can use them interchangeably.
*This method is often referred to as the Measurement Model or Repeated Subtraction Model for Division.

## SECTION 4.4 THE LINEAR MODEL FOR DIVISION

Just as with multiplication, we will explore the operation of division. We will start by looking at some models to better understand how division works.

## ACTIVITY: MODELS FOR DIVISION

Your teacher will now lead the class in an activity that reviews models of division.

You divided a class of 21 by 3 in the activity. One method involved subtracting 3 *See TE. objects at each step from the original group and counting the number of times it took to distribute all 21 objects. Another way to think about this problem is to add groups of 3 until you have 21. Skip count by 3's to accumulate objects until you have the desired number, 21. The number of skips that it takes to get to 21 is the result 21 divided by 3 .


To skip count by 3 's, count $3,6,9,12,15,18,21$, and so on. You know that $21=3 \cdot 7$ because we must skip count 7 steps by 3 's to get to 21 . The inverse is $21 \div 3=7$, which means when you divide 21 by 3 , the result is 7 because 21 is decomposed into 7 skips of 3 units per skip. This is equivalent to 7 groups of 3 . We call 3 the divisor, the quantity by which another quantity, the dividend, is to be divided. We call 7 the quotient, the end result of a division problem, and 21 the dividend, a quantity to be divided by the divisor.

When the divisor divides evenly into the dividend, or the remainder is zero, the word factor is used interchangeably with divisor. Dividing 21 by 3 is the same as looking for the missing factor $x$ that satisfies $3 \cdot x=21$. The $x$ that satisfies this equation is called the quotient and represents the number of skips of length 3 it takes to reach 21 . We call this the missing factor model. It is the reverse of the multiplication process.

We should note here that the word "factor" can be a noun that means divisor, as above where 3 is a factor of 21 . It can also be a verb. When we say, "Factor 21," we mean write 21 as a product of two or more positive integers. In this case, write $21=3 \cdot 7$ to factor 21 into a product of two numbers, 3 and 7 .

How would you model the division problem $8 \div-2$ ? We interpret this as: How many jumps of length -2 does it take to reach the location of 8 ? With 1 jump, we would land at -2 and with 2 jumps at -4 . This is the wrong direction. But with a -1 jump, we land on 2 and with a - 2 jump, we land on 4 . So, with a - 4 jump, we land on 8 . Thus, the quotient of $8 \div-2=-4$.

Using the missing factor model, this division problem is the same as looking for the missing factor $x$ that satisfies the equation: $-2 x=8$. We see that if $x$ is -4 , then $-2(-4)=8$. This pattern can be summarized by the following rule:

## RULE 4.1: POSITIVE DIVIDEND AND NEGATIVE DIVISOR

A positive number divided by a negative number yields a quotient that is negative.

Similarly, if we divide a negative number by a positive number, the quotient (the number of jumps required) must also be a negative number of jumps. For example, $-8 \div 2=-4$ because the missing factor model demonstrates that $2(-4)=-8$. This gives us another rule that states:

## RULE 4.2: NEGATIVE DIVIDEND AND POSITIVE DIVISOR

A negative number divided by a positive number yields a quotient that is negative.

Also, notice $-2(4)=-8$, and so $x=4$ is the quotient of $-8 \div-2$. Therefore:

## RULE 4.3: NEGATIVE DIVIDEND AND NEGATIVE DIVISOR

A negative number divided by a negative number yields a quotient that is positive.

## PROBLEM 1

For part (a), set up the number line with tick marks 7 units long, or set up a number line to 55 , with 5 units marked.
a.

b.

c.

d.

e.

f.


Remind your students that the portion "leftover," which is always smaller than the divisor, is called the remainder.
PROBLEM 2
a. $x=3$
b. $x=5$
c. $x=-5$
d. $x=-4$

## PROBLEM 1

Use a number line with the appropriate scale and the skip counting model to compute the following quotients: See TE.
a. $56 \div 7$
b. $\quad 91 \div 13$
c. $210 \div 15$
d. $-15 \div 3$
e. $12 \div-4$
f. $-18 \div-6$

## EXAMPLE 1

Robin has 47 feet of ribbon on a roll. She wants to cut this roll into 4 -foot strips for decorations. How many 4 -foot strips of ribbon can she make? How much ribbon will be left over, if any?

## SOLUTION

In order to make the 4-foot strips, Robin rolls out all of the ribbon and marks off 4 -foot lengths. She then skip counts the number of pieces she needs to cut and finds that $4 \cdot 11=44$. Therefore, 47 feet divided into 4 -foot pieces equals 11 pieces with 3 feet of ribbon left. In other words, $47 \div 4$ is 11 with a remainder of 3 .


Notice that when using the remainder, the solution is $47=4 \cdot 11+3$. For now, any number left after dividing, you may leave as a remainder.

## PROBLEM 2

Solve each of the following equations:
a. $4 x=12$
b. $3 x=15$
c. $4 x=-20$
d. $-2 x=8$

Have students think about this question and then discuss in small groups. Notice that this problem is different from the previous problem in two ways. We are asking students to partition, or equally share, the 20 pieces of candy among 6 children without telling them how much to give each child. In the measurement model, we specified how much to give each child; we just did not know how many children we could give that amount of candy. Listen also to see how they deal with the 2 leftover pieces.

## EXERCISES

1. a. $4,4 \cdot 8=32$
d. $7,7 \cdot 6=42$
g. $33,33 \cdot 8=264$
b. $6,6 \cdot 2=12$
e. $35,35 \cdot 1=35$
h. $57,57 \cdot 11=627$
c. $5,5 \cdot 5=25$
f. $12,12 \cdot 2=24$
i. $72,72 \cdot 24=1728$

Recall the commutative property of multiplication that states that $a \cdot b=b \cdot a$. Students may give the associated multiplication fact, for part (a) for example, as $8 \cdot 4$ or $4 \cdot 8$.

## EXPLORATION

Mr. Garza has 20 pieces of candy. He wants to divide the candy equally among 6 children. How should he distribute the candy?

One way to distribute the candy is to think of this process in steps. In step 1, give each child 1 piece of candy. This means Mr. Garza has $20-6=14$ pieces of candy left. In step 2, Mr. Garza gives each child a second piece of candy. He now has $14-6=8$ pieces of candy left. In step 3, Mr. Garza gives each child a third piece of candy. He now has $8-6=2$ pieces of candy left. He can no longer give an equal number of pieces to each of the 6 children, so he stops. It took 3 steps to equally distribute as many pieces of candy as Mr. Garza could. That means each child received 3 candies. Write this as $20=3 \cdot 6+2$. Picture this as a linear model by skip counting to divide 20 by 6 , which corresponds to the counting 3 skips of length 6: $3 \cdot 6=18 ; 2$ units short of 20 .

In division, the problem involves the dividend and the divisor, and the task is to compute the quotient. In the linear model, the dividend is the total length. There are two possible cases:
(1) Know the length of each jump, and call it the divisor. Find the quotient, which, in this case, is the number of jumps that equal the total length.
(2) Know the number of jumps, and call it the divisor. Find the quotient, which, in this case, is the length of each jump.

In multiplication, start with the length of each jump and the number of jumps. The answer is the accumulated length of all the jumps. Again, division is the reverse of the multiplication process.

## EXERCISES

1. Evaluate the following, and write the associated multiplication fact. Use the linear model or long division, if needed. See TE.
a. $32 \div 8$
b. $12 \div 2$
c. $25 \div 5$
d. $42 \div 6$
e. $35 \div 1$
f. $24 \div 2$
g. $264 \div 8$
h. $627 \div 11$
i. $1728 \div 24$
2. a. $4 \cdot 8+1=33$
d. $7 \cdot 5=35$
g. $8 \cdot 64+2=514$
b. $7 \cdot 3+3=24$
e. $10 \cdot 4+3=43$
h. $13 \cdot 52+3=679$
c. $6 \cdot 2+2=14$
f. $12 \cdot 5+3=63$
i. $33 \cdot 121+10=4003$
3. a. -3
d. $\quad-7$
g. 53
b. -6
e. $\quad-18$
h. 47
c. -2
f. -4
i. 44
4. a. between 1 and 10 c. between 10 and 100 e. between 100 and 1000
b. between 10 and 100 d . between 10 and 100 f . between 100 and 1000
5. Round 24 to 25 and 528 to 525 . There are four 25 's in every 100 , so $5(4)+1=21$ is between 10 and 50 .
6. 5 flowers for each friend; 4 will be left over.
7. Erica and her friends can make 9 macaroni necklaces. They will have 4 pieces of macaroni left over.
8. Madison can deal 17 cards to each of her friends and has 1 card left over.
9. Write the associated multiplication fact, making the remainder as small as possible. See TE.
a. $33 \div 4$
b. $24 \div 7$
c. $14 \div 6$
d. $35 \div 7$
e. $43 \div 10$
f. $63 \div 12$
g. $514 \div 8$
h. $679 \div 13$
i. $4003 \div 33$
10. Use the missing factor method to compute the following quotients: See TE.
a. $(-3) \div 1$
b. $(-12) \div 2$
c. $(-16) \div 8$
d. $84 \div(-12)$
e. $54 \div(-3)$
f. $88 \div(-22)$
g. $(-318) \div(-6)$
h. $(-705) \div(-15)$
i. $(-1848) \div(-42)$
11. Predict whether the quotient is between 1 and 10 , between 10 and 100 , or between 100 and 1000 by estimating the number of jumps. See TE.
a. $54 \div 6$
b. $195 \div 13$
c. $264 \div 4$
d. $1080 \div 18$
e. $1254 \div 6$
f. $972 \div 6$
12. Peter has 528 apples. If a sack of apples contains 24 apples, predict whether Peter can make between 1 and 10 sacks, between 10 and 50 sacks, or between 50 and 100 sacks. Show how you made this prediction. See TE.
13. Olivia has invited 6 friends to a small dinner party. Earlier this morning, Olivia picked 34 wildflowers. She wants to give each of her friends an equal number of flowers. How many flowers should she give each friend? How many flowers, if any, will be left to add to the centerpiece? See TE.
14. Erica and her friends are making macaroni necklaces. It takes exactly 11 pieces of macaroni to make one necklace. They have 103 pieces of macaroni. How many necklaces can they make? How much macaroni, if any, will be left over?
15. Madison and two of her friends decide to play a game of cards. Madison has a standard deck of 52 cards. She deals the cards so that all three have an equal number of cards. What is the maximum number of cards each of them can be dealt? How many cards, if any, will be left in this case? See TE
16. Because Mr. Scott must give all 23 students an equal amount of pens and have 11 pens left over, all answers are of the form: $23 k+11$ where $k$ is an integer. Thus, the table indicates the number of pens he had in his bag (total) and the number of pens given to each student (student):
17. Suggest students use skip counting to solve this problem. Whe 1they skip coul. ${ }^{3} 4_{\text {by }} 14$ they will see that $14 \cdot 7=98$, but $14 \cdot 8=112$. So if they skip count to multiply, 8 junaps will take théh too far, and after cutting the 98 inches of PVC pipe, there will be 10 inches left over. This is Because $98+80=103$.

If students decide to use a calculator on this problem, they may miss some of the beauty of the skip counting model. After dividing 108 by 14 (since $\frac{10}{14}$ does not result in a terminating decimal) they may say the answer is 7.71 . But what does this mean in terms of the problem? They cannot interpret the answer in its present form and still must apply skip counting principles to give a meaningful answer. Ask them what they did to get this answer. Someone should say, "Divided by 14." So to undo our problem, try doing the opposite: multiplying by 14. That is when they will get the remainder, 10 in.
11. For parts (c) and (d), given a line segment of any length, it is impossible to divide it into lengths of 0 . Given a rectangle of any area, neither its length nor width can be 0 .
a. 0
b. 0
c. undefined
d. undefined
12. а. 14
b. 37
c. 43
d. 367
13. By modeling this problem with the missing factor model, it gives students intuition about how they can scale factors by the same number and the quotient will remain the same.
a. 34
b. 34
c. 34
d. 34
e. 34
f. 34
14. Division can be simplified by dividing common factors before multiplying. For students this exercise can either show them how to change repeated addition to multiplication and scale the dividend and divisor by the same scale factor or in part (a) use the distributive property and use scale factors.
a. 18
b. 3081
c. 29
d. 516
e. 369
9. There are 23 students in Mr. Scott's math class. One day Mr. Scott comes to class with a bag full of pens. He gives an equal number of pens to each student and discovers that he has 11 pens left. He knows that his bag originally held fewer than 100 pens. What are the possible numbers of pens he could have originally had? In each case, how many pens did he give each student? See TE.
10. Annie Paul is installing a sprinkler system in her yard. The system requires 14 -inch sections of PVC pipe. She has one long 9 -foot piece of PVC. How many 14 -inch sections can she cut from this long piece? How much will be left after she cuts the sections? See TE.
11. Solve each of the following equations:
a. $2 x=12$
b. $3 x=12$
c. $\quad 4 x=12$
d. $\quad 4 x=20$
12. Calculate the following, if possible. If it is not possible, explain why.
a. $0 \div 4$
b. $0 \div(-35)$
c. $\quad 4 \div 0$
d. $-35 \div 0$
13. Evaluate the following using the missing factor model: See TE.
a. $(2 \cdot 14) \div 2$
b. $(4 \cdot 37) \div 4$
c. $(15 \cdot 43) \div 15$
d. $(29 \cdot 367) \div 29$
14. Evaluate the following using the missing factor model: See TE.
a. $(4 \cdot 17) \div 2$
b. $(8 \cdot 17) \div 4$
c. $(16 \cdot 17) \div 8$
d. $(12 \cdot 17) \div 6$
e. $(36 \cdot 17) \div 18$
f. $(108 \cdot 17) \div 54$
15. Evaluate: See TE.
a. $(18+18+18) \div 3$
b. $(3081+3081+3081+3081) \div 4$
c. $(29+29+29+29+29+29+29+29+29) \div 9$

Do you notice any patterns?
d. $(258+258+258+258) \div 2$
e. $(123+123+123+123+123+123) \div 2$

## Ingenuity

15. The product of all the positive integers $30 \times 29 \times 28 \times \ldots \times 3 \times 2 \times 1$ can be found by actually multiplying and then counting the zeros in the product. Another approach might be to ask students to think about the kinds of factors that contribute to producing zeros in the product. Students may note that multiplying by 10,20 , or 30 will produce a zero in the product. Some students may notice that $2 \times 5$ produces a zero as does $15 \times 4$. Have the students look at the list of the 30 numbers and examine the possible ways zeros can be produced after they have had a chance to explore their own strategies. There are 6 zeros in 30 !

## Investigation

16. This may lead students to discover that they can multiply on both sides of an equation by a multiplicative inverse, this is commonly referred to as dividing both sides of an equation on both sides by the same number.
a. $\frac{24}{6}$
b. $\frac{10}{2}$
c. $\frac{20}{4}$
d. $-\frac{7}{2}$
17. Ingenuity:

The factorial of a non-negative integer $n$ is the product of all positive integers less than or equal to $n$. We use the notation $n!$, which is read as " $n$ factorial," to represent this product. For example, $5!=1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$, and $6!=1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$. How many zeros are at the end of 30 ?

## 17. Investigation:

Use the skip counting model and number line to solve the following equations:
a. $6 x+2=26$
b. $2 a+1=11$
c. $4 \mathrm{n}-3=19$
d. $2 y+3=-4$

## Division Decisions

Objective: Students will decide on a method to divide a number of people or objects into a given number of groups. Students will discuss different strategies for dividing. (This is a version of the Launch for 7.1 and offers an additional option as a whole class activity.)

## Materials Needed:

20 - 30 objects per group (for option 2 only)

## Activity Instructions:

Option 1: Whole class activity. Ask 1 or 2 students to make and implement a plan to divide the class into 3 equal groups for a game. The best number of students is a multiple of 3 . Notice that we are modeling a division problem; for instance, $24 \div 3$. If the number of students in the class is not a multiple of 3 , the class will have to decide what to do. For example, if the number of students is not a multiple of 3 , there will be a remainder of either 1 or 2 . If there is a remainder of 1 , the class might appoint the extra person to be a game director. If there is a remainder of 2 , the class might also have a scorekeeper.

Option 2: Group activity. Give groups of 3 to 4 students a set of 20 to 30 objects. Ask them to make and implement a plan to divide the objects equally among the group members. Emphasize the fact that it is necessary to understand the process used in distributing the objects.

Reflect on the models your students used. Ask if there are any other methods to divide, excepting long division with pencil and paper. Make sure they discover the first two methods below.

1) Partitive Model: Create piles or groups of objects by placing one object in each pile in each round, and then count the number of rounds it takes to use all the objects, as when a dealer deals the cards in a card game.
2) Repeated Subtraction: Remove 3 objects at a time and count the number of removals it takes to remove the original objects.
3) Linear Model: Students skip count by 3 until the total is reached. This is an additive model. Be sure your students do not confuse the linear model with repeated subtraction. Ask your students to organize the piles of 3 objects into columns; create as many columns as possible and line them up to form a rectangular shape. In doing so, they have discovered how to reverse the area model of multiplication to divide.

## Section 4.5 - The Division Algorithm

## Big Idea:

Dividing using the area model

## Key Objectives:

- Understand the Division Algorithm.
- Discover how area relates to division, including remainders.


## Materials:

Calculators for checking number sense, grid paper

## Pedagogical/Orchestration:

This section re-introduces division using the area model, with remainders. It is the last preliminary to the classic long division method.

## Activity:

Division Algorithm found at the end of the section and on the CD

## Exercises:

Exercises 4 and 6 can be used as informal assessments to check for student understanding.
6th Grade: Exercises 6 and 10 can be omitted for 6th grade.

## Vocabulary:

division, divisor, quotient, remainder, dividend, algorithm

## TEKS:

$6.2(C)(D) ; \quad 7.2(C)(E)(G) ; \quad 7.9(A) ; \quad 7.13(A)(C) ; \quad 7.15(A)(B) ; \quad 8.2(C) ; \quad 8.14(A) ; \quad 8.16(A) ;$

## WARM-UPS for Section 4.5

1. A rectangle has length 8.4 cm and width 4.6 cm . Which of the following is the perimeter of this rectangle?
a. 25.6 cm
b. 25.8 cm
c. 26 cm
d. 26.2 cm
Ans: c

## Launch for Section 4.5:

Ask students to draw a model to represent and solve the following problem: "There are 125 students going on a field trip and each bus holds 40 students. The school district wants to use as few school busses as possible, so the principal will bring his 10 -passenger van to hold any extras. How many busses do they need to order, and how many students will have to ride with the principal in the van?" Look around to see what models different students are using, and discuss as a class. Ask "What do the extra students riding in the van represent?" (The remainder.) Tell your students that today they will use area models to solve problems like the school bus problem. (If no one came up with an area model to represent this problem, you can revisit it after the lesson and have students draw the area model for the problem. The area model will actually resemble 3 long school busses with one short van to hold the 5 students.)

## SECTION 4.5 THE DIVISION ALGORITHM

Another way of thinking of division is the area model. This is similar to the missing factor model. To divide 24 by 4, draw a length of 4 and ask what the width $x$ is to equal a total area of 24 . So what you are doing is looking for the missing factor: $24=4 \cdot($ what? $)$.


You know that division is the reverse operation for multiplication, just as subtraction is the reverse operation for addition. What do we mean by this? Begin with the number 12. Add 3 to get 15 . To undo the addition, you need to subtract 3 from 15 and return to the original number 12. Similarly, in the example above, you found the number 6 . Multiply by 4 to obtain 24 . That is, $24=6 \cdot 4$. To undo this multiplication, divide 24 by 4 and return to the start because $24 \div 4=6$.

## EXAMPLE 1

Using the area model, what is $20 \div 3$ ?

## SOLUTION

Begin with a length of 3 on the $y$-axis. If we mark off a length of 6 on the $x$-axis, the area of the rectangle is 18 . We compute this as $18=3 \cdot 6$. To get an area of 20 , we must add 2 more square units to the end of the rectangle. That means $20 \div 3$ has quotient 6 with remainder 2 because this corresponds to the calculation $20=3 \cdot 6+2$.


Why is the quotient 6 and the remainder 2? Why not say the quotient is 5 and the remainder is 5 ? Why not say the quotient is 4 and the remainder is 8 because $20=3 \cdot 4+8$ ? How do we decide between the different quotients and remainders?


If we think of 20 as $3 \cdot 5+5$, the picture shows that we could break up the last column into pieces of lengths 3 and 2 . Adding this extra 3 to the rectangle is represented by the calculation $3 \cdot 6+2$. The picture on the next page shows we can break up that last column into two pieces of length 3 and another piece of length 2. By adding these extra pieces of length 3 to the rectangle, we have the same calculation $3 \cdot 6+2$.

Summarize the division algorithm, but maybe not explicitly, so your students know what they are trying to find in the following examples. The quotient, $q$, is the length of the rectangle and the remainder, $r$, is the height of the "detached" column on the right so that $r$ squeezes between 0 and $b$.

In the division algorithm, where we divide two positive integers, we identify the quotient as an integer and then a remainder also as an integer. However, when we divide two numbers, integer or otherwise, we refer to the result of the division as the quotient whether the result is an integer or not. For example, $7 \div 2=3 \frac{1}{2}$ or $0.7 \div 2=0.35$.

Draw a rectangle with a length of 7 along the $x$-axis and a height of 3 along the $y$-axis, then remove one square instead of adding a square in the area model, if you wish to discuss the situation in class.


When using the area model for division, say $a \div b$, we write $a$ in the form of the calculation $a=b \cdot q+r$ where $b$ is the height of the rectangle, $q$ is the length of the rectangle along the $x$-axis and $r$ is the remainder or the height of the column added to equal $a$. This is the division algorithm where $b$ is the divisor, $q$ is the quotient and $r$ is the remainder.

Are there any restrictions on $r$ and $b$ ? Yes! When examining the above example, 20 can be written in several different ways:

$$
\begin{aligned}
& 20=3 \cdot 4+8 \\
& 20=3 \cdot 5+5 \\
& 20=3 \cdot 6+2
\end{aligned}
$$

The smaller the values of $r$ are, the closer we are to seeing if 20 is a multiple of 3 . Only when we get to $r=2$, do we see that 20 is not a multiple of 3 . There will be some remainder, namely 2 . So, one condition to put on $r$ and $b$ is that $r<b$. If $r$ is not less than $b$, then we have the situation shown in the picture above.

Is this enough to ensure exactly one answer in the division algorithm? Returning to the above example, $20=3 \cdot 7+(-1)$ also fits the division algorithm with $r<b$ as the only restriction on the remainder. Remember, $b$, the divisor in the division algorithm, is like $h$, the height in the area model. From this we realize we must add one final restriction to the remainder: $r \geq 0$.

The divisor $b$ must be positive because $r$ is not negative and $b$ is greater than $r$. We write this with our inequalities as follows: Because $r<b$, then $b>r$. And because $r \geq 0$, then $b>r \geq 0$ and $b>0$. With this added restriction, we write $20=3 \cdot 6+2$.
a. $\quad 43=6(7)+1$
b. $\quad 87=12(7)+3$
c. $\quad 148=16(9)+4$



## EXERCISES

1. a. 4 r 4
d. $9 r 1$
b. 4 r 4
e. 5 r 3
C. $12 r 11$
f. $\quad 12 r 4$
2. a. 9
e. 6
b. 90
f. 60
c. 900
g. 3
d. 9000
h. 300
i. When the dividend is multiplied by a power of 10 , the quotient is multiplied by the same power of 10 .
3. a. between 10 and 100
b. between 1 and 10
c. between 100 and 1000

We state the formal division algorithm:

## THEOREM 4.1: DIVISION ALGORITHM

Given two positive integers $a$ and $b$, we can always find unique integers $q$ and $r$ such that $a=b q+r$ and $0 \leq r<b$. We call $a$ the dividend, $b$ the divisor, $q$ the quotient and $r$ the remainder.

In our previous example with $20=3 \cdot 6+2$, the dividend $a=20$, the divisor $b=3$, the quotient $q=6$ and the remainder $r=2$.

Compute the following division problems by writing the corresponding division algorithm and sketching a picture that explains what the algorithm represents. See TE.
a. $43 \div 6$
b. $87 \div 12$
c. $148 \div 16$

## EXERCISES

1. Compute the following using the division algorithm: See TE.
a. $\quad 32 \div 7$
b. $24 \div 5$
c. $179 \div 14$
d. $\quad 37 \div 4$
e. $\quad 48 \div 9$
f. $100 \div 8$
2. Calculate the following: See TE.
a. $27 \div 3$
b. $270 \div 3$
c. $2700 \div 3$
d. $27000 \div 3$
e. $24 \div 4$
f. $240 \div 4$
g. $33 \div 11$
h. $3300 \div 11$
i. Write rules to describe any patterns you noticed in a-h. What causes these patterns?
3. Using the area model, predict whether the quotient is between 1 and 10 , between 10 and 100 , or between 100 and 1000 by estimating the length of the rectangle's base. See TE.
a. $861 \div 41$
b. $217 \div 31$
c. $1452 \div 12$

4. Alice will receive $\$ 13$ from the bank and have 1 quarter left over because 53 divided by 4 is equal to $\$ 13.25$.
5. Round 1221 to 1200 and 33 to 30 . 30 will go into 1200 a total of 40 times, which is between 10 and 50 .
6. One way to see that the two models are the same is to notice that we can convert from one to the other. For example, starting with the linear model, convert each jump to a column in the area model.

## Ingenuity

10. This exercise is a fine time to reinforce the power of simplification in division, a practice that has inexplicably fallen into disuse. It is a powerful tool for computation and later for algebraic simplification. Appropriate for 7th grade. Optional for 6th grade.
a. 7
b. 7
c. 6
d. 3
e. 10
f. 1
g. Commons factors from the divisor and the dividend simplify.
h. 12
11. If each $m p 3$ file takes up 4 MB of space, how many mp 3 files can you fit on a 700 MB CD? Will you have any space left over? 175 mp 3 files, no more space.
12. Model $35 \div 6$ using the area model. Use graph paper. See TE.
13. Alice has 53 quarters. She goes to the bank and trades them for dollar bills. How many dollar bills will she get? Will she have any quarters left? See TE.
14. We have 40 square tiles, each 1 foot by 1 foot. We would like to construct a path that is 3 feet wide. How long can we make the path? Will there be any tiles left? You can make a path 13 tiles long and have 1 tile left over.
15. Dan wants to fill 33 bags with candy. If Dan has 1221 pieces of candy, predict whether the number of pieces of candy in each bag is between 1 and 10 , between 10 and 50 , between 50 and 100, or more than 100 . Show how you made your prediction. See TE.
16. Given two positive integers $a$ and $b$, explain why the area and linear models of division for $a \div b$ give the same results. See TE.
17. Ingenuity:

Evaluate the following expressions. (Hint: you may use the missing factor method.) See TE.
a. $(3 \cdot 7) \div 3$
b. $(5 \cdot 3 \cdot 7) \div(5 \cdot 3)$
c. $(8 \cdot 2 \cdot 3) \div 8$
d. $(8 \cdot 2 \cdot 3) \div(8 \cdot 2)$
e. $(10 \cdot 12 \cdot 11) \div(12 \cdot 11)$
f. $(10 \cdot 11 \cdot 12) \div(12 \cdot 10 \cdot 11)$
g. Write rules to describe any patterns you noticed in a-f? What causes these patterns?
h. Compute $(12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6) \div(11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6)$.

## Investigation

11. a. 1 r 2
b. $2 r 1$
c. $3 r 1$
$2 r 1$
$3 r$
5 r 2
3 r 3
5 r 2
8 r 3

When dividing a sum by the same divisor, the quotient and remainder of the sum is the sum of the quotients and remainders. For example, since 13 divided by 4 was $3 r 1$ and 22 divided by 4 is 5 r 2 then $(13+22)$ divided by 4 is $(3+5) r(1+2)$ or $8 r 3$.
11.d. 2 rl
2 r 5
e. $2 r 4$
3 r 3
5
6

For (d) and (e) the quotient of the sum is equal to the sum of the quotients but the sum of the remainders is greater. So add 1 to the quotient with a new remainder, zero. For example, when dividing by 6 for part (d) the results for the first two questions are $2 r 1$ and $2 r 5$, but when we add those together using our rule from part (a) - (c) we get $4 r 6$ but we cannot have a remainder of 6 because if we had a remainder of 6 and divided by 6 we would get 1 instead of having a remainder. So, these problems are different because the sum of the remainders is equal to the divisor, 6 for this example.

Teachers, you can challenge students to extend these results further by creating examples that give remainders greater than the divisor, for example:
(f) $15 \div 4$

Answers: 3 r 3
$14 \div 4$
3 r 2
$(15+14) \div 4$
7 r 1

## 11. Investigation:

Find the quotients and remainders using the division algorithm. See TE.
a. $7 \div 5$
b. $7 \div 3$
c. $13 \div 4$
$11 \div 5$
$10 \div 3$
$22 \div 4$
$(7+11) \div 5$
$(7+10) \div 3$
$(13+22) \div 4$

What patterns do you notice?
d. $13 \div 6$
e $\quad 18 \div 7$
$17 \div 6$
$24 \div 7$
$(13+17) \div 6$

$$
(18+24) \div 7
$$

What patterns do you notice with the remainders in parts $\mathbf{d}$ and $\mathbf{e}$ ? Explain why $\mathbf{d}$ and $\mathbf{e}$ are different from $\mathbf{a - c}$.

## DIVISION ALGORITHM

Objective: Students will increase their ability and understanding of the area model of division and the division algorithm.

## Materials:

Tiles
Grid paper
Times tables (optional)
Index cards with division problems

## Activity Instructions:

Teacher will provide each student with at least 30 tiles. If teacher does not have enough tiles to give out 30 tiles per student, students will share tiles in pairs.

Students will form rectangles using tiles to show the division algorithm. For example: $33 \div 4$, students may think of $4 \times 8=32$; then form a rectangle with 4 rows of tiles and 8 columns of tiles; then students will add 1 more tile.

Each student will draw each answer on grid paper. Begin process again using other problems written on index cards provided by the teacher.

Another option is using the exercises in section 4.4 for students to solve using tiles and grid paper.

Teacher Edition Section 4.5 The Division Algorithm

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## Section 4.6 - Solving Equations

## Big Idea:

Writing and Solving Equations

## Key Objective:

- Define the variable.
- Translate the problem to an equation.
- Solve for the unknown variable.
- Check your answer.


## Pedagogical/Orchestration:

Students need to work on paper, then check work.

## Activity:

Explorations offer opportunities for group work and good class discussion.

## Vocabulary:

missing factor model, variable, expression, equation, unknown

## TEKS:

$7.5(A)(B) ; \quad 7.13(A)(B)(C)(D) ; \quad 7.15(A)(B) ; \quad 7.14(A)(B) ; \quad 8.1(B) ; \quad 8.2(A)(B)(C) ; \quad 8.5(A) ; \quad 8.14(A, B) ; \quad 8.15(A)$ ;8.16(A);

## WARM-UPS for Section 4.6

1. a. Solve the following equations: $2 x=6$ and $2 x=8$
b. What do you think is the solution of $2 x=9$ ?
c. Fill in the chart with solutions to the following equations. Do you see a pattern?

| Equation | Solution |
| :---: | :---: |
| $2 x=4$ | $x=$ |
| $2 x=5$ | $x=$ |
| $2 x=6$ | $x=$ |
| $2 x=7$ | $x=$ |
| $2 x=8$ | $x=$ |

2. If you solve the equation $4 x=83$, the solution will be
a. between 21 and 22
b. between 20 and 21
c. between 19 and 20
d. between 18 and 19

## Ans: b

## Launch for Section 4.6:

Today we will be baking random cupcakes. We will take all of the steps for baking normal cupcakes, cut them apart and place them in a hat. Then, we will pull out and perform the steps in random order. After the last step in the hat has been performed...Voila! Random cupcakes!

## Steps to make normal cupcakes:

1. Preheat oven to 350 degrees.
2. Place cupcake liners in tins.
3. Empty cupcake mix into bowl.
4. Add one egg.
5. Add $1 \frac{1}{3}$ cup of water.
6. Add $\frac{1}{3}$ cup of vegetable oil.
7. Mix on high for 2 minutes.
8. Fill cups $\frac{1}{3}$ full.
9. Bake for 18-25 minutes.
10. Let cupcakes cool to room temperature.
11. Apply icing.
12. Enjoy.

## Follow-up questions:

- Would random cupcakes be something you would like to eat? Why or why not?
- Is order important in cooking?
- Have the students put the steps in order from first to last. Are there some steps where order is not important?


## SECTION 4.6 SOLVING EQUATIONS

Now that you have seen different operations in Chapters 3 and 4, we summarize the order in which mathematical operations are performed below.

Order of Operations

- Compute the numbers inside the parentheses
- Compute any exponential expressions
- Multiply and divide as they occur from left to right
- Add and subtract as they occur from left to right


## EXPLORATION 1

When you go to Game Go and purchase three video games of equal cost, your total is $\$ 84$. What is the purchase price per game?

How could we write this problem as an equation and solve the equation using the Four Step Process to Solving Equations in Section 3.2 Exploration: Charting the Process?

## EXPLORATION 2

Consider the following expressions:
a. $9+2 \cdot 3-2$
b. $9+2 \cdot(3-2)$
c. $(9+2) \cdot 3-2$

1. What similarities and differences do you notice between expressions a and b ?
2. What similarities and differences do you notice between expressions a and c ?
3. Evaluate $a, b$, and $c$ using the order of operations.
4. Why did you get different results?

In earlier chapters, we learned about equations and how to solve equations like $x+4=6$. In section 4.4 , we studied the equation $3 x=21$ using the missing factor model. We showed how to use the operation of division to solve this equation: $x=21 \div 3=7$.

Now let us apply the knowledge we have of equations and order of operations to some real life applications.

## EXPLORATION 3

When you go to Game Go and purchase three video games of equal cost, your total is $\$ 84$. What is the purchase price per game?

How could we write this problem as an equation and solve the equation using the Four Step Process to Solving Equations in Section 3.2 Exploration: Charting the Process?

## EXAMPLE 1

A store sells CDs for $\$ 9$ each. Sandy buys some CDs for $\$ 54$. How many CDs did she buy? Write an equation for this problem and solve.

## SOLUTION

## Step 1: Define the variable.

Let $\mathrm{x}=$ the number of CDs Sandy bought.
Step 2: Translate the problem to an equation.
We know that each CD costs \$9
$9 x=$ total cost of the CDs
$9 x=54$

## Step 3: Solve for the unknown.

$9 x=54$
$9 x \div 9=54 \div 9$ or equivalently $x=\frac{9 x}{9}=\frac{54}{9}=6$
$x=6$

## Step 4: Check your answer.

Check. Is 9 multiplied by 6 equal to 54? Yes.

## PROBLEM 1

a. $x=4$
b. $x=4$
c. $a=4$
$x=\frac{9}{2}$
$x=\frac{13}{3}$
$a=\frac{17}{4}$
$x=5$
$x=\frac{14}{3}$

Watch to see if the students notice the pattern and answer the problems without any work.

## PROBLEM 1

Solve each of the following equations. Do you see a pattern?
a. $2 x=8$
$2 x=9$
$2 x=10$
b. $3 x=12$
C. $4 a=16$
$3 x=13$
$4 a=17$
$3 x=14$
$4 a=18$
$3 x=15$
$4 a=-18$

## EXAMPLE 2

Terry, Aissa, and Steve went to the snack bar to buy some snacks. They bought three slushies and a bag of Flaming Hot Chips which costs $\$ 1$. If they spent $\$ 7$ for all of this what was the cost of each slushie at the snack bar?

## SOLUTION

Using the Four Step Process, we write an equation for this problem situation and solve it.

## Step 1: Define the variable.

Let $\mathrm{c}=$ the cost of each slushie.

## Step 2: Translate the problem to an equation.

We know that they spent $\$ 7$. The Flaming Hot Chips cost $\$ 1$ added to the 3 slushies which each cost c .
$3 \mathrm{c}=$ cost of the 3 slushies
$3 c+1=$ cost of 3 slushies and chips
$3 c+1=7$
Step 3: Solve for the unknown.
This is an equation so we can use a balance scale. So the equation can be modeled as


## Step 4: Check your answer.

Check. Is 2 multiplied by 3 plus 1 equal to 7 ? Yes.

Notice that it took two steps to solve the equation: $3 c+1=7$. The first step was to subtract 1 from each side of the equation. Explain why. This results in the equivalent equation: $3 c=6$. The next step used the missing factor method to determine: 3 times what number equals 6 .

## PROBLEM 2

Show how to use this two step process to solve each of the following equations. Draw balance scale models for a and b .
a. $3 n+5=11$
b. $4 a+1=33$
c. $2 x+3=4$
d. $2 x-3=11$
e. $3 y-2=13$

As in section 3.5, we can use a number line model to show another way to think about order of the two step process in solving the equation $3 c+1=7$. Consider the number line below.


Note that the location of the value $3 c+1$ is equivalent to the location of the number 7 . Where is the location 3 c ? It is 1 unit to the left of 7 , which is the location 6.


This is equivalent to the process of subtracting 1 from $3 c+1$. The new equation is $3 c=6$. Knowing the location of $3 c$, how do we find the location of the value of c ? Using the linear model for multiplication, we can think of this situation in two ways:
a. How many jumps of length 3 does it take to reach 6 ? The answer is 2 jumps. So $\mathrm{c}=2$.
b. What length of jump would it take to reach 6 in 3 jumps? The answer is a jump length of 2 . This model is nice because it produces the following picture:


## PROBLEM 3

Use this number line method to show how to solve the equations in Problem 2, parts a, c, and d.

Number line models can also be used to solve inequalities. On the number line below, $x+1>5$ can be shown as


Note that the location of $x+1$ is above the number 5 , but because $x+1$ is greater than 5 , the circle is open and not filled in. The value of $x$ is one unit to the left of 5. Since $x$ represents all values greater than 4 , the model shows the solution as an open circle with a bold arrow pointing to the right.


## PROBLEM 4

How would the solution to the inequality $x+1<5$ differ from the solution to the above example? Model the solution on a number line.

## PROBLEM 5

Use the number line method to model and solve the inequality $2 x+3<7$.

To check that our answer is correct, we can pick a point on the graph of the inequality and plug it in. If the inequality remains true, your answer is in the set of possible solutions.

1. a. $5 n=40 ; n=8$
b. $\quad-3 n=18 ; n=-6$
c. $\quad 6=\frac{n}{4} ; n=24$
d. $\quad \frac{n}{3}=-10 ; n=-30$
e. $\quad 2 n-7=19 ; n=13$
f. $\quad \frac{1}{2} n=50 ; n=100$
2. 

a. $\quad x=4$
b. $\quad k=14$
c. $y=5$
d. $\quad z=-3$
e. $\quad m=2$
f. $\quad x=1$
g. $a=2$
h. $\quad r=3$
i. $\quad a=4$
j. $x=\frac{7}{2}$
k. $x=\frac{7}{2}$
l. $a=-\frac{5}{2}$
m. $a=\frac{13}{3}$
n. $\quad x=6$
3. Let $\mathrm{c}=$ cost of baseball
$14 c=\$ 63 ; b=\$ 4.50$
4. Let $h=$ Devyn's height
$2 h=64 ; h=32$ inches
5. Let $\mathrm{c}=$ the number of candy pieces each girl had.
$4 \mathrm{c}+12=52$
$c=10$, each girl originally had 10 pieces
Amy had $10+12=22$ pieces of candy.

## EXERCISES

1. Write an equation to represent each statement, then compute each equation.
a) five times a number is forty
b) a number times negative three is eighteen
c) six is equal to a number divided by four
d) the quotient of a number and three is negative ten
e) seven less than twice a number equals nineteen
f) one half of a number is fifty
2. Solve each equation. Check your answer and show your work through substitution. Use the balance scale or number line model for problems a-c.
a. $3 x=12$
b. $4 k=56$
c. $11 y=55$
d. $-9=3 z$
e. $14=6 m+2$
f. $4(x+1)=8$
g. $4 a+3=11$
h. $8=3 r-1$
i. $5 a-3=17$
j. $4 x+3=17$
k. $6 x+4=25$
l. $4 a+3=-7$
m. $3 \mathrm{a}-5=8$
n. $3(x-2)=12$
3. Fred buys 14 baseballs for $\$ 63$. How much does each ball cost? Write an equation for this problem and solve.
4. Devyn is half the height of her older brother Jordan who is 5 feet 4 inches. How many inches tall is Devyn? Write an equation and show how you solved.
5. Amy, Sandy, Michelle, and Christina all ended up with the same amount of Halloween candy after trick or treating. Amy snuck into the pantry and took 12 extra pieces. Now they have a total of 52 pieces. How many pieces did each girl originally have? How many did Amy have including the pieces she found? Write an equation and solve.
6. Let $\mathrm{g}=$ the number of games Mike played
$4 g+10=146$
$\mathrm{g}=34$ games played
7. Let $\mathrm{t}=$ extra texts
$0.10 \mathrm{t}+35=95$
$t=600$ extra texts
8. Let $\mathrm{w}=$ number of weeks it will take Shelly to buy a car
$40 w+2580=4100$
$w=38$ weeks
9. $2 x+5 \leq 30$

He can ride 12 or less times

10. $x \geq 70$
11. $\mathrm{y} \leq 12$
12. $\mathrm{x}=19$
$a=9$
6. For each of the inequalities below, determine which number(s) in the list $\{-4$, $-2,0,3,5\}$ satisfies(y) the inequality:
a. $x+3<5$
b. $x-2>-1$
c. $2 x+3<7$
d. $3 x-2<7$
7. Solve each of these inequalities. Represent the solutions.
a. $x+3<5$
b. $x-2>4$
c. $2 x+3<7$
d. $3 x-2<7$
8. Mike can buy a discount card for laser tag for $\$ 10$. This allows him to play laser tag for $\$ 4$ per game. After one year, Mike has spent $\$ 146$. Set up an equation to solve for how many games Mike played and solve this equation.
9. Cynthia's cell phone plan costs $\$ 35$ per month with 200 texts included. She is charged 10 cents for extra texts. Her bill for the month of January was $\$ 95$. How many extra texts did she send? Write an equation and solve.
10. Shelly is saving up for a new car for college. She has $\$ 2580$ saved already, but the car costs $\$ 4100$. She is working hard at Sandy's Sandwich Shop and saving an extra $\$ 40$ per week. Set up an equation and solve for how many weeks it will take Shelly to buy this car.
11. Todd has $\$ 30$ to spend at the fair. It costs $\$ 5$ to get in, and roller coaster rides are $\$ 2$ a piece. Write and solve an inequality to show the possible number of rides Todd can take. Model the solution to the inequality on a number line.
12. Melissa needs an average of at least 80 on her five math tests. She scored 95 , 60,85 , and 90 on her four tests. What is the minimum grade Melissa needs on her fifth test?
13. Ingenuity:

Write and solve a real-world problem corresponding to the inequality $3 y+10 \leq 46$.

## 14. Investigation:

Solve the following two equations:
a. $\frac{x+2}{3}=7$
b. $\frac{a-1}{2}=4$

1. a. -14
b. -5
c. 24
2. a. -222
b. -4144
c. 9554
3. $7(-3)=-21^{\circ}$; the temperature dropped $21^{\circ}$
4. $4(-132)=-528 ; 528$ meters below sea level
5. $11+5(-3)=-4^{\circ} \mathrm{F}$
6. $5(x+7)=5 x+35$
7. a. $5(10)+5(2)$
60
b. $3 x+3(4)$
$3 x+12$
c. $2(6)+2 x$
$12+2 x$


8. a. $4(6+3)$
4(9) 36
b. $y(7+11)$
$y(18)$
$18 y$
c. $a(9-2)$
$a(7)$
7a
9. Dimensions of rectangles: $1 \times 8 ; 2 \times 7 ; 3 \times 6 ; 4 \times 5$
10. Dimensions of rectangles: $1 \times 36 ; 2 \times 18,3 \times 12,4 \times 9,6 \times 6$
11. a. Perimeter doubles
b. Area quadruples

## REVIEW PROBLEMS

1. Draw the frog model on the number line to compute the following products.
a. $-7 \cdot 2$
b. $5 \cdot-3$
c. $-6 \cdot-4$
2. Evaluate the following products.
a. $17(-13)$
b. $\quad-56(74)$
c. $-281(-34)$
3. Suppose the temperature outside changes $-3^{\circ}$ every hour. How much will the temperature change in 7 hours?
4. A submarine is at -132 meters. If the submarine dives 4 times its initial depth, at what depth will the submarine be?
5. The temperature at $6: 00 \mathrm{p} . \mathrm{m}$. is $11^{\circ} \mathrm{F}$. The temperature decreases an average of $3^{\circ} \mathrm{F}$ for each of the next 5 hours. What is the temperature at 11:00 p.m.?
6. Write an expression that represents the area model below.

7. Use the distributive property to write the following product as a sum. Simplify if necessary. Draw a rectangle model.
a. 5 (12)
b. $3(x+4)$
c. $2(6+x)$
8. Rewrite the expressions using the distributive property. Simplify if necessary.
a. $4(6)+4(3)$
b. $7 y+11 y$
c. 9a-2a
9. List all the possible dimensions (using only integer pairs) of rectangles with a perimeter of 18 units.
10. List all the possible dimensions (using only integer pairs) of rectangles with area of 36 square units.
11. The dimensions of a rectangle are doubled:
a. What happens to the perimeter?
b. What happens to the area?
12. a. -3
c. -7
e. 7
b. -13
d. -23
f. 4
13. 6 books per shelf; 2 remaining
14. 


15. a. $67=9(7)+4$
b. $\quad 107=20(5)+7$
c. $\quad 14=3(4)+2$
16. a. $x=-1$
b. $y=6$
c. $x=-\frac{5}{2}$
d. $\frac{11}{2}$
e. $x=6$
f. $x=-3$
17. a. $2 n+3=7 ; n=2$
b. $-8=3 n-5 ; n=-1$
c. $3 n+6=12 ; n=2$
d. $5 n+2=17 ; n=3$
18. Let $\mathrm{n}=$ the number of DVD rented

$$
\begin{aligned}
& 2 n+32=68 \\
& n=18 \text { DVDs }
\end{aligned}
$$

12. Use the missing factor method to compute the following quotients.
a. $(-18) \div 6$
b. $(-65) \div 5$
c. $56 \div(-8)$
d. $46 \div(-2)$
e. $(-77) \div(-11)$
f. $(-48) \div(-12)$
13. Shelly has 26 books that she is placing on her bookshelf. Her bookshelf has 4 shelves. If Shelly puts the same number of books on the shelves, how many books will each shelf have? How many books, if any, will be remaining?
14. Model $47 \div 5$ using the area model.
15. Compute the following division problems by writing the corresponding division algorithm.
a. $67 \div 9$
b. $107 \div 20$
c. $14 \div 3$
16. Solve these equations:
a. $4 x+7=3$
b. $3 y-5=13$
c. $2 x+8=3$
d. $2 x-8=3$
e. $-2 x=12$
f. $\quad 4-2 x=10$
17. Write an equation to represent each statement, solve, and check your work by substitution. Make a balance scale for problems $a$ and $b$ and a number line model for problems c and d .
a. the sum of twice a number and 3 is 7
b. negative 8 is 5 less than three times a number
c. the product of 3 times a number, increased by 6 , is 12
d. two more than 5 times a number is 17
18. Joan is a member of the Rent a DVD Club and pays a monthly membership fee of $\$ 32$. Each DVD rental is an additional $\$ 2$. How many DVDs did Joan rent if she paid $\$ 68$ for the month of June? Write an equation and solve.

## Section 4.1: 8

Solution: Suppose that 1 is the first number crossed out. Then the order in which the numbers are crossed out is $1,5,9,2,7,12,8,4,3,6,11$, and 10 . To end on 7 , just shift everything by 3 , i.e. he started at 10 and then crossed out $2,6,11,4, \ldots, 3,8,7$.

## Section 4.2: 2 ft .

Solution: Since the cut tile covers half as much area as the uncut tile, the areas must be 2 and 4 square feet. Thus the uncut tile was 2 feet by 2 feet.

## Section 4.3: 423

Solution: The number of coins is 5 greater than a multiple of 11,3 greater than a multiple of 10 , and a multiple of 9. The smallest number that meets the last two conditions is 63 . Adding multiples of 90 will continue to produce numbers that meet both conditions. Possible values are $243,333,423, \ldots$. Dividing each by 11 shows that 423 is the smallest value that yields a remainder of 5 and meets the first condition.

## Section 4.4: 15

Solution: The quotients tell us that n is between 121 and 131 (inclusive), 120 and 129 , and 117 and 125 , so $121 \leq \mathrm{n} \leq 125$. Both 121 and 125 have a quotient of 15 when divided by 8 , so n must as well.

## CHALLENGE PROBLEMS

## Section 4.1:

The integers 1 through 12 appear on the circular face of a clock. Alex crosses out one of the numbers, then moves around the circle clockwise crossing out every fourth number that has not yet been crossed out. If 7 is the last number he crosses out, what was the next-to-last number?

## Section 4.2:

A carpenter needs to tile a 6 square foot rectangular area, but he only has square tiles that are all the same size. By using one whole tile and half of another (making only one cut) he can fill the area. How long was the tile he cut (before cutting, in feet)?

## Section 4.3:

Eleven pirates find a treasure chest. When they split up the coins in it, they find that there are 5 coins left over. They put the coins back, throw one pirate overboard, and split the coins again, only to find that there are 3 coins left over. So, they throw another pirate over and try again. This time, the coins split evenly. What is the least number of coins there could have been?

## Section 4.4:

When a number n is divided by 11 , the quotient is 11 with a possible remainder. When n is divided by 10 , the quotient is 12 with a possible remainder. When n is divided by 9 , the quotient is 13 with a possible remainder. What is the quotient when n is divided by 8 ?

## Section 5.1-Graphing on the Coordinate Plane

## Big Ideas:

Develop and explore the coordinate plane

## Key Objectives:

Locate, name and plot points on a coordinate plane.

## Pedagogical/Orchestration:

Make a big grid on the wall or floor so that students can interact with the coordinate plane. See Activities.

## Internet Resource:

Coordinate Plane- http://www.quia.com/cz/43437.html

## Activities:

Use Battleship Activity at the end of the section (also found on the CD).
A fun and useful activity is to have the whole class build a large coordinate grid with yarn on the floor or wall. It will be useful in referencing vocabulary and plotting points.

## Materials:

Handouts of coordinate grids with axes (several per student for notes and exercises), Coordinate grid for wholeclass demonstration from an overhead, a pull-down chart, or one created on chart paper, markers or map colors for exercises, Optional: dry-erase whiteboards with grids and dry erase markers for each student, globe(s) for the Investigation:

## Exercises:

12 (Investigation)—have plenty of globes handy (inflatable work well) so that students can work in groups. This is a good problem to use as a reflection of the section.

## Vocabulary:

As an activity/project, have the students develop their own vocabulary foldable (booklet or flashcards), something to help students keep a log of the vocabulary with examples, definitions, and pictures.

| coordinate | quadrant | coordinate plane |
| :--- | :--- | :--- |
| ordered pairs | Origin | $x$-coordinate |
| axes | $y$-coordinate | $x$-axis (horizontal axis) |
| lattice points | $y$-axis (vertical axis) | Cartesian coordinate system |
| point | line | plane |

## TEKS:

6.7, 7.7(A), 7.13(A)
8.7(D);
8.14(A,D);
8.16(B);

## WARM-UPS for Section 5.1

1. How can you tell just by looking at a coordinate pair what quadrant the point will lie in?
a. What quadrant does the point $(4,3)$ lie in?
b. What quadrant does the point $(-4,3)$ lie in?
c. What quadrant does the point $(4,-3)$ lie in?
d. What quadrant does the point $(-4,-3)$ lie in?
2. Chess can be played by mail or email since the movements of each chess piece can be described using the rows and columns of the chess board to tell the opponent what move has been made. List all the other examples you can think of that use a grid system to show the location of an item.

## Launch for Section 5.1:

Ask your students, "Have you ever lain in bed staring at the ceiling? Pretty boring, right? In the 1600 s there was a man named Descartes who was doing just that, but entertained himself by watching a fly that was crawling around on the ceiling. After watching the fly for awhile he started wondering if there was a way to describe the location or position of the fly so that he could tell someone else where the fly was. It occurred to him that he could describe the fly's position by measuring its distance from the walls of the room. Maybe Descartes didn't realize it at the time, but he had just invented the coordinate plane! In fact, the coordinate plane is often called the Cartesian plane in his honor." Ask your students, besides determining the location of a fly, how could a system like this be useful in real life? Possible responses could be in locating a ship at sea, or a location on a map. Inform your students, "Today we will be discovering the Cartesian plane ourselves as a way to describe the position of points." Lead your students through the description of the coordinate plane found in the beginning of Section 5.1.

# PATTERNS AND <br> F U N C T I O N S 

## SECTION 5.1 GRAPHING ON THE COORDINATE PLANE

We use the number line to represent numbers as locations. To each point on the number line we associate a number or coordinate, which is the location of that number.


For example, the number line above shows points $P$ and $Q$. To graph or plot a point $P$ with coordinate 3 on the number line, we graph the point 3 units to the right of 0 . Because point $Q$ has coordinate -2 , we graph the point 2 units to the left of 0 on the number line.

If we draw our number line horizontally as above, then positive numbers are located to the right of 0 and negative numbers are located to the left of 0 .

Suppose instead that we draw our number line vertically like a thermometer. In this case, positive numbers are above zero, and negative numbers are below zero.

On the vertical number line, we could locate a point $R$ with coordinate 3 by graphing the point 3 units above 0 . The point $S$ with coordinate -2 would be located 2 units below 0 .


Teachers, note that this section includes a lot of vocabulary that might be new to your students. Have students create a foldable Vocabulary booklet to place in their binders using graph paper. You may want to use butcher or chart paper to make a coordinate plane to display in your room. You can label the various aspects of the plane for students to reference as the year continues. You can also use sticky notes to make coordinate points for instruction. Many teachers have enjoyed using small, student-sized whiteboards with the coordinate plane printed on it. This allows students to erase mistakes easily and they can hold their boards up in unison for quick assessment.

On a number line, each point corresponds to a number. If we want to plot points on a plane, we will need to use two numbers, called coordinates, to locate the point. A coordinate plane is constructed as follows:

We begin by drawing a horizontal number line and locating the zero point, which is called the origin:


Next, draw a vertical number line through the origin of the horizontal number line, so that the two zero points coincide:


The horizontal number line is called the horizontal axis or the $\mathbf{x}$-axis; the vertical number line is called the vertical axis or the $\mathbf{y}$-axis.

Ask your students in what order the quadrants are numbered around the origin. Where is Quadrant 1? Why do they think the quadrants start where they start? Why do they think they move counterclockwise?

Answers may vary for these questions. Students might research the answers to these questions online or in the library. Encourage them to research the questions.

Remember, the point at which the two number lines, or axes, meet is called the origin and is the zero point on both number lines.

The $x$ - and $y$-axes divide the plane into four regions. Because there are four of them, we call each region a quadrant. By convention, we number the quadrants counterclockwise, starting with the upper-right quadrant. The axes don't belong to any quadrants, but rather are their boundaries.


You may wish to discuss with the students lattice structures that they have seen in a garden. The term cross grids may also require a visual accompaniment. Grid paper can provide a good example.


The coordinates for the points in the coordinate plane are always ordered pairs of numbers. The first coordinate is called the $\mathbf{x}$-coordinate; the second is the y -coordinate.

Consider the point $P$ on the coordinate plane above. To identify point $P$, we begin at the origin. First we move 4 units to the right on the $x$-axis, then we move up 3 units. We arrive at the point $P$. Therefore, the point $P$ is identified as $(4,3)$, where 4 represents the $x$-coordinate and 3 represents the $y$-coordinate.

Notice that the coordinates of the origin are $(0,0)$ because the origin lies at the zero point of both number lines. The points that contain integers, like $(4,3)$ or $(5,-2)$, are called lattice points because they fall on the cross grids, which look like a lattice.

This coordinate system with horizontal and vertical axes is called a Cartesian coordinate system. It is named after René Descartes, the French mathematician and philosopher who invented it.
1.

| $A$ | $(3,4)$ | $G$ | $(0,-2)$ |
| :--- | :--- | :--- | :--- |
| $B$ | $(4,2)$ | $H$ | $(6,0)$ |
| $C$ | $(-4,-1)$ | $I$ | $(-1,4)$ |
| $D$ | $(-4,6)$ | $J$ | $(-4,1)$ |
| $E$ | $(3,-2)$ | $K$ | $(-4,-4)$ |
| $F$ | $(2,1)$ | $L$ | $(0,0)$ |

2. 



The students may need to be reminded that "to plot a point" means to graph it on a number line or on the coordinate system.

## EXERCISES

For most of these exercises you will need coordinate planes to draw on.

1. Write the coordinates for each of the points $A$ to $L$ shown on the coordinate plane below. See TE.

2. Plot the following points on the coordinate plane. See TE.

| $M(2,5)$ | $Q(-2,-5)$ | $U(0,3)$ |
| :--- | :--- | :--- |
| $N(5,2)$ | $R(-5,2)$ | $V(4,0)$ |
| $O(-2,5)$ | $S(1,1)$ | $W(-3,0)$ |
| $P(5,-2)$ | $T(-1,1)$ | $X(0,-3)$ |

3. Answers will vary. One example of a students answers might look like this:
a. $(-1,2),(-2,4),(-5,10)$
b. $(1,-2),(2,-4),(5,-10)$
c. $(1,2),(2,4),(5,10)$
d. $(-1,-2),(-2,-4),(-5,-10)$
e. All of the points from part (a) are in Quadrant II, all of the points from part (b) are in Quadrant IV, all of the points from part (c) are in Quadrant I and all of the points from part (d) are in Quadrant III.
4. Answers will vary. One example of a students answers might look like this:
a. $(0,2),(0,4),(0,10)$
b. $(1,-2),(1,-4),(1,-10)$
c. $(0,2),(0,0),(0,-10)$
d. $(1,0),(0,0),(-5,0)$
e. All of the points from part (a) are on the positive part of the $y$-axis. All of the points from part (b) are in Quadrant IV. All of the points from part (c) are on the $y$-axis. All of the points from part (d) are on the $x$-axis.
5. Answers will vary. One example of a students response:
a. $(1,2),(2,4),(5,10),(-2,-4),(3,6)$
b. $(2,1),(4,2),(10,5),(-4,-2),(6,3)$
c. The points in part (a) lie on one line. Points in part (b) lie on another line.

Plot these on a large coordinate system by students going to the board or floor. Plot enough of them to help them see a pattern.
6. Answers will vary. An example of a student response is the following.
a. $(-1,1),(0,1),(5,1),(4,1),(-2,1)$
b. $(0,2),(2,4),(-5,10),(0,3),(9,7)$
c. $(-1,-2),(0,-4),(5,-10),(0,-5),(-9,-17)$
d. The points in part (a) fall on a line. The points in part (b) are all above the line formed by the points in part (a), more particularly the $\mathrm{y}=1$ line. These points are also in Quadrants I and II. The points in part (c) are all below the line formed by part (a).
7. Answers will vary. A student might respond in the following manner:
a. $(-3,-3),(-3,0),(-3,5),(-3,4),(-3,-2)$
b. $(-1,0),(-1,2),(-1,-5),(-1,4),(-1,9)$
c. $(-2,-2),(-2,-4),(-2,-10),(-2,-5),(-2,-17)$
d. The points in part (a) fall on a vertical line, given by the equation $x=-3$. The points in part (b) all fall on a vertical line as well, given by the equation $x=-1$. The points in part (c) all fall in the region between the line from part (a) and the line from part (b), this also happens to lie in Quadrants III and IV.
3. Find and plot 3 points that meet the following conditions: See TE.
a. Each point has a negative $x$-coordinate and a positive $y$-coordinate.
b. Each point has a positive $x$-coordinate and a negative $y$-coordinate.
c. Each point has positive $x$-and $y$-coordinates.
d. Each point has negative $x$-and $y$-coordinates.
e. What do you notice about the points in each situation?
4. Find and plot 3 points that satisfy the following conditions: See TE.
a. Each point has the $x$-coordinate equal to 0 and a positive $y$-coordinate.
b. Each point has the $x$-coordinate equal to 1 and a negative $y$-coordinate.
c. Each point has the $x$-coordinate equal to 0 , but is different from part a.
d. Each point has the $y$-coordinate equal to 0 .
e. What do you notice in each situation?
5. Find and plot 5 points that satisfy the following conditions: See TE.
a. Each point has a $y$-coordinate that is double the $x$-coordinate.
b. Each point has a $x$-coordinate that is double the $y$-coordinate.
c. What do you notice about the points in each situation?
6. Find and plot 5 points that meet the following conditions: See TE.
a. Each point has the $y$-coordinate equal to 1 .
b. Each point has the $y$-coordinates greater than 1 .
c. Each point has the $y$-coordinate less than 1 .
d. What do you notice?
7. Find and plot 5 points that meet the following conditions: See TE.
a. Each point has the $x$-coordinate equal to -3 .
b. Each point has the $x$-coordinates equal to -1 .
c. Each point has $x$-coordinates greater than -3 and less than -1 .
d. What do you notice?
8. Answers will vary. A student might respond in the following manner:
a. $(-3,-3),(0,0),(5,5),(4,4),(-2,-2)$
b. $(0,1),(2,3),(-5,-1),(-4,1),(-90,-10)$
c. $(-1,-2),(0,-4),(5,-10),(0,-5),(-9,-17)$
d. The points in part (a) fall on a line, the $y=x$ line. The points in part (b) all fall above the line from part (a). The points in part (c) all fall below the line from part (a).
9. a. $(0,1),(1,0),(0,-1),(-1,0),(0,0)$
b. $(-2,0),(-1,1),(-2,3),(-3,1),(-2,1)$
c. $(3,-1),(4,-2),(2,-2),(3,-3),(3,-2)$
d. $(-4,-1),(-4,-3),(-5,-2),(-3,-2),(-4,-2)$
e.


Notice that the students may be tempted to include such points as $(2,5)$ or $(5,2)$. These lattice points have distance greater than from the first four points because the diagonal distance is square root of 1 units from $(3,4)$ rather than 1 unit away.
10. Given the lattice point $(x, y)$ the four nearest lattice points are: $(x, y+1),(x+1, y),(x, y-1),(x-1, y)$ Ingenuity
11. Start with $(3,1)$. In one move, we go to $(2(3),-1)$ or $(6,-1)$. In the second move, we go to $(2(6),-(-1))$ or $(12,1)$. In the third move, we go to $(2(12),-1)$ or $(24,-1)$.

## Investigation

12. Make sure your students know that longitudes run vertically through the poles and latitudes run horizontal parallel to each other.
a. Answers will vary.
b. Answers will vary.
c. Look up the term "Cartesian" and find out why we call the $x y$-system the Cartesian coordinate system.
d. No
e. The latitudes and longitudes are perpendicular to each other.
f. The longitudes cross at the poles.
13. Find and plot 5 points that satisfy the following conditions: See TE.
a. Each point has the same $y$-coordinate as the $x$-coordinate.
b. Each point has the $y$-coordinate larger than the $x$-coordinate.
c. Each point has the $y$-coordinate smaller than the $x$-coordinate.
e. What do you notice?
14. Plot each of the following lattice points. For each of these points, locate and label the four nearest lattice points. See TE.
a. $(0,0)$
b. $(-2,1)$
c. $(3,-2)$
d. $(-4,-2)$
e. $(3,4)$
15. Suppose $(x, y)$ is a lattice point. What are the four nearest lattice points? See TE.
16. Ingenuity:

In a number game a move is described by the following rule: a player on a point $(a, b)$ moves to the point $(2 a,-b)$. If a player starts at $(3,1)$, where will she be after 3 moves? (24,-1)

## 12. Investigation:

Look at a globe. Lines of latitude and longitude help us locate places on the earth. This forms a coordinate system.
a. Find the coordinates of your city?
b. Find another city on the same latitude as your city?
c. Coordinate systems are often called Cartesian coordinate systems. Why?
d. Do longitude and latitude form a Cartesian coordinate system? Why?
e. What do you notice that is the same?
f. Do you notice anything that is different? Why? See TE.

## BATTLESHIP



Objective: The students will play Battleship to practice graphing and locating points on a coordinate system.

## Materials:

Battleship Folders (glue instruction on the front, coordinate planes inside folder)
Dry erase marker

## Activity Instructions:

1) Label both the $x$ and $y$ axes on both MY GUESSES and MY SHIPS AND YOUR HITS grid paper.
2) The students will choose a partner and play Battleship.
3) Each player will place 4 dots in a straight line to represent a ship- vertical, horizontal, or diagonal on their worksheet, "My Ship and Your Hits." Place 4 ships on any of the four quadrants.
4) The students will take turns guessing ordered pairs to try to find their opponent's battleships.
5) The first player calls out a coordinate pair, marking an " $X$ " on the side MY GUESSES.
6) If it is a hit, the other player marks an X on the YOUR HIT side of the folder. If it was a miss, there is no need to mark the coordinate. Each time, inform your opponent if it was a hit or a miss.
7) The game continues until one player has found all of their opponent's four ships.
8) The teacher can choose the level of difficulty for the game, based on the needs of the class:

Easy. $\qquad$ Quadrant I only
Medium.........Quadrants I and II
Hard.............All four Quadrants

Teacher Edition Section 5.1 Graphing on the Coordinate Plane

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MY SHIPS AND YOUR HITS

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## Section 5.2-Translations and Reflections

## Big Idea:

Translations and reflections transform both points and figures in the coordinate plane, Exploring Symmetry

## Key Objectives:

- Using a rule, translate a point or figure horizontally and vertically.
- Given two points, construct the rule that translated one point to the next.
- Understand and perform reflection of a point about the $x$-axis.
- Understand and perform reflection of a point about the $y$-axis
- Understand lines of symmetry
- Use lines of symmetry to reflect figures..


## Pedagogical/Orchestration::

- As with the majority of the exercises in this book, pattern development is crucial, especially in reflecting points about the $x$ - and $y$-axes. Encourage your students to look for the reflection and translation patterns and to use them when possible.
- This lesson introduces lines of symmetry and revisits reflections.
- The section asks the student to find and draw lines of symmetry, given several different types of figures and points on the coordinate plane.


## Materials:

Graph paper, Straight edge
Letters handout for Exploration 3 and Flags handout for Exploration 4 both found at the end of the section and on the CD, Paper and scissors

## Activity:

"Transformers!" from the end of the section and on the CD

## Vocabulary:

translation, reflection, lines of symmetry, congruent shapes, diagonal

## TEKS:

7.7(A,B), $7.15(\mathrm{~A}) ; 8.6(\mathrm{~B}) ; 8.7(\mathrm{D}) ; 8.14(\mathrm{D}) ; 8.15(\mathrm{~A}) ; 8.16(\mathrm{~A}) ;$

## WARM-UPS for Section 5.2

1. If the point $(\mathrm{a}, \mathrm{b})$ is in Quadrant II, where is the point ( $-\mathrm{a},-\mathrm{b}$ )? Explain your reasoning.
a. Quadrant I
c. Quadrant III
b. Quadrant II
d. Quadrant IV

Ans: d
2. Plot the points $(1,4)$ and $(4,1)$ on a coordinate system. These points are related in that each has the same coordinates as the other except in reverse order. For each point ( $x, y$ ) below, plot it and then plot the point $(y, x)$. What do you notice?
a. $(6,1)$
b. $(-2,6)$
c. $(5,-1)$
d. $(-1,-4)$
e. $(4,4)$
f. $(-5,-5)$

## Launch for Section 5.2:

Without worrying about a coordinate grid at this time, use the Launch to review vocabulary that will be used in this lesson. Draw a large $P$ on the board and then another $P$ that is above and to the left of the original $P$. Call the new $P, P^{\prime}$ (p prime) so they can get used to the terminology. Ask students, "How would you describe the movement of $P$ to $P^{\prime}$ ?" Students should say things like "it moved up and to the left." Tell students this kind of movement is a "slide" and is known as a translation. On another part of the board draw a horizontal line. Above the line draw a $W$ and the same distance below the line draw an upside down $W$ that you call $W^{\prime}$. Ask students to describe this movement of $W$ to $W^{\prime}$. Hopefully it will come out that the $W$ flipped over the horizontal line. This was not a simple slide. If students have not mentioned it, remind them that this is called a reflection, and the line it flipped over is called the line of symmetry. You can then draw a vertical line on another part of the board and several feet from the line place the letter $B$. Ask students what $B^{\prime}$ should look like after a reflection over this new line of symmetry. See if students are able to direct you to the proper placement and positioning of $B^{\prime}$. Purposefully place $B^{\prime}$ too close to the vertical line so that students can correct you and verbalize that the image has to be the same distance from the line of symmetry but on the opposite side. Tell students that today, they will be performing these kinds of transformations called translations and reflections, but they will be doing the transformations on a coordinate plane. Best of all, if they are observant they will see some neat patterns in the ordered pairs as they do the transformations.

As with all graphing, encourage your students to develop good work habits and to be as precise as possible. You will have students who are chronically out of graph paper or who want to use pen, not erasable pencil. Make sure students give each graph enough room for placing the necessary points. Discuss with your class the best place for the origin. In most cases in this section, the origin should be placed at the center of the graph paper.

## EXPLORATION 1

In Exploration 1, we want the greatest diversity possible in answers.
a. Make sure your students recognize that the point $P$ is in Quadrant IV. If you need to, review the quadrants and their respective places. Also, encourage your students to generalize that in Quadrant IV, the $y$-coordinate is always negative and the $x$-coordinate is always positive. After moving $P$ to $P^{\prime}$ and discussing how it was done, make sure students know that $P^{\prime}$ (read $P$ prime) is in Quadrant I. The notation of $P$ and $P^{\prime}$ may be new to students. This is a fairly common use for pairing an original point $P$ with its transformed or "moved" point $P^{\prime}$, as in this case.
b. Have your students compare their different points $P^{\prime \prime}$ and $P^{\prime \prime \prime}$. Make sure they talk in terms of translating the $x$ - and $y$-coordinates by adding positive or negative numbers. Again, encourage the use of pattern recognition for locating points in the different quadrants without visual help.

## SECTION 5.2 TRANSLATIONS AND REFLECTIONS

Points on the coordinate system can seem very fixed once you locate or place them. However, it is possible to "move" the points around the plane in very systematic ways. For example, the points move to the right by one unit or the points move up by 3 units. Or we move all the points an equal distance across the $y$-axis. You will examine two such ways in this section.

## EXPLORATION 1

On a piece of grid paper, draw the vertical and horizontal axes approximately centered on the paper. Locate and label the point $P(4,-2)$.
a. Move this point to the location $(3,2)$ in the first quadrant with both coordinates as positive integers. Label the new location $P^{\prime}$ (read as $P$ prime) to distinguish that this is where $P$ has moved to. Describe carefully how you did it, using directional terms (for example; up, down, right, left).
b. Now move point $P^{\prime}$ to $(-3,1)$ in the second quadrant and call it $P^{\prime \prime}$ (read as $P$ double prime). Move $P^{\prime \prime}$ to the location $(-1,-3)$ and label it $P^{\prime \prime \prime}$. Restrict the movement from grid points to grid points, not just horizontally or vertically.

With this activity, you can tell whether your students understand translations, using their own points and segments. They should be able to see the same relationship in the translated segments that they do in the translated points.

You probably observed from Exploration 1 that one way to move a point, like $P$, is to shift or slide it by adding to or subtracting from the coordinates. For example, in the figure below, one way to slide $P$ to the first quadrant is to add 6 to the $y$-coordinate. Notice point $S$ is point $P$ translated 6 units up. The translation of $P$ to $S$ adds 6 to the $y$-coordinate. In the coordinate system, translations are described as a specific horizontal motion and vertical motion.


Pick another point of your choosing on the above coordinate system and label it A. Translate A using the same translation rule as the one from P to Q in Exploration 2 and label it B. Now translate the line segment AP from the coordinate system above in the same way you did point P. Draw BQ and explain what you did.

## EXAMPLE 1

Translate triangle $A B C$, written $\triangle A B C$, below using the rule "adding 3 to the $x$-coordinate and subtracting 1 from the $y$-coordinate."


SOLUTION


## EXAMPLE 2

Describe the translation rule if rectangle $A B C D$ translates to $A^{\prime} B^{\prime} C^{\prime} D^{\prime}$ :


## SOLUTION

The translation subtracts 4 from the $x$-coordinate and adds 2 to the $y$-coordinate. Another way to describe this translation is that each point $(x, y)$ is translated to $(x-4, y+2)$.

A translation is a transformation that slides a figure a certain distance along a line in a specified direction. Notice that the original shape and the translated shape are identical except in their location in the plane. Two shapes are congruent if their size and shapes are the same

## EXPLORATION 2

a. The distance from $P$ to the $x$-axis is a vertical distance of 2 . Point $Q$ is $(4,2)$. $Q$ is a vertical distance of 2 above the $x$-axis and directly above $P$. It is identical to $P$ except for the sign of its $y$-coordinate.
b. The distance from $P$ to the $y$-axis is a horizontal distance of 4 . Point $R$ is $(-4,-2)$. $R$ is a horizontal distance of 4 to the left of the $y$-axis and directly left of $P$. It is identical to $P$ except for the sign of its $x$-coordinate.

## EXAMPLE 3

a. Encourage your students to see the pattern with the distances from the $x$-axis and the effect on the $y$-coordinates of each point.
b. Encourage your students to see the pattern with the distances from the $y$-axis and the effect on the $x$-coordinates of each point.
c. This part is a generalization of reflections, independent of the shape.

## EXPLORATION 2

a. Determine the distance from $P(4,-2)$ to the $x$-axis. Locate and label a point $Q$ in the first quadrant with first coordinate 4 that is the same distance from the $x$-axis as the point $P(4,-2)$. Describe how you chose the location for $Q$.
b. Determine the distance from $P$ to the $y$-axis. Move from point $P$ across the $y$-axis to the third quadrant to a new point $R$ that is the same distance away from the $y$-axis as $P$. Describe how you chose the location for $R$.

You can describe the moving of the point $P(4,-2)$ to the point $Q(4,2)$ as a reflection across the $x$-axis. In this case, the $x$-axis is said to be a line of reflection. The point $P(4,-2)$ is reflected across the $y$-axis to the point $R(-4,-2)$

Finally, because the line or axis also acts like a mirror, the movement described in Exploration 2 is called a reflection.

## EXAMPLE 3

Draw a reflection for each of the shapes below across the specified line of reflection.
a. Reflect the triangle below across the $x$-axis


b. Reflect the triangle above across the $y$-axis
c. Reflect the below figure across the $x$-axis.

## SOLUTION

a. Notice that each of the corresponding vertices is the same distance away from the $x$-axis:

b. Notice that each of the corresponding vertices is the same distance away from the $y$-axis:

c.


Let us look at lines of reflection without a coordinate grid. For example, the letter A is symmetric about a vertical line through the point at the top of the letter. If you reflect the left part of the A across this vertical line, you will exactly get the other half of the letter A .


## EXPLORATION 3

An alphabet sheet is included at the end of this section and on the CD.

## EXPLORATION 4

A flag sheet is included at the end of this section and on the CD.

## EXAMPLE 4

Let students attempt to draw the lines of symmetry before they see the solutions at the end of the section.

You can now examine shapes that contain the property of reflection.

## EXPLORATION 3

Examine all the capital letters in the English alphabet. Explore the letters that have lines of symmetry, that is, half of the figure is the mirror image of the other half. For each letter, determine how many lines of symmetry the letter has. Identify and draw these symmetries.

## EXPLORATION 4

Examine flags from different countries. Include the flags of the US, Japan, South Korea, Israel, Mexico, Australia, France, Brazil, Switzerland and Egypt. You may add more, if you wish. Determine which flags have lines of symmetry or symmetries, how many symmetries they have, and where the lines are situated in the figure.

## EXAMPLE 4

Examine the following examples of the indicated shapes to determine whether they have lines of symmetry, how many they have, and where the lines are situated.
a. Equilateral triangle:

d. Parallelogram:

b. Scalene right triangle:

e. Trapezoid:

c. Rectangle: $\qquad$

## EXERCISES

1. a. $(3,3) \rightarrow(5,1)$
b. $\quad(-1,5) \rightarrow(1,3)$
c. $(5,0) \rightarrow(7,-2)$
2. a. $\quad(2,3) \rightarrow(-2,6) \quad$ b. $\quad(-1,-1) \rightarrow(-5,2) \quad$ c. $\quad(2,-1) \rightarrow(-2,2)$
3. a. $(-2,3)$
b. $(-1,-4)$
c. $(2,4)$
d. $(3,-2)$

## SOLUTION

a. The equilateral triangle has 3 lines of symmetry:

b. The scalene right triangle has 0 lines of symmetry:

c. The rectangle has 2 lines of symmetry:

d. The parallelogram has 0 lines of symmetry:

e. The trapezoid has 1 line of symmetry: $\square$

Translations and reflections are two transformations that move shapes to other locations in the plane. When you see two identical shapes in different parts of the plane, ask how the shapes are related and what transformations relate one to the other.

## EXERCISES

1. Plot the following points and translate each by using the rule "add 2 to the $x$-coordinate and subtract 2 from the $y$-coordinate." See TE.
a. $(3,3)$
b. $(-1,5)$
C. $(5,0)$
2. Plot the following points and translate each by using the rule "add -4 to the $x$-coordinate and add 3 to the $y$-coordinate." See TE.
a. $(2,3)$
b.
$(-1,-1)$
c.
$(2,-1)$
3. Plot the following points and reflect each point about the $y$-axis. See TE.
a. $(2,3)$
b. $(1,-4)$
C. $(-2,4)$
d. $(-3,-2)$
4. a. $(2,-3)$
b. $(1,4)$
c. $(-2,-4)$
d. $(-3,2)$
5. a. $A^{\prime}(6,0), B^{\prime}(7,3), C^{\prime}(9,-1)$
b. $\quad A^{\prime}(-4,4), B^{\prime}(-3,7), C^{\prime}(-1,3)$
c. $A^{\prime}(2,-2), B^{\prime}(3,-5), C^{\prime}(5,-1)$
d. $A^{\prime}(-2,2), B^{\prime}(-3,5), C^{\prime}(-5,1)$
6. add 4 to the $x$-coordinate and add -2 to the $y$-coordinate.

This is a good place to point out that the order in which you do a transformation matters.
7. a. $A^{\prime}(2,3), B^{\prime}(4,3), C^{\prime}(4,0)$
b. $A^{\prime}(0,-7), B^{\prime}(-2,-7), C^{\prime}(-2,-4)$
9. Lines of Symmetry: A, B, C, D, E, H, I, K, M, O, T, U, V, W, X, Y No Lines of Symmetr: F, G, J, L, N, P, Q, R, S, Z
10. Make sure to assign and discuss problem 10 with students.
10. a. Reflecting about the $x$-axis reverses the $y$-coordinate, as points are "flipped upside down"
b. Reflecting about the $y$-axis reverses the $x$-coordinate.
c. Two reflections on both axes reverses both coordinates.
4. Reflect the points from Exercise 3 about the $x$-axis. See TE.
5. Draw $\triangle A B C$ with the vertices $A(2,2), B(3,5)$ and $C(5,1)$. See $T E$.
a. Translate $\triangle A B C$ using the rule "adding +4 to the $x$-coordinate and adding -2 to the $y$-coordinate." What are the new vertices?
b. Translate $\Delta A B C$ by translating each point $(x, y)$ to the point $(x-6, y+2)$. What are the new vertices?
c. Reflect $\triangle A B C$ about the $x$-axis. What are the new vertices?
d. Reflect $\triangle A B C$ about the $y$-axis. What are the new vertices?
6. Describe the translation rule that transforms $\triangle A B C$ to $A^{\prime} B^{\prime} C^{\prime}$ where the vertices of $\triangle A B C$ are $A(-3,1), B(0,5)$ and $C(1,-1)$ and the vertices of $\Delta A^{\prime} B^{\prime} C^{\prime}$ are $A^{\prime}(1,-1), B^{\prime}(4,3)$ and $C^{\prime}(5,-3)$. See TE.
7. Draw $\triangle A B C$ with the vertices $A(0,3), B(-2,3)$ and $C(-2,0)$. See TE.
a. Reflect $\triangle A B C$ about the line $x=1$. What are the new vertices? Draw this transformed triangle $A^{\prime} B^{\prime} C^{\prime}$. What do you notice about the relationship between $\triangle A B C$ and $\triangle A^{\prime} B^{\prime} C^{\prime}$ ?
b. Reflect the original $\triangle A B C$ about the line $y=-2$. . Draw and label the new vertices $A^{\prime \prime} B^{\prime \prime} C^{\prime \prime}$.
8. Perform the following transformations.
a. Reflect the point $(3,-2)$ about the $x$-axis then translate the point by adding 5 to the $x$-coordinate and adding 2 to the $y$-coordinate. $(8,4)$
b. Do the two transformations return the point $(3,-2)$ to the same ending point? No.

9 Find 5 capital letters of the alphabet that have lines of symmetry. Find 2 letters that do not have lines of symmetry. See TE.
10. Perform the following transformations. See TE.
a. What transformation reflects the point $(x, y)$ to $(x,-y)$ ? Explain.
b. What transformation reflects the point $(x, y)$ to $(-x, y)$ ? Explain.
c. What transformations reflect the point $(x, y)$ to $(-x,-y)$ ? Explain.
11. a. Students can plot the point $(5,3)$ and use reflections across the $x$-axis and also across the $y$-axis to find other points whose distance from the origin is the same. The lattice points will include $(5,-3),(-5,3)$ and $(-5,-3)$.
11. b. There are other such points, but they do not have integer coordinates. This foreshadows the distance formula and Pythagorean theorem.
12. There are 4 lines of symmetry: 2 side bisectors and 2 diagonals.
13. There are infinitely many lines of symmetry; any diameter is a line of symmetry.
14. a. 5 lines of symmetry
b. 6 lines of symmetry
c. 8 lines of symmetry
15. a. 1 line of symmetry
b. 4 lines of symmetry
c. none
d. none
16. There are 2 lines of symmetry, one vertical and one horizontal.
11. On a coordinate plane,
a. Find 3 points whose distances from the origin are identical to the distance from the point $(5,3)$ to the origin. See TE.
b. Find 3 points whose distances from the origin are identical to the distance from the point $(4,4)$ to the origin. Explain whether it is possible to find more points. $\quad(4,-4),(-4,-4),(-4,4)$; See TE.
12. Draw a square and find all of the lines of symmetry.
13. How many lines of symmetry does a circle have?
14. Determine all the lines of symmetry for the figures below:
a.

b.

c.

15. Identify the figures that have lines of symmetry. For the figures which have symmetry draw the appropriate line(s) of symmetry.
a.

C.

d.

b.


16. Draw all the lines of symmetry for the figure below.


Ingenuity
17. The only possible letters they can use are $\mathrm{A}, \mathrm{H}, \mathrm{I}, \mathrm{M}, \mathrm{O}, \mathrm{T}, \mathrm{V}, \mathrm{W}, \mathrm{X}$, and Y . We don't assume the code "words" they make are real words. There are 11 choices for the first and second letters. The third must be the same as the first, so the number of code words is $(11)(11)(1)=121$.

## Investigation:

18. a. Graph with points $P(4,5), A(-2,4), B(0,7)$, and $C(-4,6)$. Also dot in horizontal line $y=3$. Then label and plot $P^{\prime}(4,1), A^{\prime}(-2,2), B^{\prime}(0,-4)$, and $C^{\prime}(-4,0)$.
d. The vertical distance from the point to the line is the same as the vertical distance from the reflection to the line.

## 17. Ingenuity:

How many 3-letter code words, like "AHA," have a vertical line of symmetry?
For simplicity, assume all the letters are capitalized.
18. Investigation: See TE.
a. Graph the point $P(4,5)$ and the line $y=3$ on a coordinate plane.
b. Find and label a point $P^{\prime}$ that is the reflection of point $(4,5)$ about the line $y=3$.
c. Repeat part $\mathbf{b}$ for the points $A(-2,4), B(0,7)$ and $C(-4,6)$.
d. What does every pair of points have in common?

## TRANSFORMERS



Objective: Students will practice translating an object on a coordinate plane.

## Materials:

Copies of the coordinate plane attached
Copies of the Transformers Worksheet

## Activity Instructions:

1) Make a copy of the coordinate plane and the worksheet, one copy per student. It is best to NOT copy this as a two-sided document.
2) Explain to the students that they will be practicing the skill of translating an object on a coordinate plane. They will need to be very careful when graphing their coordinates in order to answer the ques tions correctly.
3) Recommend that they use a pencil when first working on the worksheet, but if they want to go back and color the picture afterwards, they are more than welcome.

Teacher Edition
Section 5.2 Translations and Reflections


Name $\qquad$

## Transformers Worksheet

On the coordinate plane provided, a basic arrow has been placed on the grid and the direction of the arrow is facing right. The vertices of the arrow are currently on the following coordinates:

Start: $(-6,4),(-6,6),(-2,6),(-2,7),(0,5),(-2,3),(-2,4),(-6,4)$ Finish.

1) Translate this basic arrow across the coordinate plane so that each $x$-value moves to the right 5 spaces and each $y$-value moves up 3 spaces. Record your new coordinates below, and sketch the new arrow on the coordinate plane. Label this basic arrow T1.

2) Using your coordinates from image T 1 , change each so that the value in the x position is now the value in the y position, and the value in the y position is now the value in the x position. For example, if my coordinate was ( $-2,3$ ), then my new coordinate would be (3, -2). Record your new coordinates below, and sketch the new arrow on the coordinate plane. Label this basic arrow T2.

3) Using your coordinates from image T2, change each value to its opposite. For example, if my coordinate was (5, $-4)$, then my new coordinate would be $(-5,4)$. Record your new coordinates below, and sketch the new arrow on the coordinate plane. Label this basic arrow T3.

4) If you have done everything correctly, you should have translated the original arrow so that T1 is facing right, T2 is facing up, and T3 is facing down. What would you need to do to the coordinates from T3 in order to create a new image that is facing left? Use the space below to explain your answer.

Letters Worksheet for 5.2 Exploration 3


427 ()

Flags Worksheet for 5.2 Exploration 4


## Section 5.3 - Functions

## Big Idea:

Develop the idea of function: a one-to-one relationship in which one input value yields only one output value.

## Key Objectives:

- Discover that inputs are first coordinates (from domain), and outputs are second coordinates (from range).
- Use function rule to generate ordered pairs.
- Discover the use of tables to see patterns.


## Materials:

- Sticky Notes
- Handouts of coordinate grids with axes (several per student for notes and exercises)
- Coordinate grid for whole-class demonstration
- Markers or map colors


## Pedagogical/Orchestration:

- Reinforce also that given an input, it generates only one specific output.
- Reinforce that the notation $f(x)$ means: the output that you get when you input $x$.
- There is often some initial confusion between the parenthetical use in multiplication and functions. You need to make sure your students don't think $\mathrm{F}(\mathrm{x})$ is F times x . Remind them that, many times the same word, like "love" means different things on a date and on the tennis court. It all depends on the context-where you are. In the same way, students need to be aware of when they are dealing with functions and when they are dealing with multiplication.


## Activities:

"Functions, Functions, Functions" and "Guess my Function" from the end of the section and on the CD

## Vocabulary:

function, domain, range, notation, rule, table, graph

## TEKS:

6.5; 6.11(A,C); 6.12(A,B); 7.1(C); 7.2(E,F); 7.4(A,B,C); 7.7(A); 7.13(C); 7.14(A); 8.2(A); 8.4(A) ;8.7(D); 8.14(C,D); 8.15(A,B)

## WARM-UPS for Section 5.3 (Functions)

1. Each package of a certain brand of granola bars contains 6 bars.
a. How many bars are there in 4 packages? 9 packages? Ans: 4 packages has 24 bars. 9 packages has 54 bars.
b. How can you organize all the possible pairs of number of packages/number of bars? Ans: You can make a list, either vertical or horizontal. You can make a table or a chart with one column the number of packages and the second column the number of bars in these packages.
c. If a school principal wants to provide each of its 342 students with a granola bar at recess, how many packages will she need to obtain? Ans: Let $\mathbf{x}=$ \# packages need for 342 students. So $6 \mathrm{x}=342$. So, x $=57$ packages of bars.
2. Jacob gets $\$ 14$ allowance each week for cleaning his room and helping around the house and yard. Which of the following is the number of days it will take Jacob to get $\$ 96$ ?
a. 45 days
b. 46 days
c. 47 days
d. 48 days

Ans: $d$ because he gets $\$ 2$ per day and if $2 x=96$, then $x=48$.

## Launch for Section 5.3:

Tell students today they will be creating a machine that performs a function. Ask for two volunteers and have the students stand in front of the class with arms raised and hands touching, "London Bridge" style. These students are the function machine. There will be subjects, called "inputs," that will go into the machine. The machine will perform a function on the inputs and change them in some way. When they leave the machine, the inputs have been transformed and are now an "output." The machine always performs the same function on each input. Ask for volunteers brave enough to go through the function machine. As students line up to go through the function machine, ask audience students to guess what the function of the machine is. Have a large coordinate grid set up to graph the input and output values as coordinate pairs. Give the first input student a sticky note with a 4 on it. As he/she comes out of the machine, switch the 4 with a 12 . Have a student plot the point $(4,12)$ on the grid. Have the input student place the sticky notes with the 4 and 12 on the board under the titles of input and output. Do this with several students plotting the input and output values of $(2,6),(1,3)$ and others on the graph and placing the sticky notes on the table. Guide students to the idea that rearranging the sticky notes so the input values are in numerical order helps in noticing patterns. Students will hopefully guess that the function is to multiply the input by 3. Have a discussion of the arrangement of the points on the grid. Students should notice the linear pattern. Ask students if the pattern they see will continue for all inputs going through the function machine. Tell students, "Pay close attention to the lesson today, and apply what you learned about function machines to the mathematical functions we will discuss."

Have the students write these pairs of numbers, i.e. (3 days, 6 planes) on pieces of paper or cards. Have individual students put these on the board with tape. Don't organize the data for them. After putting up 7-9 pairs, reflect on what they notice. Ask if there is any way to "organize" the information in such a way that could help them see any patterns.

Things to listen for: The table starts with 0 in the first column. The numbers in the first column increase by 1 for the first few slots and then they make some jumps. The numbers in the second column are increasing by 2 for the first few lines.

Teachers, if students are having trouble organizing information, then you might want to re-copy the table on the board so that you can write the corresponding pair notation next to each row in the table. Realize that the end goal is to get students to develop tables themselves. This will help your students to visually link the table and the pairs.

For example:

| Days | Total Number of Planes | Coordinates |
| :---: | :---: | :---: |
| 0 | 0 | $(0,0)$ |
| 1 | 2 | $(1,2)$ |
| 2 | 4 | $(2,4)$ |
| 3 | 6 | $(3,6)$ |
| 5 | 10 | $(5,10)$ |
| 10 | 20 | $(10,20)$ |
| $x$ | $y=2 x$ | $(x, 2 x)$ |

## SECTION 5.3 FUNCTIONS

In our daily lives, we often encounter situations in which we receive a set of instructions and then perform certain tasks based on those instructions. In mathematics, this is the role of a function. A function is a rule that assigns a unique output value to each number in a set of input values.

## EXPLORATION 1

Sarah builds model airplanes. She makes two airplanes each day. How many airplanes will she make in 4 days? 10 days? Organize the information to reveal a pattern in the number of airplanes she makes in a given number of days.

How did you organize information in the exploration above? Do you see a pattern in the number of airplanes she can make in a given number of days?

One way to organize such information is to build a table such as the one to the right. Notice that the first column is the number of days, and the second column is the total number of airplanes that she can make in the corresponding number of days.

What do you notice about this table? Why is this a good way to organize the information? If you are given an input of $x$,

| Days | Total Number of Planes |
| :---: | :--- |
| 0 |  |
| 1 |  |
| 2 |  |
| 3 |  |
| 5 |  |
| 10 |  |
| $x$ | $y=$ | what is an equation for the corresponding output $y$ ?

This is an example of a function. There is a rule, or function, to determine how many planes Sarah has produced based on the number of days she has worked. We can think of a function as a machine with inputs and outputs. The input is the number of days Sarah has worked. The output is the number of planes produced.


Now that we have an idea of what a function does, let's go ahead and make a more formal definition.

## DEFINITION 5.1: FUNCTION

A function is a rule which assigns to each member of a set of inputs, called the domain, a member of a set of outputs, called the range.

Functions are often expressed by equations, like $y=2 x$ from Exploration 1. The input $x$ is called the independent variable, and the output $y$, which depends on the value of $x$, is called the dependent variable. These terms can be a little confusing, since given an output $y$, we could solve for x and find that $x=\frac{y}{2}$. However, this would be like running our function machine backwards. In function form, $y$ traditionally takes the place of the function $f(x)$ and is set equal to an expression involving $x$.

For example, consider again Sarah's function from Exploration 1. The domain is the set of non-negative integers $0,1,2,3, \ldots$ and the range is the set of even non-negative integers $0,2,4,6 \ldots$.


Notice that a function produces pairs of numbers. Let's call Sarah's function $F$. From the table and the picture above, we can see that $(0,0),(1,2),(2,4),(3,6)$, and $(5,10)$ are some of the pairs that belong to the function $F$, where the $x$, or first-coordinate is, the input, and the $y$, or second-coordinate, is the output. In other words, the ordered pairs are of the form (input, output) or ( $x, F(x)$ ). Each

Emphasize that when you write $f(x)$, it is an output for the input $x$. The name of the function is $f$. Finally, take time to discuss the relationship between the inputs and outputs. This is especially powerful in the coordinate notation. If you take the $x$ coordinate and double it, you will get the $y$ coordinate. As students progress through this chapter, keep them looking for patterns between the $x$ and $y$ coordinates because knowing one can help them find the other.

## EXPLORATION 2

| lbs of peanuts | Cost in dollars |
| :---: | :---: |
| 0 | 0 |
| 1 | 4 |
| 2 | 8 |
| 3 | 12 |
| 4 | 16 |
| 10 | 40 |
| 26 | $C(x)=y=4 X$ |
| $x$ |  |

pair of numbers can be thought of as a point on the coordinate system, so we can also talk about the graph of a function. The graph of the function is the pictorial representation of the function.

In mathematics, a notation is a technical system of symbols used to represent unique objects. We can write "The function $F$ pairs the number 1 with 2" symbolically as " $F(1)=2$." We read this as " $F$ of 1 equals 2." This means $F$ sends the input 1 to the output 2 . Note: We write $F(x)$ to denote the value of the output with input $x$. So $F(1)=2$ since $F$ represents 1 to 2 . Be careful, because $F(x)$ is NOT $F \cdot x . F$ is a function, not a variable. Similarly, because the function $F$ pairs the number 2 in our domain with the number 4 in the range to give us the pair $(2,4)$, we write " $F(2)=4$."

So $F(x)=y$ the number of planes that can be produced in $x$ days. We can express this rule in general as $F(x)=2 x$. We also say the function has the equation $y=2 x$.

## EXPLORATION 2:

Juan sells peanuts for $\$ 4$ per pound. Fill in the table below:

| Ibs of peanuts | Cost in dollars |
| :---: | :---: |
| 0 |  |
| 1 |  |
| 2 |  |
| 3 |  |
| 4 | $C(x)=y=$ |
| 10 |  |
| 26 |  |
| $x$ |  |

a. What is an equation for the cost $C$ ? $\quad \mathbf{y}=4 \mathbf{x}$
b. Why is $x=-1$ not an input for this test function? you can't have a amount negative
c. Is $x=1.5$ a possible input? What is the value of $C(1.5)$ ? $\quad$ yes; $\$ 6$
d. What is the domain of this cost function? all positives and 0 .
e. What is the independent variable in the cost table?
f. What is the dependent variable in the cost table?

PROBLEM 1

| Rides | Cost |
| :---: | :---: |
| 0 | 5 |
| 1 | 7 |
| 2 | 9 |
| 3 | 11 |
| $x$ | $2 x+5$ |

## EXERCISES

1. 



For 3 more pairs of points that belong to $F(x)$, answers will vary.
2. a.

| $x$ | $R(x)$ |
| :--- | :--- |
| 0 | $R(0)=0$ |
| 1 | $R(1)=6$ |
| 2 | $R(2)=12$ |
| 3 | $R(3)=18$ |
| 5 | $R(5)=30$ |
| 10 | $R(10)=60$ |
| $x$ | $R(x)=6 x$ |

c. $D:\{a l l$ non-negative integers $\}$

R: \{all non-negative integer multiples of 6\}
b.


## PROBLEM 1

Mary pays $\$ 5$ to get into the carnival. It costs $\$ 2$ per ride. Make a table with the number of rides as inputs and the total cost as the output. What is the equation for the cost function?

## EXERCISES

1. This problem deals with function $F$ from Exploration 1.
a. What is the value of $F(4)$ ? $F(6)$ ?
8, 12.
b. To represent $F$ by a table of values, copy and fill in the adjacent table. See table.
c. Rewrite each row of the adjacent table as an ordered pair, of the form $(x, F(x))$, and plot the corresponding points on a coordinate plane. Produce three more pairs that belong to this list.

| $x$ | $F(x)$ |
| :--- | :--- |
| 0 | $F(0)=0$ |
| 1 | $F(1)=2$ |
| 2 | $F(2)=4$ |
| 3 | $F(3)=6$ |
| 5 | $F(5)=10$ |
| 10 | $F(10)=20$ |
| $x$ | $F(x)=2 x$ | See TE.

d. What is the independent variable in the cost variable?
e. What is the dependent variable in the cost table?
f. What is the relation between the two variables?
2. Sally sells pencils for 6 cents each. If Sally sells 3 pencils, then she earns 18 cents. In other words, Sally's revenue for selling 3 pencils is 18 cents.
a. Let $R(x)=$ Sally's revenue for selling $x$ pencils. Make a table showing $R(0), R(1), R(2), R(3), R(5), R(10)$, and $R(x)$. See TE.
b. Make a list of the first six ordered pairs from part (a), and plot these points on the coordinate plane. $\quad(0,0),(1,6),(2,12),(3,18),(5,30),(10,60)$
c. What are the domain and range for $R(x)$ ? See TE.

d. This is an opportunity to solve this graphically or solve an equation. For example, $4 x+20=44 . \rightarrow 4 x=24$ $\rightarrow \mathrm{x}=6$.
e. $4 \mathrm{x}+20=68 \rightarrow 4 \mathrm{x}=48 \rightarrow \mathrm{x}=48 \div 4 \rightarrow \mathrm{x}=12$
f. This part requires a fractional or decimal answer, so some students may have a bit of trouble with this.
4.
g. ( $\mathrm{x}, \mathrm{H}(\mathrm{x})$ )
$(1,4),(4,13),(6,19),(22,67),\left(3 \frac{2}{3}, 12\right),\left(7 \frac{1}{3}, 23\right)$
3. In order to sell lemonade, Juan paid $\$ 20$ for a sign and a table. It then cost him $\$ 4$ for each gallon of lemonade he made.
a. What was Juan's total cost for making 2 gallons, including the \$20 set-up cost? \$28
b. Complete the adjacent cost table, where $C(x)=$ the total cost to produce $x$ gallons of lemonade. See table.
c. Plot these ordered pairs on the coordinate plane. See TE.
d. Suppose that we do not know how many gallons Juan made, but we

| $x=\#$ of <br> gallons | $C(x)=$ Cost <br> of $x$ gallons |
| :---: | :--- |
| 0 | 20 |
| 1 | $24=20+4$ |
| 2 | $28=20+4+4$ |
| 3 | $32=20+4(3)$ |
| 5 | $40=20+4(5)$ |
| 10 | $60=20+4(10)$ |
| $x$ | $4 x+20$ | know the cost. Can we determine the number of gallons that can be produced at that cost? Complete the adjacent table. See table.

e. How many gallons of lemonade can be made for $\$ 68$ ?
f. How many gallons of lemonade can be made for $\$ 30$ ? 2.5 gallons
4. Consider the function H given by the rule $H(x)=3 x+1$. Compute the following:

| $x=\#$ of <br> gallons | $C(x)=$ Cost <br> of $x$ gallons |
| :--- | :---: |
| 1 | 24 |
| 2 | 28 |
| 3 | 32 |
| 6 | 44 |
| 8 | 52 |
| 10 | 60 |
| 20 | 100 |

a. $H(0)$
1
d. $H(-1)$
$-2$
b. $H(1)$
4
e. $H(5)$
c. $H(2)$

7
f. $\quad H(3)$
g. Complete the table to the right
using the function $H(x)$ as given above.
16
10

| $x$ | $H(x)$ |
| :---: | :---: |
| 1 | 4 |
|  | 7 |
|  | -2 |
|  | 1 |
|  | 16 |
|  | 10 |

5. a. 0
b. 8
c. -16
d. 6
e. -24
f. 18

| $x=$ Time <br> (min.) | $M(x)=$ Number <br> of M\&Ms left |
| :---: | :---: |
| 0 | 9 |
| 1 | 8 |
| 2 | 7 |
| 3 | 6 |
| 4 | 5 |
| 5 | 4 |
| 6 | 3 |
| 7 | 2 |
| 8 | 1 |
| 9 | 0 |

c. $\quad M(x)=9-x$

d. Terry will run out of $M \& M^{\prime} s$ after 9 minutes.
e. $D:\{0,1,2,3,4,5,6,7,8,9\}, R:\{0,1,2,3,4,5,6,7,8,9\}$

Teachers should stress the different representations of multiplication and its distinction with the functional notation. For example, Multiplication notation: $2 \times 3,2 * 3,(2)(3)$ and $2 \cdot 3$. Function notation: $2 \cdot g(x)$ and $2 g(3)$. Or even noting that $g(3)$ is not a multiplication of a variable $g$ with 3 ; though with $3 x$ or $x 3$, with $x$ as a variable, could have been viewed as $3(x)$ or $x(3)$, and think of this as multiplication. BEWARE.
7. j. D: \{all integers\}, R: \{all non-negative even integers\}
5. Consider the function $J$ given by the rule $J(x)=-2 x$. Compute the following:
a. $\mathrm{J}(0)$
d. $J(-3)$
b. $J(-4)$
e. J(12)
c. $J(8)$
f. J(-9)
6. Terry has 9 M\&M's. He eats them very slowly; in fact, he takes 1 minute to eat each one. See TE.
a. Make a table for the number of M\&M's Terry has left after x minutes. Use zero for the starting time.
b. Consider the table of inputs and outputs as a table of points ( $x, y$ ) where $x=$ minutes and $y=M \& M s$ left after $x$ minutes. Graph the points from this table.
c. Find the function $M(x)$ that gives the number of M\&M's Terry has left after $x$ minutes. What do you notice?
d. When will Terry run out of M\&M's?
e. What are the domain and range for $M(x)$ ?
7. Let the function $h$ be defined by $h(x)=x+|x|$ for all integers $x$. Compute the following:
a. $h(0) \quad 0$
b. $h(2) 4$
c. $h(4) 8$ f. $h(-2) \quad 0$
d. $\quad h(-4) \quad 0$
e. $\quad h(-6) \quad 0$
g. Plot points from parts a-f on a coordinate plane.
h. Compute $h(y)$, where $y$ is a positive integer. $2 y$
i. Compute $h(y)$, where $y$ is a negative integer. 0
j. What is the domain and range for $h(x)$ ?
8. Let the functions $f$ and $g$ be defined by $f(x)=x+5$ and $g(x)=4 x$. Compute each of the following:
a. $f(0)+g(0) \quad 5$
b. $f(1)+2 \cdot g(3) \quad 30$
c. $f(2) \cdot g(2) \quad 56$
9. $F(x)=4 x+1$
10. $C(x)=7 x+2$

| Number of Games $(x)$ | Total Cost $C(x)$ |
| :---: | :---: |
| 1 | 9 |
| 2 | 16 |
| 3 | 23 |
| 4 | 30 |
| $x$ | $7 x+2$ |

11. For $f(x): D:\{a l l ~ i n t e g e r s\}, ~ R: ~\{a l l ~ i n t e g e r s ~ g r e a t e r ~ t h a n ~ o r ~ e q u a l ~ t o ~ 5\} ~$ For $g(x)$ : D: \{all integers\}, R: \{all non-negative integers\}
12. Find the rule that can be used to determine the value of any term described in the table below.

| Position | Term |
| :---: | :---: |
| 1 | 5 |
| 2 | 9 |
| 3 | 13 |
| 4 | 17 |
| $x$ | $?$ |

10. Jason is playing miniature golf. He must pay $\$ 2.00$ to rent the putter and $\$ 7.00$ for each game he plays. Fill in the outputs in the following table. Determine a rule to calculate the cost of playing $x$ games, including the putter rental.

| Number of Games $(x)$ | Total Cost $C(x)=y$ |
| :---: | :--- |
| 1 |  |
| 2 |  |
| 3 |  |
| 4 |  |
| $x$ |  |

a. What is the independent variable in the table?
b. What is the dependent variable in the table?
c. What is the relation between the two variables?
11. Suppose that the functions $f$ and $g$ are defined by $f(x)=|x|+5$ and $g(x)=|x+5|$ for all integers $x$. What are the domain and range of $f$ ? What are the domain and range of $g$ ? See TE.
12. Let the function $j$ be defined by $j(x)=x-|x|$ for all integers $x$. Find the domain and range of $j$. $D:\{a l l$ integers\}, $\mathrm{R}:\{$ non-positive even integers\}

## 13. Ingenuity:

a. Write a formula of a function that has domain equal to all integers, and the range is all integers greater than or equal to 4. $\quad F(x)=|x|+4$
b. Write a formula for a function that has domain equal to all integers and, the range is all integers less than or equal to $-7 . \quad F(x)=-|x|-7$

## 14. Investigation 1

Although you don't have to make the decision here, one of the more important issues when considering functional relationships involves which variable causes the change in the other variable. For instance, does the change in temperature cause a change in time, or does the change in time cause a change in temperature? Your students will choose the second of the two possibilities. That determines whether the function is written as $T(x)$ or $X(t)$.
a. Because $x$ is the number of hours after 7 P.M., $x=-3$ means 3 hours before $7 . T(-3)=22$.
b. The $x$-value that corresponds to 11 P.M. is $x=4$. The temperature at 11 P.M. is $1^{\circ} \mathrm{F}$. The relationship between temperature and time using function notation is $T(x)$ where $x$ is the time and temperature varies as a function of time.
c. The temperature $28^{\circ} \mathrm{F}$ is reached at 2 P.M.
d. To use multiplication for (a): $13+(-3)(-3)=22 \quad$ To use multiplication for $(b): 13+(-3)(4)=1$
e. Equation: $T(x)=13+-3 x$

| Time | 1 P.M. | 2 P.M. | 3 P.M. | 4 P.M. | 5 P.M. | 6 P.M. | 7 P.M. | 8 P.M. | 9 P.M. | 10 P.M. | 11 P.M. | 12 P.M. | 1 P.M. |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $x$ | -6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $T(x)$ | $31^{\circ} \mathrm{F}$ | $28^{\circ} \mathrm{F}$ | $25^{\circ} \mathrm{F}$ | $22^{\circ} \mathrm{F}$ | $19^{\circ} \mathrm{F}$ | $16^{\circ} \mathrm{F}$ | $13^{\circ} \mathrm{F}$ | $10^{\circ} \mathrm{F}$ | $7^{\circ} \mathrm{F}$ | $4^{\circ} \mathrm{F}$ | $1^{\circ} \mathrm{F}$ | $-2^{\circ} \mathrm{F}$ | $-5^{\circ} \mathrm{F}$ |

## Investigation 2

15. $g$.


## 14. Investigation 1:

In Peoria, Illinois, the temperature drops an average of $3^{\circ} \mathrm{F}$ per hour over a twelvehour period from 1 P.M. to 1 A.M. The temperature at 7 P.M. is $13^{\circ}$ F. Let $T(x)$ represent the temperature in Peoria $x$ hours after 7 P.M. See TE.
a. What does $x=0$ and $x=-2$ mean in the context of this problem? What is $T(0), T(1)$, and $T(-2)$ ?
b. What $x$-value corresponds to 11 P.M.? What is the temperature at 11 P.M.?
c. Start to make a table of the time $x$ from 1 P.M. to 1 A.M. and the corresponding temperature $T(x)$.
d. Write the relationship between temperature and time using function notation. In other words, find an equation for $T(x)$.
e. When is the temperature equal to $4^{\circ} \mathrm{F}$ ? $28^{\circ} \mathrm{F}$ ?
f. Is it possible to use multiplication to help determine the temperatures in parts $\mathbf{a}$ and $\mathbf{b}$ ? If so, explain how.
15. Investigation 2:

For each positive odd integer $n$, let $s(n)$ be equal to the sum of all the first $n$ odd positive integers. Calculate the following:
a. $s(1)$
1
d. $s(4) \quad 16$
b. $s(2)$
4
e. $s(5) \quad 25$
c. $s(3)$

9
f. What patterns do you notice about this function? Write a formula for the function s. $\quad s(n)=n^{2}$
g. Represent this function in a geometric way that reinforces the pattern. See TE.

## Functions, Functions, Functions



Objective: The students will use their understanding of functions to find the value of a function given one input and its corresponding output.

## Materials:

Functions, Functions, Functions Worksheet (one copy per student)

## Activity Instructions:

1) Divide your class into equal groups. Four students per group would work best, if possible.
2) Make one copy of the function worksheet for each student.
3) Explain to the students that they are to find a function that will work for each table given one input and output per table. The challenge is that each member of the group must have a different function.
4) After discovering a function that works, each student will complete her table with at least 3 more values for each input and output.
5) It is expected that the group will work together, as a whole, even though they each have their own worksheet. If one student in the group opts to take the easy way out and find a simple function that works, she is still responsible for helping her group complete their worksheets.

Name $\qquad$ Date $\qquad$

## FUNCTIONS, FUNCTIONS, FUNCTIONS

Directions: You will need to consider the input and output that is given for each table, and use this information to discover a function that will work for the data given. After finding and recording a function that works, find at least 3 more input and output values for each table using your function.

|  | Table 1 |  |
| :---: | :---: | :---: |
| Input (x) | Function | Output (y) |
| 1 |  | 7 |
|  |  |  |
|  |  |  |
|  |  |  |


|  | Table 2 |  |
| :---: | :---: | :---: |
| Input (x) | Function | Output (y) |
| 2 |  | 11 |
|  |  |  |
|  |  |  |
|  |  |  |


|  | Table 3 |  |
| :---: | :---: | :---: |
| Input (x) | Function | Output (y) |
| 3 |  | 15 |
|  |  |  |
|  |  |  |
|  |  |  |


|  | Table 4 |  |
| :---: | :---: | :---: |
| Input (x) | Function | Output (y) |
| 4 |  | 19 |
|  |  |  |
|  |  |  |
|  |  |  |


|  | Table 5 |  |
| :---: | :---: | :---: |
| Input (x) | Function | Output (y) |
| 5 |  | 11 |
|  |  |  |
|  |  |  |
|  |  |  |

# Guess My Function 



Objective: Students will work in small groups to find patterns in numbers to discover functions. Students will state all functions using function notation.

Materials:
Set of Guess My Function cards
Scratch paper or small dry erase boards
Dice (one die per group)
Activity Instructions:
The class will need to be divided into small groups of 4 to 6 students. Each group will be given a set of the Guess My Function cards and one die. The students in each group will roll the die to see who will go first.
Once it has been established who will go first in each group, the student who goes first will take the top card from the deck. The student will then use the information on this card to give clues to the person to his left. For example, if the card reads, "add five," the student holding the card will tell the player to his left, $f(2)=7$ and $f(0)=5$. (This reads, " $f$ of two equals seven," and " $f$ of zero equals five.") The player on the left will use these clues to try to guess the function. Once the player on the left thinks he knows the function, he will give a third value for $x$. For example, he could state that $f(3)=8$ (read as $f$ of three equals 8). The student holding the card will tell this player whether or not he is correct, but the round isn't over until this player can state the correct function. If the player can state that the function is $\mathrm{f}(\mathrm{x})=$ $x+5$ or $f(x)=5+x$, then that player gets a point.

It might be a good idea to set a time limit on each round to make sure that each player in the group has time to play. A good time limit might be 2 minutes per person. You can either use a sand timer, or have a group member watch the clock. Within this two minute time period, the player that is guessing the function can have unlimited guesses to try to find the correct function.

The game will continue in a clockwise direction until either all Guess My Function cards are used or until the designated time for the game is up. The player with the most points at the end of the game wins.

| Add four $f(x)=x+4$ | Subtract six $f(x)=x-6$ | Multiply by two $f(x)=2 x$ |
| :---: | :---: | :---: |
| Divide by seven $f(x)=\frac{x}{7}$ | Multiply by three, then add five $f(x)=3 x+5$ | The absolute value of $f(x)=\|x\|$ |
| Add three, then divide by two $f(x)=\frac{x+3}{2}$ | Add ten $f(x)=x+10$ | Subtract five $f(x)=x-5$ |


| Multiply by four | Divide by three | Multiply by five, <br> then subtract <br> two |
| :---: | :---: | :---: |
| $f(x)=4 x$ | $f(x)=\frac{x}{3}$ | $f(x)=5 x-2$ |
| Multiply by six | Add twelve | Subtract eight |
|  |  |  |
| $f(x)=6 x$ | $f(x)=x+12$ | $f(x)=x-8$ |
| Add six, then <br> divide by three | Add twenty | Subtract nine |
| $f(x)=\frac{x+6}{3}$ | $f(x)=x+20$ | $f(x)=x-9$ |

## Section 5.4-Graphing Functions

## Big Idea:

Develop a logical process for graphing techniques, and apply these techniques for graphing a straight line.

## Key Objectives:

- Plot points from a function table into a coordinate plane.
- Identify function rules for function tables being graphed.


## Materials:

Graph paper, Class Graph Board, Individual Graph White Boards if available, Graphing Utility if available

## Pedagogical/Orchestration:

Students should have previous experience with the terminology for graphing on coordinate plane (domain, range, input, output, quadrant, axes, function table, etc.).

Make sure students are exposed to the terminology "solve for $y$ in terms of $x$ ".
Throughout the lesson and exercises, make sure students label the lines with the equations.

## Activities:

"Secret Message" at the end of the section and on the CD.

## Vocabulary:

linear equation, function table

## TEKS:

7.4(A, B, C);
7.13(A,C);
7.14(A); 7.15(A,B);
8.4(A);
8.7(D);
8.12(C)
8.13(A, B);
8.14(A, B,C);
8.15(A,B); 8.16(A,B);

## WARM-UPS for Section 5.4

1. Fernando has a cell phone. He pays a flat fee of $\$ 30$ for 400 minutes per month and $\$ 0.20$ for each minute used over 400 minutes. Which of the following would be his bill if he used 437 minutes in a month?
a. $\$ 37$
b. $\$ 37.40$
c. $\$ 37.80$
d. $\$ 38$
2. Suppose $f$ is the function given by the rule $f(x)=y=12-2 x$. Complete the table below:

| Input | Output |
| :---: | :---: |
| -3 | $12-2(-3)=18$ |
| -2 | $12-2(-2)=16$ |
| -1 | $12-2(-1)=14$ |
| 0 | $12-2(0)=12$ |
| 1 | $12-2(1)=10$ |
| 2 | $12-2(2)=8$ |
| 3 | $12-2(3)=6$ |
| 5 | $12-2(5)=2$ |
| 6 | $12-2(6)=0$ |
| 7 | $12-2(7)=-2$ |

## Launch for Section 5.4

This is a good activity for dry erase boards with grids if you have them; if not, grid paper will suffice. For consistency and comparison, have all the students work on the same-sized grid. Tell your students they need to prepare a coordinate grid in which the $x$-values range from -20 to positive 20 and the $y$-values range from -30 to positive 200. Tell them to draw and scale their $x$ and $y$ axes, putting thought into what their intervals should be.

Make sure students realize that the scale for the $x$-axis does not have to be the same as the scale for the $y$-axis. For instance, the $x$-axis intervals can go by 2's whereas the $y$-axis intervals can go by 10 's. Do make sure that students do not change the interval spacing on a given axis and are consistent with their spacing for the whole axis. Have students compare their coordinate plans to each others' planes. More examples can be given if there is time.

Another good example is to give them -2 to +2 on the $x$-axis where they can use fractional intervals and 0 to 100 on the $y$-axis where negative $y$-values are not needed. Ask them what quadrants would be necessary for this grid and have them scale the axes. Tell your students that careful planning and good graphing techniques will help them in today's lesson of graphing lines.

This Exploration is about how to scale the $y$-axis when you plot a collection of points.

1. Inputs $\{0,1,2,3,4,5,6,7,8\}$

Outputs $\{0,5,10,15,20,25,30,35,40\}$
2. Only quadrant 1 is necessary. Check student drawings.
3. No, the values would not fit.
$x$-axis: units of 1
$y$-axis: units of 5

## SECTION 5.4 GRAPHING FUNCTIONS

In section 5.3 we plotted points on a coordinating system where the first coordinate was the input and the second coordinate was its corresponding output. The following exploration will help you develop a set of good graphing techniques.

## EXPLORATION 1: GRAPHING POINTS

To graph the following data, follow a logical process:

| $x$-coordinate | $y$-coordinate |
| :---: | :---: |
| 0 | 0 |
| 1 | 5 |
| 2 | 10 |
| 3 | 15 |
| 4 | 20 |
| 5 | 25 |
| 6 | 30 |
| 7 | 35 |
| 8 | 40 |

1. What are the inputs (domain)? What are the outputs (range)?
2. Based on your answer to question 1 , which quadrants are necessary for the graph? What will your graph look like? Sketch an example.
3. Based on your answer to question 1 , will all the points fit on a $10 \times 10$ graph if the scale is 1 for each axis? If not, what should each graph unit be equal to, on the horizontal axis? on the vertical axis? Notice that the two axes need not have the same scale.
4. Now that you have carefully considered an appropriate scale for the axes, graph the points given in the data.

Now that we can plot points on the coordinate plane, let's look at more complex structures.
A. $(1,3),(3,5)$

Answers may vary. All should lie on the line $y=x+2$
Suppose the points are $(0,2),(1,3),(2,4),(3,5)$ and $(4,6)$
You can recognize the pattern that the $y$-value is always two more than the $x$-value, so $(5,7)$ and $(6,8)$ will also work.
B. Make sure your students' column headings are properly labeled: $x$ and $y$.
C. Order is very important in finding patterns. Make sure your students have checked their work with the line.
D. At this point all of your students, we would hope, had noticed that the $y$ values are always two more than the $x$ values.
E. $(40,42)$ and $(38,40)$
F. This can be written as $(x, x+2)$
$y=x+2$

## EXPLORATION 2

Consider the line below:

A. Identify the coordinates of the two labeled lattice points. Find three other lattice points that are also on the line. Make a list of all five points. How can you find more lattice points on the line without looking back at the graph?
B. One way to organize this list is to make a table. Make a table and fill in the points from your list.
C. Did you put the points in your table in a certain order? Without looking at the graph, find three more points on the line using your table. Check that these points are actually on the line.
D. Describe any patterns you see in the points of this line.
E. What point on the line has 40 as its first coordinate? What point has 40 as its second coordinate?
F. If the point $(x, y)$ is on this line, what is the relationship between $x$ and $y$ ? Write this as an equation.
G. In the graph above, what is the independent variable?
H. What is the dependent variable?

[^0]
## EXPLORATION 3


A. Identify the coordinates of the four labeled lattice points on lines $a$ and $b$. Find three other lattice points that are also on each line. Make a list for each of these two lines. How can you find more lattice points on each line without looking back at the graph?
B. One way to recognize this list is to make a table. Make a table for each line and fill in the points from your list for each line.
C. Did you put the points in your table in a certain order? Without looking at the graph, find three more points on the line using your table. Check that these points are actually on the lines.
D. Describe any patterns you see in the points of these lines.
E. What point on line $a$ has 20 as its first coordinate? What point on line $b$ has -10 as its second coordinate?
F. If the point $(x, y)$ is on line $a$, what is the relationship between $x$ and $y$ ? Write this as an equation.
G. If the point $(x, y)$ is on line $b$, what is the relationship between $x$ and $y$ ? Write this as an equation.
2. This window is not square. One unit horizontally is not the same as one unit vertically. You can use "windows" and \#4 to square the screen (or \#5).
3. You can disable a formula without clearing it. Place the cursor on the " $=$ " in " $y=$ " window. If you hit enter, it will be diabled but not erased. If you hit "graph" button, it will not graph. To turn it back on, repeat the steps. You can have the students practice this procedure with the first 3 equations.
7. You must use the right arrow to move over to that part of the table. Use the left arrow to move back.

## GRAPHING CALCULATOR ACTIVITY

Objective: To explore the graphing calculator as a tool to study lines.
We will use the calculator to graph the lines given by the equations:

$$
y=x \quad y=x+2 \quad y=2 x
$$

1. Turn on the calculator and press the " $\mathbf{y =}=$ " button. Type in each formula using the first 3 slots. That is, the " $y_{1}=$ ", " $y_{2}=$ " and " $y_{3}=$ ".
2. Next, push the "zoom" button on the top row and use the down arrow to move the cursor to \#6. Push "enter". Push "graph". You shuld have three graphs on your screen.
3. What do you notice about these lines? What are the similarities? What are the differences?
4. Explore these lines given by these formulas: $y=2 x+3, y=x+3, y=-x+3$ What do you notice about these lines? What are the similarities? What are the differences?
5. We will now explore the first 3 lines using the table button. Type the equations in again if you cleared them. Be sure to disable or clear the other equations.

6 First push " $2^{\text {nd }}$ " button and then the "tableset" button. Make sure the "tblstart=" slot has the value 0 , which tells the table to start at 0 . Next, make sure the button with a " $\Rightarrow$ tbl=" has the value of 1. This will have the first coordinate column ( " $\mathbf{x}$ ") change by 1 unit as you move down the table.
7. Next, push the "table" button. Notice that these are columns of numbers just like the tables we have made. The first column is called the " $\mathbf{x}$ " column and it contains the first coordinates. The other columns are labeled by " $\mathbf{y}_{\mathbf{1}}=$ "', " $y_{2}=$ " and " $y_{3}=$ ". Using this table, what similarities and differences do you notice about these lines? Compare these observations with what you noticed about their graphs. Explore the table by using the up and down arrow and the right and left arrow. Can you find the values for the line given by the equation in the " $y_{3}=$ " slot?
8. Push the "tableset" button again. Type in the number 0 in the "tblstart=" slot. Now change the $\Leftrightarrow$ tbl" to 0.5 so that the first coordinates will change by only $\frac{1}{2}$ as we move up or down the " $x$ " column.
9. Explore the tables if you push the "tableset" button and reset the " $\Rightarrow \mathrm{tbl}=$ " to 0.1
10. This activity is just to show that slightly change in equation can produce big changes in graphs. Have students compare the takes with graphs. Which one tells you more about the functions?
10. Explore these different formulas that produce curves. For example, $y_{1}=x \cdot x$

$$
=x^{2}, y_{2}=x^{2}+4 x, y_{3}=\frac{1}{x}, y_{4}=x^{3}, y_{5}=x^{4}-4 x, y_{6}=4 x-x^{2}
$$

| $x$ | $y_{1}$ | $y_{2}$ | $y_{3}$ |
| :--- | :--- | :--- | :--- |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
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|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |


| $x$ | $y_{4}$ | $y_{5}$ | $y_{6}$ |
| :--- | :--- | :--- | :--- |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
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|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |

1. a. Answers may vary.
b. Answers may vary. Although students may find themselves completing part (b) before they complete part (a) and may choose the coordinates from the table given.
2. Your students might have trouble with the instruction "write an equation defining $y$ in terms of $x$." The instruction means to solve for $y$, with all $x$ terms on the other side of the equation.

## EXERCISES

Plot the points $(1,2)$ and $(3,4)$ on a coordinate plane. Carefully draw a straight line through these two points. Label this line $L$. Exercises $1-4$ deal with line $L$. Remember that the $x$-coordinate is the first-coordinate. The $y$-coordinate and the second-coordinate are also interchangeable terms.

1. a. Write a list of four more points on line $L$.
b. Include these points in the table to the right and fill in the second coordinates that are missing.
2. a. What point has first coordinate 30? $(30,31)$
b. What point has second coordinate 30 ? $(29,30)$
3. If $(x, y)$ is a point on this line, write an equation defining $y$ in terms of $x . \quad y=x+1$
4. a. If $p$ is the first coordinate of a point on the line, what is the second coordinate of the point in terms of $p$ ? ( $p, p+1$ )
b. If $q$ is the second coordinate of a point on the line, what is the first coordinate of the point in terms of $q$ ? ( $q-1, q)$

| First <br> Coordinate | Second <br> Coordinate |
| :---: | :---: |
| -2 | -1 |
| -1 | 0 |
| 0 | 1 |
| 1 | 2 |
| 2 | 3 |
| 3 | 4 |
| 4 | 5 |
| 5 | 6 |
| 10 | 11 |
| 20 | 21 |
|  |  |
|  |  |
|  |  |
|  |  |

Plot the points $(2,6)$ and $(4,8)$ on a coordinate plane. Draw a straight line through these two points and label this line $M$. We will explore points on line $M$ in Exercises 5-7.
5. a. Write a list of four points on line M. Answers will vary.
b. Expand your list to a table like the one above. Answers will vary.
c. What point has first coordinate 10 ? $(10,14)$
d. What point has second coordinate 24 ? $(20,24)$
6. a. What point has first coordinate $x$ ? $(x, \underline{x+4})$
b. What point has second coordinate $y$ ? $(y-4, y)$
8. Recommend students use a table to organize their information. When using a table or graph, find the coordinate part asked for on the graph and find the other corresponding coordinate on the table or graph. Graph of $y=$ $4 x$.

9.

10. a. Answers may vary, although students may do part b before they do part a and use points from part (b) in their answer to part (a).
b. Answers may vary. But students are likely to complete the chart provided.
11. a. $y=6-x=-x+6$ or $x+y=6$
b. $(12,-6)$
c. $(-4,10)$
7. What is the equation that describes line $M$ ? Write your equation in the same form as your answer to Exercise 3, in terms of a general point $(x, y) . \quad y=x+4$
8. Plot the graph line that satisfies the equation $y=4 x$. For $\mathbf{b}-\mathbf{e}$, how could you find this point using the table, equation, or graph? See TE.
a. What point on the line has 5 as the first coordinate?
b. What point on the line has 24 as the second coordinate?
c. What point on the line has 56 as the second coordinate?
d. What point on the line has 92 as the second coordinate?
e. What point on the line has -36 as the second coordinate?
f. In your graph, what are the independent and dependent variables?
9. Plot the graph line that satisfies the equation $y=-6 x$. See TE.
a. What point on the line has 5 as the first coordinate?
b. What point on the line has -18 as the second coordinate?
c. What point on the line has -42 as the second coordinate?
d. What point on the line has 12 as the second coordinate? $\quad(-2,12)$

Plot the points $(1,5)$ and $(4,2)$ on a coordinate plane. Draw a straight line through these two points and label this line N. Exercises 10 and 11 deal with this line. See TE.
10. a. Write a list of 4 more points that lie on line $N$.
b. Include these points in the table to the right and fill in the second coordinates that are missing.
11. a. Use the pattern you see in the table to write an equation for this line involving a typical point $(x, y)$, solving for $y$.
b. Use this equation to find the point that has 12 as its first coordinate.
c. Use this equation to find the point that has 10 as its second coordinate.

| First <br> Coordinate | Second <br> Coordinate |
| :---: | :---: |
| -2 | 8 |
| -1 | 7 |
| 0 | 6 |
| 1 | 5 |
| 2 | 4 |
| 3 | 3 |
| 4 | 2 |
| 5 | 1 |
| 10 | -4 |
| 20 | -14 |
|  |  |
|  |  |
|  |  |
|  |  |

12. 

| $x$ | -6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $y$ | 18 | 15 | 12 | 9 | 6 | 3 | 0 | -3 | -6 | -9 | -12 | -15 | -18 |

13. Encourage your students to make a table that shows if $x=-2$, then $y=(-3)(-2)=6$. Even though your students have not reviewed order of operations, remind them that in computing $-3 x+13$ for different values for $x$, you multiply -3 by $x$ first and then add 13 .
a.

| $x$ | $y$ |
| :--- | :--- |
| -3 | 22 |
| -2 | 19 |
| -1 | 16 |
| 0 | 13 |
| 1 | 10 |
| 2 | 7 |
| 3 | 4 |
| 4 | 1 |
| 5 | -2 |
| 6 | -5 |


c. You might remind students about how the number multiplied by $x$ affected the graph of $y$ in Section 5.1.
d. $-3 x+13=-17 ;-3 x=-17+-13 ;-3 x=-30 ; x=10$.
14. a.

| $x$ | $y$ |
| :---: | :---: |
| -8 | 28 |
| -7 | 26 |
| -6 | 24 |
| -5 | 22 |
| -4 | 20 |
| -3 | 18 |
| -2 | 16 |
| -1 | 14 |
| 0 | 12 |


| $x$ | $y$ |
| :---: | :---: |
| 1 | 10 |
| 2 | 8 |
| 3 | 6 |
| 4 | 4 |
| 5 | 2 |
| 6 | 0 |
| 7 | -2 |
| 8 | -4 |

b.

c. If $y=30$, then $x=-9$ then $-2 x+12=30 ;-2 x=18$;

$$
x=-9
$$

12. For the line given by the equation $y=-3 x$, make a table of points with first coordinates ranging from -6 to +6 . Plot these points on a coordinate plane. See TE.
13. a. Make a table of points for the line given by the equation $y=-3 x+13$, with first coordinates ranging from -3 to +6 . Show how to compute the second coordinates for these points. See TE.
b. Plot these points on a coordinate plane.
c. What is the difference between the table in this question and the table in problem 10?
d. What is the $x$-coordinate, or the first coordinate, when the $y$-coordinate is $\mathbf{- 1 7}$ ? Is there a way to solve for the $x$-coordinate using the table? If so, how?
14. For the line given by the equation $y=-2 x+12$, See $T E$.
a. Make a table of points with first coordinates ranging from -8 to +8 .
b. Plot these points on a coordinate plane.
c. What is the $x$-coordinate, or the first coordinate, when the $y$-coordinate is 30 ? Show how to solve for the $x$-coordinate.
15. Perform the following transformations.
a. Reflect the line given by the equation $y=3$ about the $x$-axis. This gives the line $y=-3$
b. Take the result from part a and reflect it about the $y$-axis. This gives the same line.
c. Describe what happens.

## 16. Ingenuity:

Heather has $\$ 0.30$ in her pocket in pennies and nickels.
a. Make a table of all the possible combinations of coins Heather could have, with number of pennies, $P$, in the left column and number of nickels, $N$, in the right column. See TE on next page.
b. Plot these points on a graph, using number of pennies $P$ as the first coordinate. Find an equation which relates $N$ and $P$. See $T E$.

## Ingenuity

16. a.

| Number of <br> Pennies | Number of <br> Nickels |
| :---: | :---: |
| 0 | 6 |
| 5 | 5 |
| 10 | 4 |
| 15 | 3 |
| 20 | 2 |
| 25 | 1 |
| 30 | 0 |

b.

18. a.


18b. Answers may vary, but students should indicate a difference in the rate of change at which the lines rise. The coefficient of x causes the differences between the lines.
18. c. The line $y=8 x$ should look very steep. The line $y=0 x$ or $y=0$ is horizontal.

18d. Make sure your students can graph all of the graphs in Exercise 11 in the same graphing window with their graphing utility.
18. Investigation:
a. Draw the lines represented by the equations $y=x, y=2 x$ and $y=4 x$ on the same coordinate plane. See TE.
b. What are the similarities between these lines? What are the differences? What causes these differences? See TE.
c. What do you think the line $y=8 x$ would look like? What about the line $y=0 \cdot x$ or $y=0$ ? See TE.
d. Check your work with your graphing utility.

## SECRET MESSAGE



Objective: The students will complete function tables and use the ordered pairs to graph linear functions on a coordinate plane.

## Materials:

Secret Message Worksheet (one per student)

## Activity Instructions:

1) Make a copy of the Secret Message worksheet for each student.
2) The students will first complete all of the function tables, filling in all missing values.
3) The students will graph five individual line segments, one per table. The table will provide the $x$ value of each line segment's endpoints.
4) When finished, the students should see a secret message.

Name $\qquad$ Date $\qquad$

## SECRET MESSAGE

Directions: You will need to complete each table and fill in all missing values. After you have completed each table, you will use the ordered pairs from each table to make five different line segments. The $x$-values given in each table represent the $x$-coordinate of each line segment's end point. When you are all finished, you will see a secret message that should make you feel more confident about your ability to graph linear functions.

| $x$ | $y=4 x+24$ | $y$ |
| :---: | :---: | :---: |
| -4 |  |  |
|  |  |  |
|  |  |  |
| -8 |  |  |


| $x$ | $y=-3 x-4$ | $y$ |
| :---: | :---: | :---: |
| -4 |  |  |
|  |  |  |
|  |  |  |
| 2 |  |  |


| $x$ | $y=-4$ | $y$ |
| :---: | :---: | :---: |
| -7 |  |  |
|  |  |  |
|  |  |  |
| 3 |  |  |


| $x$ | $x=5$ | $y$ |
| :---: | :---: | :---: |
|  |  | 6 |
|  |  |  |
|  |  |  |
|  |  | -4 |

## Section 5.5 - Applications of Linear Functions

## Big Idea:

Use linear functions in everyday life.

## Key Objective:

Develop and use concepts: Revenue, Expenses, Profit, Break-Even Point

## Pedagogical/Orchestration:

Make sure students realize difference between "revenue" and profit.
Internet Resource:
Jeopardy Game: Percent, Interest, Discount, and Sale Price- http://www.quia.com/cb/55701.html

## Activity:

"DVD" Sales at the end of the section and on the CD

## Vocabulary:

revenue, expenses, cost of production, profit, break-even point

## TEKS:

7.4(A,B,C); 7.13(A,B,C); 7.14(A); 7.15(A,B); 8.3(B); 8.4(A); 8.5(A); 8.7(D); 8.12(C); 8.13(A,B); 8.14(A,B,C,D); 8.15(A,B); 8.16(A,B);

## WARM-UPS for Section 5.5 (Applications of Linear Functions)

1. Suppose we have a function satisfying the following input/output table:

| Input | Output |
| :---: | :---: |
| 8 | 28 |
| 10 | 34 |

Which of the following could be the rule for $f$ ?
a. $y=4 x-4$
c. $y=3 x$
b. $y=4 x-6$
d. $y=3 x+4$
Ans: d
2. Given the following ordered pairs of inputs/outputs, find a rule for these points. Use your rule to fill in the missing outputs. How do you know that your rule works?

| Input | Output |
| :---: | :---: |
| -1 |  |
| 0 | 3 |
| 1 | 5 |
| 2 | 7 |
| 3 | 9 |
| 4 |  |
| 5 |  |
| 6 |  |
| 7 |  |

## Launch for Section 5.5:

This lesson will be about applying linear functions to everyday life. This would be a good time to make an application using an actual situation in your students' lives. Here is an example. "Yearbooks are sold for $\$ 30$ each at our school yearbook booth. You can buy extra year book signature pages for $\$ 0.50$ each. Think about what some possible inputs and outputs would be, if the input is the number of extra pages bought and the output is the total spent at the yearbook booth. As students give you various inputs and outputs write them helter-skelter all over the board with no apparent order. Tell them, "This is really confusing. What would be a good way to organize these inputs and outputs?" Hopefully students will suggest a table. This is a good time to ask them what patterns they are seeing and even to ask for a rule for the total cost, $c$, that would be a function of the number of purchased extra pages, $x$ : $c(x)=30+0.50 x$. Even if students are unable to come up with the rule, tell students that today they will be learning how certain situations can be represented as linear functions, and how we use linear functions in our everyday life. You can always revisit this situation at the end of the lesson and see if your students are then able to come up with the rule.

## EXPLORATION 1

a.

| Input | Outputs |  |  |
| :---: | :---: | :---: | :---: |
| $x$ | $f(x)=y=2 x$ | $g(x)=2 x+2$ | $h(x)=2 x-1$ |
| -1 | -2 | 0 | -3 |
| 0 | 0 | 2 | -1 |
| 1 | 2 | 4 | 1 |
| 2 | 4 | 6 | 3 |
| 3 | 6 | 8 | 5 |

b. Parallel Lines
c. As input increases by 1 , the output increases by 2

Questions

1. $m=2$
2. increase of output with input increase by 1
3. With input $x=0$, the output is $y=m(0)+b=b$. Also, it is where line intersects the $y$-axis.

## PROBLEM 1

Parallel, b. on top, c. in the middle, and a. on bottom

## SECTION 5.5 APPLICATIONS OF LINEAR FUNCTIONS

## EXPLORATION 1

a. Fill in the table for each of these function given in the table below. Plot these prints and draw the lines.
b. What do the graphs of these lines have in common?
c. What do their tables have in common?

| Input | Outputs |  |  |
| :---: | :---: | :---: | :---: |
| $x$ | $f(x)=y=2 x$ | $g(x)=2 x+2$ | $h(x)=2 x-1$ |
| -1 |  |  |  |
| 0 |  |  |  |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |

The rules that produce straight line graphs all have a common form:

$$
y=m x+b
$$

The functions are called linear functions, that is, they have equations of the form $y=m x+b$.

Questions:

1. In Exploration 1 , what is the value of $m$ for each of the lines?
2. What effect does the number $m$ have on the table for each line?
3. What effect does the number $m$ have on the graph for each line?
4. What role does the number $b$ and the $y$-intercept play for each line?

Notice that the value of the number $m$ in each of the linear functions above is the amount of change in the second coordinate when the first coordinate increases by 1 unit. This number $m$ is the constant rate of change of $y$. The value of $m$ determines how fast the value of $y$ increases or decreases in the table and on the graph.

## EXPLORATION 2

a. $y=4 x$

| $\mathbf{x}$ | $\mathbf{y}$ | $\mathbf{y} \div \mathbf{x}$ |
| :---: | :---: | :---: |
| -2 | -8 | 4 |
| -1 | -4 | 4 |
| 0 | 0 | NA |
| 1 | 4 | 4 |
| 2 | 8 | 4 |
| 3 | 12 | 4 |
| 10 | 40 | 4 |

b. Yes, for x not equal 0 , the third column is 4 for all other values of x .4 is the constant of proportionality here.

## PROBLEM 1

Consider the following three linear function:
a. $y=3 x-2$
b. $\quad y=3 x+4$
c. $\quad y=3 x+11$

Predict how their graphs will be related. Use a graphing calculator to check your predictions
The following exercises use the mathematics you have learned and apply them to practical problems. Before you begin, recall the word cost. You may have thought about the cost of a new pair of shoes, or some candy, or even a movie ticket.
People or companies that produce, make, or sell these items also have costs. An important idea that we use in the exercises is that cost can have two parts: a fixed cost and a variable cost. By fixed cost, we mean the part of the cost that does not depend on how many items a company or person produces. Variable cost, on the other hand, is the cost that depends on the number of items the company or person makes. Usually, the more items you produce the more it costs. In the following problems, we consider the Total cost = Variable cost + Fixed cost.
Another important idea is the amount of money that a company or person makes for selling the items. We call this amount the revenue. Keep these ideas in mind as you work on the following problems that include making and selling lemonade and birdhouses.

## EXPLORATION 2

Functions $F$ and $G$ are defined as $y_{1}=F(x)=4 x$ and $y_{2}=G(x)=x+4$.
a. Fill in the table for $F$ and $G$ with $x=\{0,1,2,3,4,5\}$.

| $\boldsymbol{x}$ | $F(x)=y_{1}$ | $\boldsymbol{y} \div \boldsymbol{x}$ | $\mathbf{G}(\boldsymbol{x})=\boldsymbol{y}_{2}$ | $\boldsymbol{y} \div \boldsymbol{x}$ |
| :---: | :---: | :---: | :---: | :---: |
| -2 | -8 |  |  |  |
| -1 | -4 |  |  |  |
| 0 | 0 |  |  |  |
| 1 | 4 |  |  |  |
| 2 | 8 |  |  |  |
| 3 | 12 |  |  |  |
| 10 | 40 |  |  |  |


2. c. Starting on the $y$-axis at 18 , every time you move to the right one unit you also move up one unit vertically. 3. parts $b$ and c involve setting up equations and solving for x , the input: $x+18=30$ and $x+18=46$
b. Is there a pattern for the third column? For the fifth column? Explain.

If a function has a rule in the form $y=K x$, then for any input $x \neq 0$, the quotient of $\frac{y}{x}$ will always have the value $K$. The number $K$ is called the constant of proportionality, and the function is proportional.

## EXERCISES

Sarah opens a lemonade stand. It costs her $\$ 18$ to set up the stand and \$1 for the ingredients to make each gallon of lemonade.

Problems 1 through 6 deal with Sarah's lemonade stand.

1. a. How much does it cost Sarah to make 1 gallon of lemonade? How much does it cost Sarah to make 2 gallons? \$19, \$20
b. Copy and complete the following chart.
c. How many gallons of lemonade can she make for \$18? 0 gallons
2. a. Plot the data from Problem 1 as points on a coordinate plane with the first coordinate representing the number of gallons and the second coordinate representing the cost in dollars. Label the $x$-axis as the "number of gallons"

| Number <br> of Gallons | Cost <br> in \$ |
| :---: | :---: |
| 0 | 18 |
| 1 | 19 |
| 2 | 20 |
| 3 | 21 |
| 4 | 22 |
| 5 | 23 |
| 6 | 24 |
| 7 | 25 |
| 8 | 26 |
| 9 | 27 |
| 10 | 28 | and the $y$-axis as the "cost in $\$$." See TE.

b. What patterns do you notice in the points plotted in part a? They fall in a straight line.
c. How would you describe the way the points appear to someone who cannot see the graph? See TE.
3. a. If she made $x$ gallons, what is her cost $y$ ? Write an equation for the cost $y$ in terms of gallons $x$. What is the constant rate of change of the linear function? $y=x+18$
b. If she has $\$ 30$ to spend on making lemonade, how many gallons can she make? 12 gallons
c. If she has $\$ 46$ to spend on making lemonade, how many gallons can she make? 28 gallons
4. Sarah wants to sell the lemonade for $\$ 3.00$ per gallon. Copy and complete the table below. How much money will she collect selling $x$ gallons of lemonade?

| Number of Glasses | Sales (Revenue) in \$ |
| :---: | :---: |
| 0 | 0.00 |
| 1 | 3.00 |
| 2 | 6.00 |
| 3 | 9.00 |
| 5 | 15.00 |
| 7 | 21.00 |
| 10 | 30.00 |
| 15 | 45.00 |
| 20 | 60.00 |
| 30 | 90.00 |

a. If she sells 10 gallons of lemonade, what is the amount of her sales or revenue? If she sells 20 gallons? 30 gallons? $\$ 30.00, \$ 60.00, \$ 90.00$
b. If she wants her revenue to be $\$ 36$, how many glasses does she need to sell? 12 glasses
c. Describe the relationship between the first and second coordinates of these points.
d. Write an equation for this line, the graph of the revenue function. What is the constant rate of change of the linear function? $\quad y=3 x$
5. Determine whether each of the following linear functions has a constant of proportionality. Explain the differences in the graphs that lead to your decision.
a. $y=3 x$
b. $y=x+1$
c. $y=-2 x$
d. $y=x-2$
7. b. Key: Big dot is for Total cost, Small dot is for Variable cost.

e. $2 x+4=160$
$2 x=156$
$2 x+4=70$
$2 x=66$
$x=78$ birdhouses
$x=33$ birdhouses
6. Sarah wants to compare her cost of production with her revenue. So, to compare, plot these data points on the coordinate plane from Exercise 2. What does each coordinate represent for the new points? Compare the graphs of the cost function with the revenue function. Where does the graph of the revenue function intersect the graph of the cost function? What do the coordinates of this intersection point represent?
7. Rent-It-All charges $\$ 25$ for the deposit on a tool rental plus $\$ 15$ an hour.
a. Write the function of the total cost $y$ in terms of the number of hours $x . \quad y=15 x+25$ What is the constant rate of change of the linear function?
b. Fred and Jane rent a sander to redo their floors. They check the sander out at 10 A.M. and return it at 5 P.M. How much do they pay? \$130

Juan decided to build birdhouses to sell. He made an initial investment of $\$ 4$ to purchase tools. He then went to the store to buy wood to make his birdhouses. It cost him $\$ 2$ for the wood for each birdhouse, after his initial investment. Juan decided to sell his birdhouses for \$3 each. Exercises 8, 9, 10, and 11 deal with Juan's birdhouse business.
8. Begin by making a chart of how much it would cost Juan to make various numbers of birdhouses. There are two types of cost we want to consider: variable costs and fixed costs. The variable cost is the cost of making the birdhouses without counting Juan's initial $\$ 4$ investment. The fixed cost is the $\$ 4$ that Juan spent initially for tools. The total cost is the sum of the variable cost and the fixed cost.
a. Copy and complete the adjacent table.
b. Plot these data for the variable cost as points on a coordinate system with the $x$ coordinate representing the number of birdhouses and the $y$ coordinate representing the cost in dollars. Label the $x$-axis as the "number of birdhouses" and the $y$-axis as the "cost in $\$$. "

| Number <br> of Houses | Variable <br> Cost in \$ | Total <br> Cost in \$ |
| :---: | :---: | :---: |
| 0 | 0 | 4 |
| 1 | 2 | 6 |
| 2 | 4 | 8 |
| 5 | 10 | 14 |
| 10 | 20 | 24 |
| 20 | 40 | 44 |
| 50 | 100 | 104 |
| 100 | 200 | 204 |
| $x$ | $2 x$ | $2 x+4$ |

8. Stress to your students that "revenue" means "sales" and not "profit." To figure profit, we would have to take cost into consideration and subtract it from the revenue or sales, a more difficult proposition.
9. b. $\quad R=3 x$
c.

10. b. Answers will vary. The cost and revenue are equal or even, hence the phrase break-even point.
11. c. The first coordinate represents the number of birdhouses that need to be sold for the cost to be equal to the revenue.
c. Now plot the points that show the total cost on the same coordinate plane.
d. Write an equation for the total cost, $T$, to build $x$ birdhouses. Is $T$ a linear function? What is $m$ ? What is $b$ ? $\quad T=2 x+4 ;$ Yes; $m=2, b=4$
e. How many bird houses can Juan build for $\$ 160$ ? $\$ 70$
12. a. Make a table of revenue collected for selling different quantities of birdhouses. Complete the table below.

| Number of Houses | Revenue $R$ in \$ <br> $y=R(x)$ | $y \div x$ |
| :---: | :---: | :---: |
| 0 |  |  |
| 1 |  |  |
| 2 |  |  |
| 5 |  |  |
| 10 |  |  |
| 20 |  |  |
| 50 |  |  |
| 100 |  |  |
| $x$ |  |  |

b. Write an equation involving $R$ and $x$, where $R$ is the revenue from selling $x$ birdhouses. Is $R$ a linear function? What is $m$ ? What is $b$ ? Does $R$ have a constant of proportionality?
c. Graph your revenue line on the coordinate plane you used in the previous problem.
10. a. Now consider the total cost and the revenue. At what point do the two graphs intersect? $(4,12)$
b. The point where the cost and revenue intersect is called the break-even point. Why do you think it's called this? See TE.
c. What does this point represent? See TE.
d. Can you find the intersection of the two lines algebraically?
11. How is Juan's profit represented on the graph? See TE.
10. Juan's profit is represented by the vertical difference between the revenue line and the total cost line.

Profit $=$ Revenue $-\operatorname{Cost} . P(x)=R(x)-C(x)=3 x-(2 x-4) ; P(x)=x-4$
11. a. yes; 4 d. yes; -3

## Ingenuity

12. If the students graph the lines, they should be able to determine the point of intersection to be $(3,21)$. Since the third line must also have this point of intersection, the point $(3,21)$ must satisfy the equation, $y=3 x+k$. The equation becomes $21=3(3)+k$ and $k=12$.
13. For each of the following linear functions, determine the $y$-intercept and if there is a constant of proportionality, what is its value?
a. $y=4 x$
b. $y=4 x+3$
c. $y=6 x-2$
d. $y=-3 x$
14. Ingenuity:

The three lines represented by the equations $y=6 x+3, y=-x+24$, and $y=3 x+k$ intersect at the same point. What is the value of $k$ ? 12

## 14. Investigation:

In mathematics, a palindrome is a number that reads the same backward or forward. For example, 12321, 5665 and 11 are all palindromes.
a. Find all the two digit palindromes. How many are there? 9
b. How many three digit palindromes are there? Try to list them, using a pattern. 90
c. How many four digit palindromes are there? 90. Following the pattern in part (b), 1001, 2002, 3003, ..., 1111, 2112, 3113, ..., 1221, 2222, $3223, \ldots$, there will be the same number of four digit palindromes as there are three digit palindromes.


Objective: To reinforce the students ability to solve an application problem involving linear functions.

## Materials:

Graph Paper
Notebook Paper
Overhead transparency of the problem

Activity Instructions: The students will work in small groups to figure out the solution to the problem. Each group will need to provide a graphical representation of the linear functions and an answer to each part of each question.

## Applications of Linear Functions - Cost, Revenue, Profit

$$
\begin{aligned}
& P(x)=\text { Profit from sales of } x \text { units } \\
& R(x)=\text { Total revenue from sales of } x \text { units } \\
& C(x)=\text { Total Cost of producing } x \text { units } \\
& \text { Profit }=\text { Revenue }- \text { Cost }
\end{aligned}
$$

1) A company manufactures DVD players and sells them for $\$ 80$ each. The fixed costs of producing the DVD players are $\$ 30,000$ plus a variable cost of $\$ 20$ for each DVD player produced.
a) Find the cost and revenue functions.
b) Find the profit function. What is the profit if 10,000 DVD players are sold?
c) Find the break even point.

Teacher Edition
Section 5.5 Applications of Linear Functions

|  |  |  |  |  |  |  |  |  |  |  |  | T |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
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## Section 5.6 - Patterns and Sequences

## Big Idea:

To develop a sequence or pattern of numbers as a function with inputs

## Key Objective:

- Identify patterns in arithmetic sequences
- Write an expression to find the $\mathrm{n}^{\text {th }}$ term
- Identify the expression when given terms in a sequence


## Materials:

Paper/Pencil

## Pedagogical/Orchestration:

- Students need to work on paper


## Activity:

Explorations offer opportunities for group work and good class discussion

## Vocabulary:

sequence, arithmetic sequences, position, term, geometric sequences

## TEKS:

7.4 (C); 7.2 (C); 8.5(B)

## WARM-UPS for Section 5.6 (Patterns and Sequences)

1. For each of the following lists of number, predict what the next three members could be. For each list, explain how you made your prediction.
a. $4,8,12,16$, $\qquad$ , __ , -_

Ans: 20, 24, 28 : adding 4
b. $5,9,13,17$, $\qquad$ , $\qquad$ , Ans: 21, 25, 29 : adding 4
c. $4,8,16,32$,__, , __, , -

## Ans: 64, 128, 256 : multiplying by 4

d. $1,2,4,7,11,16$, $\qquad$ , - , $\qquad$ Ans: 22, 29, 37 : at each step, you add a number that is one more than at the previous step. The general formula is $a(n)=a(n-1)+n-1$
2. a. At a weather station in the Arctic, the temperature at sundown is $14^{\circ} \mathrm{F}$. If the temperature drops by $4^{\circ} \mathrm{F}$ per hour, what will be the temperature 10 hours after sundown? Write a rule for the temperature x hours after sundown.
a. $54^{\circ} \mathrm{F}$
b. $-10^{\circ} \mathrm{F}$
c. $-26^{\circ} \mathrm{F}$
d. $-24^{\circ} \mathrm{F}$

Ans: c because $14-4(10)=14-40=-26^{\circ} \mathrm{F}$
b. Write a rule for the temperature $x$ hours after sundown.

Ans: The temperature after x hours is given by $\mathrm{y}=14 \mathbf{- 4 x}$.

## Launch for Section 5.6:

For the following lists of numbers, determine what you think is the next number in the list. Explain how you made your choice.
a. $4,8,12,16$, $\qquad$
b. $1,3,5,7, \ldots, \ldots$
c. $3,6,12,24$, $\qquad$

For sequences, it is a tradition to use n as the input (instead of x ).

## SECTION 5.6 PATTERNS AND SEQUENCES

There are many areas of mathematics in which we study lists of numbers like those in Exploration 1. We call these lists of numbers sequences. Looking at the table of a function where the inputs start at 0 or 1 and increase by 1 unit, the second column is a list of outputs which can be written horizontally in a list. We have looked for patterns in these lists of outcomes in order to find the rule for the function. When the inputs of a function are restricted to positive integers or non-negative integers, the function is called a sequence. Most of the sequences that we will study have a pattern that can be described by a rule.

## EXPLORATION 1

For the following list of numbers, determine what you think is the next number in the list. Explain how you made your choice.
a. $\quad 4,8,12,16$, $\qquad$
b. $\quad 1,3,5,7, \ldots, \ldots$
c. $3,6,12,24, \ldots, \ldots$

## EXAMPLE 1

Consider the sequence defined by the rule that for each input n , representing the place in the list, the output is given by $\mathrm{y}=2 \mathrm{n}$. Making a horizontal table of outputs, we get

| Input | 1 | 2 | 3 | 4 | 5 | 6 | 10 | 25 | n |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Output | 2 | 4 | 6 | 8 |  |  |  |  | 2 n |

Thus the sequence can also be written as the list: $2,4,6,8, \ldots$. Each member of a sequence is called a term of the sequence. The three dots at the end of the list means that the sequence continues for all positive integers. It is common to
think of the first term in the list as the output you get with input of $\mathrm{n}=1$. If we use function notation, then the first term is denoted by a(1) or $\mathrm{a}_{1}$. Organizing the sequence vertically, we get

$$
\begin{aligned}
& a(1)=a_{1}=2 \\
& a(2)=a_{2}=4 \\
& a(3)=a_{3}=6 \\
& a(4)=a_{4}=8 \\
& a(n)=a_{n}=2 n
\end{aligned}
$$

Another way to represent the information above is:

| $a_{1}$ | $a_{2}$ | $a_{3}$ | $a_{4}$ | $a_{n}$ |
| :---: | :---: | :---: | :---: | :---: |
| 2 | 4 | 6 | 8 | $2 n$ |

The input tell us the position of the output in the list. Thus, $\mathrm{a}(3)=6$ means that the third number in the sequence is 6 . Some typical questions about this sequence are:
a. What are the next 3 terms? That is, what are $a_{5}, a_{6}$, and $\mathrm{a}_{7}$ ? The answers are 10,12 , and 14 .
b. What is the value of the 25 th term of the sequence, that is, what is $\mathrm{a}_{25}$ ? The answer is $a(25)=a_{25}=2 \cdot 25$.
c. What is the $n^{\text {th }}$ member of the sequence? This is another way of asking what the rule for the sequence is. The answer is $a(n)=a_{n}=2 n$.

Of course the question in part c is obvious because we gave the rule in the beginning. But we are often given a sequence of a list of numbers with the first 3 to 5 terms in the list. In this case, the question in part c can be challenging.

You probably noticed that given a term of this sequence, you can determine the next term by adding 2 . In other words, we can write

## PROBLEM 1

a. $a(5)=15, a(6)=18 . c=3$
b. $\quad a(5)=16, a(6)=19 . c=3$
c. $a(6)=16, a(7)=20 . c=4$
d. $\quad a(9)=14$. Not an arithmetic sequence.

## Method 1

$$
\begin{gathered}
a_{2}=a_{1}+2=4=2 \cdot 2 \\
a_{3}=a_{2}+2=6=2+2+2=2 \cdot 3 \\
a_{4}=a_{3}+2=8=2+2+2+2=2 \cdot 4 \\
a_{5}=2 \cdot 5 \\
\text { and so on. }
\end{gathered}
$$

## Method 2



DEFINITION 5.2: ARITHMETIC SEQUENCE
A sequence $a_{1}, a_{2^{\prime}}, a_{3^{\prime}} a_{4^{\prime}} \ldots$ is an arithmetic sequence if there is a number $c$ such that for each positive integer $n, a_{n+1}=a_{n}+c$, that is, $a_{n+1}-a_{n}=c$.

Thus, our sequence above is an arithmetic sequence because for each positive integer $n, a(n+1)=a(n)+2$.

## PROBLEM 1

Based on the portion of the sequences shown below, write the next two terms. Use either method 1 or method 2 from above to determine which of the sequences is an arithmetic sequence and, if it is, what is the constant difference between any two consecutive terms.

PROBLEM 2
b. $\quad a_{n}=a(n)=4+3(n-1)=3 n+1$
c. $\quad a_{n}=a(n)=3 n$
a. $\quad 3,6,9,12, \ldots$
b. $\quad 4,7,10,13, \ldots$
c. $\quad 2,6,10,14,18, \ldots$
d. $2,4,5,7,8,10,11,13, \ldots$

How do we determine a rule or formula for the sequence in part a. in Problem 1? One way to discover the rule is to list the sequence carefully and look for a pattern.

For the sequence $3,6,9,12, \ldots$, we make a vertical list:

$$
\begin{aligned}
& a_{1}=3 \\
& a_{2}=3+3=(2) 3=3(2) \\
& a_{3}=3+3+3=3(3) \\
& a_{4}=3+3+3+3=3(4)
\end{aligned}
$$

What is $a_{10}$ ? The answer is $a_{10}=3(10)$. So, what is the rule for $a_{n}$ ? The answer is $a_{n}=3 n$.

You can also see this patterns using a table:

| Input | Output |
| :--- | :--- |
| 1 | $\mathrm{a}_{1}=3$ |
| 2 | $\mathrm{a}_{2}=3+3=(2) 3=3(2)$ |
| 3 | $\mathrm{a}_{3}=3+3+3=3(3)$ |
| 4 | $\mathrm{a}_{4}=3+3+3+3=3(4)$ |
| 10 | $\mathrm{a}_{10}=3(10)$ |
| $n$ | $\mathrm{a}_{\mathrm{n}}=3 \mathrm{n}$ |

## PROBLEM 2

Determine a rule for each of the sequences in parts b. and c. from Problem 1.

PROBLEM 3


Design 4

| Design number | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Total blocks | 7 | 12 | 17 | 22 | 27 | 32 | 37 |

$B(x)=5 x+2$
1.
a. $5,7,9,11,13$
b. $2,5,8,11,14$
c. $2,6,10,14,18$

## PROBLEM 3

Use the pattern blocks to make the next two designs. Sketch the designs and record your work in the table on your answer sheet. Write a function that describes the sequence.


| Design Number | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Total Blocks | 7 | 12 |  |  |  |  |  |

## EXERCISES

1. For each of the following rules for a function, write out the first 5 terms of the corresponding sequence.
a. $\quad a_{n}=a(n)=2 n+3$
b. $\quad a_{n}=a(n)=3 n-1$
c. $\quad a_{n}=a(n)=4 n-2$
2. 

$$
\begin{array}{ll}
\text { a. } & a(5)=13, a(6)=16 . c=3 \\
\text { b. } & a(5)=23, a(6)=28 . c=5 \\
\text { c. } & a(6)=32, a(7)=38 . c=6
\end{array}
$$

3. a. $\quad a(n)=5 n$
b. $\quad a(n)=9+5(n-1)=5 n+4$
c. $\quad a(n)=5+2(n-1)=2 n+3$
d. $a(n)=1+4 n=4 n+1$
4. a. $1,3,5,7,9$
b. $\quad S_{n}=1+2(n-1)=2 n-1$
c. $\quad T_{n}=4,10,16, \ldots$
5. For each of the following sequences, write the next two terms. Use either method 1 or method 2 from above to determine which of the sequences is an arithmetic sequence and, if it is, what is the constant difference between any two consecutive terms.
a. $3,6,9,12,15 \ldots$
b. $7,15,21,28, \ldots$
c. $-4,-8,-12,-16, \ldots$
d. $1,4,7,10, \ldots$
e. $3,8,13,18, \ldots$
f. $2,8,14,20,26, \ldots$
g. Determine a rule for the $\mathrm{n}^{\text {th }}$ term for each of these sequences.
6. For each of the following sequences, write the next 2 terms. Determine a rule for each sequence.
a. $5,10,15,20, \ldots$
b. $9,14,19,24, \ldots$
c. $5,7,9,11, \ldots$
d. $5,9,13,17, \ldots$
7. Suppose you make the following sequence of figures with tooth picks. Let $S_{n}$ be the number of squares at the $\mathrm{n}^{\text {th }}$ stage.

a. Write out the first 5 terms of this sequence.
b. Write a formula for $S_{n}$.
c. Write a sequence for the number of tooth picks $\mathrm{T}_{\mathrm{n}}$ used at the $\mathrm{n}^{\text {th }}$ stage.

## 5. a. $1,4,9,16,25$

b. $\quad S_{n}=n^{2}$
c. $\quad T_{n}=4 n$
$T_{n}=4,8,12,16, \ldots$
5. Suppose you make the following sequence of figures with tooth picks. Let $S_{n}$ be the number of squares at the $\mathrm{n}^{\text {th }}$ stage.
$\square$

a. Write out the first 5 terms of this sequence.
b. Write a formula for $S_{n}$.
c. Write a sequence $\mathrm{P}_{\mathrm{n}}$ for the number of tooth picks on the outside (perimeter).
d. Write a formula for $P_{n}$.
1.
A. Quadrant I
B. Quadrant IV
C. Quadrant I
D. Quadrant II
E. Quadrant III
F. $x$-axis

2.
a.

d. the $y$ value goes up by 3 every time the $x$ value increases by 1
e. $y=3 x+2$
f. $(11,35)$
g. $(-5,-13)$
3.
a. 7
b. -1
c. 1
d. 9
e. 3
f. -3
9.

## REVIEW PROBLEMS

1. Plot the following points on a coordinate plane and determine their quadrants.
A (1,5)
$B(4.5,-4)$
$C(3,11)$
$D(-2,0.5)$
$E(-6,-2)$
$F(1,0)$
2. a. On the another coordinate plane, draw the line that passes through points $A(1,5)$ and $C(3,11)$.
b. Label 3 other points on the line.
c. Make a chart of the points below that lie on the same line by filling in the table below.
d. Using the table, describe the pattern in the points.
e. Write an equation for this line in terms of $x$, solving for $y$.
f. Find the point that has 11 as its $x$-coordinate.
g. Find the point that has -13 as its $y$-coordinate.

| $x$ | $y$ |
| :---: | :---: |
| -5 | -13 |
| -3 | -7 |
| -1 | -1 |
| 0 | 2 |
| 1 | 5 |
| 2 | 8 |
| 3 | 11 |
| 4 | 14 |
| 5 | 17 |
| 10 | 32 |
| 20 | 62 |
| $P$ | $3 p+2$ |

3. Function $g$ is defined as $g(x)=2 x+5$. Compute the following:
a. $g(1)$
b. $\quad g(-3)$
c. $\quad g(-2)$
d. $g(2)$
e. $g(-1)$
f. $g(-4)$
g. Graph all the ordered pairs $(x, g(x))$ from parts a through $f$.
4. a. Answers may vary
b. $\quad T(x)=3 x+6$
C.

g. Domain: zero and all the positive integers, Range: all integers greater than or equal to 6
5. a. $c(x)=x+14$
e.

b.

d. $x=16$
f. $(7,21)$;This is the point where total cost is the same as revenue. After this point he begins to make a profit.
6. a. 0
b. 0
c. 0
d. 6
e. 12
7. Manuel decided to build jewelry boxes. He purchased tools for $\$ 6$. He went to the hardware store for wood for the jewelry boxes. After the initial investment, Manuel discovered it cost him $\$ 3$ for the wood necessary to make each jewelry box. So he decided to sell each jewelry box for $\$ 5$.
a. Make a table for the variable $x$, the number of jewelry boxes, and the total cost $T(x)$ in dollars. Fill in three values for $x$ and $T(x)$.
b. Write a rule that computes the total cost, $T(x)$, for making $x$ jewelry boxes.
c. Plot the data for the total cost on a coordinate system.
d. Make a second table for revenue collected from selling jewelry boxes.
e. Write a rule that computes the revenue $R(x)$ for selling $x$ jewelry boxes.
f. Plot the data for revenue earned on the same coordinate system as part (c). At what point do they intersect?
g. What is the domain and range of $T(x)$ ?
8. Victor decides to make cakes. He invests $\$ 14$ for startup supplies. He went to the store to buy ingredients for the cakes. After the initial investment, Victor discovers it cost him $\$ 1$ to make each cake. He decides to sell each cake for \$3.
a. Write a rule for the total cost to make $x$ items.
b. Write a rule for the revenue earned by selling $x$ items.
c. How many cakes can he make for a cost of $\$ 30$ ? $x=16$
d. How many cakes does he need to make to produce a revenue of $\$ 48$ ?
e. Plot the total cost and revenue.
f. Where do the 2 graphs intersect? What does the point of intersection mean in the problem?
9. Let the function $f$ be defined by $f(x)=|x|-x$ for all integers $x$. Compute the following:
a. $f(0)$
b. $f(4)$
c. $f(1)$
d. $f(-3)$
e. $f(-6)$
10. a. $y=55 x+75$
b. 405
c. $55 \mathrm{x}+75=625$
$55 x=550$
$x=10$ hours
11. 



10. 5,4 , and 1 line(s) of symmetry respectively.
11. a. $6,7,8,9,10, \ldots$
b. $1,4,7,10,13, \ldots$
c. $13,18,23,28,33, \ldots$
12. a.
..., 16, 19
b. ..., 17, 20
c. ..., 20, 23
$a_{1}=4=1+3$
$a_{2}=7=1+3+3$
$a_{3}=10=1+3+3+3$
$a_{1}=2$
$a_{2}=2+3$
$a_{3}=2+3+3$
$a_{4}=2+3(3)$
$a_{5}=2+3(4)$
$a_{6}=2+3 n-3$
$a_{n}=3 n-1$
7. Logan rents a boat from Rent-lt Boats. They charge $\$ 55$ per hour with a deposit of \$75.
a. Write the function of the cost, $y$, in terms of the number of hours $x$.
b. Logan goes deep-sea fishing from 8 A.M. to 2 P.M. How much does he pay to rent the boat?
c. If they are charged $\$ 625$, how long did they use the boat?
8. Plot the following points and translate each by using the rule "add 5 to the $x$-coordinate and add -2 to the y-coordinate."
a. $(4,4)$
b. $(0,4)$
c. $(-4,1)$
9. Plot the following points and reflect each point about the $x$-axis.
a. $(4,4)$
b. $(3,-6)$
C. $(-1,4)$
d. $(-2,-2)$
10. Determine all lines of symmetry for the figures below.

11. For each of the following rules for a function, write out the first 5 terms of the corresponding sequence.
a. $\quad a_{n}=a(n)=n+5$
b. $\quad a_{n}=a(n)=3 n-2$
c. $\quad a_{n}=a(n)=5 n+8$
12. For each of the following sequences, write the next 2 terms and determine a rule for each.
a. $4,7,10,13, \ldots$
b. $2,5,8,11,14, \ldots$
C. $6,10,14, \ldots$

Section 5.1: 35
Solution: The grid below shows how many paths there are to each point. Note that we just add the number of ways to the point below and the point to the left, starting with 1 way at $(0,0)$.


In general there are $\binom{a+b}{b}=\frac{(a+b) \text { ! }}{a!b!}$ paths from the origin to $(a, b)$, since we take $a+b$ steps and can choose any $b$ of them to be vertical.

## Section 5.2: 204

Solution: There are $4^{4}=256$ functions from $S$ to $S$, since there are 4 choices for the image of each of $1,2,3$, and 4. There are 4 functions that map all four to the same value. If we map three to the same value, then there are 4 choices for the three (or choices for the remaining one), 4 values they can map to, and 3 choices for the image of the remaining number, for a total of 48. That leaves 256-4-48=204.

Alternatively, there are $4 \cdot 3 \cdot 2 \cdot 1=24$ functions that map every number to a different range value, $6 \cdot \frac{4 \cdot 3}{2}=36$ that map two numbers to one value and two to another ( 6 ways to choose 2 ), and $6 \cdot 4 \cdot 3 \cdot 2=144$ that map two to one value and the others to different values. That gives a total of $24+36+144=204$.
Section 5.3: 84
Solution: Let's count the paths that start with a step to the right, which by symmetry will be half of the total. From the grid below, we see there are 42 such paths, for a total of 84 .


## CHALLENGE PROBLEMS

## Section 5.1:

Alan, an ant, starts at the origin in the coordinate plane. Every minute he can crawl one unit to the right or one unit up, thus increasing one of his coordinates by 1 . How many different paths can Alan take to the point $(4,3)$ ?

## Section 5.2:

Let $f$ be a function whose domain is $S=\{1,2,3,4\}$ and whose range is contained in S . How many such functions do not map more than two domain values to the same range value?

## Section 5.3:

Alan, an ant, is back at the origin. Every minute he can crawl one unit to the right or one unit up, but after the first step he will not cross the line $y=x$ (he can touch the line, though). How many such paths can Alan take to the point $(5,5)$ ?

## Section 5.4:

Phone company A charges 23 cents per minute. Company B charges 11 cents per minute with a $\$ 10$ monthly fee. Company $C$ charges a flat rate of $\$ 30$ per month with unlimited calls. What are the minimum and maximum number of minutes you can use each month such that company $B$ will be the cheapest?

Answer: 84 and 181
Solution: Let x be the number of minutes used. Company A is the cheapest for 0 minutes. It catches up to Company B when $23 x=11 x+1000$, or $x=88 \frac{1}{3}$, so Company $B$ becomes cheaper at 84 minutes. It catches up to Company $C$ when $11 x+1000=3000$, or $x=181 \frac{9}{11}$, so Company B is only cheapest up to 181 minutes.

## Section 6.1 - Decimals

## Big Ideas:

- Comparing and order decimals on a number line
- Adding and subtracting decimals


## Key Objectives:

- Review place value.
- Transfer integer understanding of the number line to decimals.
- Review decimals and extend to the negatives.
- Connect adding negative decimals and subtracting positive decimals.


## Pedagogical/Orchestration:

- Exploration is good practice on constructing number lines for decimal values.
- Use "Shopping Trip" activity at end of section.
- Paragraph just before Exercises is foreshadowing scientific notation.


## Materials:

Number line handout from the CD, Graph paper, Rulers

## Activity:

"Shopping Trip" at end of section and on the CD
"Place Value Spinning" at end of section and on the CD
"Place Value Die" at end of section and on the CD

## Exercises:

Exercise 3 uses multiplication and division to convert measurement units.
Exercise 7 connection to Chapter 3.
Exercises 9, 10, 11, 12 connection to Chapter 4.
Students could work in groups to create poster size solutions of Exercises 9 \& 10, graphing both lines on the same grid. They can compare and talk about the steepness of the lines for each exercise.

## Vocabulary:

denomination, place value, estimate

## TEKS:



## WARM-UPS for Section 6.1

1. Which of the following numbers is closest in distance to the number . 496 ?
a. 0.5
b. 0.502
c. 0.49
d. 0.489

Ans: a with the distance of .004
2. a. Is the number 1.6 closer to 1 or 2? Explain your reasoning. Ans: 2
b. Is the number 4.71 closer to 4.5 or 4.95 ? Explain your reasoning. Ans: 4.5
c. Is the number 0.209 closer to 0.2 or 0.25 ? How much closer? Ans: The distance from 0.209 to 0.2 is 0.009 and from 0.209 to 0.25 is 0.041 . So, 0.2 is closer.

## Launch for Section 6.1:

Ask your students if they know the definition of a digit. Listen to responses and make sure that they understand that a digit is a numeral in the decimal number system. Ask the students, "How many digits are in this number system?" They have many misconceptions about this, so make sure they understand there are exactly 10 digits in our number system: 0-9. Ask students, "Why do you think that 10 digits were chosen for our number system base? Why not 8, or 11 ? (You can even show them how counting in a Base 8 system would look: $0,1,2,3,4,5,6,7,10,11,12,13,14,15,16,17,20 \ldots)$ The answer to the question of why a base 10 system is of course because of our 8 fingers and 2 thumbs also known as "digits." Let students know that even the root of the word decimal, "decem," means 10 in Latin. Ask them what other words they know have this root and have something to do with the number 10. (decade, decimeter, even December which at one time was the 10th month of the year in the old Roman calendar.) Remind students, "Because of place value and powers of 10, we are able to represent any number you want using only these few 10 digits. (This is easier than the Mayan culture which had a vigesimal system with 20 symbols in its number system - presumably including toes with the fingers and thumbs.) After that, you might discuss what they know about place value, both before and after the decimal point. Finally, let students know that today is all about working with decimal numbers and understanding how the place value works.

Make sure for the difficult-to-draw number lines that you use the number lines on the CD that are created for this section.

# D E CIMAL <br> R EPRESENTATION <br> AND OPERATIONS 



## SECTION 6.1 DECIMALS

We have all been concerned about the cost of an item when we shop or go out to eat. For example, a cheeseburger might cost $\$ 3.85$ at a restaurant, a pair of jeans might cost $\$ 18.97$ at a store or the entrance fee to an amusement park might cost $\$ 27.00$. All of these prices use decimal notation, a common way of writing numbers that include parts that are less than 1 . Instead of $\$ 27.00$, we can write $\$ 27$ instead. This means 2 tens and 7 ones. However, when we write $\$ 27.00$, the zeros to the right of 7 give us more information: that the price is exact and includes no cents. In the number 43.26, the 3 is in the one's place which is one-tenth the ten's place occupied by the 4 . Then 2 is in the tenth's place which is one-tenth the one's place. In an example with money, notice that the 8 in the cost of the $\$ 3.85$ cheeseburger is in the dime's place or the tenths-of-a-dollar place and 5 is in the penny's place or the hundredth's place. It takes 10 dimes to make a dollar and it takes 10 pennies to make a dime. Therefore it takes 100 pennies to make a dollar.

## EXPLORATION: LOCATING DECIMAL NUMBERS ON A NUMBER LINE.

If we think of 1 on the number line as $\$ 1.00$, where would we locate half a dollar or $\$ 0.50$ ? Because there are 10 dimes in a dollar, where would $\$ 0.10$ be located on the number line? $\$ 0.20$ ? $\$ 0.30$ ? Can you locate $\$ 0.01$, or more simply 0.01 , on the number line, knowing that there are 10 pennies in a dime?

We know when we write 0.30 that there is another way that this decimal can be written. Thirty cents, or hundredths, can be written as 0.3 . How could you show the two numbers are really equivalent to each other on the number line? We know three dimes or 0.3 has the same value as 30 pennies or 0.30 .

Make sure your students discover the strategy for determining which number is located to the right of the other on the number line, i.e., 0.4 is to the right of 0.27 because $0.4=0.40$ and 40 is greater than 27. As a general rule, write all numbers so they are exact to the same decimal place, adding zeros when necessary. Then start with the largest place value and order the digits, moving to the smaller place values. If the digits are equal, just go to the next smaller decimal place.

Addition and Subtraction of Decimals
The number sense skill that is demonstrated to the right is one of the most useful for estimates. It involves figuring the addition or subtraction necessary to get from the estimate to each of the numbers being estimated. Encourage your students to understand and use it.

Use a number line like the one below to estimate the locations of the following decimal numbers. Notice that 0 and 1 are labeled on the number line.
a. 0.4
b. 0.27
c. 0.68
d. 0.7
e. 0.374
f. 0.307
g. 0.397


Discuss how you approximated the specific location. What strategy did you use to determine which number is greater than or less than another? In general, what strategy can you use to compare decimal numbers?

One strategy that you can use to compare decimals is to write the numbers so that they have the same smallest place value. For example, 0.2 and 0.27 can be written as 0.20 and 0.27 . When we compare the hundredths place, clearly 27 hundredths is more than 20 hundredths because 27 is greater than 20 .

For each pair of numbers, determine which is greater. Justify your answer using the number line.
a. 0.68 and 0.7
b. 0.34 and 0.339
c. $\quad 0.268$ and 0.271

## ADDITION AND SUBTRACTION OF DECIMALS

Betty is about to take a trip. She fills her car with gas for $\$ 29.90$ and buys a map for $\$ 3.49$, a drink for $\$ 1.09$ and a pack of gum for $\$ 0.99$. Estimate the cost of her purchase before taxes. Is $\$ 40.00$ enough to pay for the purchase, excluding tax?

If you calculated $\$ 30.00+\$ 3.50+\$ 1.00+\$ 1.00$ to get $\$ 35.50$ you had a good estimate of her cost. To get the actual cost, however, you must add $\$ 29.90+\$ 3.49+\$ 1.09+\$ 0.99$ or take the estimated cost and subtract the excess of $10 \phi+1 \phi+1 \not \subset$ from the estimated cost, then add in $9 \not \subset$ for the underestimation of $\$ 1.09$. You overestimated by $3 申$ so you should subtract $3 申$ from $\$ 35.50$ for the actual cost.

In the problem, there are 27 pennies, 22 dimes, and 33 dollars.

Linear Model
b. On the number line, it is natural to add from left to right. For example, to compute $0.38+0.47$, locate 0.3 and add 0.4 to get 0.7 . Then add 0.08 to 0.7 to get 0.78 to get, and finally add 0.07 more to get 0.85 .
a. 8.9 billion $=8,900,000,000$
b. 1.5 million $=1,500,000$
c. 0.5 million $=500,000$

When you subtract $3 \notin$ from $\$ 35.50$, you are really subtracting $\$ 0.03$ from $\$ 35.50$ to get $\$ 35.47$. As with addition, it is important to keep in mind the place value and subtract the hundredths from the hundredths, the tenths from the tenths, and so forth. You might have heard the phrase "line up the decimals." This vertical method assures that the place values also line up to do the calculation.

One way to find a sum of money is to first add the hundredths or pennies, to get 27 pennies or 0.27 dollars, then add the tenths, dimes, to get 22 dimes or 2.2 dollars and finally add the 33 dollars. So, $\$ 0.27+\$ 2.2+\$ 33=\$ 35.47$. Explain how this is the same as the traditional method of stacking the numbers to line up the decimal points and adding from right to left.

## Linear Model:

How do you use the number line to add decimal numbers? Compute the following sums using the number line, and then add using the traditional stacking method.
a. $0.2+0.06$
b. $0.38+0.47$
c. $0.23+0.54$
d. $0.26+0.31$

Explain how the number line can help to estimate a sum before you calculate the actual total.

How do you use the number line to subtract decimals? Compute the following differences using the number line, and then subtract using the traditional stacking method.
a. $0.63-0.47$
b. $0.2-0.06$

There are times when we need to write really big numbers but want to avoid writing too many zeros. For example, we write 2 million instead of 2,000,000 or 30 million instead of $30,000,000$. If we write 700 thousand, we mean 700,000 .
a. Write 8.9 billion as a whole number.
b. What is another way of writing 1.5 million?
c. What is another way of writing 0.5 million?
2.
a.

b.

c. -0.109 is not located between -0.11 and -0.24 . Ordering negative numbers is counter intuitive for many students who really understand ordering positive numbers. Try to get them to think about how far from zero the negative number is. The further from zero, the smaller (more negative).

3. You may wish to review metric conversions of meters to centimeters and visa-versa with your students.
a. 2.35 m
b. 5467.2 cm

## EXERCISES

1. Order the following sets of four numbers by locating them on a small segment of the number line.
a. $0.7,0.08,0.69,0.16$
$0.08,0.16,0.69,0.70$
b. $0.109,0.098,0.23,0.228$
$0.098,0.109,0.228,0.230$
c. $4.08,3.16,4.2,3.61$
3.16, 3.61, 4.08, 4.20
2. In working each of the following exercises, be careful to scale your number line appropriately.
a. Draw and label 0.1 and 0.3 on a number line. Plot and name 4 additional points between these two points. Answers will vary.
b. Draw and label 0.29 and 0.307 on a number line. Plot and name 4 additional points between these two points. Answers will vary.
c. Draw and label -0.11 and -0.24. Plot -.0109 on your number line? Is it between -0.11 and -0.24 ? Explain. Next, plot and label 4 additional points. Answers will vary. See TE.
3. Convert the following measurements: See TE.
a. 235 centimeters to meters
b. 54.672 meters to centimeters
4. Compute the following, using the car model from Chapter 2, if necessary:
a. $0.31+0.45$
0.76
d. $0.74+-0.57$
0.17
b. $0.3+0.78$
1.08
e. 2.3-1.06
1.24
c. $1.307+2.46$
3.767
f. $0.603-0.045$
0.558
5. Determine which of the following pairs of numbers is closer together.

| a. | 0.4 and 0.5 | or | 0.23 and 0.27 |
| :--- | :--- | :--- | :--- |
| b. | 0.57 and 0.61 | or | 0.592 and 0.608 |
| c. | 0.12 and 0.27 | 0.592 and 0.608 |  |
| or | 0.089 and 0.11 | 0.089 and 0.11 |  |

6. Bill is making cookies from a European recipe. The recipe calls for 336 grams of sugar. Bill discovers if he is short 1.7 grams of sugar or if he adds 1.7 grams of sugar too much, the taste of the cookies is unchanged. Between what amounts of sugar will the taste remain unchanged? 334.3 and 337.7
7. Connection to Chapter 3.
8. 

a. $0.5 x$
b. $0.2 x$
10. Connection to Chapter 4.
11. Label points a-f on the graph also.
$\bullet(-7.5,-5.5)$
12.
$\bullet(-4.5,-3.25)$

b. 0.135
c.
7. Solve the following equations.
a. $x+0.35=0.46$
$x=0.11$
c. $m+4.03=3.565 \quad m=-0.465$
b. $y+1.2=0.7$
$y=-0.5$
d. $z+0.02=0.64 \quad z=0.62$
8. Simplify the following expressions:
a. $0.2 x+0.3 x$
b. $0.3 x-0.1 x$
9. Ms. Trent has the same number of quarters as she has dimes. Write an expressions that describes the total amount of money Ms. Trent has. $=$
10. Sandra has a plant that is 1.354 meters tall after a growth spurt. If it grew 0.296 meters, how tall was it before the growth spurt? 1.058 m
11. Draw a coordinate plane and plot and label the following points: See TE.
a. $(0.5,0.5)$
b. $(-1,0.5)$
c. $(-1.25,0.75)$
d. $(7.5,-5.5)$
e. $(-4.5,-3.25)$
f. $(-7.5,-5.5)$
12. Use graph paper to draw the graph of the line $y=2 x+1.5$. If necessary, set up a table of values. See TE.
13. Ricky fills a 12 liter bottle in the lab by adding 9.37 liters of hydrochloric acid. How much acid did he have in the bottle in the beginning? 2.63 liters
14. Bobby is conducting two independent chemistry experiments. To conduct these experiments, he needs 0.887 grams of silver nitrate for Experiment A and a total of 1.438 grams of silver nitrate for both Experiment A and B. How much silver nitrate does he need for Experiment B? 0.551 grams
15. Jon is feeling sick. He decides to go to the doctor who finds that his temperature is $102.37^{\circ} \mathrm{F}$. If normal body temperature is $98.6^{\circ} \mathrm{F}$, what was the increase in his body temperature? $3.77^{\circ} \mathrm{F}$

## 16. Ingenuity:

Find the midpoint between each of the following pairs of numbers, showing your work on the number line.
a. 0.3 and 0.4
b. 0.12 and 0.15
c. $\quad 0.3$ and 0.37

Investigation
17.

a. 0.5 , pt. $(0.5,1)$
b. $1.5, \mathrm{pt} .(1.5,3)$
c. 2.5, pt. $(2.5,5)$
d. -0.5 , pt. $(-0.5,-1)$

## 17. Investigation:

Use a graphing calculator for the following using the table function. Scale by .5. Draw the graph of the line $y=2 x$. See TE.
a. Find the first coordinate of a point on this line whose second coordinate is 1 by looking at the graph. Check using a table on your calculator.
b. Find the first coordinate of a point on this line whose second coordinate is 3 by looking at the graph. Check using a table on your calculator.
c. Find the first coordinate of a point on this line whose second coordinate is 5 by looking at the graph. Check using a table on your calculator.
d. Find the first coordinate of a point on this line whose second coordinate is -1 by looking at the graph. Check using a table on your calculator.

## SHOPPING TRIP



Objective: The students will answer questions about a shopping trip to reinforce and practice their skills with adding, subtracting, and estimating with decimals.

## Materials:

Shopping Trip Worksheet
Pencil
Calculator (for teacher only)

## Activity Instructions:

1) Pass out one worksheet per student. Feel free to let the students work in groups or in partners, if they want.
2) While students are working on their worksheet, the teacher should be walking around the room and monitoring their progress. You might want to have a calculator in your hand while you are roaming. If you notice that a student has a wrong answer, have them check it on the calculator. Once they have made a mistake, see if you can get them to discover where they went wrong.
3) Once all worksheets are completed, let students wander the room and compare answers with others. This is an excellent way to encourage students to defend their answer choices. If two students have answers that do not agree, then they should both discuss their methods and see if they can figure out who made the mistake and why.

## ANSWER KEY:

1) $\$ 10.74$
2) $\$ 6.82$
3) Answers will vary.
4) $\mathrm{No}, \$ 20$ is not enough. He would need $\$ 7.15$ more to make all of those purchases.

Name $\qquad$

## Shopping Trip Worksheet



Jacob's dad went grocery shopping this week at the local supermarket. Below is a list of some of the items that he was interested in buying. Use the table of items to answer the questions below about his shopping trip.

| Corny Flakes Cereal | $\$ 4.59$ | Chocolate Chip Cookies | $\$ 3.45$ |
| :--- | :--- | :--- | :--- |
| Orange Juice | $\$ 4.65$ | Whole Milk | $\$ 2.70$ |
| Wheat Bread | $\$ 1.79$ | Peanut Butter | $\$ 2.89$ |
| Dozen Eggs | $\$ 0.99$ | Strawberry Ice Cream | $\$ 3.15$ |
| Hot Dogs | $\$ 1.35$ | Bagels | $\$ 1.59$ |

1) If Jacob's dad only bought cookies, cereal, and milk, how much money did he spend?
2) If Jacob's dad only bought hot dogs, eggs, bagels and peanut butter, how much money did he spend?
3) If Jacob's dad had a $\$ 20$ bill, choose 5 items that he could buy. Then, tell me how much change he would have left after making this purchase.
4) If Jacob's dad decided to buy everything in the chart, would $\$ 20$ be enough to pay for it all? If so, how much extra money will he have left over? If not, how much more does he need?

## PLACE VALUE DIE

Objective: The student will be able to identify place value of numbers with place values from the thousands to the ten-thousandths.

## Materials:

Place value chart
10-sided die
Notebook Paper

## Activity Instructions:

1) Students will work in small groups of 3 or 4 .
2) Each student in their working group will roll the die 4 times to get 4 numbers which they may place in any order they wish. The fifth roll of the die will determine where the decimal point will be placed, according to the following guide:

- Rolling 0 or $1=$ decimal point goes all the way to the right of the 4th digit.
- Rolling 2 or $3=$ decimal point goes between the 3rd and 4th digits.
- Rolling 4 or $5=$ deciaml point goes between the $2 n d$ and 3 3rd digits.
- Rolling 6 or $7=$ decimal point goes between the 1 st and 2 2nd digits.
- Rolling 8 or $9=$ decimal point goes to the left of the 1st digit.

Digit Key:

3) In each group, students collect as many numbers as teacher chooses to assign them. Then, teacher asks students to share which student had the greatest number out of the entire class. The group with the greatest number out of the entire class gets a reward (teacher's choice).


## PLACE VALUE SPINNING

Objective: The student will be able to identify place value of numbers with place values from the thousands to the ten-thousandths.

## Materials:

Place value chart
Place value spinner
Paper clips
Notebook paper

## Activity Instructions:

1) Teacher will engage students in quick review of the place value chart by giving them numbers with the thousands down to the ten-thousandths place value. Each student will write the number in a copy of his/her place value chart. Teacher may continue giving them more numbers as she/he monitors and checks for accuracy. Then, teacher will ask students to write the greatest number in the thousands down to the ten-thousandths that they can think of. Each student simply writes his/her numbers in a copy of their place value chart.
2) Students will work on the place value spining activity in small groups of 3 or 4 . There will be 4 spins. The first spin, students will use digits from the smallest circle which will represent the 10ths place. The second spin, students will use digits from the next largest circle, which will represent the 100ths place. The third spin, students will use digits from the next largest circle, which will represent the 1,000ths place. The fourth spin, students will use digits from the largest circle, which will represent the 10,000ths place.
3) In each group, students collect as many numbers as teacher chooses to assign them. Students write their numbers on a sheet of paper. Then, teacher asks students to share which student had the greatest number out of all. The group with the greatest number out of the entire class gets a reward (teacher's choice).
4) The game may continue with students working individually to create numbers, if the teacher prefers.

## PLACE VALUE SPINNER



## Section 6.2-Multiplication of Decimals

## Big Idea:

Explore multiplication of Decimals with Linear and Area Models.

## Key Objectives:

- Understand Linear Model of Multiplication on number line.
- Understand Area Model of Multiplication of Decimals.
- Model (0.2)(0.3)


## Materials:

Calculator, graph paper

## Pedagogical/Orchestration:

Students need to be able to demonstrate their understanding of the Area Model of Multiplication of Decimals by drawing the model on graph paper.

## Activities:

Explorations offer opportunities for group work and good class discussion.

## Exercises:

Exercise 3 - students will need a calculator.
Exercie 4-7 involve setting up proportions.
Exercise 8-9 involve solving equations.

## Vocabulary:

linear model, area model

## TEKS:

7.2(A,B,C,D,F,G); 8.1(A); 8.2(A,B,C); 8.5(A); 8.14(A); 8.15(A); 8.16(A

## WARM-UPS for Section 6.2

1. Which of the following is equivalent to the product: (.03)(.4)? Explain your answer.
a. 0.12
b. 0.012
c. 0.0012
d. 1.2
Ans: b
2. Compute the following products. What do you notice?
a. (1.2)(14) Ans: 16.8
b. (12)(1.4) Ans: 16.8
c. (1.2)(1.4) Ans: 1.68
d. (0.2)(4) Ans: 0.8
e. (2)(0.4) Ans: 0.8

## Launch for Section 6.2:

Use the Exploration 1 to compare these products:
(3)(2)
(0.3)(2)
(0.3)(0.2)

## SECTION 6.2 MULTIPLICATION OF DECIMALS

## EXPLORATION 1

Use the linear model to show how to compute the following products:
a. $3 \cdot 2$
b. $(0.3) \cdot 2$
c. $(0.3) \cdot(.2)$

The first product is simply 2 jumps of length 3 which yields a product of 6 . The second product is 2 jumps of length 0.3 , as shown below:


Two jumps of length 0.3 gives us the location of 0.6 .
However, modeling the third product, ( 0.3 )(0.2), is not clear. What do we mean by 0.2 jumps? For this product, the area model may be more helpful.

Using the area model for (3)(2), we find the area of a 3 by 2 rectangle. How do we use the area model for (0.3)(2)?

Draw a rectangle that has length 0.3 and width 2 . When drawing this rectangle it is helpful to use grid paper and choose an appropriate scale. In this case, we need to measure both 0.3 and 2 . Using a grid, assign each small square a length of 0.1

Outline a 1 by 1 square using the 0.1 grid. How many 0.1 by 0.1 small squares are in a 1 by 1 square? Now outline a 0.3 by 2 rectangle on the grid. With a picture of the rectangle and its dimensions, you can see that the area is made up of 60 small squares, each with an area of $0.1 \times 0.1=0.01$. So the product is $(0.3)(2)=(60)(0.01)=0.60=0.6$.


We can rearrange the model above to look like the following:


Thus, $(60)(0.01)=0.60$ since there are 60 hundredths shaded above. However, we can write 0.60 as 0.6 , or 6 tenths, which is modeled below.


PROBLEM 1
What patterns do you notice?
a. $\quad 0.08$
b. $\quad 0.72$
c. $\quad 0.18$
d. 2

How do we compute the product $(0.3)(0.2)$ using the area model? Consider the grid below. Each side of the square has length 1 . Note that the length of each little square is 0.1 . Use this grid to model the product (0.3)(0.2).


One way to show the product (0.3)(0.2) is to shade a rectangle within the grid that is 0.3 long (horizontally) and 0.2 wide (vertically). The result is a small rectangle with 6 little squares. What is the area of each little square? Since the large square has area 1 and there are 100 little squares, the area of each little square is 0.01 or $1 / 100$. So the area of 6 little squares is 0.06 or $6 / 100$.

## PROBLEM 1

Compute the following products using the grid.
a. $(0.4)(0.7)$
b. $(0.8)(0.9)$
c. $(0.6)(0.3)$
d. (0.5)(4)

PROBLEM 2

| $(4)(1)=4$ | $(0.4)(2)=0.8$ | $(0.8)(7)=5.6$ |
| :---: | :---: | :---: |
| $(4)(0.1)=0.4$ | $(0.4)(0.2)=0.08$ | $(0.8)(0.7)=0.56$ |
| $(4)(0.01)=0.04$ | $(0.4)(0.02)=0.008$ | $(0.8)(0.07)=0.056$ |
| $(4)(0.001)=0.004$ | $(0.4)(0.002)=0.0008$ | $(0.8)(0.007)=0.0056$ |
| $(0.4)(0.1)=0.04$ | $(0.04)(0.2)=0.008$ | $(0.08)(0.07)=0.0056$ |

PROBLEM 3

| $(2.4)(3.1)=7.44$ | $(562)(7)=3934$ | $(0.483)(27)=13.041$ |
| :---: | :---: | :---: |
| $(0.24)(3.1)=0.744$ | $(562)(0.7)=393.4$ | $(4.83)(27)=130.41$ |
| $(0.024)(3.1)=0.0744$ | $(56.2)(0.7)=39.34$ | $(48.3)(27)=1304.1$ |
| $(0.24)(0.31)=0.0744$ | $(5.62)(0.7)=3.934$ | $(4.83)(2.7)=13.041$ |
| $(24)(0.31)=7.44$ | $(0.562)(7)=3.934$ | $(4.83)(0.27)=1.3041$ |

## PROBLEM 2

Compute the following groups of products. Look for patterns that can help us multiply decimal numbers. You may use a calculator.

| $(4)(1)=$ | $(0.4)(2)=$ | $(0.8)(7)=$ |
| :---: | :---: | :---: |
| $(4)(0.1)=$ | $(0.4)(0.2)=$ | $(0.8)(0.7)=$ |
| $(4)(0.01)=$ | $(0.4)(0.02)=$ | $(0.8)(0.07)=$ |
| $(4)(0.001)=$ | $(0.4)(0.002)=$ | $(0.8)(0.007)=$ |
| $(0.4)(0.1)=$ | $(0.04)(0.2)=$ | $(0.08)(0.07)=$ |

What patterns do you notice?

## PROBLEM 3

Compute the following groups of products. Look for patterns that can help us multiply decimal numbers. You may use a calculator.

| $(2.4)(3.1)=$ | $(562)(7)=$ | $(0.483)(27)=$ |
| :---: | :---: | :---: |
| $(0.24)(3.1)=$ | $(562)(0.7)=$ | $(4.83)(27)=$ |
| $(0.024)(3.1)=$ | $(56.2)(0.7)=$ | $(48.3)(27)=$ |
| $(0.24)(0.31)=$ | $(5.62)(0.7)=$ | $(4.83)(2.7)=$ |
| $(24)(0.31)=$ | $(0.562)(7)=$ | $(4.83)(0.27)=$ |

a. What patterns do you notice?
b. How many decimal places does each factor have?
c. How many decimal places are in each product?
d. What is the connection between these two for each product?

This is another problem that requires the use of a graphing calculator. An alternative might be an Excel spreadsheet with an accompanying line graph. If you do not have access to graphing calculators and you have exhausted the usual channels for borrowing them, you can do this by hand, but it is much harder and more tedious.

## PROBLEM 4

Compute the following groups of products.

| $(0.2)(0.35)=$ | $(6.2)(4)=$ | $(0.48)(0.25)=$ |
| :---: | :---: | :---: |
| $(0.35)(0.2)=$ | $(4)(6.2)=$ | $(0.25)(0.48)=$ |

a. What patterns did you notice?
b. State a Commutative property for all numbers.

## EXPLORATION 2

Explore the change in the $y$ values in the graph of the line $y=8 x$ using the table of a graphing calculator with an increment of 1 , then 0.5 , then 0.25 and finally 0.125 . What did you find?

When the increment is 1 , the $y$ values change by 8 . If the increment is 0.5 , the change in $y$ values is 4 . There is a 0.25 increment with associated change of 2 , and a 0.125 increment with a change of 1 .

## EXPLORATION 3


a. What is the area of the shaded rectangle with sides of length 0.2 and $x$ ? What is the area of the shaded rectangle with sides of length 0.3 and $x$ ? What is the total area of both rectangles?

PROBLEM 5
$0.6 y$ and $0.8 y$
Total area $=0.6 y+.8 y=(0.6+0.8) y=1.4 y$

## EXERCISES

2. a. between 0.1 and $1 \quad$ b. between 0.01 and $0.1 \quad$ c. between 10 and 100
3. Appropriate for 7th grade
a. 0.66
b. 3.072
c. 0.18432
b. Compute the following sum: $0.3 x+0.2 x$. Draw a picture to illustrate how this answer makes sense.

In the Exploration above, the area of each shaded rectangle is computed by the formula of length times width. The area of the first rectangle is $(0.2)(x)=0.2 x$ and the area of the second rectangle is $(0.3)(x)=0.3 x$. The total area of the two rectangles is the sum of these two areas: $0.2 x+0.3 x$. Do you see that this is the same as $0.5 x$ ?

## PROBLEM 5

Compute the area of each of the following rectangles. Then, compute the total area of both rectangles.


## EXERCISES

1. Use the area model and the distributive property to compute the following products. Indicate the area of each interior part in your model.
a. (3) (5.2)
15.6
c. $(0.26)(4)$
1.04
b. $(4.7)(3.2)$
15.04
d. $(37)(0.24)$
8.88
2. Predict whether the product is between 0.01 and 0.1 , between 0.1 and 1 , between 1 and 10 or between 10 and 100. Explain your reasoning. See TE.
a. $(0.73)(0.44)$
b. $(0.245)(0.2)$
c. $(31.7)(0.77)$
3. Compute the following: See TE.
a. $(0.2)(3.3)$
b. (1.28)(2.4)
c. $(0.512)(0.36)$
4. 

a. 0.9 x
b. 1.2 a
c. 1.3 x
d. 0.5 b
e. 2.4 A
f. $0.8 x$
g. 1.4 y
5. Appropriate for 7 th grade

Be sure to tell the students this problem needs to be drawn to scale using a ruler. It will take up quite a bit of room on their paper to draw the rectangle with length 25 cm , but it will fit. The rectangles have the same area because $2.5(12)=25(1.2)=30$.
8. Your students might want to use an appropriate number line to help visualize where the answer will be. Encourage them to round at least one of the numbers, if necessary. For instance, in part (a), round 37 to 40 . Then the product will be 920, close enough for a correct answer.
a. between 100 and 1000
d. between 10 and 100
b. between 0.1 and 1
e. between 0.1 and 1
c. between 10 and 100
f. between 0.1 and 1
9.

| 0.63 | 4 | 5561 | 504 |
| :---: | :---: | :---: | :---: |
| 6.3 | 4 | 5.561 | 50.4 |
| 6.3 | 0.4 | 55.61 | 5.04 |
| 0.063 | 0.04 | 55.61 | 5.04 |
| 0.63 | 0.4 | 0.5561 | 5.04 |
| 6.3 | 0.004 | 5.561 | 0.504 |

4. Compute the following sums:
a. $0.2 x+0.7 x$
b. $0.7 a+0.5 a$
c. $x+0.3 x$
d. $0.8 b-0.3 b$
e. $3 A-0.6 A$
f. $x-0.2 x$
g. $y+0.4 y$
5. Draw rectangle $A$ with length 2.5 cm and width 12 cm . Draw rectangle $B$ with length 25 cm and width 1.2 cm . Explain why these rectangles have the same area. See TE.
6. Ramses has 38 framed posters he plans to sell for $\$ 24.49$ each. Estimate the amount of money Ramses will make. $\$ 950$, the actual value is $\$ 930.62$
7. Draw a number line from -5 to 5 divided into tenths and use it to find the following products:
a. $(0.3)(4)$
1.2
d. $(-0.4)(3)$
$-1.2$
b. $(0.3)(-6)$
-1.8
e. $(-1.2)(4)$
-4.8
c. $(1.2)(3)$
3.6 f. $(-0.3)(-7)$
2.1
8. Predict whether each of the following products is between 0.1 and 1 , between 1 and 10 , between 10 and 100 or between 100 and 1,000. Explain how you made your decision.
a. $\quad(23)(37)$
b. $\quad(0.012)(9)$
c. $(13.75)(4)$
d. $(0.47)(110)$
e. $(0.023)(5)$
f. $(0.002)(412)$
9. For each group of products, compute the first product and check your answer with a calculator. Then fill in the other products. Check your answers with a calculator. See TE.

| $(0.7)(0.9)=$ | $(0.5)(8)=$ | $(83)(67)=$ | $(56)(9)=$ |
| :---: | :---: | :---: | :---: |
| $(0.7)(9)=$ | $(5)(8)=$ | $(0.83)(6.7)=$ | $(56)(0.9)=$ |
| $(7)(0.9)=$ | $(0.5)(0.8)=$ | $(8.3)(6.7)=$ | $(56)(0.09)=$ |
| $(0.7)(0.09)=$ | $(0.5)(0.08)=$ | $(83)(0.67)=$ | $(0.56)(9)=$ |
| $(7)(0.9)=$ | $(0.05)(8)=$ | $(0.83)(0.67)=$ | $(5.6)(0.9)=$ |
| $(70)(0.09)=$ | $(0.05)(0.08)=$ | $(0.083)(67)=$ | $(0.056)(9)=$ |

13. $(\$ 4.75)(\$ 1.29)=\$ 6.1275 \quad$ approximately $\$ 6.13$
14. 

| Input | -0.8 | -0.6 | -0.4 | -0.2 | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 | 1.4 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Output | -2.4 | -1.8 | -1.2 | -0.6 | 0 | 0.6 | 1.2 | 1.8 | 2.4 | 3.0 | 3.6 | 4.2 |

15. 

| Input | -2 | -1.5 | -1 | -0.5 | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Output | -0.8 | -0.6 | -0.4 | -0.2 | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 | 1.2 |

16. When the increment is 1 , the $y$ values change by 8 . If the increment is 0.5 , the change in $y$ values is 4 . There is a 0.25 increment with associated change of 2 , and 0.125 increment with change of 1 .
This is another problem that requires the use of a graphing calculator. An alternative might be an Excel spreadsheet with accompanying line graph. If you do not have access to graphing calculators and you have exhausted the usual channels for borrowing them, you can do this by hand, but it is much harder and more tedious.
17. The area model is a rectangle 3.6 vertical dimension by 2.9 horizontal dimension. The area is 10.44 square miles.

18. Brandon went to the Farmer's Market and purchased 3.2 pounds of bananas that cost $\$ 0.48$ per pound. How much does he pay for these bananas? \$1.54
19. Donna bought a toy for each of her 17 cats. Each toy costs $\$ 2.35$. What was the cost of these toys before tax? \$39.95
20. Juanita averaged 6.5 minutes per mile in a race that covered 8.5 miles. How long did it take her to run this race? 55.25 minutes
21. Sophia bought 4.75 kilos of beans from the store at the price of $\$ 1.29$ per kilo. How much did she pay for the beans? See TE.
22. Fill in the outputs in the table for the function given by the equation
$y=3 x$. See TE.

| Input | -0.8 | -0.6 | -0.4 | -0.2 | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 | 1.4 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Ouput |  |  |  |  |  |  |  |  |  |  |  |  |

15. Fill in the outputs in the table for the function given by the equation $y=0.4 x$. See TE.

| Input | -2 | -1.5 | -1 | -0.5 | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Output |  |  |  |  |  |  |  |  |  |  |  |

16. Explore the change in the $y$ values in the graph of the line $y=8 x$ using the table of a graphing calculator with an increment of 1 , then 0.5 , then 0.25 and finally 0.125 . What did you find? See TE.
17. Investigation: A bird refuge is in the shape of a rectangle 3.6 miles long and 2.9 miles wide. Draw a visual representation of this refuge on grid paper, using a 0.1 scale, and use it to determine the area of this rectangle. Explain how you use the grid to compute the area. Multiply 3.6 miles by 2.9 miles using the traditional algorithm only after you have an answer using the visual representation. See TE.

## Section 6.2 - Multiplication of Decimals

## Big Idea:

Modeling of long division using the scaffolding method

## Key Objectives:

- Relate the long division process to the linear and area models of multiplication.
- Discover patterns in division and relate to multiplication.
- Understand the long division process


## Materials:

Graph paper, Calculators starting with Exercise 9

## Pedagogical/Orchestration:

- This is a basic introduction to the true meaning of division and the long division process.
- Have students check the answer as often as possible by performing the reverse of multiplication to get the original dividend back.


## Activities:

"Bags of Stuff" at the end of the section and on the CD.
"Percent Composition of Sugar in Chewing Gum" at the end of the section and on the CD.

## Exercises:

Until Exercise 9, discourage the use of calculators. The division problems are designed to review basic skills and build confidence through the recognition of patterns that aid computation.

## Vocabulary:

dividend, quotient, divisor, remainder, scaffolding

## TEKS:

6.2 (C)(D); 7.1(A)(B); 7.2(A)(B)(C)(D)(G); 7.3(B); 7.13(A)(C); 7.14(A); 7.15(A)(B); 8.2(C,D); 8.3(B); 8.14(A); 8.15(A); 8.16(A)

## WARM-UPS for Section 6.2

1. Bear Foot, the zoo keeper has 36 penguins and 108 frozen fish popsicles. does he have enough to distribute the fish popsicles evenly? If not, how many are left over? If so, how many does each penguin get?

Ans: Yes, he can distribute them evenly. Each penguin gets 3 fish-sicles.
2. How many tiles, each of length 14 inches, can be laid end to end in a room that is 30 feet long?
a. 24 tiles c. 26 tiles
b. 25 tiles d. 27 tiles
c. 26 tiles
d. 27 tiles

# Ans: (b) because $30 \mathrm{ft}=360 \mathrm{in}$. So, skipping counting by 14 , we get $(14)(10)=140,(14)(20)=$ 280 plus 5 more 14 's is 70 . Thus, $(14)(25)=350$ and you can fit 25 tiles lengthwise into the room with 10 inches of space left over. 

## Launch for Section 6.2:

Tell your students, "An uncle has 6 nephews and nieces. If he has $\$ 28.56$ in loose change and wants to distribute this money equally among them, how much will he give to each nephew and niece?" Allow students to use their own strategies to determine the answer. Expect the students to use different ways to model this problem. Once students have worked on the problem, show any correct visual models the students have drawn that show the answer to be $\$ 4.76$. Tell your students, "The number you found is called the quotient of the problem, and we will be using this method throughout the lesson to check our division process and to understand the true meaning of division."

## SECTION 6.3 LONG DIVISION

We have seen how closely related multiplication and division are. For example, we know $8 \div 4=2$ because $4 \times 2=8$. Also recall that in the long division form, the multiplication fact is rewritten as

The first product is simply 2 jumps of length 3 which yields a product of 6 . The second product is 2 jumps of length 0.3 , as shown below:

We have the dividend 8 "under" the quotient 2 , and the divisor 4 is to the left of the dividend.

By changing the dividend to 9 , our problem becomes $4 \longdiv { 9 }$. Because $8 \div 4=2$, $9 \div 4$ must be more than 2 . In the long division form,
4) $\frac{2}{9}$ The area model looks like this:
$-8$


The quotient is 2 and the remainder is 1 .
Consider the problem $4 \longdiv { 8 0 }$, where the dividend is not simply 8 but 8 tens. The quotient is then 2 tens or 20 because $4 \times 20=80$. In the long division form,
The area model is:
-80
0


Use the long division form to evaluate the problem $4 \longdiv { 8 0 0 }$ with 8 hundreds. Do you agree the answer is 2 hundreds or 200 ? In the long division form,
200
4) 800
The area model is: $\square$

Note that the above picture is not to scale. If you were to draw it to scale and leave the height unchanged, imagine how long the rectangle would be.
A more complex problem is $4 \longdiv { 8 4 }$. One way to think of this problem is to notice the place values and observe that $84=80+4$. You know that

$$
4 \longdiv { 2 0 } \text { and } 4 \longdiv { 4 }
$$

Putting these together gives

$$
\frac{21}{4 \longdiv { 8 4 }}
$$

Here is the area model for this problem:

or

$$
\begin{array}{r}
1 \\
20 \\
4 \longdiv { 8 4 } \\
\hline-80 \\
\hline 4 \\
-\frac{-4}{0}
\end{array}
$$

This is called the scaffolding method because the different partial quotients are first computed and stacked, then combined, much as a scaffold is used in constructing a building.

Reflect with class about different ways this could have been scaffolded, for example, $10+10+10+10+4+2$ +1 with remainder.

## EXAMPLE 1

Use scaffolding method to compute the division problem $567 \div 12$.

## SOLUTION

There is more than one way to implement the scaffolding method in division. here is one way:

$$
\begin{array}{r}
2 \\
5 \\
20 \\
20 \\
12 \lcm{567} \\
\frac{-240}{87} \\
\frac{-60}{27} \\
\frac{-24}{3}
\end{array}
$$

so $20+20+5+2=27$. Therefore, $567 \div 12=27$ with remainder 3 .

## PROBLEM 1

Use this scaffolding method to compute the following quotients. You may sketch a picture of the corresponding area model if it helps.
a. $380 \div 14$
b. $950 \div 6$

In division, we know that it is more common to start with the largest place value to determine the first digit of the quotient and then gradually include the smaller place values. We will now compute the following division problem using both the traditional method side by side with the scaffolding method. Compare the two approaches.

## EXAMPLE 2

Use both the scaffolding method and area model to compute the division problem $552 \div 15$.

## EXAMPLE 3

If we follow the model of skip counting, we would say "how many jumps of 6 would reach 3?" This would be a $\frac{1}{2}$ of a jump. However, this process gets tricky if there are, say, 6 candy bars for $\$ 1.80$. This is an example of where we would switch the roles of 6 from the length of the jump to the number of jumps and the missing number becomes the length of the jump instead of the number of jumps.
This picture illustrates $x \cdot 6=3$ instead of $6 \cdot x=3$ like the model normally suggests.
The reason why we can do this is because multiplication is commutative.

## Solution

Scaffolding Method Traditional Method:

| 1 | 36 |
| ---: | ---: |
| 5 | $1 5 \longdiv { 5 5 2 }$ |
| 10 | $\frac{-45}{102}$ |
| 20 | $\frac{-90}{12}$ |
| $\frac{-300}{252}$ |  |
| $\frac{-150}{102}$ |  |
| $\frac{-75}{27}$ |  |
| $\frac{-15}{12}$ |  |

## PROBLEM 2

Use both the scaffolding and traditional models to compute the following division problem: $3528 \div 13$

## EXAMPLE 3

Sara spent \$3 on 6 Hershey bars. How much did each candy bar cost?

## SOLUTION

This is a typical division problem where the cost of each candy bar is $3 \div 6$. You might expect rouble because the divisor is greater than the dividend. Using the linear skip counting model, how long does each skip need to be to travel a distance of 3 units, or dollars in this case, in 6 skips?

a. To divide the length of $\$ 1$ dollar into 5 equal jumps, students must try by trial and The correct jump length is 0.2 . So, $1 \div 5=0.2$.

To use the scaffolding method in part (a), you must place a decimal point in the dividend and then add 0's as necessary to complete the long division until you get a remainder of 0 . Remind students that in the traditional method, when dividing by a whole number they always line the quotient with the dividend and place the decimal point in the quotient above the decimal point in the dividend. Point out that this results in the same placement of the decimal points that the linear model predicts.

You can see that each jump is $\$ 0.50$ or half a dollar. This represents the fact that each candy bar costs $\$ 0.50$. If necessary, verify this using the calculator by computing $3 \div 6$ or adding six skips 0.50 long. The long division method gives us the same result, because there is a decimal point before the 5 . You might write the problem like step 1 below. The divisor is greater than the dividend, so modify the long division process by placing a decimal point. Add the two zeros in the dividend because we are working with money and we know that $\$ 30=0 \$ 3.00$ where $\$ 0.00$ represents no cents. Where does the decimal place appear in the quotient? Why does this make sense?
Step 1:
$6 \longdiv { 3 }$

Step 2:
0.50

6 $\lcm{3.00}$
$-3.00$
0.00

## PROBLEM 3

Compute the following division problems by using an abbreviated number line from 0 to 2 , like the one below. Find the quotient using the skip-counting method. Then use the scaffolding method to verify your answer. Make sure the decimal point in the quotient makes sense in the context of the problem. Then use the calculator to confirm your work, if necessary.
a. $\$ 1 \div 5$
b. $\$ 1.60 \div 8$
c. $\$ 1.20 \div 4$


When long division involves two-digits numbers, the skip-counting model becomes more difficult. We will now use the traditional method (or the scaffolding method) to compute division problems. At this point knowing the multiplication facts and how to use them make life much simpler.

In developing the area model with 4 columns, we use the partitive model of division in which each column represents a distribution to each nephew, giving each a dollar. After 6 rounds, Mr. Garza has $\$ 2$ left over. We then distribute $\$ 0.50$ to each nephew on the last round.

The area model should look similar to the $3 \longdiv { 8 1 3 }$ problem. However, the students should notice that the rectangles equal two $3 \times 10$, seven $3 \times 1$ and one sliver that equals $3 \times(0.1)$.

Discuss the importance of noting the decimal place in the dividend and correspondingly noting it in the quotient. Ask students how they can judge the reasonableness of their answer.
0.813 divided by 3 is 2.71 . 8130 divided by 3 is 271 . The place values change but the digits stay the same.

## EXERCISES

1. a. 9
b. 62
c. 32
90
620
320
0.9
6.2
3.2
0.62
0.32

## PROBLEM 4

Mr. Garza has some money in his pocket that he intends to divide equally among his four nephews. How much will each nephew receive if he has
a. $\$ 26$ in his pocket,
\$6.50
b. $\$ 27.40$ in his pocket
$\$ 6.85$

Recalling our earlier problem 12) 567 let us consider a related problem.

Draw a representation for this problem to see the long division process. Then use the long division algorithm to see how the quotient is obtained numerically as well.

Find the quotient $1 2 \longdiv { 5 6 . 7 }$ and include a visual representation, although you might have to think small. Do the process and your picture correspond?

Notice the magnitude of your quotient changes by a power of 10 . What do you think the quotient of $1 2 \longdiv { 0 . 5 6 7 }$ is?

What about $1 2 \longdiv { 5 6 7 0 }$ ? What changes between the two quotients and what remains the same? See TE.

## EXERCISES

1. Compute the following quotients using scaffolding long division, if necessary. Verify your answer using multiplication. You might also want to check using your calculator, if you are unsure. See TE.
a. $72 \div 8$
$720 \div 8$
b. $868 \div 14$
c. $736 \div 23$
$7.2 \div 8$
$8680 \div 14$
$7360 \div 23$
$8.68 \div 14$
$7.36 \div 23$
2. 

a. 0.25
b. 0.4
0.25
0.4
0.25
0.4
0.25
0.4
3.
a. 0.2
b. 0.4
c. 0.6
d. $4 \div 5$ will be 0.8
e.

4.
a. 0.4
b. 1.4
c. 2.4
d. $17 \div 5$ is 3.4
5. a. $\$ 12$
b. $\$ 1.20$
c. $\$ 0.12$
6. Your students should perform 2 divisions to determine the cost per ounce. Reward them if they can do the computation in their heads, as long as they can explain what they did. Because $\$ 2.40$ for 40 ounces results in paying $\$ 0.06$ per ounce and $\$ 2.50$ for 50 ounces results in paying $\$ 0.05$ per ounces the Martel Brand rice has a better value.
2. Compute the following quotients using long division and check your answer with multiplication. See TE.
a. $1 \div 4$
b. $4 \div 10$
$2 \div 8$
$16 \div 40$
$3 \div 12$ $20 \div 50$
$4 \div 16$ $28 \div 70$
3. Compute the following quotients using the method of long division and check with a calculator, if necessary. See TE.
a. $1 \div 5$
b. $2 \div 5$
c. $3 \div 5$
d. Predict what $4 \div 5$ equals.
e. Plot and label these quotients on a number line like the one below:

4. Compute the following quotients and check your answer: See TE.
a. $2 \div 5$
b. $7 \div 5$
c. $12 \div 5$
d. Predict what $17 \div 5$ equals.
5. a. Donna paid $\$ 60$ for five CD's of equal cost. How much did each CD cost?
b. Shirley paid $\$ 6$ for five hot dogs. How much did she pay for each hot dog?
c. Gary paid $\$ 0.60$ for five pieces of candy. How much did he pay for each piece? See TE.
6. Mindy is shopping for groceries at the store. The Jones Brand rice sells for $\$ 2.40$ for 40 ounces and the Martel Brand rice sells for $\$ 2.50$ for 50 ounces. Which brand is a better value and why? See TE.
7. a. See table for costs of $1 \mathrm{lb} ., 2 \mathrm{lb} ., 3 \mathrm{lb}$. and 4 lb . of coffee.
b. The cost of 3.5 lb of coffee would appear in between the cost for 3 lb . and 4 lb . of coffee, $\$ 14$.

c. You can buy 8 lb . of coffee with $\$ 32$. This becomes the division problem $32 \div 4$.
d. You can buy 2.25 lb . of coffee with $\$ 9$. This cost value would appear halfway between $\$ 8$, which corresponds with a 2 lb . and $\$ 10$, which corresponds to 2.5 lb .
e. You can buy 10.75 lb . of coffee with $\$ 43$.
8.
a. 0.10
b. 0.05 c. 0.10
d. 0.15
e. $4 \div 20$ is 0.2

Teachers, for difficult number lines like these, look in the CD for duplicating masters..
f.

7. A small grocery store sells coffee for $\$ 4.00$ per pound. As you did in previous sections, let $x$ be the number of pounds of coffee and $y$ be the cost of $x$ pounds of coffee. The equation $y=4 x$ describes the relationship between pounds of coffee $x$ and the cost $y$. See TE.
a. Copy and fill the adjacent table for the line given by the equation $y=4 x$. Plot the points in this table on a coordinate system.

| Pounds of <br> Coffee (x) | Cost (y) |
| :---: | :---: |
| 0 | 0 |
| 0.5 | 2 |
| 1 | 4 |
| 1.5 | 6 |
| 2 | 8 |
| 2.5 | 10 |
| 3 | 12 |
| 3.5 | 14 |
| 4 | 16 |
| 4.5 | 18 |
| 5 | 20 |

b. Use the table to determine the costs of $1 \mathrm{lb} ., 2 \mathrm{lbs} ., 3 \mathrm{lbs}$., and 4 lbs . of coffee. What is the cost of 3.5 pounds of coffee? See TE.
c. How much coffee can you buy with $\$ 32.00$ ?
d. How much coffee can you buy with $\$ 9.00$ ?

Where would this appear in the table?
e. How much coffee can buy with $\$ 43.00$ ?
8. Compute the following quotients using division and check the answer with a calculator, if necessary.
a. $1 \div 10$
b. $1 \div 20$
c. $2 \div 20$
d. $3 \div 20$
e. Use the pattern to predict what $4 \div 20$ is.
f. Plot these quotients on a number line like the one below. Label each point plotted.

9. Teachers, there are several good ways to compute this with a graphing calculator. Make sure your students see as many as possible. Make sure they know how to use the table function on their calculators.
a. $(2,10),(4,20),(6.5,32.50),(7.2,36)$
b. A customer can buy 1.5 lb . with $\$ 7.50$.
c. A customer can buy 1.86 lb . with $\$ 9.30$.
d. A customer can buy 4.57 lb . with $\$ 22.85$.
10.
a. 1.25
b. 12.5
c. 1.25
d. 0.125
e. 0.125
f. 0.25
g. 0.375
h. 0.5
i. 0.625

11. This problem foreshadows reciprocals otherwise known as multiplicative identities.
a. 1
b. 1
c. 1
d. 1
e. 1
f. 1
g. 1
h. 1
i. 1
12. Encourage your students to make conjectures about winning strategies and play many games to test them out. Also have students write down any observations they make about winning strategies and even the number of toothpicks each player had. It may be easier to make conjecture based on their results if they develop a good method for keeping score and keeping track of winning positions.
b. If there are two toothpicks left and it is your turn you should take away one toothpick, this forces your opponent to take the last toothpick and secures victory. If there are five toothpicks left you should take away 4 toothpicks, this also guarantees victory.
c. If there are six toothpicks and you must take at least one toothpick and can, at maximum, take 4 toothpicks, then you will lose. If you take one toothpick then there will be five toothpicks left and your opponent can use the strategy from part (b). If you take four toothpicks then your opponent can still use the strategy from part (b) and guarantee victory. It doesn't matter what you do if your opponent knows your strategy from part (b). You will lose.
d. If there are seven toothpicks, you should take one toothpick to force your opponent into the situation described in part(c). If there are eight toothpicks, you should take two toothpicks to give your opponent six toothpicks and make them lose. For nine toothpicks, take three and if you have ten toothpicks you should take four. You will win if you force your opponent to have six toothpicks.
e. If you continue the strategy pattern, having $1,6,11,16,21$, or 26 toothpicks will guarantee a loss if your opponent keeps in mind a modified strategy from part (b). Any other number of toothpicks can secure a victory.
g. Finish labeling the tenths and multiples of 5 hundredths on this number line.
9. A farmer sells his pecans for $\$ 5$ per pound Assign $x$ as the number of pecans and $y$ the price of $x$ pounds of pecans. The equation $y=5 x$ describes the relationship between pounds of pecans and the cost in dollars.
Enter the equation $y=5 x$ in the graphing utility. See TE.
a. Compute the prices of $2 \mathrm{lb} ., 4 \mathrm{lb}$., 6.5 lb . and 7.2 lb . of pecans. Check your answers by using the table function on the calculator or by tracing the graph of the line.
b. How many pounds of pecans can a customer buy with $\$ 7.50$ ?
c. How many pounds of pecans can a customer buy with $\$ 9.30$ ?
d. How many pounds of pecans can a customer buy with $\$ 22.85$ ?
10. Use long division on the following division problems until you see a pattern: See TE.
a. $1000 \div 8$
b. $100 \div 8$
c. $10 \div 8$
d. $1 \div 8$

Use the number line in Exercise 8, part c , to plot and label the following quotients. See TE.
e. $1 \div 8$
f. $2 \div 8$
g. $3 \div 8$
h. $4 \div 8$
i. $5 \div 8$
11. Compute the following: See TE.
a. $2(1 \div 2)$
b. $4(1 \div 4)$
c. $5(1 \div 5)$
d. $8(1 \div 8)$
e. $10(1 \div 10)$
f. $10(1 \div 20)$
g. $(4 \div 5)(5 \div 4)$
h. $(5 \div 8)(8 \div 5)$
i. $(10 \div 4)(4 \div 10)$
12. Ingenuity:

Consider the following game that is played with two people and a row of 30 toothpicks. The rules are that on each turn a player can take away at most 4 toothpicks, but must take away at least 1 . The person who takes away the last toothpick loses. See TE.
a. Play a few rounds with a friend.
b. Suppose there are two toothpicks left and it is your turn. How many toothpicks should you take away? What should you do if there are five toothpicks left? If you follow your strategy are you guaranteed a victory?
c. What should you do if there are six toothpicks left? If your opponent knows your strategy from part $\mathbf{b}$ can you still win?
d. What should you do if there are $7,8,9$, or 10 toothpicks left? Who will win?
e. Can you think of a strategy for any number of toothpicks? Are there some numbers where you are guaranteed a victory?

## PERCENT COMPOSITION OF SUGAR IN CHEWING GUM

Objective: Students will use their knowledge of percents and fractions along with scientific procedure to determine the percent composition of sugar in chewing gum.

## Materials

1 stick of chewing gum per student
1 paper cup per group
Triple beam balance

## Activity Instructions:

For this activity we're assuming chewing gum has only 2 components: gum and sugar. The sugar will dissolve when you chew it, leaving behind only the gum. Thus with this information, we can calculate the percent of sugar in the gum.

The class should divide up into groups of 4-5 students. Each student gets a piece of gum and each group gets a paper cup.

First, find the mass of a paper cup. Then, unwrap the gum and put all the sticks of gum for the group in the cup and record its mass. (This is "total" gum mass - because it includes gum and sugar).

Then, each students should chew his/her piece of gum for 5 minutes. At the end of 5 minutes, each member of the group puts their chewed gum back into the cup. The new mass of the cup and 5 pieces of gum is recorded as the mass of the "chewed gum."

In order to calculate the mass of sugar in the gum students should subtract the mass of the chewed gum from the mass of the total gum. Note that his mass also includes the mass of the cup so be sure that your students subtract the mass of the cup from the mass of the sugar in the chewing gum.
(total gum mass - mass of chewed gum) - mass of paper cup = mass of sugar
In order to calculate the percent sugar in the gum students should take the mass of the sugar and divide by the total gum mass; this calculation will give a decimal value which should be multiplied by 100 to give the percent value.
mass sugar/ total gum $\times 100$

## Activity Worksheet

## Percent Composition of Chewing Gum

## Measurements:

A. Mass of paper cup $=$ $\qquad$
B. Mass total gum (before you chew it) + paper cup $=$ $\qquad$
C. Mass of chewed gum + paper cup $=$ $\qquad$

## Calculations:

D. Mass of total gum $(B-A)=$ $\qquad$
E. Mass of chewed gum $(C-A)=$ $\qquad$
F. Mass of sugar $(D-E)=$ $\qquad$
G. Percent comp. of sugar in chewing gum ((D/E) x 100) $=$ $\qquad$

## Conclusion Questions:

1) Would a dentist recommend chewing this gum? Why or why not?
2) Would changing the number of pieces of gum change the results of the lab? Explain.

## BAGS OF STUFF

Objective: This activity will help reinforce long division strategy.

## Materials:

Long division exercises
Stories about division

## Activity Instructions:

Teacher will guide students in filling in the boxes to solve long division application problems. Students will think of 539 oranges that need to be stored in bas of 4 oranges. They will figure out how many bags will be needed to store all oranges.

First write the dividends 'inside' the corner, and the divisor outside:

4 $\begin{array}{r}100 \\ 539\end{array}$
4) 539

| Hundreds | Tenths | Ones |
| :--- | :--- | :--- |
| How many 4's fit into 539? | How many 4's fit into 139? | How many 4's fit into 19? |
|  |  | 4 |
| 100 |  |  |
| $4 \longdiv { 5 3 9 }$ | 30 | 30 |
| $\frac{-400}{139}$ | $4 \boxed{539}$ | 100 |
|  | $\frac{-400}{139}$ | $4 \longdiv { 5 3 9 }$ |
|  | $\frac{-120}{19}$ | $\frac{-400}{139}$ |
|  |  | $\frac{-120}{19}$ |
|  |  | $\frac{-16}{3}$ |

The teacher will remind students and help them connect long division to the model given in the book. So they may write the above problem as follows:

Connection to division scaffolding:


So the final answer is 134 bags of oranges with 3 oranges left over.

## Section 6.4 - Division of Decimals

## Big Idea:

Using patterns in learning to divide with decimals

## Key Objectives:

- Use number sense and discover patterns that aid in division with decimal numbers.
- Understand how changes in the place value of the dividend affect the quotient.
- Understand how changes in the place value of the divisor affect the quotient.
- Recognize and use the symbols for repeating decimals.
- Use division to solve word problems and equations.


## Materials:

Calculators

## Pedagogical/Orchestration:

This section is really a number sense section with division. Encourage your students, especially the ones who are numero-phobic, to try to look for patterns and use them to make computation simpler and more understandable.

## Activity:

"Repeating Decimal Game" at the end of the section and on the CD. This can be done at end of lesson as culminating activity or as informal assessment.
"The Better Bargain" at the end of the section and on the CD.
"Big Water Bottle" at the end of the section and on the CD.
"Decimal Patterns with Division" at the end of the section and on the CD.

## Vocabulary:

quotient, dividend, divisor, repeating decimals, terminate

## TEKS:

7.1(B);
7.2(B); 7.3(B);
7.13(A); 7.15(A)(B);
8.14(A);
8.16(A)

## WARM-UPS for Section 6.4

1. a. How many $\$ 0.10$ (dimes) are tehre in $\$ 3.70$ ?
b. How many $\$ 0.25$ (quarters) are there in $\$ 14.50$ ?

Ans: 37
Ans: 58
c. How are the questions in parts a and b related to division?

Ans: $(37)(0.10)=3.70$ and $(58)(0.25)=14.50$

## Extra questions:

d. How many times does 0.2 divide into 2.8? Ans: 14 time because (14)(0.2) $=2.8$
e. How many times does 0.2 divide into 0.9 ? Ans: 4.5 times because (4.5)( 0.2 ) $=0.9$. You can think about this problem as how many $\$ 0.20$ are tehre in $\$ 0.90$. The answer is 4 with a half of a \$0.20 left over.
2. An amusement park manager is given $\$ 96$ to pay for tickets for rides on the Ferris Wheel for students who made the honor roll. If tickets cost $\$ 2.75$ each, how many students can have a free ride?
a. 32 students
b. 34 students
c. 35 students
d. 36 students

Ans: (c) because $96 \div 2.75=35$ with a remainder of $\$ 0.75$

## Launch for Section 6.2:

How many quarters are there in $\$ 1.25$ ? How many ways can you think about this problem? Notice part b in warm-up problem 1.

Expect to use different ways to model this problem.
(1) Skip counting to get 5 quarters.
(2) Divide 125 cents by 25 cents.
(3) Divide $\$ 1.25$ by $\$ 0.25$.

The first two ways students can already compute. The third way is DIVISION WITH DECIMALS.

## SECTION 6.4 DIVISION OF DECIMALS

In previous sections, we have used visual models of division to reexamine the process of long division. All of the division problems we have examined have had divisors that were integers.

## EXPLORATION

A. What is the effect of increasing the dividend by a factor of 10 ?

Complete the table below by performing the division indicated. For the last two calculations in each column, if you see a pattern, fill in the answer and check with the calculator, if necessary.

$$
\begin{aligned}
& 1 \div 4=0.25 \\
& 10 \div 4=2.5 \\
& 100 \div 4=25 \\
& 1000 \div 4=250
\end{aligned}
$$

What is the pattern?
B. What is the effect of increasing the divisor by a factor of 10 in the following sequence of division problems?

Complete the table below by performing the division indicated. For the last two calculations in each column, if you see a pattern, fill in the answer and check with the calculator, if necessary.

$$
\begin{aligned}
& 2 \div 1=2 \\
& 2 \div 10=0.2 \\
& 2 \div 100=0.02 \\
& 2 \div 1000=0.002
\end{aligned}
$$

What pattern do you notice?
C. What is the effect of decreasing the dividend by a factor of 10 ?

Complete the table below by performing the division indicated. For the last two calculations in each column, if you see a pattern, fill in the answer and check with the calculator

| $112 \div 7=16$ | $3276 \div 14=234$ |
| :--- | :--- |
| $11.2 \div 7=$ | $327.6 \div 14=$ |
| $1.12 \div 7=$ | $32.76 \div 14=$ |
| $0.112 \div 7=$ | $3.276 \div 14=$ |

What pattern do you notice? How does this pattern compare to the patterns you observed in parts A and B ?
D. What is the effect of decreasing the divisor by a factor of 10 ?

| $117 \div 9=13$ | $450 \div 18=25$ |
| :--- | :--- |
| $117 \div 0.9=$ | $450 \div 1.8=$ |
| $117 \div 0.09=$ | $450 \div 0.18=$ |
| $117 \div 0.009=$ | $450 \div 0.018=$ |
| $117 \div 0.0009=$ | $450 \div 0.0018=$ |

The trickiest long division problems are those from part D in which the divisor is a decimal number. It might not be clear where to place the decimal point in the quotient. There ia a connection between division by a decimal number and the patterns you observed in parts $A, B, C$, and $D$.

From the patterns you have observed, the quotient $0.36 \div 0.2$ is related to the quotient $3.6 \div 2=18$. IN part B, increasing the divisor by a factor of 10 decreases the quotient by a factor of 10 . In part A, increasing the dividend by a factor of 10 increases the quotient by a factor of 10 . When both the dividend and the divisor increase by a factor of 10 . the quotient remains the same as the original division problem.

The advantage of the second division, the transformed division problem, is that the divisor is a whole. If each of a and $b$ is a number and $b$ is not 0 , then

$$
\frac{a}{b}=\frac{10 a}{10 b}
$$

| Original division problem <br> with decimal divisor | Transformed division problem <br> with whole number divisor |
| :--- | :--- |
| $48 \div 0.4=$ | $480 / 4$ |
| $192 \div 1.2=$ | $1920 / 12$ |
| $0.324 \div 3.6=$ | $3.24 / 36$ |
| $14 \div 0.25=$ | $1400 / 25$ |
| $4.452 \div 0.84=$ | $445.2 / 84$ |

You have previously learned that a fraction is equivalent to division. IN the following tables, we will use the fractional way to represent division to help us see the patter of where the decimal point is located in the quotient.

| $\frac{342}{9}$ | $\frac{360}{80}$ | $\frac{5508}{34}$ |
| :---: | :---: | :---: |
| $\frac{34.2}{0.9}$ | $\frac{36}{8}$ |  |
| $\frac{34.2}{0.09}$ | $\frac{3.6}{0.8}$ |  |
| $\frac{3420}{90}$ | $\frac{0.36}{0.08}$ |  |

For the following division problems, find an appropriate transformed division problem by multiplying both the original dividend and divisor by the same power of 10. Then compute the answer to the transformed problem using division and check you answer for both with a calculator, if necessary.

| Original division problem <br> with decimal divisor | Transformed division problem <br> with whole number divisor |
| :--- | :---: |
| $48 \div 0.4=$ |  |
| $192 \div 1.2=$ |  |
| $0.324 \div 3.6=$ |  |
| $14 \div 0.25=$ |  |
| $4.452 \div 0.84=$ |  |

You might have noticed that all of the long division problems that we have considered have had a very nice property in common: at the end, every remainder is 0 . The quotients terminate or stop. This type of decimal number is called a terminating decimal. Of course, in real computations this is seldom the case. Look at the following division problems:
a. $1 \div 3$
b. $2 \div 3$
c. $5 \div 6$

Using the long division process,

| 0.3333... | 0.6666... | 0.8333... |
| :---: | :---: | :---: |
| 3) $1.0000 \ldots$ | 3) $2.0000 \ldots$ | 6 $5.0000 \ldots$ |
| -0 | -0 | -0 |
| 1.0 | 2.0 | 5.0 |
| -0.9 | -1.8 | -4.8 |
| 0.10 | 0.20 | 0.20 |
| -0.09 | -0.18 | -0.18 |
| 0.010 | 0.020 | 0.020 |
| $\underline{-0.009}$ | -0.009 | -0.018 |
| 0.0010 | 0.0020 | 0.0020 |
| -0.0009 | -0.0018 | -0.0018 |
| 0.0001 | 0.0002 | 0.0002 |
| . | . |  |

As you can see, the long division process in the three problems above never has a remainder of 0 . The decimals forms of the quotients of the division problems above are

$$
\begin{aligned}
& 1 \div 3=0.3333 \ldots=0.33 \overline{3}=0 . \overline{3} \\
& 2 \div 3=0.6666 \ldots=0.66 \overline{6}=0 . \overline{6} \\
& 1 \div 6=0.1666 \ldots=0.166 \overline{6}=0.1 \overline{6}
\end{aligned}
$$

These decimals number are called repeating decimals. When dividing 1 by 2 , the decimal form of the quotient is $1 \div 2=0.500$ $\qquad$ However, we do not include the zeros and o not call 0.5 a repeating decimal because the last remainder is 0 . For what positive integers $m$ and $n$ will the quotient $m \div n$ equal a repeating decimal? You can explore this question in the Repeating Decimal Game.

## EXERCISES

1. 

| $350 \div 10=35$ | $160 \div 3.2=50$ | $63 \div 0.75=84$ |
| :--- | :--- | :--- |
| $350 \div 100=3.5$ | $160 \div 32=5$ | $63 \div 7.5=8.4$ |
| $350 \div 1000=0.35$ | $160 \div 320=0.5$ | $63 \div 75=0.84$ |
| $350 \div 1000=0.035$ | $160 \div 3200=0.05$ | $63 \div 750=0.084$ |

2. 

a. 9
b. 62
c. 32
62
32
9
62
32
9
62
32
3.
a. 0.4
b. $0.3 \overline{3}$
0.4
$0.3 \overline{3}$
0.4
$0.3 \overline{3}$
0.4
$0.3 \overline{3}$
4. Remind your students that the answer to the transformed problem is the answer to the original problem

| Original division problem <br> with decimal divisor | Transformed division problem <br> with whole number divisor |
| :--- | :--- |
| $84 \div 2.4=$ | $840 \div 24=35$ |
| $56 \div 4.5=$ | $560 \div 45=12.4$ |
| $37.6 \div 0.16=$ | $3760 \div 16=235$ |
| $52 \div 1.25=$ | $5200 \div 125=41.6$ |
| $1.86 \div 0.027=$ | $1860 \div 27=68.8 \ldots$ |

5. 

a. $0.166 \overline{6}$
b. $0 . \overline{33}$
c. 0.5
d. Consider that $4 \div 6=(2+2) \div 6=(2 \div 6)+(2 \div 6)$ by a previous exercise.
Similarly, $5 \div 6=(2+3) \div 6=(2 \div 6)+(3 \div 6)$
Thus, $4 \div 6=0.1 \overline{66}$ and $5 \div 6=0.8 \overline{33}$

## EXERCISES

1. Complete the table below by performing the division indicated. For the last two calculations in each column, if you see a pattern, fill in the answer and check with the calculator, if necessary. See TE

| $350 \div 10=$ | $160 \div 3.2=$ | $63 \div 0.75=$ |
| :--- | :--- | :--- |
| $350 \div 100=$ | $160 \div 32=$ | $63 \div 7.5=$ |
| $350 \div 1000=$ | $160 \div 320=$ | $63 \div 75=$ |
| $350 \div 1000=$ | $160 \div 3200=$ | $63 \div 750=$ |

2. Compute the following quotients. Explain the results in each set of problems. See TE
a. $72 \div 8$
b. $868 \div 14$
c. $736 \div 23$
$720 \div 80$
$8680 \div 140$
$7360 \div 230$
$7.2 \div 0.8$
$86.8 \div 1.4$
$73.6 \div 2.3$
$0.72 \div 0.08$
$8.68 \div 0.14$
$7.36 \div 0.23$
3. Compute the following quotients. Leave the quotient in decimal form and check your answers with a calculator, if necessary. What do you notice about each set of problems? How can this help you compute the quotients quickly? See TE
a. $2 \div 5$
b. $1 \div 3$
$4 \div 10$
$2 \div 6$
$8 \div 20$
$3 \div 9$
$16 \div 40$
$4 \div 12$
4. For the following division problems, first find an appropriate transformed division problem. Then, perform division on the transformed problem and check your answer with a calculator. See TE

| Original division problem <br> with decimal divisor | Transformed division problem <br> with whole number divisor |
| :--- | :---: |
| $84 \div 2.4=$ |  |
| $56 \div 4.5=$ |  |
| $37.6 \div 0.16=$ |  |
| $52 \div 1.25=$ |  |
| $1.86 \div 0.027=$ |  |

5. Compute the following quotients and check with a calculator, if necessary. See TE
a. $1 \div 6$
b $2 \div 6$
c. $3 \div 6$
d. Predict what $4 \div 6$ and $5 \div 6$ are.
6. a. Variables: $n=$ number of yards of fabric Sally purchased.

Translate: $64=2.5 n$
Solve: $n=25.6$
b. Variables: $m=$ Meters of computer cable Maria purchased.

Translate: $6.4=2.5 \mathrm{~m}$
Solve: $m=2.56$
Check: (2.56)2.25 $=6.4$
c. Variables: $\mathrm{c}=$ pounds of candy Gary purchased.

Translate: $0.64=2.5 \mathrm{c}$
Solve: $\mathrm{c}=0.256$
Check: (0.256)2.5 $=0.64$
7.

| Pounds of <br> Cheese $(x)$ | Cost <br> $(y)$ |
| :---: | :---: |
| 0 | 0 |
| 0.25 | 1 |
| 0.5 | 2 |
| 0.75 | 3 |
| 1 | 4 |
| 1.5 | 6 |
| 2 | 8 |
| 2.5 | 10 |
| 3 | 12 |

a. $y=4 x$
b. The cost for 1.25 lb . of cheese would be found between the cost for 1 lb . of cheese and 1.5 lb . of cheese, it would be $\$ 5$.
d. You can buy 0.375 lb . of cheese with $\$ 1.50,1.25 \mathrm{lb}$. with $\$ 5,1.75 \mathrm{lb}$. with $\$ 7$.
e. You can buy 2.21 lb . with $\$ 8.84$.
8. a. Sandy can buy 26 candy bars. Her change is 10 cents.
b. Sandy can buy 3 bags of chips. Her change is 34 cents.
9.
a. 60
b. 40
c. 60
60
40
60

## Ingenuity

10. Consider evaluating $1 \div 7$ by long division. Each step give one of 6 possible remainders - the positive integers less than 7 - or remainder 0 , which means the decimal terminates. As soon as the remainder repeats, the decimal quotient repeats as well. Extending this logic to the general case, there are $n-1$ positive remainders less than $n$, so the longest possible cycle of $1 \div n$ is $n-1$.
11. Write an equation for each problem and solve it. See TE
a. Sally bought $\$ 64$ worth of fabric. Each yard cost $\$ 2.50$. How many yards of fabric did Sally purchase?
b. Maria paid $\$ 6.40$ for some computer cable. The computer cable cost $\$ 2.50$ per meter. How many meters of cable did she buy?
c. Gary paid $\$ 0.64$ for a bag of candy. The candy sold for $\$ 2.50$ per pound. How many pounds of candy did he buy?
12. A small grocery store cells cheddar cheese for $\$ 4.00$ per pound. See TE
a. Let $x$ be the number of pounds of cheese and $y$, the cost of $x$ pounds of cheese. What is the equation for $y$ that describes the relationship between pounds of cheese and cost?
b. What is the cost of 1.25 pounds of cheese? Where would you find this cost in a table for the line given by the equation in part a?
c. Copy and fill the adjacent table. Plot these points on a coordinate system.
d. If you shop at the store, how much cheese can you buy with \$1.50? \$5.00? \$7.00?

| Pounds of <br> Cheese $(x)$ | Cost <br> $(y)$ |
| :---: | :---: |
| 0 |  |
| 0.25 |  |
| 0.5 |  |
| 0.75 |  |
| 1 |  |
| 1.5 |  |
| 2 |  |
| 2.5 |  |
| 3 |  |

e. How much cheese can you buy with $\$ 8.84$ ?
8. A store sells candy bars for $\$ 0.65$ each and small bags of chips for $\$ 0.49$ each. Sandy has $\$ 17.00$ to spend. See TE
a. How many candy bars can she buy? What is her change?
b. How many bags of chips can she buy? What is her change?
9. Solve the following equations: See TE
a. $25 x=1500$
$0.25 x=15$
b. $4 x=160$
$0.4 x=16$
c. $5 x+600=900$
$0.05 x+6=9$

## 10. Ingenuity:

The quotient $1 \div 7$ is the repeating decimal $0.142867142857142857 \ldots$ and has a 6 -digit cycle. Explain why the quotient $1 \div 7$ could not be a repeating decimal with a cycle longer than 6 . If $n$ is a positive integer, what is the longest possible cycle $1 \div n$ ? See TE

## Investigation

11. Using this technique, we are able to take the repeating decimal and convert the equation to one with two integers, so that the repeating decimal is the quotient of the two integers.
a. $9 \longdiv { 5 }$
b. $9 9 \longdiv { 2 5 }$
c. 99) 12
d. $9 \longdiv { 1 . 5 }$ or $1 8 \longdiv { 3 }$
e. 9) 9 or 1
12. b. Hint If $x=0.2525 \ldots$, then $100 x=25.2525 \ldots$
13. e. Another way to prove that $1=0.999 \ldots$

$$
\begin{gathered}
\frac{1}{3}+\frac{2}{3}=0.333 \ldots+0.666 \ldots \\
1=0.999 \ldots
\end{gathered}
$$

## 11. Investigation:

In this section, you discovered that the quotient $10 \div 3$ is ten times the quotient $1 \div 3$. If we let $x=0.33 \ldots$, then $10 x=3.33 \ldots$. Subtracting equal quantities from both sides,

$$
\begin{aligned}
10 x & =3.333 \ldots \\
-x & =0.333 \ldots \\
9 x & =3.000 \ldots
\end{aligned}
$$

So $9 x=3$. Remember the missing factor form, which says that $x$ is the quotient $3 \div 9$. When we divide 3 by 9 , we compute the quotient $3 \div 9=0.333 \ldots$. Using the same technique, find a division problem that equals the repeating decimal. See TE
a. $0.55 \overline{5} \ldots$
b. $0.25 \overline{25} \ldots$
c. $0.12 \overline{12} \ldots$
d. 0.999...

## BIG WATER BOTTLE

Objective: The students will use all three models of division to solve a problem.

## Materials:

Notebook paper

## Activity Instructions:

The teacher will give the students the following problem:

A 32 oz. bottle of water needs to be evenly distributed between five kids. How much water will each child get, and how much will be left over?

The students will show the solution to this problem using all three models described in their math books (Linear Model, Area Model, and Long Division.)

As a challenge the teacher could change the size of the bottle or the number of kids that will be sharing the water. Students could display their solutions on poster size paper, and then share their solutions with the class.

## THE BETTER BARGAIN

Objective: Students will use real world applications in determining the better bargain using division and multiplication of whole numbers and decimals.

## Materials:

"To Bur or Not to Buy" student copy (1 per student)
"To Buy or Not to Buy" team copy (1 per team)
3 cereal boxes of choice-including its cost (1 per team member)
$\frac{1}{2}$ sheet poster board
Tape

## Activity Instructions:

1) Student are put in groups of 3 or 4 .
2) Students are asked to bring to class a box or bag of their favorite cereal.
3) Students answer the questions on their activity worksheet.
4) Students tape or staple the cereal boxes and activity worksheet to the $\frac{1}{2}$ sheet poster board for presentation and display for their class.

## To Buy Or Not To Buy <br> (Team Copy)

1. Examine the cereal boxes and predict which cereal is the better bargain
$\qquad$ Explain your thinking: $\qquad$
$\qquad$
2. Complete the student copy. Record each team member's results in the table.

| Team Member | Cereal Name | Size (Ounces) | Cost | Number of <br> Servings | Cost Per <br> Serving |
| :--- | :--- | :--- | :--- | :--- | :--- |
|  |  |  | $\$$ |  | $\$$ |
|  |  |  | $\$$ |  | $\$$ |
|  |  |  | $\$$ | $\$$ |  |
|  |  |  | $\$$ | $\$$ |  |

3. Discuss your findings.
4. Your team decides to buy all the boxes of cereal, how much will you pay the cashier?
5. Which cereal was the better bargain?

Was the team prediction correct?
Yes
or
No
6. List 3 things that can affect a consumer's decision when buying cereal or any other product:
A.
B. $\qquad$
C. $\qquad$

To Buy Or Not To Buy
(Student Copy)

| Team Member | Cereal Name | Size (Ounces) | Cost | Number of <br> Servings | Cost Per <br> Serving |
| :--- | :--- | :--- | :--- | :--- | :--- |
|  |  |  |  |  |  |
|  |  | $\$$ |  |  | $\$$ |

1. Record the following in the table:
A. Cereal
B. Total cost of cereal
C. Total serving size in ounces. (See front or side of box)
D. The number of servings per box
E. Cost per serving: *to find the cost per serving, divide the price of the cereal by the total number of servings per box.
2. Which operations(s) did you use? $\qquad$
3. If you paid the cashier with a $\$ 20$ bill to pay for your box of cereal, how much change would you receive?

$$
\$
$$

$\qquad$
4. If you wanted to buy 3 boxes of your cereal, how much would you pay the cashier?
\$ $\qquad$
5. If you paid the cashier with a $\$ 20$ bill to pay for all the team boxes of cereal, would you have enough money?

YES or No (Circle One)
If yes, how much change would you receive? \$ $\qquad$
If no, how much more money would you need? \$ $\qquad$
6. If you wanted to buy 5 boxes of each cereal and share the cost equally with your team, how much would each of you have to pay?
\$ $\qquad$

## DECIMAL PATTERNS WITH DIVISION

Objective: Use the skip counting and the linear model on the number line to perform division. This activity is meant to be a supplement to Explorations A and B.

## Material:

A number line from 0 to 1 subdivided into hundredths.

## Activity Instructions:

Use the skip counting and the linear model on the number line to perform the division in problems $a, b$ and $c$. Predict the answer for problem d. Check your answers with the calculator, if necessary.
a. $1 \div 1=$ $\qquad$ b. $1 \div 0.1=$
c. $1 \div 0.01=$
d. $1 \div 0.001=$

Teachers, remind your students that $a \div b$ means a divided by b. Expect them to skip count by 0.1 and 0.01 in parts $b$ and $c$.

Do you notice a pattern? Compare this pattern with the pattern discussed in the previous examples.
As the divisor is decreased by a factor of 10, the quotient increases by a factor of ten.
Extension 1: What is the effect of changing the dividend in the problems above? Use the linear model on the same number line to perform the division problems $\mathrm{a}, \mathrm{b}$ and c . Predict the quotient in part d and check with a calculator, if necessary.
a. $0.2 \div 0.1=$
b. $0.43 \div 0.01=$ $\qquad$ c. $0.07 \div 0.01=$ $\qquad$
d. $0.27 \div 0.001=$ $\qquad$

Extension 2: Using the same number line, perform the division in problems $\mathrm{a}, \mathrm{b}$ and c . Predict the answer for problem d. Check your answers with the calculator, if necessary.
a. $1 \div 5=$ $\qquad$
b. $1 \div 0.5=$ $\qquad$
c. $1 \div 0.05=$ $\qquad$
d. $1 \div 0.005=$ $\qquad$

## REPEATING DECIMAL GAME

Objective: The students will find division problems that produce repeating decimals, with the use of a calculator. Students will be challenged in this activity to find different division problems that will produce a variety of number of digits that repeat.

## Material:

Notebook paper

## Activity Instructions:

The teacher will divide the class up into groups of three to four students. Each group will be asked to find division problems $n \div m$, with $n$ and $m$ positive integers, so that the quotient of $n \div m$ is a repeating decimal. Each team must show why the quotient is repeating without using a calculator. Challenge teams to discover division problems that produce quotients which have cycles of 2 digits, i.e., $14 \div 33=0.424242$... Can they find quotients which have cycles of 3 digits? 4 digits? ...and so on. Note that for example that $1 \div 7=$ $0.142857142857142857 \ldots$ has a 6 digit cycle.

To make this activity competitive for your class, challenge the groups to first find how many different cycles of repeating digits they can find. Once you have a group that wins this challenge, then you can encourage them to find as many division problems as they can per cycle. The group that has the most division problems for each particular cycle will then be able to win too.

1. a. between 0.1 and 1
c. between 0.1 and 1
b. between 10 and 100
d. between 100 and 1000
2. $(\$ 3.95)(7.4)=\$ 29.23$
3. $(\$ 5.60)(2.4)=\$ 13.44$
4. a. $7(\mathrm{rO})$
c. 35 (r3)
b. 10 (r10)
d. 3 (r10)
5. a. $4(\mathrm{rO})$
c. 3 (r8)
b. 7 (r8)
d. 8 (r18)
6. a. 72 (r0)
b. 25 (r19)
c. 537 (r17)
7. a. 52 (r2)
b. 24 (r29)
c. $\quad 985(\mathrm{r} 21)$
8. a. $103(\mathrm{r} 2)$
b. 163 ( r 3 )
c. 107 (r202)
9. a. $122(r 1)$
b. 284 (r4)
c. $250(\mathrm{r} 22)$

## REVIEW PROBLEMS

1. Predict whether the product is between 0.01 and 0.1 , between 0.1 and 1 , between 1 and 10, between 10 and 100, or between 100 and 1000. Then, compute each and describe the pattern.
a. (0.24)(3.8)
c. $\quad(24)(0.038)$
b. $(0.024)(3800)$
d. (2.4)(380)
2. Jimmy filled his car last Monday. Gas cost $\$ 3.95$ per gallon and he bought 7.4 gallons. How much did Jimmy pay?
3. Tim is at the fish market. Grouper sells for $\$ 5.60$ a pound and Tim buys 2.4 pounds of it. How much does Tim pay for the fish?
4. Use the linear model to find the quotient that makes the remainder as small as possible:
a. $21 \div 3$
b. $120 \div 11$
c. $143 \div 4$
d. $94 \div 28$
5. Use the area model to find the quotient that makes the remainder as small as possible:
a. $24 \div 6$
b. $71 \div 9$
c. $44 \div 12$
d. $210 \div 24$
6. Find the quotient and remainder using scaffolding.
a. $216 \div 3$
b. $819 \div 32$
c. $39218 \div 73$
7. Find the quotient using scaffolding.
a. $210 \div 4$
b. $989 \div 40$
c. $78821 \div 80$
8. Find the quotient and remainder using long division.
a. $311 \div 3$
b. $2122 \div 13$
c. $34121 \div 317$
9. Find the quotient using long division.
a. $611 \div 5$
b. $3412 \div 12$
c. $12022 \div 48$
10. 11 feet of ribbon $=132$ inches of ribbon. 132 in $\div 14$ in $=9$ bows ( 6 inches of ribbon left over).
11. $\$ 1.35 \div \$ 3.60=0.375$ pounds of candy
12. a. $x=27 \div 4=6.75$
d. $3.18 x=1.06, x=1.06 \div 3.18=0.333 \ldots$
b. $6 x=96, x=96 \div 6=16$
e. $x=20.93 \div 2.3=9.1$
c. $x=1.275 \div 6.25=0.204$
f. $x=19.1 \div-4=-4.775$
13. a. $2100 \div 42=50$
d. $4000 \div 125=32$
b. $560 \div 15=37.333 \ldots$
e. $3720 \div 54=68.888 \ldots$
c. $3760 \div 32=117.5$
f. $6.4 \div 512=0.0125$
14. a. no
b. yes
c. no
d. yes
15. Mary has an 11 -foot piece of ribbon. She is making a number of hair bows each 14 inches long. How many hair bows can she make?
16. Alex pays $\$ 1.35$ for a bag of candy. The candy sold for $\$ 3.60$ a pound. How many pounds of candy did he buy?
17. Solve the following equations:
a. $4 x=27$
b. $6 x-7=92$
c. $6.25 x=1.275$
d. $3.18 x-1.5=6.45$
e. $2.3 x=20.93$
f. $-4 x=19.1$
18. Transform the division problems into problems with whole number divisors and then compute the quotient.
a. $210 \div 4.2$
b. $56 \div 1.5$
c. $37.6 \div .32$
d. $40 \div 1.25$
e. $3.72 \div 0.054$
f. $0.064 \div 5.12$
19. Determine which of the following quotients is equivalent to the quotient $42.65 \div 3.84$.
a. $4.265 \div 3.84$
b. $4.265 \div .384$
c. $426.5 \div 384$
d. $426.5 \div 38.4$
 to

$$
\begin{aligned}
& \frac{15}{\frac{23}{20}} \\
& \frac{30}{30} \\
& \frac{30}{=}
\end{aligned}
$$

$\stackrel{1}{=}$
$-\frac{3}{2}$
$-\frac{0}{0}$
$=0$
$\frac{1 \mathrm{e}}{\underline{\mathrm{f}} \frac{3}{2 \underline{g}}}$
$\frac{\mathrm{~h} 0}{\frac{\mathrm{i} 0}{2}}$
0
make the final difference 0 . Now $f$ can be 2 with $a=5$ and $g=0$ or $f$ can be 3 with $a=7$ and $g=8$. If $a$ is 7 , then

$\underline{\underline{1 e}}$
$\frac{2}{2} \frac{3}{2}$
$\frac{\text { h0 }}{\frac{h 0}{0}}$
0
We see that $h$ is 3 , so $d$ is 6 , leaving us only 3 values to determine. Now e can be 0 with $b=2$ and $c=2$, or e can be 5 with $\mathrm{b}=7$ and $\mathrm{c}=3$. Since one number has to be 7 , we conclude that e is 5 and we're done.

Section 6.4: $\mathrm{a}=24$ and $\mathrm{d}=6$
Solution: Since $a=\frac{1}{0.041 \overline{\mathrm{~d}}}$, we compute $\frac{1}{0.014}=24.39 \ldots$ and $\frac{1}{0.042}$, thus a must be 24. Finally, $\frac{1}{24}=0.041 \overline{6}$

## CHALLENGE PROBLEMS

## Section 6.3:

While performing a trick of long division, a mathemagician made some of his digits disappear. Alas, he cannot reconstruct the missing digits... until he remembers that one of them is a 7 . Fill in the blanks to complete the calculation.


## Section 6.4:

If $1 \div a=0.041 \bar{d}$ for a positive integer $a$ and $a$ single digit $d$, find $a$ and $d$.

## Section 7.1 - Divisibility, Factors and Multiples

## Big Idea:

Discovering relationship of factors, divisors, multiples and division

## Key Objectives:

Introduce the primes by discovery and not by formal definition at this time.

## Review:

Focus on Fourth Paragraph of 7.1 where we define "divisible," "factor," and "multiple."
What is a factor?
What are multiples?
Divisibility Rules

## Materials:

Graph paper, Blackline masters for Possible Rectangles and Sieve of Eratosthenes, Map colors or Markers for Sieve activity

## Pedagogical/Orchestration:

Two big activities in this section: Possible Rectangles and Sieve of Eratosthenes.

- Prime number is not defined until the reflection in Section 7.2. This section foreshadows the ideas of prime and composite numbers. Students may try to articulate the properties of these two types of numbers.
- Exploration 1, step 2, third sentence talks about "number of rectangles possible with area n." Remind the students about what area is. The concept of area is taught in 5th grade, maybe earlier.
- Exploration 1, step 9 foreshadows primes: numbers that have exactly 2 factors; the factor 1 and the factor itself.
- Exploration 1, step 10 foreshadows perfect squares.
- Another pattern is factor pairs. The dimensions created in the Possible Rectangle Dimensions column are called factor pairs. When these factor pairs are multiplied, they result in the value of column $n$.


## Activities:

"Possible Rectangles"; "Sieve of Eratosthenes"; "Divisibility War" (play at end of Section 7.1), "Product Bingo" and "Divisibility Rules" at the end of the section and on the CD

## Vocabulary:

divisible, factor, multiple, factor pair, simplified, product, common factor, common multiple, sieve

## TEKS:

$6.1(\mathrm{E})(\mathrm{F}) ; \quad 7.13(\mathrm{C})(\mathrm{D}) ; \quad 7.14(\mathrm{~A}) ; \quad 7.15(\mathrm{~A}) ; \quad 8.15(\mathrm{~A}) ; \quad 8.16(\mathrm{~A})$;

## WARM-UPS for Section 7.1

1. Valerie has 144 pennies. At the moment she has one pile with 144 pennies. She wants to explore different ways she can place the coins into piles of equal size. Which of the following is not a possible number of equal sized piles?
a. 36 piles
b. 24 piles
c. 20 piles
d. 18 piles

Ans: d
2. Which pairs of numbers are equivalent?
a. $\frac{2}{5}, \frac{8}{20}$ Answer: yes
b. $\frac{3}{7}, \frac{10}{28}$ Answer: no
c. $\frac{9}{12}, \frac{15}{20}$ Answer: yes, both are equivalent to $\frac{3}{4}$

## Launch for Section 7.1:

Tell your students, "Today's big idea is divisibility, factors and multiples. I am going to show you a definition of divisibility that is in the book and we are going to try to make sense of it." Write or project the following on the board:

> | DEFINITION 7.1: DIVISIBILITY |
| :--- |
| Suppose that $n$ and $d$ are integers, and that $d$ is not 0 . The number |
| $n$ is divisible by $d$ if there is an integer $q$ such that $n=d \cdot q$. Equiva- |
| lently, $d$ is a factor of $n$ or $n$ is a multiple of $d$. |

Ask your students to think about the definition for a minute. Then tell your students that at first the definition may sound like it's in a foreign language, but they are going to be given a way to help them translate it. Write the variables $n, d$, and $q$ on the board and create a table for values that you try. Ask students for some integers to substitute for $n$ and $d$. Pick some easy integers such as $n=3$ and $d=6$, and say, "We are checking to see if $n$ is divisible by $d$, or in this case if 3 is divisible by 6 . Is there an integer $q$ such that $3=6 \cdot q$ ? Since there is not, put the word "none" under $q$. What if we interchanged the values for the variables and let $n=6$ and $d=3$. This time the equation will read $6=3 \cdot q$. The value of $q$ is 2 and so using the above definition, 6 is divisible by 3 and equivalently, 3 is a factor of 6 , and 6 is a multiple of 3 . Put the values in the table and try some other values for $n$ and $d$. Once the students have a grasp of the definition, tell them, "Today you will be using this definition to connect factors to division and multiples, and to make a table of some special numbers."

You can revisit the Linear Model developed in Chapter 4 with division or use 3 groups of 4 marbles with 2 left over.

Teacher Tip: There are a lot of new words and phrases introduced in this section, make a chart to correlate the relationships between factor, multiple and divisible.

## SECTION 7.1 DIVISIBILITY, FACTORS AND MULTIPLES

One of the most important concepts in mathematics is the idea of divisibility. Suppose you have 14 marbles, and you want to give the same number of marbles to each of three friends. Is it possible to give each friend the same number and have none left?

You can let each person have 4 marbles, but there are two left. This process is called division. You divide 14 by 3 to get the quotient 4 with remainder 2 . This is equivalent to the calculation $14=3 \cdot 4+2$. In building the multiplication table for 3 , your skip counting by 3 , starting at 0 , does not list 14 .

On the other hand, if you have exactly 12 marbles, you can give each friend 4 marbles, and everybody has an equal number of marbles. This process corresponds to $12=3 \cdot 4$.

What does this have to do with divisibility? We know that 14 objects cannot be divided equally among 3 people. Another way to say this is, " 14 is not divisible by 3, " or " 3 is not a factor of 14 ." Note that this is equivalent to " 14 is not a multiple of 3 ." On the other hand, we can divide 12 things equally among 3 people. Mathematically, " 12 is divisible by 3 " means " 3 is a factor of 12 ." The last statement is equivalent to " 12 is a multiple of 3." Although there is no integer that you can multiply by 3 to equal 14 , there is the integer 4 that you can multiply by 3 to equal 12 .

## DEFINIIION 7.1: DIVISIBILITY

Suppose that $n$ and $d$ are integers, and that $d$ is not 0 . The number $n$ is divisible by $d$ if there is an integer $q$ such that $n=d \cdot q$. Equivalently, $d$ is a factor of $n$ and $n$ is a multiple of $d$.

As a lead into Exploration 1, review the idea of a rectangle. Point out that we are picking $n$ to be the area and are trying to find the length and width that will give us this area of $n$ square units.

## EXPLORATION 1: THE POSSIBLE RECTANGLE MODEL

Materials: Teachers, make copies of the Possible Rectangle Table on the CD, or the black line master in the Teacher's Supplement

1. In small groups, you can have different groups do different values of $n$. Students can draw rectangles on graph paper and post on the wall as the number n increases. They can also use tiles to physically make the rectangles in groups. Have your students share their results in small groups. After an initial period, have a class discussion about several possible values for $n$, like 6, 7, 8 and 9 . Then continue either individually or in small groups through 20. For each $n$, give students who have discovered rectangles record their results on a bulletin grid paper in a prominent place. Record on Class Chart as groups report. Reflect with them what the different columns of the chart are telling us.

For example, we know that 12 is divisible by 3 because $12=3 \cdot 4$. We can show this using our marble example:


$$
\begin{aligned}
\text { Factor } \cdot \text { Factor } & =\text { Product } \\
3 \cdot 4 & =12
\end{aligned}
$$

Notice that the twelve marbles are in a neat, rectangular array. This suggests another way for us to see the divisors of a positive integer. We'll experiment with this in the following activity.

## EXPLORATION 1: THE POSSIBLE RECTANGLE MODEL

Materials: You will need graph paper for this activity.

1. For each positive integer $n$ from 1 to 20 , make as many rectangles with integer side lengths as you can that have area equal to $n$ square units.
2. It might be good to point out that the values of the sides are factors of $n$ and that they come in pairs. You might dramatize this by connecting the pairs with an arch.

| $n$ | Number of <br> Possible <br> Rectangles | Possible Rectangle <br> Dimensions | Possible Side <br> Lengths (in <br> increasing order) |
| :---: | :--- | :--- | :--- |
| 5 | 1 | $1 \times 5$ | 1,5 |
| 6 | 2 | $1 \times 6,2 \times 3$ | $1,2,3,6$ |
| 7 | 1 | $1 \times 7$ | 1,7 |
| 8 | 2 | $1 \times 8,2 \times 4$ | $1,2,4,8$ |
| 9 | 2 | $1 \times 9,3 \times 3$ | $1,3,9$ |
| 10 | 2 | $1 \times 10,2 \times 5$ | $1,2,5,10$ |
| 11 | 1 | $1 \times 11$ | 1,11 |
| 12 | 3 | $1 \times 12,2 \times 6,3 \times 4$ | $1,2,3,4,6,12$ |
| 13 | 1 | $1 \times 13$ | 1,13 |
| 14 | 2 | $1 \times 14,2 \times 7$ | $1,2,7,14$ |
| 15 | 2 | $1 \times 15,3 \times 5$ | $1,3,5,15$ |
| 16 | 3 | $1 \times 16,2 \times 8,4 \times 4$ | $1,2,4,8,16$ |
| 17 | 1 | $1 \times 17$ | 1,17 |
| 18 | 3 | $1 \times 18,2 \times 9,3 \times 6$ | $1,2,3,6,9,18$ |
| 19 | 1 | $1 \times 20$ | 1,19 |
| 20 | 3 |  | $1 \times 10,4 \times 5$ |
|  |  | $2,4,5,10,20$ |  |


$\left.$| $n$ | Number of <br> Possible <br> Rectangles | Possible Rectangle |
| :--- | :--- | :--- | :--- |
| Dimensions |  |  |$\quad$| Possible Side |
| :---: |
| Lengths (in |
| increasing order) | \right\rvert\,

3. We are foreshadowing the idea of composites and primes. The definition of these are in the next section. Don't spoil students' attempts at articulating what the properties of these numbers are. They might notice (1) some $n$ have only 1 possible rectangle because they are primes or (2) a few have an odd number of possible side lengths because they are perfect squares. If your students don't notice any patterns yet, the following steps will lead them to discover the patterns. If your class has already discovered the factors, go to Step 6.
4. If students are confident with filling in the table fo numbers $1-20$, you may skip \#4 since a completed version of this chart is in Section 7.2.
5. Organize the data in a table provided by your teacher. In the first column, write the positive integer $n$. In the second column, write the number of rectangles possible with area $n$. In the third column, list all the possible dimensions of the rectangles. In the fourth column, list all the possible lengths of sides of the rectangles, in increasing order. For example, we have filled in the results for the value of $n=4$ on the table.

| $n$ | Number of <br> Possible <br> Rectangles | Possible Rectangle <br> Dimensions | Possible Side <br> Lengths (in <br> increasing order) |
| :---: | :--- | :--- | :--- |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  | $1,2,4$ |
| 4 | 2 |  |  |
| 5 |  |  |  |
| 6 |  |  |  |
| 7 |  |  |  |
| 8 |  |  |  |
| 9 |  |  |  |
| 10 |  |  |  |
| 11 |  |  |  |
| 12 |  |  |  |
| 13 |  |  |  |
| 14 |  |  |  |
| 15 |  |  |  |
| 16 |  |  |  |
| 17 |  |  |  |
| 18 |  |  |  |
| 19 |  |  |  |
| 20 |  |  |  |

3. What do you notice in the table so far? Answers will vary.
4. Without drawing rectangles, continue the table for $n$ from 21 to 40 .
5. Looking at the extended table, do the patterns continue?
6. Each is a list of all the factors of $n$.
7. Factors or divisors. Have students draw lines above the list in the last column connecting factor pairs. This forms a factor rainbow. The only exception is for perfect squares, where there is a factor in the middle with no pair, such as $1,2,4,8,16$ for $n=16$. This foreshadows the question in step 10 , where the answer is perfect squares with the odd factor in the middle of the list.
8. This might be a good time to talk about how important and unique the number 1 is in multiplication. That is why mathematicians call it the multiplicative identity. Multiplying any number, $n$, by 1 gives you the identically same number $n$. In the next exploration, your students will find that 1 is also the only number that is neither prime nor composite.
9. $3,5,7,11,13,17$, and 19 all generate only one rectangle. They are called primes. A formal definition will be given in Section 7.2. These numbers each have exactly 2 factors, the number itself and 1.
10. They are the perfect squares. You might ask your students why the number is odd. This could lead to the discussion of multiple factors. We say that 2 is a factor of 4 with multiplicity 2.

## EXAMPLE 1

Note the different ways a question can be asked. It is very important to make students comfortable with the language of mathematics.
6. Looking at a given number $n$, what do you notice about the numbers in the last column for this value of $n$ ? See TE.
7. What do we call the numbers in the last column in relation to $n$ ? For each rectangle, the dimensions form a factor pair, such as 3 and 6 for $n=18$. If you put all the factors in the last column in order, such as $1,2,3,4,6,12$ for $n=12$, how do the factor pairs line up? See TE.
8. What do you notice about the number 1? Find any other numbers that have this same property, if possible. No
9. Circle the values of $n$ that generate only one rectangle. See TE.

How many factors does each of these have? See TE.
How would you describe the circled numbers, excluding 1? See TE.
10. Use a different color pen or marker to box the values of $n$ that have an odd number of positive divisors. See TE

Notice that all of the factors in our chart are positive. Generally when talking about factors, we just mean the positive factors.

Let's explore some strategies for finding all the factors or divisors of a given positive integer $n$. In particular, if $n$ is a positive integer and $k$ is a positive integer, how do we determine whether $k$ is a factor of $n$; or equivalently, whether $n$ is a multiple of $k$ ? The method used in the Possible Rectangle Activity works well for small numbers but does not work as well for larger numbers. So we want to find a method that can be used when we have to deal with large numbers.

## EXAMPLE 1

Is 13 a factor of the number 798? Equivalently, is 798 a multiple of 13 ?

## SOLUTION

Starting with the second question, we could skip count by 13 to determine whether 798 is a multiple of 13 . However, this is not an efficient strategy. Instead, ask if there is a positive integer $q$ such that $798=13 \cdot q$. You can answer this using long division, which results in a quotient of 61 and a remainder of 5 . This means

## PROBLEM 1

The remainder is not 0 in the non-divisible case and the remainder is 0 in the divisible case. We usually say that 15 divides "evenly into 825 " rather than "into 825 with remainder 0 ."
a. One way to answer this question is for students to divide 825 by 15 . The students will find that $825 \div 15=55$. Therefore, we conclude 825 is divisible by 15 because $15 \cdot 55=825$. Another way to ask this question is what times 15 equals 825 ? Students may try different numbers and come up with 55 through some process of trial and error. The students should note that we say 825 is divisible by 15 because there is an integer that when multiplies by 15 equals 825 .
b. Any number divisible by 15 is also divisible by 3 and 5 . Why should this be true? One way to note this is that $825=15 \mathrm{n}$ for some integer n . Notice that $15=3 \cdot 5 \cdot n$. if $825=15 \mathrm{n}$, then $825=3 \cdot 5 \cdot n$ and we can write that $825=3 \mathrm{~m}$ for some integer m and $825=5 \mathrm{r}$ some other integer r .

## EXPLORATION 2: SIEVE OF ERATOSTHENES

A sieve is a meshed appliance that filters and separates particles usually by size, letting small particles pass through the mesh while capturing larger particles. The Sieve of Eratosthenes is not an appliance, but is a process that separates numbers, in our case removing certain numbers and keeping others. This is a way to separate and keep the prime numbers while removing the composite numbers.

1. You can get this from the black line masters in the Teacher Supplement.
2. Do not refer to prime and composite yet because we have not defined them. Just say that we cross out the number " 1 " because it is special (the multiplicative identity).
3. Ask your students to look for visual patterns or short cuts as they go through this activity. At this point in the activity, half of the numbers should be marked out from the columns headed by the even numbers.
4. The first number is 3 because 3 is the second prime. Again, as always with mathematics, ask your students to be alert for patterns.
5. The circled numbers $2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89,97$ are called primes. Students may notice that each is not a multiple of any natural number that comes before it in the original list except for 1 . Ask your students to notice any pattern that involved these same numbers in Possible Rectangle Activity. They were the numbers that had only one rectangle. Make sure your students know that these are the prime numbers. We will give a careful definition in Section 7.2.
that $798=13 \cdot 61+5$. The goal is to find an integer $q$ so that $798=13 q$. If you skip counted by 13 , you would not land on 798 . That is, there is no integer $q$ for which $798=13 q$. So 13 is not a factor of 798 .

## PROBLEM 1

a. Is the number 825 divisible by 15 ?
b. Is every number divisible by 15 also divisible by 3 and 5? Explain.

What distinction in the remainders did you notice between the examples of the divisible case and the not divisible case? Check with a few other examples to confirm that the distinction holds in those cases.

## EXPLORATION 2: SIEVE OF ERATOSTHENES

This exploration is based on an ancient method attributed to a famous Greek mathematician, Eratosthenes of Cyrene. The process involves letting a certain kind of number pass through the sieve leaving only another kind of number left in the sieve. Try the exploration and see for yourself.

1. Use the grid of the first 100 natural numbers in the rows of ten handout.
2. Mark out the number 1 . We will see why in the next section.
3. Using a colored pencil or marker, circle the number 2 and then mark out every remaining multiple of 2 until you have gone through the whole list. What is a mathematical term for the marked out numbers?
4. From the beginning, with a different colored pencil or marker, circle the first number that is not marked out and not circled. Then mark out all remaining multiples of that number.
5. Repeat this process until you have gone all the way through the list.
6. Make a new ordered list of all the circled numbers. What do these numbers have in common? How is this list of numbers related to patterns from the possible rectangle activity?
7. For students who have a difficult time understanding what the question is asking, a teacher might ask, "are these numbers also a multiple of another number?"
$6,12,18, \ldots$ These numbers are also multiples of 2 , hence why they were already marked out. Since they are multiples of 2 and multiples of 3 , they are multiples of 6 .

Use Divisibility War at the end of this section or during the Exercises as it is appropriate for practice.

## EXERCISES

These problems encourage students to use number sense instead an exhaustive process involving long division.
Share possible ways to compute answers such as:

1. a. 5 is not a factor of 34 because when you count by 5 's, you will not get 34 or land on 34 on the number line.
b. $152 \div 19=8$ with remainder 0 .
c. 4 is a factor of 68 because 4 is a factor of 64 and if you add 4 , you get 68 .

Etc.
2.
a. 85 because $3 \cdot 85=255$
b. 67 because $2 \cdot 67=134$
c. 143 because $7 \cdot 143=1,001$
7. You might have noticed that in the third round, some of the multiples of 3 were already crossed out in the second round. Find 3 such numbers. Why did this happen?
8. What kind of numbers did you mark out exactly twice? How many factors do these numbers have?
9. What kind of numbers did you mark out once?
10. After what number do we just circle the rest of the numbers on the list?

## EXERCISES

1. In each of the following problems, values of $n$ and $d$ are given. Determine whether $d$ is a factor of $n$. Explain how you use any patterns or prior knowledge that help you answer the questions. Use long division only on the starred items.

|  | $d$ (factor) | $n$ | Is $d$ a factor of <br> $n ? ~ E x p l a i n . ~$ |
| :--- | :---: | :---: | :---: |
| a. | 5 | 34 | no |
| *b. | 19 | 152 | yes, $8(19)=152$ |
| c. | 4 | 68 | yes, $17(4)=68$ |
| ${ }^{*}$ d. | 13 | 300 | no |
| e. | 9 | 81 | yes, $9(9)=81$ |
| f. | 4 | 86 | no |
| g. | 3 | 57 | yes, $19(3)=57$ |
| h. | 99 | 0 | yes, $0(99)=0$ |
| *i. | 17 | 788 | no |
| j. | 1 | 44 | yes, $44(1)=44$ |

2. Find the largest factor of each of the following numbers, excluding the number itself.
a. $n=255$
b. $n=134$
c. $n=1,001$
d. For a-c, sketch and label the sides of a rectangle in which $n$ represents area, and the largest factors are the lengths. Using these rectangles, find the corresponding factor to determine the width.
3. a. Encourage students who see that 25 is the smallest multiple of 5 greater than 23 without having to divide manually.
b. Do the same for 63 here. Encourage students to use their knowledge of the multiplication tables to shorten their work in math. Let them know that this tradeoff between knowing a system and tedious algorithmic work continues through high school math.
4. Answer: c
5. Answer: d
6. a. What is the least multiple of 5 that is greater than 23 ? 25
b. What is the least multiple of 7 that is greater than 59? 63
c. A small boat is used to cross a river. The boat can carry 5 passengers. If 23 people want to cross, how many trips does the boat need to take? 5
d. Buses are used for school field trips. Each bus can fit 22 people. How many buses are needed if 317 students want to go on the field trip? 15
7. a. What is the least multiple of 22 that is greater than 317 ? 330
b. What is the least multiple of 18 that is greater than 350 ? 360
8. Sally wrote two number patterns, as shown below.

$$
\begin{aligned}
& \text { Set } R=\{2,4,6,8,10, \ldots .\} \\
& \text { Set } T=\{4,8,12,16,20, \ldots .\}
\end{aligned}
$$

If these patterns continue, which of the following numbers would appear in both Set $R$ and Set $T$ ? Select the best choice and explain your answer.
a. 46
b. 30
c. 52
d. 70
6. The numbers in Set $R$ share a common characteristic.

$$
\text { Set } R=\{48,54,6,66,12,24\}
$$

The numbers in Set $S$ do not share this characteristic with set $R$.

$$
\text { Set } S=\{9,20,39,15,63,27,44\}
$$

Which best describes the characteristic that only the numbers in Set R share? Select the best choice and explain your answer.
a. Numbers less than $70 \quad 3$
c. Numbers that are composite
b. Numbers greater than 5
d. Numbers divisible by 6
7. a. How many positive multiples of 3 are less than or equal to 30 ? 10
b. How many positive multiples of 4 are less than or equal to 30 ? 8
c. Suppose $0<x<30$. How many positive multiples of $x$ are less than or equal to 30 ? $30 / \mathrm{x}$
8. a. Yes, because 65 is a multiple of $5(65=13 \cdot 5)$, so she can pay with 5 nickels.

Encourage your students to use their number sense every time they can to save time and effort. Ask if any student had an easier way of working the problem.
b. No, since dimes cannot be used to get 65 cents evenly. 10 is not a factor of 65 .
c. Use the divisibility rules when you can, noting when multiples end in 5 and when they end in 0 .
9. Note that the equal lengths apply to both rolls. The factors of 48 are $1,2,3,4,6,8,12,16,24,48$. The factors of 72 are $1,2,3,4,6,8,9,12,18,24,36,72$. Thus, the numbers that are factors of both, called common factors, are $1,2,3,4,6,8,12,24$. So the possible lengths Ms. Ellis can cut are: $1,2,3,4,6,8,12,24$ feet.
10. Answer: b
11.

| Columns | 1 | 2 | 3 | 4 | 5 | 6 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Rows | 60 | 30 | 20 | 15 | 12 | 10 |

12. Greatest Common Factor. There will be either 5 or 6 people at the party. List multiples of 5 and multiples of 6 . The first common multiple will be the first number of cookies that will serve both 5 or 6 people evenly, with no cookies left over.
13. a. Serena has a large number of nickels in her purse. Can she pay with exact change for a pack of gum that costs 65 cents? If so, how many nickels does she need? Explain your answer in terms of factors and/or multiples. Yes, 13 nickels
b. Serena has only dimes instead of nickels in her purse. Can she use exact change for the same pack of gum? If so, how many dimes does she need? If not, what is the problem? Explain your answer in terms of factors and/or multiples. See TE.
c. Explain your answers to these problems in terms of factors and multiples. See TE
14. Ms. Ellis has 72 feet of red ribbon and 48 feet of green ribbon. She wants to make a wall decoration using pieces of ribbon all cut the same whole number length of both colors. What lengths of ribbon are possible if she does not want any ribbon left over? Are these possible lengths factors or multiples of 72 and 48? See TE.
15. A teacher has 32 students in her class. She wants to put the students into groups so that each group has the same number of students. Which of the following does NOT represent the number of students she could put into groups? Select the best choice and explain your answer..
a. 4
b. 10
c. 8
d. 16
16. A high school newspaper is printing the names of students who are on the math team. The editor of the newspaper wants to divide the names into a certain number of columns so that each column contains the same number of names. There are 60 students on the math team, and the number of columns has to be fewer than 10. What are the different formats, by rows and columns, the newspaper can use? See TE.
17. Thomas is baking cookies for a party. Alice, Brad, Carlos and Diana have told Thomas that they definitely plan to attend. Eric has told Thomas that he would like to attend, but he is not sure he can make it. Thomas wants to bake enough cookies so that he and all his guests can have the same number of cookies, whether or not Eric shows up. What is the smallest number of cookies that Thomas can bake for his party? 30
18. b. The pairs are $(1,30),(2,15),(3,10)$ and $(5,6)$. Ask your students if they discovered a technique they can use so that they do not skip any factor. If they haven't, lead them to the technique. Have students show how they found these pairs, such as a T-chart or connecting the pairs with an arching line above the list to create a rainbow effect. 1 connects to 30,2 connects to 15,3 connects to 10 , and 5 connects to 6 .
19. In pairing the factors, 1 is paired with 36,2 with 18,3 with 6,4 with 9 , but 6 is left by itself. You could pair 6 with 6 . 36 has an odd number of factors. This is a property every perfect square has.
20. a. 4 k for some integer k
b. 3 k for some integer k
c. 6 k for some integer $\mathrm{k} ; 3$ is a factor of n because 3 is a factor of 6
21. $q=p k$, where $k$ is an integer
$r=q m$, where $m$ is an integer
Therefore, $r=p k m$
22. The idea is to use factor pairs. For each of the factors given in the list, find the other member of its pair. The pairs are $(1,140),(2,70),(4,35),(7,20),(10,14)$. Since 11,12 , and 13 are not factors and 14 is, this means we have all of the factors because any factor greater than 14 would be in a pair with a factor less than 14 . So if the list of factors we started with is all of the factors from 1 to 10 , then we now have them all.
23. a. Using your data from the Possible Rectangle Exploration, write all the factors of 30 . 1, 2, 3, 5, 6, 10, 15, 30
b. There is a natural way to pair up the positive factors of 30 . Explain how to do this in terms of the rectangle model. What pairs do you get?
24. Is it possible to pair up the positive factors of 36? Explain. See TE.
25. a. Consider the statement " $m$ is a multiple of 4". Write an algebraic expression for $m$.
b. Consider the statement " 3 is a factor of $n$ ". Write an algebraic expression for $n$.
c. Suppose 6 is a factor of $n$. Write an algebraic expression for $n$. Is 3 a factor of $n$ ? Explain.
26. If $p$ is a factor of $q$, and $q$ is a factor of $r$, show that $p$ is a factor of $r$. Try this with numbers. Prove it in general.
27. The numbers $1,2,4,5,7$ and 10 are six factors of 140 in numerical order. How can you use this information to find larger factors of 140 ? Find all those factors. $140,70,35,28,20,14$
28. Ingenuity: See answer on next TE page.

Five brothers want to divide 100 cookies among themselves. They decide to divide the cookies like this:
a. The oldest brother suggests a way to divide the cookies among the 5 siblings.
b. The 5 brothers vote on the proposal. If $50 \%$ or more agree with the proposal, the cookies will be divided as the oldest brother proposed.
c. If fewer than $50 \%$ agree with the proposal, the oldest brother will leave the game, can get no cookies, and can no longer vote. Then the next oldest brother will suggest a way to divide the cookies.
d. This process is repeated until a proposal is accepted and the cookies are divided among the brothers.

Assuming that all the brothers are extremely clever, greedy, and want at least one of the cookies, what would the oldest brother suggest? The brothers have no choice in their behavior and must act according to these qualities.

The following table shows the rounds that correspond to the strategy of working backwards if the table is read from the bottom up for Problem 18.

| Votes needed | Round | E: Youngest | D | C | B | A: Oldest |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | 1 | 1 | 0 | 1 | 0 | 98 |
| 2 | 2 | 0 | 1 | 0 | 99 |  |
| 2 | 3 | 1 | 0 | 99 |  |  |
| 1 | 4 | 0 | 100 |  |  |  |
| 1 | 5 | 100 |  |  |  |  |

18. One problem solving approach that students may be familiar with is to work backwards in order to find an ideal solution. For this problem we can model a situation and work backwards so the older brother can get the best deal possible. For the sake of this explanation, let us label the brothers from oldest to youngest as: Brother A, $B, C, D$, and $E$.
The three qualities these brothers have are that they are clever, greedy, and want at least one of the cookies. Because the brothers are clever, then they cannot be easily fooled and can figure out what the next brother will propose. Because they are greedy, then they will try to get as many cookies as possible for themselves and will not distribute the cookies evenly. Because they want at least one of the cookies, they are pleased (at the absolute least) with 1 cookie.
comes down to only having 3 people then brother D is left out of having any cookies so brother D can be won over with 1 cookie. Brother E does not matter because we already have the required number of votes, so he gets no cookies.

## 19. Investigation:

A band of five pirates finds a chest containing a number of gold coins. The pirates try to divide the stash equally, but one pirate ends up with one coin more than each of the other pirates. The other pirates become jealous, take his coins, and toss him overboard. The remaining four pirates again try to divide the stash equally, but one pirate ends up with three more coins than the other three pirates, so the three take his coins and toss him overboard. The three remaining pirates are able to divide the stash equally. If the pirates have between 40 and 60 coins all together, how many coins do they have? 51

Step 4: Since Brother A is clever, he knows of brother B's plan. With 5 people voting he needs 3 votes total. Brother B cannot be bought out with cookies, because if he votes "no" then he will secure 99 cookies, so brother A does not need his vote and can give him no cookies. Brother C was pushed out of the last decision and so would be happy with 1 cookie so we give him 1 cookie. Brother E was also given no cookies in the proposed brother B situation, so he would be happy with 1 cookie as well and this is how we can buy out his vote. Brother A will propose to give cookies away as follows: $98,0,1,0$, and 1 .
What does it mean to work backwards for this problem?
Step 1: First, let us consider the situation where there are only 2 brothers left who are competing for cookies.
Since there are only 2 brothers left, brother D and E , then brother D is proposing a situation and only needs 1 vote, his own, to secure that his situation will be accepted. Since he is greedy and does not need brother E's vote then he will take all 100 cookies and leave no cookies for brother E.
Step 2: Brother E wants at least 1 cookie, so if you were brother C and wanted to make sure that you got some cookies you would have this situation in mind and appease brother E easily so you could get his vote. Brother C would take as many cookies as he could and give brother E one cookie to secure his vote. Brother D he would not care about because he only needs 1 other vote, as 2 out of 3 votes is more than $50 \%$. So brother C would propose to keep 99 cookies for himself, give brother D no cookies and give brother E only 1 cookie to get his vote.
Step 3: Because brother B is clever he knows this is exactly what brother C would do, so he wants to improve upon his situation and get cookies too. Brother B only needs 1 more person to vote for his situation, so you should keep this in mind. Brother $C$ will not vote for the situation described by brother $B$ because if he votes "no" then he can get 99 cookies. Knowing this, brother B does not give brother C any cookies. If the situation
19. You should always encourage students to read the problem carefully before beginning and to write down all of the information they know at the beginning of the problem. After they write down all they know about this problem they may come up with these pieces of information:

- When the pirates divide the coins among 5 pirates, one of the pirates has an extra coin.
- When the pirates divide the coins among 4 pirates, one of the pirates has 3 extra coins.
- When the pirates divide the coins among 3 pirates, they can easily divide the coins.
- The number of coins we are looking for is between 40 and 60 coins.

Keeping in mind that the number of coins the pirates have is between 40 and 60 coins and because the number of coins they are looking for must be 1 more than a multiple of 5 , then the choices for the number of coins the pirates can have are: $41,46,51$, and 56 .
Either students will check to see which of these numbers have a remainder of 3 when divided by 4 or students will look for the multiples of 3 first. If students look for a multiple of 3 first, which is easier for students who are familiar with divisibility rules, then they will quickly find that 51 is the answer. If students look for a remainder of 3 when the number of coins is divided by 4 , then they will also commit to 51 as the answer. Students can discuss which strategy they used and how essential each piece of information was to their solution.

## POSSIBLE RECTANGLES TABLE

| n | Number of Possible Rectangles | Possible Rectangle Dimensions | Possible Side Lengths (in increasing order) |
| :---: | :---: | :---: | :---: |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |
| 4 | 2 | $1 \times 4,2 \times 2$ | 1, 2, 4 |
| 5 |  |  |  |
| 6 |  |  |  |
| 7 |  |  |  |
| 8 |  |  |  |
| 9 |  |  |  |
| 10 |  |  |  |
| 11 |  |  |  |
| 12 |  |  |  |
| 13 |  |  |  |
| 14 |  |  |  |
| 15 |  |  |  |
| 16 |  |  |  |
| 17 |  |  |  |
| 18 |  |  |  |
| 19 |  |  |  |


| n | Number of Possible Rectangles | Possible Rectangle Dimensions | Possible Side Lengths |
| :---: | :---: | :---: | :---: |
| 20 |  |  |  |
| 21 |  |  |  |
| 22 |  |  |  |
| 23 |  |  |  |
| 24 |  |  |  |
| 25 |  |  |  |
| 26 |  |  |  |
| 27 |  |  |  |
| 28 |  |  |  |
| 29 |  |  |  |
| 30 |  |  |  |
| 31 |  |  |  |
| 32 |  |  |  |
| 33 |  |  |  |
| 34 |  |  |  |
| 35 |  |  |  |
| 36 |  |  |  |
| 37 |  |  |  |
| 38 |  |  |  |
| 39 |  |  |  |
| 40 |  |  |  |

## SIEVE OF ERATOSTHENES

(student copy)

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
| 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |

## SIEVE OF ERATOSTHENES

(answer key)

| 1 | $\mathbf{2}$ | $\mathbf{3}$ | 4 | $\mathbf{5}$ | 6 | $\mathbf{7}$ | 8 | 9 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{1 1}$ | 12 | $\mathbf{1 3}$ | 14 | 15 | 16 | $\mathbf{1 7}$ | 18 | $\mathbf{1 9}$ | 20 |
| 21 | 22 | $\mathbf{2 3}$ | 24 | 25 | 26 | 27 | 28 | $\mathbf{2 9}$ | 30 |
| $\mathbf{3 1}$ | 32 | 33 | 34 | 35 | 36 | $\mathbf{3 7}$ | 38 | 39 | 40 |
| $\mathbf{4 1}$ | 42 | $\mathbf{4 3}$ | 44 | 45 | 46 | $\mathbf{4 7}$ | 48 | 49 | 50 |
| 51 | 52 | $\mathbf{5 3}$ | 54 | 55 | 56 | 57 | 58 | $\mathbf{5 9}$ | 60 |
| $\mathbf{6 1}$ | 62 | 63 | 64 | 65 | 66 | $\mathbf{6 7}$ | 68 | 69 | 70 |
| $\mathbf{7 1}$ | 72 | $\mathbf{7 3}$ | 74 | 75 | 76 | 77 | 78 | $\mathbf{7 9}$ | 80 |
| 81 | 82 | $\mathbf{8 3}$ | 84 | 85 | 86 | 87 | 88 | $\mathbf{8 9}$ | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | $\mathbf{9 7}$ | 98 | 99 | 100 |

Prime numbers in bold.

## DIVISIBILITY WAR

Objective: This game is designed to reinforce the skill of divisibility. Students will practice divisibility rules on a random set of $2-, 3$ - and 4 -digit numbers.

## Materials:

One set of Divisibility Cards per group
One copy of Divisibility War Key per group
Two color counting chips or any similar item. Each group will need about 50 .

## Activity Instructions:

To begin, make the Divisibility Cards. To do that, write the 2- to 4-digit numbers listed on the Divisibility War Key on a set of index cards, one set per group.

Play the game with 2, 3, 4 or 6 players in the following manner:

1) The dealer shuffles the deck of cards.
2) The dealer deals the pack equally and face down.
3) Each player stacks his cards into a pile face down in front of him.
4) The player to the left of the dealer takes the first turn.
5) During a player's turn, he should:
(a) place the top card of his stack face up for all to see,
(b) make a statement about the divisibility of his number. For instance, state that the number on the card he has placed is evenly divisible by $2,3,4,5,6,8,9$ and/or 10 or none of these numbers, and
(c) place zero to eight chips, depending on the number of divisibility tests that hold for the number on the card, beside the placed card. For example, if a player states that the number on his card is divisible by $2,4,5$ and 10 , he should place four chips beside his card because 4 divisibility tests, 2,4,5 and 10, hold for the number on the card.
6) The player to the left of the last player takes his turn next.
7) The next player may challenge a previous player's statement of divisibility after the player places his chips and before the next player's turn begins. A challenge entails the following:
(a) A challenger must make a statement of divisibility. A challenger may not repeat the player's statement of divisibility.
(b) The player to the left of a challenger may challenge any challenger and the original player as long as the latest challenger does not repeat the statement of divisibility of the original player or any other challenger.
(c) Every player may challenge any previous challenger and original player once before the next player takes his turn.
(d) To settle the challenge, players should reference the Divisibility War Key.
(e) If the Divisibility War Key proves any player completely correct, the player or challenger with the correct statement of divisibility gets as many chips as tests that hold for the number on the card in question.
(f) If the Divisibility War Key proves no player completely correct, no player gets any chips.
8) Each player takes his turn to complete a round.
9) At the end of each round, the player with the most chips wins the round.
10) The winner of the round collects all cards placed during the round and places these cards at the bottom of his pack. If at the end of a round, two or more players have the same number of chips, there is a war. The following must be done during a war:
(a) The players with the same number of chips each play three cards face down and a fourth face up.
(b) The player with the largest number on his face-up card wins all of the cards in the war.
11) Rounds continue until time allotted for the game runs out or any player runs out of cards. The player with the most cards wins.

DIVISIBILITY WAR HELPER

| A number is divisible by: | IF: |
| :---: | :--- |
| 2 | the last digit is $0,2,4,6$, or 8 |
| 3 | the sum of the digits is divisible by 3 |
| 4 | the last two digits are divisible by 4 |
| 5 | the last digit is 0 or 5 |
| 6 | the number is divisible by both 2 and 3 |
| 8 | the last 3 digits are divisible by 8 |
| 9 | the sum of the digits is divisible by 9 |
| 10 | the last digit is 0 |



Section 7.1 Divisibility, Factors and Multiples
DIVISIBILITY WAR KEY

| 24 | $\{2,3,4,6,8\}$ | 285 | $\{3,5\}$ | 2225 | $\{5\}$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 34 | $\{2\}$ | 308 | $\{2,4\}$ | 2318 | $\{2\}$ |
| 35 | $\{5\}$ | 309 | $\{3\}$ | 2416 | $\{2,4,8\}$ |
| 36 | $\{2,3,4,6,9\}$ | 335 | $\{5\}$ | 2997 | $\{3,9\}$ |
| 44 | $\{2,4\}$ | 338 | $\{2\}$ | 3000 | $\{2,3,4,5,6,8,10\}$ |
| 46 | $\{2\}$ | 385 | $\{5\}$ | 3335 | $\{5\}$ |
| 48 | $\{2,3,4,6,8\}$ | 408 | $\{2,3,4,6,8\}$ | 5256 | $\{2,3,4,6,8,9\}$ |
| 55 | $\{5\}$ | 429 | $\{3\}$ | 5952 | $\{2,3,4,6,8\}$ |
| 56 | $\{2,4,8\}$ | 438 | $\{2,3,6\}$ | 6000 | $\{2,3,4,5,6,8,10\}$ |
| 57 | $\{3\}$ | 447 | $\{3\}$ | 6525 | $\{3,5\}$ |
| 60 | $\{2,3,4,5,6,10\}$ | 495 | $\{3,5,9\}$ | 6720 | $\{2,3,4,5,6,8,10\}$ |
| 62 | $\{2\}$ | 524 | $\{2,4\}$ | 7552 | $\{2,4,8\}$ |
| 65 | $\{5\}$ | 567 | $\{3,9\}$ | 8135 | $\{5\}$ |
| 72 | $\{2,3,4,6,8,9\}$ | 625 | $\{5\}$ | 8228 | $\{2,4\}$ |
| 74 | $\{2\}$ | 657 | $\{3,9\}$ | 8872 | $\{2,4,8\}$ |
| 75 | $\{3,5\}$ | 666 | $\{2,3,6,9\}$ | 8886 | $\{2,3,6\}$ |
| 80 | $\{2,4,5,8,10\}$ | 669 | $\{3\}$ | 9297 | $\{3,9\}$ |
| 84 | $\{2,3,4,6\}$ | 700 | $\{2,4,5,10\}$ | 9927 | $\{3,9\}$ |
| 98 | $\{2\}$ | 731 | $\{3,9\}$ | 9954 | $\{2,3,4,6,9\}$ |
| 115 | $\{5\}$ | $\{5\}$ |  |  |  |
| 117 | $\{3,9\}$ | 715 | 728 | $\{2,4,8\}$ |  |
| 128 | $\{2,4,8\}$ | 735 | $\{3,5\}$ |  |  |
| 130 | $\{2,5,10\}$ | 741 | $\{3\}$ |  |  |
| 140 | $\{2,3,5,10\}$ | 770 | $\{2,5,10\}$ |  |  |
| 150 | $\{2,3,5,6,10\}$ | 771 | $\{3\}$ |  |  |
| 160 | $\{2,4,5,8,10\}$ | 849 | $\{3\}$ |  |  |
| 171 | $\{3,9\}$ | 888 | $\{2,3,4,8,9\}$ |  |  |
| 175 | $\{5\}$ | $\{3,5\}$ |  |  |  |
| 190 | $\{2,5,10\}$ | 960 | $\{2,3,4,5,6,8,10\}$ |  |  |
| 196 | $\{2,4\}$ | 1115 | $\{5\}$ |  |  |
| 200 | $\{2,4,5,8,10\}$ | 1135 | $\{5\}$ |  |  |
| 216 | $\{2,3,4,6,8,9\}$ | 1280 | $\{2,4,5,8,10\}$ |  |  |
| 240 | $\{2,3,4,5,6,8,10\}$ | 1324 | $\{2,4\}$ |  |  |
| 256 | $\{2,4,8\}$ | 2204 | $\{2,4\}$ |  |  |
| 260 | $\{2,4,5,10\}$ | 2220 | $\{2,3,4,5,6,10\}$ |  |  |
|  |  |  |  |  |  |

## DIVISIBILITY RULES

Objective: Students will connect divisibility with factors and multiples

## Materials:

Divisibility Rules Chart
Index cards with random numbers between 20 and 1000
Index cards with characteristics of divisibility duplicated as many times as needed.

## Activity Instructions:

Teacher guides students in a Divisibility Rules flash card game by discussing patterns. Class should be divided into groups of 4 to 5 students.

1) Teacher prepares index cards with characteristics of divisibility duplicated several times and index cards with random numbers from 20 to 1000 (each group will need one of each set of cards)
2) Students will match each number card with its corresponding characteristic card(s). See chart below. The teacher may want to display this chart in the classroom as a guide.
3) Each group records their matched up pairs on a sheet of paper.

DIVISIBILITY RULES CHART

| If this is true about a number: | Then number is divisible by: |
| :--- | :---: |
| Last digit is even | 2 |
| The sum of the digits is divisible by 3 | 3 |
| The last digit is 0 or 5 | 5 |
| The number is divisible by 2 and 3 | 6 |
| The sum of the digits is divisible by 9 | 9 |
| The number ends in 0 | 10 |

Note: If a number is divisible by two prime numbers, then it is divisible by the products of the two numbers. Since 70 is divisible by both 5 and 7 then, it is also divisible by 35 .

## PRODUCT BINGO

Objective: This game is designed to reinforce factors, multiples and prime \& composite numbers; it will also help students review multiplication facts.

## Materials:

One Product Bingo card per student
2 sets of number cards with the numbers 1-10 written on them
Timer/stop watch
Color pencils

## Activity Instructions:

1) Make number cards. Write numbers 1-10 on a set of index cards, two sets per group.
2) Play the game with $2,3,4$, or 5 players.
3) The dealer shuffles the deck of number cards and deals the pack equally, facing down.
4) Decide who goes 1st, 2nd, 3rd, etc.
5) Each player selects a card from his/her hand and places it face up on the table for all to see. Each card will be the 1st factor for each individual.
6) Player 1 gets to select one card from his/her hand to be the 2nd factor. Players multiply their own individual factor to the common factor and make their product which they can then mark off if it is on their Bingo Cards. (Note that players are allowed to see each others Bingo Cards in order to help strategize what factors to select).
7) Play progresses to the next player, repeating steps 5-7 and re-shuffling and re-dealing as needed.
8) The winner is the player who crosses out four numbers connected in a row vertically, horizontally, or diagonally.

## PRODUCT BINGO

| 2 | 22 | 53 | 14 | 5 | 60 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 27 | 8 | 9 | 10 | 11 | 12 |
| 33 | 44 | 55 | 66 | 77 | 88 |
| 59 | 20 | 42 | 52 | 83 | 54 |
| 25 | 26 | 27 | 28 | 29 | 30 |
| 30 | 32 | 33 | 34 | 95 | 100 |


| 6 | 22 | 24 | 34 | 44 | 56 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 76 | 12 | 88 | 84 | 25 | 90 |
| 98 | 84 | 14 | 5 | 68 | 28 |
| 100 | 42 | 82 | 16 | 20 | 56 |
| 12 | 11 | 10 | 9 | 8 | 7 |
| 32 | 8 | 15 | 33 | 27 | 20 |


| 15 | 45 | 35 | 65 | 75 | 85 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 12 | 20 | 18 | 16 | 42 | 22 |
| 16 | 28 | 42 | 49 | 70 | 63 |
| 56 | 64 | 8 | 40 | 32 | 48 |
| 54 | 81 | 36 | 27 | 9 | 27 |
| 36 | 60 | 54 | 30 | 50 | 100 |


| 16 | 20 | 24 | 21 | 28 | 36 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 20 | 25 | 30 | 35 | 40 | 45 |
| 24 | 30 | 36 | 42 | 48 | 54 |
| 28 | 35 | 42 | 49 | 56 | 63 |
| 32 | 40 | 48 | 56 | 64 | 72 |
| 36 | 45 | 54 | 63 | 72 | 81 |


| 15 | 45 | 35 | 65 | 75 | 85 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 12 | 20 | 18 | 16 | 42 | 22 |
| 16 | 28 | 42 | 49 | 70 | 63 |
| 56 | 64 | 8 | 40 | 32 | 48 |
| 54 | 81 | 36 | 27 | 9 | 27 |
| 36 | 60 | 54 | 30 | 50 | 100 |


| 16 | 20 | 24 | 21 | 28 | 36 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 20 | 25 | 30 | 35 | 40 | 45 |
| 24 | 30 | 36 | 42 | 48 | 54 |
| 28 | 35 | 42 | 49 | 56 | 63 |
| 32 | 40 | 48 | 56 | 64 | 72 |
| 36 | 45 | 54 | 63 | 72 | 81 |


| 4 | 6 | 8 | 10 | 12 | 14 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 6 | 9 | 22 | 15 | 18 | 21 |
| 8 | 32 | 16 | 20 | 24 | 21 |
| 10 | 15 | 20 | 25 | 30 | 42 |
| 14 | 21 | 28 | 35 | 63 | 81 |
| 50 | 30 | 40 | 60 | 80 | 100 |


| 10 | 12 | 14 | 16 | 28 | 36 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 15 | 18 | 21 | 28 | 40 | 30 |
| 24 | 35 | 42 | 49 | 56 | 63 |
| 27 | 9 | 24 | 60 | 100 | 20 |
| 8 | 7 | 50 | 55 | 63 | 81 |
| 90 | 32 | 16 | 20 | 70 | 72 |

## Section 7.2 - Prime and Composite Numbers

## Big Idea:

Understanding and comparing prime and composite numbers. Discover the divisibility rules for 2,5,10,3,6,9.

## Key Objectives:

Introduce formal definitions of prime and composite.

## 7th Grade Strategy:

Go back to the Sieve of Eratosthenes (from Section 7.1)
Review Definition 7.2
Exercise 4 and 5
Use Exercise 10 as a discussion problem

## Materials:

Sieve of Eratosthenes Chart from Section 7.1.

## Pedagogical/Orchestration:

- Reflection on the Number of Rectangles Chart from Section 7.1.
- The number 1 is neither a prime nor a composite number.
- The Identity of Multiplication: For any number $n, n \times 1=n$.


## Activity:

"Prime Magic Square" and "Factor Bingo" at the end of the section and on CD

## Vocabulary:

prime number, composite number, multiplicity

## TEKS:

6.1(D); 7.2(F)(G);
7.13(C)(D);
$7.14(A)(B) ; \quad 7.15(A)(B) ;$
8.14(B);
8.15(A);
8.16(A)

## WARM-UPS for Section 7.2 (Prime and Composite Numbers)

1. You are asked to guess a number and are told it is a multiple of 5,6 and 7 . What is the smallest positive number you should guess? What is the largest number less than 1000 that you should guess?
2. Which of the following fractions is not equivalent to $\frac{8}{12}$ ?
a. $\frac{4}{6}$
b. $\frac{6}{9}$
C. $\frac{10}{15}$
d. $\frac{3}{4}$
Ans: d

## Launch for Section 7.2:

Ask your students, "Who can find the largest number that is less than 100 and has the fewest factors?" Once students have thought about the question, allow them to answer: the number they are looking for is 97. Ask students if any of them made use of the Sieve of Eratosthenes Exploration from yesterday. Ask them how that would be helpful. Ask the students to look at the numbers that were left in the Sieve, and to discuss what those numbers had in common. Tell students, "In today's class, we will give a definition for these special numbers and study patterns to learn more about them."

## SECTION 7.2 PRIME AND COMPOSITE NUMBERS

In the Possible Rectangle Model and the Sieve of Eratosthenes Exploration in Section 7.1, you examined factors and multiples. In the Possible Rectangle Model Exploration, you found the positive factors of the integers from 1 to 40 . The results are below. You might have noticed that some numbers had many positive factors, while some had only two. The integers with two factors are highlighted in the table.

| $n$ | Number <br> of <br> rectangles <br> of area $n$ | Dimensions of the rectangles | Side lengths of rectangles |
| :---: | :---: | :--- | :--- |
| 1 | 1 | $1 \times 1$ | 1 |
| $\mathbf{2}$ | 1 | $1 \times 2$ | 1,2 |
| $\mathbf{3}$ | 1 | $1 \times 3$ | 1,3 |
| 4 | 2 | $1 \times 4,2 \times 2$ | $1,2,4$ |
| $\mathbf{5}$ | 1 | $1 \times 5$ | 1,5 |
| 6 | 2 | $1 \times 6,2 \times 3$ | $1,2,3,6$ |
| $\mathbf{7}$ | 1 | $1 \times 7$ | 1,7 |
| 8 | 2 | $1 \times 8,2 \times 4$ | $1,2,4,8$ |
| 9 | 2 | $1 \times 9,3 \times 3$ | $1,3,9$ |
| 10 | 2 | $1 \times 10,2 \times 5$ | $1,2,5,10$ |
| $\mathbf{1 1}$ | 1 | $1 \times 11$ | 1,11 |
| 12 | 3 | $1 \times 12,2 \times 6,3 \times 4$ | $1,2,3,4,6,12$ |
| $\mathbf{1 3}$ | 1 | $1 \times 13$ | 1,13 |
| 14 | 2 | $1 \times 14,2 \times 7$ | $1,2,7,14$ |
| 15 | 2 | $1 \times 15,3 \times 5$ | $1,3,5,15$ |
| 16 | 3 | $1 \times 16,2 \times 8,4 \times 4$ | $1,2,4,8,16$ |
| $\mathbf{1 7}$ | 1 | $1 \times 17$ | 1,17 |
| 18 | 3 | $1 \times 18,2 \times 9,3 \times 6$ | $1,2,3,6,9,18$ |
| $\mathbf{1 9}$ | 1 | $1 \times 19$ | 1,19 |
| 20 | 3 | $1 \times 20,2 \times 10,4 \times 5$ | $1,2,4,5,10,20$ |
| 21 | 2 | $1 \times 21,3 \times 7$ | $1,3,7,21$ |
| 22 | 2 | $1 \times 22,2 \times 11$ | $1,2,11,22$ |
|  |  |  |  |


| $n$ | Number <br> of <br> rectangles <br> of area $n$ | Dimensions of the rectangles | Side lengths of rectangles |
| :--- | :---: | :--- | :--- |
| 23 | 1 | $1 \times 23$ | 1,23 |
| 24 | 4 | $1 \times 24,2 \times 12,3 \times 8,4 \times 6$ | $1,2,3,4,6,8,12,24$ |
| 25 | 2 | $1 \times 25,5 \times 5$ | $1,5,25$ |
| 26 | 2 | $1 \times 26,2 \times 13$ | $1,2,13,26$ |
| 27 | 2 | $1 \times 27,3 \times 9$ | $1,3,9,27$ |
| 28 | 3 | $1 \times 28,2 \times 14,4 \times 7$ | $1,2,4,7,14,28$ |
| 29 | 1 | $1 \times 29$ | 1,29 |
| 30 | 4 | $1 \times 30,2 \times 15,3 \times 10,5 \times 6$ | $1,2,3,5,6,10,15,30$ |
| 31 | 1 | $1 \times 31$ | 1,31 |
| 32 | 3 | $1 \times 32,2 \times 16,4 \times 8$ | $1,2,4,8,16,32$ |
| 33 | 2 | $1 \times 33,3 \times 11$ | $1,3,11,33$ |
| 34 | 2 | $1 \times 34,2 \times 17$ | $1,2,17,34$ |
| 35 | 2 | $1 \times 35,5 \times 7$ | $1,5,7,35$ |
| 36 | 5 | $1 \times 36,2 \times 18,3 \times 12,4 \times 9,6 \times 6$ | $1,2,3,4,6,9,12,18,36$ |
| 37 | 1 | $1 \times 37$ | 1,37 |
| 38 | 2 | $1 \times 38,2 \times 19$ | $1,2,19,38$ |
| 39 | 2 | $1 \times 39,3 \times 13$ | $1,3,13,39$ |
| 40 | 4 | $1 \times 40,2 \times 20,4 \times 10,5 \times 8$ | $1,2,4,5,8,10,20,40$ |

The numbers that have only two positive factors play a special role in mathematics and have a special name.

## DEFINITION 7.2: PRIME AND COMPOSITE

A prime number is an integer $p$ greater than 1 with exactly two positive factors: 1 and $p$. A composite number is an integer greater than 1 that has more than two positive factors. The number 1 is the multiplicative identity; that is, for any number $n, n \cdot 1=n$. The number 1 is neither a prime nor a composite number.

## PROBLEM 1

Reflect with your class/students the different approaches they attempted. Did anyone try finding possible rectangles with area 17? 1? 119? It is not prime because $119=7 \cdot 17$. Also, talk about using the sieve and number sense. Why can students eliminate the primes 2,3 , and 5 immediately? The next prime is 7 and it works.

Your work with the Possible Rectangle Table (PR Table) allows you to see the relationships between a given number, $n$, the number of rectangles possible with $n$ as its area, and the number of factors $n$ has. In particular, you can see from your PR Table that the prime numbers between 1 and 40 , in increasing order, are $2,3,5,7,11,13,17,19,23,29,31$ and 37 . These are the only integers greater than 1 and less than or equal to 40 that had exactly one rectangle possible and hence exactly two factors. Let's consider a larger number and determine whether it is prime or not.

## PROBLEM 1

Is 119 prime or composite?

Here is another approach that is not as geometric, but it is systematic.

## EXAMPLE 1

Is the number 171 prime?

## SOLUTION

To see if there are any other factors of 171 between 1 and 171 , begin to divide 171 by numbers less than 171, in increasing order beginning with 2.

| Divide by | Quotient | Remainder | Factor? |
| :---: | :---: | :---: | :---: |
| $171 \div 2$ | 85 | 1 | No |
| $171 \div 3$ | 57 | 0 | yes |

Dividing 171 by 3 gives quotient 57 and remainder 0 , showing that 3 is a factor of 171. This is enough information to conclude that 171 is composite because 171 has not just 1 and itself as factors, but it also has 3 and 57 as factors.

You might have noticed that your Investigation in Section 7.1 involving the Sieve of Eratosthenes allowed you to find the prime numbers less than 100 quickly. Can you explain how the process worked? At what point is it possible to know that all the numbers left are prime?

## PROBLEM 2

Your students will be working on this a long time if they work alone. After a few factors, ask if it is possible to divide the task into parts for small groups or individuals to share. Depending on your class, the students might use calculators. What they will find is that 127 , as composite as it looks, is prime. No numbers work between 1 and 127. That is not the point of this exploration, however. The questions that follow suggest that the observant student can make this work much easier. Again, let your students discover the shortcut for themselves, if possible. This is the foundation for the Ingenuity problem.

## EXPLORATION 1

Have students make a table of the Divisibility Rules that they recall from their previous experiences.

| A number is divisible by: | If |
| :---: | :---: |
| 2 | The last digit is $0,2,4,6$, or 8. |
| 3 | The sum of the digits is divisible by 3. |
| 5 | The last digit is 0 or 5. |
| 6 | The number is divisible by 2 or 3 |
| 10 | The last digit is 0. |

## EXPLORATION 2

There are many shortcuts. Expect students to discover or to know the fact that it is only necessary to check prime factors between 1 and $n-1$ to find factors of $n$. Be especially alert to see if any student notices that it is only necessary to check prime factors between 1 and the $\sqrt{ }$. Don't spoil this discovery. There are some later exercises designed to lead the students to this fact. If, however, one of your students does discover the $\sqrt{ } n$ strategy shortcut, verify it and then celebrate that discovery. Other techniques include seeing the factor 2 in even numbers, the factor 5 in numbers ending in 5 or 0 and the factor 3 in numbers whose digit-sum is a multiple of 3 . Also look for multiplicity of factors, like $2^{3}$.
a. composite, $3 \cdot 29$
b. prime
c. composite, $17 \cdot 19$

The number 171 does not appear on the Sieve of Eratosthenes chart. Instead of using the possible rectangle model, we approached the problem by finding factors. Why was that smart? We could have shortened our search even more. Because 171 is odd, we know that 2 is not a factor of 171 .

## PROBLEM 2

Is the number 127 prime?

## EXPLORATION 1

- How do you know that a number is divisible by 2?
- How do you know that a number is divisible by 5 ?
- How do you know that a number is divisible by 10 ?
- How do you know that a number is divisible by 3 ?
- How do you know that a number is divisible by 6?

| A number is divisible by: | If |
| :---: | :---: |
| 2 |  |
| 3 |  |
| 5 |  |
| 6 |  |
| 10 |  |

## EXPLORATION 2

In small groups or individually, determine whether the following numbers are prime or composite. Try to devise as many time-saving strategies as you can, so you don't have to check every integer between 1 and the target number.
a. 87
b. 131
c. 323

## EXERCISES

1. Continue to reward students who find an easier way using patterns. Have them share their strategies. You might even set up a list that you display in the room.
a. $1, \mathbf{2}, \mathbf{3}, 6$
b. 1,11
c. $1, \mathbf{2}, \mathbf{3}, 6,9,18$
d. $1, \mathbf{2}, 4, \mathbf{5}, 10,20$
e. $1, \mathbf{3}, 9,27$
f. $1, \mathbf{2}, \mathbf{3}, \mathbf{5}, 6,10,15,30$
g. 1,41
h. $\mathbf{1 , 5 , 1 1}, 55$
i. $\mathbf{1}, \mathbf{2}, \mathbf{3}, 4, \mathbf{5}, 6,10,12,15,20,30,60$
j. $\mathbf{1}, \mathbf{5}, \mathbf{1 3}, 65$

Prime factors are indicated with bold above.
2. The numbers in $b$ (11) and $g$ (41) are prime.
4. a. $1,2,3,4,6,9,12,18,36$
b. $1,2,3,4,6,8,12,16,24,48$
c. $1,2,4,8,16,32,64 ; 36$ and 64 are perfect squares and have odd numbers of factors. 48 is not a perfect square and has an even number of factors.

## EXERCISES

1. For each of the following numbers, find all of the factors of the number. Circle the prime factors in each. You may want to use the data you collected in Exploration 1 in this chapter or the Sieve of Eratosthenes. See TE.
a. 6
C. 18
b. 11
d. 20
e. 27
f. 30
g. 41
h. 55
i. 60
j. 65
2. Are there any numbers given in Exercise 1 that are prime? If so, how did you determine they are prime? See $T E$.
3. Valerie used square tiles to make 4 designs, as shown below.


Which design is composed of a prime number of tiles? Design 2
4. For each of the following numbers, find all the factors of $N$. List them in order from least to greatest. Draw a line above the list from one factor to its factor pair.
a. $\quad N=36$
b. $\quad N=48$
c. What is different about the list of factors for 36 and 48 ?
d. Find the factors for $N=64$. Compare this list to the lists for 48 and 36. What do you notice?
5. $S$ is the set $\{4,8,12,16,20,24, \ldots\}$, and $T$ is the set $\{2,4,6,8,10,12,14, \ldots\}$. Remember, the ellipsis .. means that the set continues infinitely in the same way it started. Because of that understanding, what is the next number not shown in $S$ ? In $T$ ? Draw a Venn Diagram showing the relationship between $S$ and $T$. Is $S$ a subset of $T$ ? Is $T$ a subset of $S$ ? Explain.
6. Now assume that $S$ is the set of multiples of 2 and $T$ is the set of multiples of 6 . Write $S$ and $T$ in set notation. Draw a Venn diagram showing the relationship between $S$ and $T$. Is $S$ a subset of $T$ ? Is $T$ a subset of $S$ ? Explain.
7. a. $101,103,107,109$
e. $251,257,263,269,271,277,281,283,293$
b. $113,127,131,137,139,149$
f. $307,311,313,317,331,337,347,349$
c. $151,157,163,167,173,179,181,191,193,197,199$
d. $211,223,227,229,233,239,241$
8. a. composite
b. composite
c. composite
d. prime
9. a. $30=2 \cdot 3 \cdot 5$
d. $26=2 \cdot 13$
b. $66=2 \cdot 3 \cdot 11$
e. $24=2 \cdot 2 \cdot 2 \cdot 3$
c. $28=2 \cdot 2 \cdot 7$
10. Check $2,3,5,7,11,13,17,19$ and 23 because the factors greater than 23 will have a factor less than 23 and it would already have been found. Use the T-chart to show this pattern. So, any factor of 503 greater than 23 would be on the right side of the T-chart for 503 and its factor pair pattern could be smaller and on the left side of the T-Chart so we would already have found it. (23)(23) = 529 > 503)
11. a. 151 is prime.
b. 117 is composite, $9 \cdot 13$.
c. $\quad 107$ is prime.
d. 121 is composite, $11 \cdot 11$.
e. 143 is composite, $11 \cdot 13$.
f. 221 is prime.
12. $10=2,5 ; 100=2 \cdot 2 \cdot 5 \cdot 5 ; 1,000=2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5 ; 10,000=2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5 \cdot 5$. Students may notice increasing powers of the factors 2 and 5 . We know that $50=2 \cdot 5 \cdot 5,2$ has multiplicity 1 and 5 has multiplicity 2. The other factors on the list have the same multiplicity for 5 and 2.
15. For parts (a) and (b) we break up each number into a multiple of 100 , a multiple of 10 and the ones place will stand alone. For 261 , we can write 261 as $2 \cdot 100+6 \cdot 10+1$. Then we use the fact that $100=99+1$ and we know that $10=9+1$ to write 261 as $2 \cdot(99+1)+6 \cdot(9+1)+1$. After we distribute we see 261 as 2 $\cdot 99+2 \cdot 1+6 \cdot 9+6 \cdot 1+1$ which, after some application of the commutative property of addition, becomes $2 \cdot 99+6 \cdot 9+2 \cdot 1+6 \cdot 1+1$.
So, $261=2 \cdot 99+6 \cdot 9+2 \cdot 1+6 \cdot 1+1=2 \cdot 99+6 \cdot 9+2+6+1$ which is also equal to $2 \cdot 99+6 \cdot 9$ +9 or $99(2)+9(6+1)$. If you look at just the value of each of the digits, 2,6 , and 1 and add those up then you can always test to see whether or not a number is divisible by 9 and students can prove it using this method.
7. Try to find a prime between the following pairs of numbers. Verify that your choice is a prime. See TE.
a. 100 and 110
b. $\quad 110$ and 150
c. $\quad 150$ and 200
d. 200 and 250
e. 250 and 300
f. 300 and 350
8. Determine whether the following are prime or composite numbers. Explain the strategies used to determine your solution.
a. 125
b. 321
c. 122
d. 127
9. Factor each of the following numbers using only prime factors. See TE.
a. 30
b. 66
c. 28
d. 26
e. 24
10. In determining the factors of 503 , what possible divisors do you need to check to determine whether it is prime or composite?
11. Determine as efficiently as possible whether each of the following numbers is prime or composite. See TE.
a. 151
b. 117
c. 107
d. 121
e. 143 f. 211
12. Write each of the following numbers as products of only prime factors: 10 , $100,1,000,10,000$. Find the prime factors of each. What do you notice? Show that 50 and 25 have the same prime factors as the numbers in the first list. Ignore multiplicity, or repetition of a factor. For example, since 2 is a factor of 24 and $24=2 \cdot 2 \cdot 2 \cdot 3$ we say that 2 is a factor of 24 with multiplicity of 3 . Using multiplicity, how are the factors of 50 different from the factors of the numbers in the list? See TE.
13. Numbers like 15 and 140 are divisible by 5 . State a conjecture about what numbers divisible by 5 look like. Numbers divisible by 5 end in either 5 or 0 .
14. State a conjecture about what numbers divisible by 10 look like.
15. Investigation:

The numbers $9,99,999$ and 9,999 are divisible by 9 . One way to see why is to convert the number into a sum in which each term is a multiple of 9 . For example, $99=90+9$ and $9,999=9,000+900+90+9$. You can use this pattern to determine whether other numbers are multiples of 9 . Explain why each of the numbers below is a multiple of 9 by observing the way in which it is decomposed. Note that 9 is a factor of each of the addends.

Numbers divisible by 10 end in 0 .
15. c. i. $3+4+2=9$, divisible by 9
ii. $8+3=11$, not divisible by 9
iii. $9+5+4=18$, divisible by 9
iv. $2+7+6=15$, not divisible by 9
v. $2+5+7+4=18$, divisible by 9
15. d. If the sum of the digits of a number is divisible by 9 then the number is divisible by 9 .
16. The rule of 3 is that if the sum of the digits of a number $n$ is divisible by 3 , then the number $n$ is divisible by 3 . Students can use decomposition yet again to prove whether or not a number is divisible by 3 . Let us consider the number 2574 which we can write as the sum $2 \cdot 1000+5 \cdot 100+7 \cdot 10+4 \cdot 1$. After rewriting each place value as a multiple of 3 with a remainder of 1 we see that $2574=2 \cdot(999+1)+5 \cdot(99$ $+1)+7 \cdot(9+1)+4 \cdot 1$.
We must use distribution to see that $2574=2 \cdot 999+2 \cdot 1+5 \cdot 99+5 \cdot 1+7 \cdot 9+7 \cdot 1+4 \cdot 1$. Students may ask, how does this relate to 3 ? But they do know that $9=3 \cdot 3$, so any number that is a multiple of 9 is also a multiple of 3 (but not all numbers that are multiples of 3 are multiples of 9 . Your students can check to see that a number like 2469 is a multiple of 3 but not a multiple of 9 ). This means that because 2574 can be written as $2(3 \cdot 333)+5(3 \cdot 33)+7(3 \cdot 3)+2+5+7+4$ and because the first 3 addends are multiples of 3 , all we have to do is check the sum $2+5+7+4$, which is the digit sum of 2574 . Because $2+5+7+$ $4=18$ and $18=3 \cdot 6,2574=2(3 \cdot 333)+5(3 \cdot 33)+7(3 \cdot 3)+3 \cdot 6$, and 2574 is a multiple of 3 .
18. This is a good chance to talk about even numbers and odd numbers. Your students might notice that every even number greater than 2 , up to 26 , appears in the table. On the other hand, only a few odd numbers appear in the table. There is a long-standing conjecture in mathematics, the Goldbach Conjecture, that claims that any even number greater than 2 can be written as a sum of two primes. By the way, check to see if all your students know the difference between even and odd. If any of them seem to be confused, talk about even socks and odd socks.
a. $261=2 \cdot 100+6 \cdot 10+1$

$$
\begin{aligned}
& =2(99+1)+6(9+1)+1 \\
& =2 \cdot 99+2+6 \cdot 9+6+1 \\
& =2 \cdot 99+6 \cdot 9+9
\end{aligned}
$$

b. $846=8 \cdot 100+4 \cdot 10+6$

$$
\begin{aligned}
& =8(99+1)+4(9+1)+6 \\
& =8 \cdot 99+8+4 \cdot 9+4+6 \\
& =8 \cdot 99+4 \cdot 9+18 \\
& =8 \cdot 99+4 \cdot 9+2 \cdot 9
\end{aligned}
$$

c. Use the same method to determine whether each of the following numbers is divisible by $9: 342,83,954,276$ and 2,574 . See TE.
d. Make a conjecture for a rule to determine whether or not a number is divisible by 9. See TE.

## 16. Ingenuity:

a. Using a process similar to that in the investigation, to show why the rule to determine whether or not a number is divisible by 3 works. See TE.
17. $p$ and $q$ are primes.
a. List all of the factors of $p$. 1 and $p$
b. Let $n=p q$. List all of the factors of $n . \quad 1, p, q$, and $p q$
18. Copy and complete the following addition table, with each row and each column headed by a prime number. Then list all the integers that appear as sums in your addition table. Make as many observations as possible.

| + | 2 | 3 | 5 | 7 | 11 | 13 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 4 | 5 | 7 | 9 | 13 | 15 |
| 3 | 5 | 6 | 8 | 10 | 14 | 16 |
| 5 | 7 | 8 | 10 | 12 | 16 | 18 |
| 7 | 9 | 10 | 12 | 14 | 18 | 20 |
| 11 | 13 | 14 | 16 | 18 | 22 | 24 |
| 13 | 15 | 16 | 18 | 20 | 24 | 26 |

19. This exercise should reinforce the Commutative Property of Multiplication. See if students realize that the table is symmetric about the diagonal. They might be able to see this, especially if you remind them about symmetry or if you tilt the table. The last question foreshadows the idea of unique prime factorization in the Fundamental Theorem of Arithmetic. That is, a positive integer can be written as a product of primes in only one way. In this case, there are many numbers that do not appear in the table, because we are taking products of only two primes.

## Ingenuity

20. It is sufficient to check until the prime factors are less than $\sqrt{ } 103$. Since factors of any number come in pairs, after checking the prime factors less than $\sqrt{ } 103$ any other factors of 103 will be less than the primes you have already checked. For all $n$, it is sufficient to check the primes $2,3,5, \ldots, \sqrt{ } n$.

## Investigation

21. c. Discuss what conjecture means with students. In general, a mathematical conjecture is a statement that is thought to be true but has not been formally proven. The conjecture is often based on observations and patterns. Students might use the area model to show that if they put an $n$ by $k$ rectangle adjacent to an $n$ by $m$ rectangle, they get an $n$ by $(k+m)$ rectangle. Algebraically, this is $n \cdot k+n \cdot m=n \cdot(k+m)$ or an example of the Distributive Property.
22. Copy and complete the following multiplication table, in which each row and each column is headed by a prime number. Are there any products in the table that appear only once? Are there any products that appear in the table more than twice? See TE.

| $x$ | 2 | 3 | 5 | 7 | 11 | 13 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 4 | 6 | 10 | 14 | 22 | 26 |
| 3 | 6 | 9 | 15 | 21 | 33 | 39 |
| 5 | 10 | 15 | 25 | 35 | 55 | 65 |
| 7 | 14 | 21 | 35 | 49 | 77 | 91 |
| 11 | 22 | 33 | 55 | 77 | 121 | 143 |
| 13 | 26 | 39 | 65 | 91 | 143 | 169 |

## 20. Ingenuity:

In the process of determining whether a number $n$ is prime or composite, we discussed how checking for any prime factors less than $n$ is sufficient. Let's explore if it is sufficient to check a smaller set of numbers. For example, consider the number 103 and explore whether it is necessary to check the entire list of numbers less than 103 to determine if it is prime or whether it is sufficient to check a smaller set of numbers. Explain your conclusion. Consider what you might be able to conclude about checking whether any number $n$ is or is not prime. What set of numbers must one check to determine whether $n$ is prime? See TE.

## 21. Investigation:

a. The number 152 is the sum of 120 and 32 . Both 120 and 32 are multiples of 4 . Is 152 a multiple of 4 ? Yes
b. The number 231 is the sum of 140 and 91 . Both 140 and 91 are multiples of 7 . Is 231 a multiple of 7 ? Yes
c. Make a conjecture based on the pattern in these two examples. Show why your conjecture always works.
Using part (a) as an example, because of the distributive property:
$120+32=4 \times 30+4 \times 8=4(30+8)=4(38)=152$

## PRIME MAGIC SQUARE

Objective: Students will use the prime numbers found from the Sieve of Eratosthenes activity in section 3.2 to complete a magic square.

## Materials:

Copy of Prime Magic Square
List of first 25 prime numbers

## Activity Instructions:

Make a copy of the incomplete magic square on the next page, or ask students to make a copy from the board or overhead.

Tell your students that they will use their list of the first 25 primes from the Sieve of Eratosthenes activity to complete this magic square.

Unlike most magic squares they have seen in the past, this magic square is unique. This magic square is only comprised of Prime Numbers less than 100, with no repeated numbers. Another unique quality of this magic square is that it is not necessary that each row, column and diagonal have the same sum. It is just necessary that each row, column and diagonal have a sum in the range of 211 to 213.

We've provided the students with five of the twenty-five boxes already filled in. If you need to modify this activity to fit the needs of your students, you can add more answers to make it easier or take some of the answers away to make it harder. For a really great challenge, don't provide them with any boxes already filled in.

The solution to this magic square is:

| 41 | 79 | 17 | 13 | 61 |
| :---: | :---: | :---: | :---: | :---: |
| 53 | 3 | 83 | 67 | 7 |
| 59 | 97 | 5 | 23 | 29 |
| 11 | 31 | 37 | 89 | 43 |
| 47 | 2 | 71 | 19 | 73 |

## PRIME MAGIC SQUARE

Directions: Complete the magic square below so that each row, column or diagonal has a sum in the range of 211 and 213. Each box below must be filled in with exactly one of the first 25 prime numbers less than 100, with no number repeated. Some boxes have already been filled in for you. Good luck and have fun!


## FACTOR BINGO

Objective: This game is designed to reinforce factors, multiples and prime \& composite numbers; it will also help students review multiplication facts.

## Materials:

One Factor Bingo card per student
One set of number cards 1-36
Timer/stop watch
Color pencils

## Activity Instructions:

1) Make number cards. Write numbers 1-36 on a set of index cards, one set per group.
2) Play the game with $2,3,4$, or 5 players.
3) The dealer shuffles the deck of number cards and deals the pack equally, facing down.
4) The time is set for $1-2$ minutes
5) Each player starts at the same time and goes through as many cards as they can, marking off the factors of their number cards on their Factor Bingo card.
6) The winner is the player who crosses off the most factors by the end of the $1-2$ minutes.
7) To break a tie, players may re-shuffle number cards.

Variation: To continue reinforcing multiplication facts, students my play again for speed by decreasing the amount of minutes allowed for play.

## FACTOR BINGO

| 1 | 2 | 3 | 4 | 5 | 6 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 7 | 8 | 9 | 10 | 11 | 12 |
| 13 | 14 | 15 | 16 | 17 | 18 |
| 19 | 20 | 21 | 22 | 23 | 24 |
| 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 |


| 36 | 35 | 34 | 33 | 32 | 31 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 30 | 29 | 28 | 27 | 26 | 25 |
| 24 | 23 | 22 | 21 | 20 | 19 |
| 18 | 17 | 16 | 15 | 14 | 13 |
| 12 | 11 | 10 | 9 | 8 | 7 |
| 6 | 5 | 4 | 3 | 2 | 1 |


| 19 | 11 | 20 | 7 | 23 | 1 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 12 | 21 | 8 | 24 | 2 | 28 |
| 22 | 9 | 25 | 3 | 29 | 13 |
| 10 | 26 | 4 | 30 | 14 | 33 |
| 27 | 5 | 31 | 15 | 34 | 17 |
| 6 | 32 | 16 | 35 | 18 | 36 |


| 1 | 22 | 7 | 27 | 11 | 30 |
| :---: | :---: | :---: | :---: | :--- | :---: |
| 17 | 2 | 23 | 8 | 28 | 12 |
| 33 | 18 | 3 | 24 | 9 | 29 |
| 14 | 34 | 19 | 4 | 25 | 10 |
| 31 | 15 | 35 | 20 | 5 | 26 |
| 13 | 32 | 16 | 36 | 21 | 6 |


| 1 | 7 | 13 | 19 | 25 | 31 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 8 | 14 | 20 | 26 | 32 |
| 3 | 9 | 15 | 21 | 27 | 33 |
| 4 | 10 | 16 | 22 | 28 | 34 |
| 5 | 11 | 17 | 23 | 29 | 35 |
| 6 | 12 | 18 | 24 | 30 | 36 |


| 21 | 26 | 30 | 33 | 35 | 36 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 15 | 20 | 25 | 29 | 32 | 34 |
| 10 | 14 | 19 | 24 | 28 | 31 |
| 6 | 9 | 13 | 18 | 23 | 27 |
| 3 | 5 | 8 | 12 | 17 | 22 |
| 1 | 2 | 4 | 7 | 11 | 16 |


| 31 | 25 | 19 | 13 | 7 | 1 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 32 | 26 | 20 | 14 | 8 | 2 |
| 33 | 27 | 21 | 15 | 9 | 3 |
| 34 | 28 | 22 | 16 | 10 | 4 |
| 35 | 29 | 23 | 17 | 11 | 5 |
| 36 | 30 | 24 | 18 | 12 | 6 |


| 16 | 15 | 13 | 14 | 17 | 18 |
| :---: | :---: | :---: | :---: | :--- | :---: |
| 10 | 9 | 7 | 8 | 11 | 12 |
| 4 | 3 | 1 | 2 | 5 | 6 |
| 22 | 21 | 19 | 20 | 23 | 24 |
| 28 | 27 | 25 | 26 | 29 | 30 |
| 34 | 33 | 31 | 32 | 35 | 36 |

## Section 7.3 - Exponents and Order of Operations

## Big Idea:

Developing an understanding of Exponents, Order of Operations, Perfect Squares and Perfect Square Roots

## Key Objectives:

- Discover patterns for simplifying operations with exponents.
- Use of Order of Operations


## Materials:

Graphing calculators, Colored paper

## Pedagogical/Orchestration:

- Students need to work problems on paper first, then check work with calculator.


## Internet Resource:

Rags to Riches: Order of Operations- http://www.quia.com/rr/116044.htm|

## Activities:

"PEMDAS," "Order of Operations," and "I Have-Who Has" at the end of the section and on the CD
Explorations 1 and 2 offer opportunities for group work and good class discussion.

## Vocabulary:

exponential notation, power, base, exponent, order of operations

## TEKS:

6.2(E);
6.13(A); 7.2(E)(F)(G);
8.1(A,D,E);
8.2(B);
8.16(A

## WARM-UPS for Section 7.3

1. Compute the following sums. Can you find a short cut for each computation?
a. $-3+-3+-3 \quad$ Ans: $(-3)(3)=-9$
b. $-4+-4+-4+-4 \quad$ Ans: $(-4)(4)=-16$
c. $-5+-5+-5+-5+-5 \quad$ Ans: $(-5)(5)=\mathbf{- 2 5}$
d. $-6+-6+-6+-6+-6+-6$ Ans: $(-6)(6)=-36$
e. $-1+-2+-3+-4+-5+-6$ Ans: -21
f. $1+-2+3+-4+5+-6+7+-8+9+-10$ Ans: -5
g. $1+-2+3+-4+5+\ldots+99+-100$ Ans: -50
2. Predict which is the greater of each of the following pairs of numbers. Then compute each pair and determine if you prediction is correct or not and why.
a. $(7+8)^{2}$ and $7^{2}+8^{2}$
b. $(9+11)^{2}$ and $9^{2}+11^{2}$
c. $(a+b)^{2}$ and $a^{2}+b^{2}$

Ans: If $a>0$ and $b>0$, then $(a+b)^{2}=a^{2}+2 a b+b^{2}>a^{2}+b^{2}$ because $2 a b>0$.

## Launch for Section 7.3

Ask the students, "What would you do if you had this decision to make? Your uncle is asking you what you would prefer for your birthday gifts and gives you the choice of $\$ 50$ every year on your birthday, or start with $\$ 1.00$ for this birthday and double the amount every year thereafter." Allow students some time to discuss their preference, but do not give an answer or explanation at this time. After students discuss their ideas tell them, "Pay attention to the examples we do in class today and see if any of them will help you decide what the best option would be. Our lesson is about exponents so see if you can relate today's lesson to the birthday present decision." After the lesson, remember to relate back to this launch and ask students if they have made a decision on which present they would choose. It is interesting to note that if the student chose the $\$ 50$ option, on the 21 st year he would have received $\$ 50$ for a twenty-one year total of $\$ 1050$, whereas on the doubling dollar option, on the 21 st year alone, the student would receive over one million dollars. Hopefully by the end of today's lesson the student will have begun to appreciate the power of the exponent.

## SECTION 7.3 EXPONENTS AND ORDER OF OPERATIONS

In Section 4.1, we modeled multiplication by repeatedly adding an integer to itself. There are also situations in which it is useful to multiply a number repeatedly by itself.

## EXAMPLE 1

Escherichia coli bacteria are more commonly known as E. coli. One of the bacteria lives in a petri dish. The number of bacteria in the dish doubles each hour. How many bacteria are living in the dish after 1 hour? 2 hours? 3 hours? 5 hours? $n$ hours?

## SOLUTION

Because the number of bacteria doubles each hour, after 1 hour there will be 2 bacteria. After 2 hours there will be $2 \cdot 2=4$ bacteria. We can write this information using exponential notation as

$$
\begin{aligned}
& 1^{\text {st }} \text { hour }=2=2^{1} \\
& 2^{\text {nd }} \text { hour }=2 \cdot 2=2^{2}=4 \\
& 3^{\text {rd }} \text { hour }=2 \cdot 2 \cdot 2=2^{3}=8
\end{aligned}
$$

Continuing this pattern, $5^{\text {th }}$ hour $=2 \cdot 2 \cdot 2 \cdot 2 \cdot 2=2^{5}=32$ bacteria are living in the dish after 5 hours. If we let $\mathrm{n}=$ number of hours, there are $2^{n}$ bacteria after $n$ hours.

## DEFINITION 7.3: EXPONENTS AND POWERS

Suppose that $n$ is a whole number. Then, for any number $x$, the $n^{\text {th }}$ power of $x$, or $x$ to the $n^{\text {th }}$ power, is the product of $n$ factors of the number $x$. This number is usually written $x^{n}$. The number $x$ is usually called the base of the expression $x^{n}$, and $n$ is called the exponent.
2. This is a perfect time to use the graphing calculator as an exploratory tool. First, on the home screen, enter a 2 . Mentally count that as 1 . Then press $\times 2$ and enter. The answer should be 4 . Mentally count that as 2 . Continue to press enter as you count. For instance, the next time you press enter, your should count 3 and see 8. What you are seeing is the powers of 2 you are counting. Continue to press enter, counting and looking for a number that is greater than 10,000. The 13th time you press enter, you should see 8192 on the screen. The 14th time you press enter, you should see 16384, which is larger than 10,000. You can show them how to use the " $\wedge$ " in computing powers of 2 , such as $2 \wedge 5=32$ and $2 \wedge 7=128$.

Exploration $\mathbf{1}$ and $\mathbf{2}$ might be a fine time to use the power of the calculator. Even though it is tempting to express numbers as powers only, it is a good exercise in the power of exponents to see how quickly the numbers grows, within reason. Have students work out the calculations in both Explorations first and then check with a calculator.

## Exploration 1

The students may observe a pattern for $x^{n} \cdot x^{m}=x^{(n+m)}$. The objective here is not to get to this rule of exponents, but you may find the students will natrually discover this rule.

Continuing Example 1, use a calculator to answer the following questions:

1. How many bacteria will there be after 10 hours? 1024
2. We need at least 10,000 bacteria for an experiment. When can we harvest this many bacteria? between the 13 th and 14 th hour

## EXPLORATION 1

By using the definition of exponential notation and multiplication, we see that:

$$
3^{4} \cdot 3^{6}=(3 \cdot 3 \cdot 3 \cdot 3) \cdot(3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3)=3^{10}=3^{4+6} .
$$

Compute the following products, showing all your work.
a. $3^{2} \cdot 3^{3}$
$3^{5}$ or 243
c. $3^{3} \cdot 3^{2}$
$3^{5}$ or 243
b. $2^{2} \cdot 2^{3}$
$2^{5}$ or 32
d. $10^{3} \cdot 10^{5}$
$10^{8}$ or $100,000,000$

Does this calculation agree with your rule? Use this pattern to check your answers in Exploration 1. Answers will vary, but we hope yes.

Do you see a pattern for multiplying numbers in exponential form with the same base? Use your observation to determin the product in the problem below.

## PROBLEM 1

Compute the product: $2^{5} \cdot 2^{4} \quad 2^{9}$
The pattern leads to the multiplication property for exponents.

PROPERTY 7.1: MULTIPLICATION OF POWERS
Suppose that $x$ is a number and $a$ and $b$ are whole numbers. Then

$$
x^{a} \cdot x^{b}=x^{a+b}
$$

Include pattern justification for 4 raised to the 0 . Ask students how they would reach conclusion that 4 to the zero power is 1 . Then ask if this works for every number? You might want to talk about the problem with $0^{0}$, depending on your class. $0^{\circ}$ is undefined. There are a couple of ways to convince students of this. First, take any number $n$ to the 0th power. Then $n^{0}=n^{x-x}=n^{x} / n^{x}$. But no one can divide by 0 , so $n$ cannot be 0 . Or ask what your students think $0^{0}$ is. There are two logical answers: $0^{0}=1$ because any number to the 0th power $=1.0 \mathrm{OR} 0^{0}=0$ because 0 to any power is 0 . No number can have two values, $500^{\circ}$ is undefined.

## PROBLEM 2

Use this property to rewrite the following products and then compute them. Use a calculator to compute both equivalent forms.
a. $4^{3} \cdot 4^{5} \quad 4^{8}=65536$
c. $10^{3} \cdot 10^{4}$
$10^{7}=10000000$
b. $2^{5} \cdot 2^{5} \quad 2^{7}=1024$
d. $3^{2} \cdot 4^{3}$
576

Special Cases: What do $4^{1}$ and $4^{0}$ equal?
We note that $4 \cdot 4=4^{2}=4^{1+1}=4^{1} \cdot 4^{1}$, so $4^{1}$ must be the same as 4 . We can use the same process for any number $x$ : $x \cdot x=x^{2}=x^{1+1}=x^{1} \cdot x^{1}$, so $x^{1}=x$.

What does $4^{0}$ equal? Because $4 \cdot 4^{0}=4^{1} \cdot 4^{0}=4^{1+0}=4^{1}=4=4 \cdot 1$, we see that multiplying by $4^{0}$ is the same as multiplying by the number 1 . We therefore assume that for any positive integer $n, n^{0}=1$.

## Place Value

Consider the number 4,638. Using place value and our notation of exponents, we can rewrite 4,638 in the following way:

$$
4 \cdot 1,000+6 \cdot 100+3 \cdot 10+8 \cdot 1=4 \cdot 10^{3}+6 \cdot 10^{2}+3 \cdot 10^{1}+8 \cdot 10^{0}
$$

Start with the expression $4 \cdot 10^{3}+6 \cdot 10^{2}+3 \cdot 10^{1}+8 \cdot 10^{0}$, or in calculator notation, $4 \times 10^{3}+6 \times 10^{2}+3 \times 10^{1}+8 \times 10^{0}$. In what order can we perform the calculations in this expression so the sum equals 4,638 ?

## PROBLEM 3

Compute the following:
a. $7 \cdot 10^{2}+3 \cdot 10^{1}+2 \cdot 10^{0} \quad 732$
b. $6 \cdot 10^{3}+5 \cdot 10^{2}+9 \cdot 10^{1}+3 \cdot 10^{0} \quad 6,593$
c. $6 \cdot 10^{3}+9 \cdot 10^{1}+3 \cdot 10^{0} \quad 6,093$

In this section, exponents will be included in problems for order of operations.
a. $1+2^{3}=1+8=9$
b. $3+2 \cdot 5=3+10=13$
c. $2 \cdot 4^{3}=2 \cdot 64=128$

What we are looking for here is the idea of order of operations. That is, the calculator does not simply multiply or add from left to right, as we read.

Parentheses can be used when necessary to help us keep the order of operations straight in our minds. For example, $(7 \cdot 8)-(6 \div 2)$ is the same as $7 \cdot 8-6 \div 2$.

For example, to compute $7 \cdot 8-6 \div 2$, the rule tells us that we perform any multiplication and division before any addition or subtraction, from left to right. So in the above example, we compute $56-3$ or 53 as our answer; the parentheses $(7 \cdot 8)-(6 \div 2)$ are understood. However, if there were any parentheses in the problem, we would compute them first. For example, for $7 \cdot(8-6) \div 2$ the order of operations is $7 \cdot 2 \div 2$ or $14 \div 2$ or 7 . The operation in the parentheses is performed before any of the other operations.

## EXPLORATION 2

Use the problem in Exploration 2 in the following way.

1. Pose the problem to students to solve using prior knowledge.
2. List their answer choices
3. Ask "Why are there several answers to the same problem?"
4. Lead students to the conclusion that order of operations is important.
5. Explain that the conventional process for order of operations will be examined in the following activity and it will lead to the correct answer to this problem. Then guide students through the Order of Operations Activity found at the end of this section in the Teacher's Edition.

You may wish to use simpler computations that highlight the difference in the obtained values depending onthe order of operations. For example, consider the sequence: $3+4 \times 5,3 \times 4+5,3 \times 2^{2},(3 \times 2)^{2}$ and discuss multiplication before addition, exponents before multiplication, parenthesis before exponenets and other rules int he order of operations.

We are using PEMDAS to compute $20-10 \div 2+3 \wedge 3-9=20-10 \div 2+27-9=20-5+27-9=33$

## ORDER OF OPERATIONS

To compute $2^{4}$ on a calculator, we enter $2 \wedge 4$. To multiply 3 by 5 , we enter $3 \times 5$. Suppose we enter the following into a calculator:
a. $1+2 \wedge 3$
b. $3+2 \times 5$
c. $2 \times 4 \wedge 3$

What will the results be? Can you explain what the calculator is doing? You might wonder why the calculator does not perform these calculations from left to right, as we read them. We can see that the order the calculator uses, which is called the order of operations, is natural by examining our place value system.

Remember from Section 4.6, we have the order in which mathematical operations are performed, as shown below.

## Order of Operations

- Compute the numbers inside the parentheses
- Compute any exponential expressions
- Multiply and divide as they occur from left to right
- Add and subtract as they occur from left to right

There are different forms of grouping symbols: parentheses, ( ), brackets, [ ], and braces, $\{$ \}. Absolute value symbols are treated as a type of grouping symbol.

Why do these two problems have different solutions?
a. $7 \cdot 8-6 \div 2$
b. $7 \cdot(8-6) \div 2$

## EXPLORATION 2

Compute the following, showing all your work.

$$
20-10 \div 2+3^{3}-9
$$

There are different forms of grouping symbols: parentheses, ( ), brackets, [ ], and braces, $\{$ \}. Absolute value symbols are treated as a type of grouping symbol.
1.
a. $5^{3}=125,3^{5}=243,3^{5}>5^{3}$
b. $9^{4}=6561,3^{6}=729,9^{4}>36$
c. $5^{4}=625,10^{3}=1000,10^{3}>5^{4}$
d. $5^{3}=125,2^{7}=128,2^{7}>5^{3}$
2. a. 10
b. 25
c. 1000
d. 16
4.
a. 15 d. 11
b. 3
e. 166
c. $\quad-9$
f. 1744

## PROBLEM 4

Compute the following:
a. $6 \div 2\left(4^{2}+7\right)$
b. $\quad 4 \cdot|7-3| \div 2$
c. $4+2^{3} \times 3-(17-5) \times 3+(17-5) \div 2$
d. $\quad 10+\left(5-2^{2}\right) \cdot|-9+8|$

## EXERCISES

1. Expand and compute the answer of the following. Decide which is greater or if the two numbers are equal.
a. $5^{3}$ or $3^{5}$
C. $5^{4}$ or $10^{3}$
b. $9^{4}$ or $3^{6}$
d. $5^{3}$ or $2^{7}$
2. Evaluate the following expressions:
a. $\quad 2^{3}+2$
b. $3^{2}+4^{2}$
c. $2^{3} \cdot 5^{3}$
d. $2^{3} \cdot 2$
3. Evaluate the following expressions:
a. $2^{3} \cdot 2^{4}$ and $2^{7}$

128
b. $3^{3} \cdot 3^{2}$ and $3^{5}$243
c. $2^{3} \cdot 3^{2} \quad 72$
d. $2^{3} \cdot 3^{3}$ and $6^{3}$

216
4. Evaluate the following numerical expressions using Order of Operations.
a. $17-2^{3}+6$
b. $17-\left(2^{3}+6\right)$
c. $4-(8+5)$
d. $4-8+5 \cdot 3$
e. $7+5 \cdot 3-16 \cdot 3^{2}$
f. $28(5+3)^{2}-3\left(2^{4}\right)$
5. Go over the first step in the process to make sure the students understand the tripling process resulting from the cutting and the stacking.
7. a. 8096
b. 12024
c. 378
d. 4592
e. 4092
8. Answer: c
9. Notice that each student must eat a number of pieces of candy that is a power of 5 : the first student must eat $5^{0}$, the second $5^{1}$, the third $5^{2}$, and so on. The sixth must eat $5^{5}=3125$
5. Rhonda has one sheet of paper. She cuts it into thirds and stacks the three sheets. If she completes this process a total of 5 times, how many sheets thick will the resulting stack be? By the way, she only has to complete the process 27 times before the stack reaches the moon. 243 units
6. Calculate the following:
a. $10^{1}$
10
c. $10^{3} \quad 1,000$
b. $10^{2}$
100
d. $10^{8}$
100,000,000
e. Explain how you would calculate $10^{200}$ Write 1 followed by 200 zeros.
7. Evaluate the following numerical expressions using Order of Operations.
a. $3 \cdot 10^{2}-204+\left(8 \cdot 10^{3}\right)$
b. $8,034+\left(4 \cdot 10^{2}\right) \cdot 10-10^{2} \div 10$
c. $3 \cdot 10^{2}+7 \cdot 10+8 \cdot 10^{0}$
d. $4 \cdot 10^{3}+5 \cdot 10^{2}+9 \cdot 10^{1}+2 \cdot 10^{0}$
e. $4 \cdot 10^{3}+9 \cdot 10^{1}+2 \cdot 10^{0}$
8. What is the value of the expression below? Select the best choice and explain your answer.

$$
5+5(9 \div 3)^{2}
$$

a. 35
b. 90
c. 50
d. 230
9. Six students decide to have a candy eating contest. The first student eats 1 piece of candy. Each student must then eat 5 times as many pieces of candy as the previous student. How many pieces of candy must the sixth student eat? See TE.
10. Evaluate the following numerical expressions using Order of Operations.
a. $2 \times 8+48 \div 3-4^{2} \times 2 \quad 0$
b. $4^{3}+100-4(7-4)^{3} \div 18 \quad 162$
11. a. -9
b. 11
c. 22
d. - 11
e. 9
f. 642
g. 4,837
h. 5,904
12. Answer: b
13.
a. $\quad(7)(7)(7)$
d. $(12)(12)(12)$
b. $\quad(8)(8)(8)$
e. $(10)(10)(10)$
c. $(5)(5)(5)(5)$
f. (6)(6)(6)(6)
14. B
11. Compute the following:
a. $4-(8+5)$
b. $4-8+5 \cdot 3$
c. $8(8+3) \div 2^{2}$
d. $5^{2}+6(-10+4)$
e. $4+|9-1| \div 2^{2}+3$
f. $\quad 6 \cdot 10^{2}+4 \cdot 10^{1}+2 \cdot 1$
g. $4 \cdot 10^{3}+8 \cdot 10^{2}+3 \cdot 10^{1}+7 \cdot 10^{0}$
h. $5 \cdot 10^{3}+9 \cdot 10^{2}+4 \cdot 10^{0}$
12. What is the value of the expression below? Select the best choice and explain your an

$$
10+7 \cdot 8^{2} \div 2
$$

a. 61
b. 234
c. 544
d. 66
13. Compute the following:
a. $(3+4)^{3}$
b. $(3+5)^{3}$
c. $(2+3)^{4}$
d. $(3 \cdot 4)^{3}$
e. $(2 \cdot 5)^{3}$
f. $(2 \cdot 3)^{4}$
14. Part I: A mistake was made in simplifying the expression below.

$$
\begin{aligned}
& \text { Simplify: } 3+4(5+2)-2^{3} \\
& \text { Step 1: } 3+4(7)-2^{3} \\
& \text { Step 2: } 7(7)-2^{3} \\
& \text { Step 3: } 49-2^{3} \\
& \text { Step 4: } 49-8 \\
& \text { Step 5: } 41
\end{aligned}
$$

In which step did the first mistake occur?
a. Step 1
b. Step 2
c. Step 3
d. Step 4

Part II: What was the mistake made?
15. $\quad P_{1}=2$
$P_{2}=2(2)$
$P_{3}=2^{3}$
$P_{n}=2^{n}$
Ingenuity
16. Most students will want to answer $2^{3}$, but if you take $2^{3}$ (or 8 ) cookies and $2^{3}$ cookies and put them together, the result will be 16 , or $2^{4}$ cookies. Since we want to divide $2^{6}$, or 64 , cookies evenly we will want to give each part 32 cookies, or $2^{5}$ cookies.
15. Look back at Example 1. If the number of bacteria initially is $P$, how many bacteria will there be after 2 hours? After 3 hours? After $n$ hours?
16. Ingenuity:

If $2^{6}$ cookies are divided evenly between two people, how many will each person receive? Explain how you reached your answer. Each person will receive $2^{5}$ or 32 cookies. Answers will vary.

## 17. Investigation:

We can get $x^{10}$ from $x$ in 5 multiplications of previously generated powers, e.g. $x \rightarrow x^{2} \rightarrow x^{4} \rightarrow x^{5} \rightarrow x^{9} \rightarrow x^{10}$. What is the minimum number of such multiplications required to get $x^{100}$ from $x$ ? 8

## PEMDAS

Objective: Students will be able to solve number expressions that may include the four basic operations (+ - * $\div$ ), parentheses, and/or exponents.

## Materials:

PEMDAS Chart
Index cards

## Activity Instructions:

1. Teacher will have students work in groups to create number expressions with numbers $0-9$ written in index cards, including each of the basic operations (+-* $\div$ ), parentheses, and exponents 2 and 3.
2. Each group will solve number expressions created by each student. As each expression is solved, students in the groups check each other's solutions.
3. Students will refer back to the PEMDAS (Please Excuse My Dear Aunt Sally) chart. Students need to remember that multiplication and division are equal level. Addition and subtraction are also in the same level.
4. After you have done parentheses and exponents, work from left to right and multiply or divide whichever comes first. Do the same if there is an addition and a subtraction.

PEMDAS Chart

| $P$ | Parentheses first |
| :---: | :--- |
| E | Exponents (ie Powers and Square Roots, etc.) |
| MD | Multiplication and Division (left-to-right) |
| AS | Addition and Subtraction (left-to-right) |

## Order of Operations

Activity Instructions: Students will create a PEMDAS flip chart in the following manner:

1. Take a single sheet of card stock or other similar paper.
2. Fold the page in half lengthwise. The top will be the folded side and the bottom is the open side.
3. Measure a 1 inch section at the top (near the fold) to create the longest rectangle. This will be the title section.
4. Divide the remaining section into 4 windows. Remember to cut only the top layer.

Order of Operations $18+7^{2} \times(8-2) \div 3+8 \quad$ (This will be written in the title section)

| Window 1 | Window 2 | Window 3 | Window 4 |
| :--- | :--- | :--- | :--- |
|  |  |  |  |

Write the following information on the cover and inside each window in order from left to right.

| Window: | Outside: | Inside: |
| :---: | :--- | :--- |
| 1 | P | $18+7^{2} \times(8-2) \div 3+8$ |
|  | Simplify (solve) | $18+7^{2} \times 6 \div 3+8$ |
|  | grouping symbols |  |
|  | like Parenthesis |  |
| 2 | E | $18+7^{2} \times(8-2) \div 3+8$ |
|  | Find the value | $18+7^{2} \times 6 \div 3+8$ |
|  | of powers (exponents) | $18+7 \times 7 \times 6 \div 3+8$ |
|  | $18+49 \times 6 \div 3+8$ |  |
| 3 | MD | $18+7^{2} \times(8-2) \div 3+8$ |
|  | Multiply and/or | $18+7^{2} \times 6 \div 3+8$ |
|  | divide in order from | $18+7 \times 7 \times 6 \div 3+8$ |
|  | left to right | $18+49 \times 6 \div 3+8$ |
|  |  | $18+294 \div 3+8$ |
|  |  | $18+98+8$ |
| 4 | AS | $18+7^{2} \times(8-2) \div 3+8$ |
|  | Add and/or subtract | $18+7^{2} \times 6 \div 3+8$ |
|  | in order from | $18+7 \times 7 \times 6 \div 3+8$ |
|  | left to right | $18+49 \times 6 \div 3+8$ |
|  |  | $18+294 \div 3+8$ |
|  | $18+98+8$ |  |
|  | $116+8$ |  |
|  |  | 124 |

# I HAVE - WHO HAS 



Objective: The students will play rounds of the I Have - Who Has game to reinforce skills learned in section 7.3 about exponents.

## Materials:

I Have-Who Has set of cards (copied on cardstock, then cut out)
Scratch paper (to work out the solutions)

## Activity Instructions:

Shuffle the cards and then pass them out, one each to all of the students. If you have extra cards, ask for volunteers in the class who are willing to work with more than one card at a time. It is necessary for all of the cards to be passed out before you begin to play.

Choose a student to go first and have that student read the top of their card. The game continues going around the room until everyone has had an opportunity to read their card and it comes back to the original student. *Please remind students to read their cards carefully. The card -25 reads "negative of 2 raised to the fifth power" while the card ( -2 ) 5 reads "the quantity negative 2 raised to the fifth power."

This game should be timed, and repeated as necessary until the class is able to finish a whole round in a specified amount of time. Let the students set their own time goal to make it more meaningful for them.

|  |  | I have 13..... | I have 5...... |
| :---: | :---: | :---: | :---: |
| Who has $2^{4}+1$ ? | Who has $(-3)^{2}+4 ?$ | Who has $1^{3}+(-2)^{2}$ ? | Who has $10^{3}+2 ?$ |
| I have $1,002 \ldots$ | I have 728.... | I have 20.... | I have 72.... |
| Who has $9^{3}-1$ ? | Who has $(-5)^{2}-5 ?$ | Who has $4^{3}+2^{3}$ ? | Who has $(10-4)^{2} ?$ |
| I have 36.... | I have - $\mathbf{3 0}$.... | I have 32.... | I have - $4 . . .$. |
| Who has $(-3)^{3}-3 ?$ | Who has $2^{(4+1)} ?$ | Who has $(-4)^{2}-20 ?$ | Who has $1^{5}-1^{4} ?$ |
| I have $\underline{\mathbf{0}} . . . .$. | I have -5..... | I have $\underline{\mathbf{4 0}} \ldots$ | I have 81... |
| Who has $-3^{2}+4$ ? | Who has $-5^{2}-15$ ? | Who has $\left(3^{2}\right)^{2}$ ? | Who has $\left(-2^{3}\right)^{2}-4 ?$ |


| I have 60..... | I have 44..... | I have 91..... | I have 40... |
| :---: | :---: | :---: | :---: |
| Who has $6^{2}-(-2)^{3} ?$ | Who has $10^{2}+-3^{2} ?$ | Who has $(-7)^{2}-9 ?$ | Who has $-2^{5}-2 ?$ |
| I have - $\mathbf{3 4} .$. | I have -28.... | I have 1.... | I have 49..... |
| $(-2)^{5}+2^{2} ?$ | Who has $\left(-4^{3}\right)^{0}$ ? | Who has $3^{0}+(-7)^{2} ?$ | Who has $8^{2}-10^{0} ?$ |
| I have 63.... | I have 43... | I have 16.... | I have 64...... |
| Who has $4^{3}-4^{2}-4^{1}-4^{0}$ ? | Who has $2^{3} \cdot 2^{1}$ ? | Who has $4^{2} \cdot 4^{1}$ ? | Who has $(-12)^{2}-4 ?$ |
| I have 140... | I have $1,024 \ldots . .$ |  |  |
| Who has $2^{10}$ ? | Who has $4 \cdot 4^{0}$ ? |  |  |

## Section 7.4 - Square Numbers and Square Roots

## Big Idea:

Develop an understanding of Perfect Squares and Perfect Square Roots

## Key Objectives:

- Use an area model to discover Square Numbers and their Square Roots
- Understand that Square Numbers and Square Roots are inverses of each other.


## Materials:

- Graph paper
- Graphing calculators
- Colored pencils


## Pedagogical/Orchestration:

- Students need to work problems on paper first, then check work with calculator.


## Vocabulary:

perfect square number, square number, square root, radical notation, radicand

## TEKS:

7.1(C); 8.1(C)

## WARM-UPS for Section 7.4

1. a. How would you simplify this fraction? $\frac{10+15}{2+3}$
b. Which of these is a correct way to answer part a? Explain your answer.
i. $\frac{10}{2}+\frac{10}{3}=5+5=10$
ii. $\frac{10}{2}+\frac{15}{3}=5+5=10$
2. Which of the following numbers is equivalent to $987,654,321$ ?
a. $987.654321 \times 10^{3}$
b. $98.7654321 \times 10^{7}$
c. $9.87654321 \times 10^{1}$
d. $9.87654321 \times 10^{9}$

Ans: b

## Launch for Section 7.4:

Students will make a number line from 0 to 100, using a scale of 1 labeling only the Perfect Square Numbers. Make an area model below every Perfect Square Number on the number line.

| Area | Dimensions |
| :---: | :---: |
| 1 | $1 \times 1$ |
| 4 | $2 \times 2$ |
| 9 | $3 \times 3$ |
| 16 | $4 \times 4$ |
| 25 | $5 \times 5$ |
| $\ldots$ | $\ldots$ |
| 625 | $25 \times 25$ |

## SECTION 7.4 SQUARE NUMBERS AND SQUARE ROOTS

## SQUARE NUMBERS AND SQUARE ROOTS

Identify the dimensions of the squares below and their areas.


You found that the:
$1 \times 1$ square has area $1^{2}=1$ square unit
$2 \times 2$ square has area $2^{2}=4$ square units
$3 \times 3$ square has area $3^{2}=9$ square units
$4 \times 4$ square has area $4^{2}=16$ square units
$5 \times 5$ square has area $5^{2}=25$ square units
Square numbers or perfect squares are whole numbers, $n$, that can be written as the square of a whole number, b : That is $\mathrm{n}=\mathrm{b}^{2}$.

9 is a perfect square because $9=3^{2}$. Identify other perfect squares less than 9. How many perfect squares are there less than or equal to 100 ? Identify and locate these perfect squares on the number line.

Create a table similar to the one below, and identify the perfect squares associated to squares of dimensions $1 \times 1$ to $25 \times 25$.

| Area | Dimensions |
| :---: | :---: |
|  | $1 \times 1$ |
|  | $2 \times 2$ |
|  | $3 \times 3$ |
|  | $4 \times 4$ |
|  | $5 \times 5$ |
|  | $\ldots$ |
|  | $25 \times 25$ |

Just as we say the numbers $1,4,9,16$ etc. are perfect squares, $1,2,3,4$, are their corresponding square roots. We use the radical notation $\sqrt{n}=b$, read square root of $n$ is $b$, to mean that $b$ when squared equals $n$.

For example, $\sqrt{16}=4$ because $4^{2}=16$. What is $\sqrt{25}$ ?
It would be correct that $\sqrt{25}=5$ because $5 \cdot 5=25$. Do you see a relationship between 5 and 25 ? Between 4 and 16 ? Find all the square roots of numbers less than or equal to 100 that are whole numbers. How does your table above relate perfect squares and square roots? Explain.

What patterns do you observe about the square numbers, the square roots of numbers, and square numbers and their corresponding square roots? (Odd square numbers have odd square roots, even square numbers have even square roots; difference between two consecutive square numbers is an odd number).

## PROBLEM 1

Draw a square that illustrates each of the following and explain:
a. A side with a length of $\sqrt{144}$ units $(\sqrt{144}=12$, therefore draw a square with dimensions $12 \times 12$ ).
b. A side with a length of $\sqrt{625}$ units $(\sqrt{625}=25$, therefore draw a square with dimensions $25 \times 25$ ).
c. An area of 289 square units ( $\sqrt{289}=17$, therefore draw a square with dimensions 17×17).

You identified perfect squares such as $1,4,9,16,25,36,49,64,81,100$. In addition, you identified square roots such as $\sqrt{1}, \sqrt{4}, \sqrt{9}, \sqrt{16}, \sqrt{25}, \sqrt{36}$, $\sqrt{49} \sqrt{64}, \sqrt{81}, \sqrt{100}$ and saw a relationship between the perfect squares and their square roots. Notice that these are very special whole numbers. As an extension to this topic, you may wish to consider what it would mean to talk about $\sqrt{2}$ or $\sqrt{3}$.

## SCIENTIFIC NOTATION

In studying the real world, we often have to use large numbers. For example, the earth's circumference is $40,000,000$ meters. There are even larger numbers that play an important role in understanding the world we live in. The earth's
mass is approximately $5,973,600,000,000,000,000,000$ metric tons. We can use exponents to help us write and compare such large numbers. For example, we can write the following numbers in equivalent ways:

$$
\begin{aligned}
& 3,500=3.5 \times 1000=3.5 \times 10^{3} \\
& 35,000=3.5 \times 10,000=3.5 \times 10^{4} \\
& 35,000,000=3.5 \times 10^{7}
\end{aligned}
$$

We call writing numbers in this form scientific notation. What does the exponent in each of these represent? How do you determine the exponent for each of these numbers? Notice that the first part of a number written in scientific notation is always greater than or equal to 1 and less than 10 . The advantage of converting a number in standard notation to scientific notation is apparent in converting the earth's mass from standard notation to scientific notation: 5.9736 $\times 10^{21}$ metric tons.

## PROBLEM 2

Write the distances from each of the following planets to the sun using scientific notation.

| Planet | Distance in kilometers | Distance from sun in kilome- <br> ters using scientific notation |
| :--- | :--- | :--- |
| Mercury | $57,900,000 \mathrm{~km}$. | $5.79 \times 10^{7}$ |
| Venus | $108,200,000 \mathrm{~km}$. | $1.082 \times 10^{8}$ |
| Earth | $149,600,000 \mathrm{~km}$. | $1.496 \times 10^{8}$ |
| Mars | $227,900,000 \mathrm{~km}$. | $2.279 \times 10^{8}$ |
| Jupiter | $778,300,000 \mathrm{~km}$. | $7.783 \times 10^{8}$ |
| Saturn | $1,427,000,000 \mathrm{~km}$. | $1.427 \times 10^{9}$ |
| Uranus | $2,871,000,000 \mathrm{~km}$. | $2.871 \times 10^{9}$ |
| Neptune | $4,497,100,000 \mathrm{~km}$. | $4.4971 \times 10^{9}$ |

In some situations, we need to work with very small numbers. The pattern of converting a number such as 0.00035 into scientific notation is to count the number of decimal places that you need to shift in the number 0.00035 to get to the form 3.5. So,

$$
0.00035=\frac{0.00035 \times 10^{4}}{10^{4}}=\frac{35}{10^{4}}=3.5 \times 10^{-4}
$$

## EXERCISES

1. 

13 | 13 |
| :---: |

b.

c.

2. 100 and 400 are perfect squares because $100=10^{2}$ and $400=20^{2}$.
4. a.

21

12


An application of this type of conversion is in converting the width of an electron into scientific notation. The width of an electron (one of the particles that form an atom) is .0000000000000028 meters. So,

$$
0.0000000000000028=\frac{0.0000000000000028 \times 10^{15}}{10^{15}}=\frac{2.8}{10^{15}}=2.8 \times 10^{-15}
$$

## PROBLEM 3

Fill in the chart by converting the numbers to scientific notation or standard notation.

| Standard Notation | Scientific Notation |
| :---: | :---: |
| 0.00000378 |  |
| 0.00000000024 |  |
|  | $4.973 \times 10^{-5}$ |
|  | $9.831 \times 10^{-3}$ |

## EXERCISES

1. Draw a square with side lengths given below. Write the dimensions as whole numbers and also indicate the area of each square.
a. $\sqrt{169}$
b. $\sqrt{49}$
c. $\sqrt{400}$
2. Which of the numbers below are perfect squares? Explain your reasoning.
a. 100
b. 200
c. 300
d. 400
3. What is another way of writing $\sqrt{0}$ ? Explain. 0 , because $0^{2}=0$
4. Draw a model for the following squares with:
a. area 1 square unit
b. area 441 square units
5. $D$
$\begin{array}{llll}\text { 8. a. } 4.8 & \text { b. } 8.1 & \text { c. } 10.1 & \text { d. } 19.4\end{array}$
c. side length $\sqrt{144}$ units
d. area $x^{2}$ square units, for a positive number $x$.
e. area $9 x^{2}$ square units, for a positive number $x$.
f. side length $2 x$ units, for a positive number $x$. What is the area?
6. The model below is a square with an area of 144 square units.


Which of these equations can be used to determine $s$, the side length of this model in units?
a. $s=\sqrt{12}$
b. $s=144$
c. $s=12^{2}$
d. $s=\sqrt{144}$
6. A concert needs to set up 625 chairs on the floor level. If the chairs are placed in a square arrangement, how many should be in each row? 25
7. If the area of a square is 289 square inches. What is the side length of the square? 17
8. Estimate the following to the nearest tenth. Check your solution with a calculator.
a. $\sqrt{23}$
b. $\sqrt{66}$ c. $\sqrt{103}$ d. $\sqrt{376}$
9. $p$ is a prime number.
a. List all of the factors of $p^{2} .1, p, p^{2}$
b. List all of the factors of $p^{3}$. $1, p, p^{2}, p^{3}$
c. What is the square root of $p^{6}$ ? $\sqrt{p^{6}}=p^{3}$
10. Write the following number using scientific notation:
a. 45,700
$4.57 \times 10^{4}$
b. $507,300,000,000$
$5.073 \times 10^{11}$
c. $289,000,000,000,000,0002.89 \times 10^{17}$
d. 0.00036
$3.6 \times 10^{-4}$
e. 0.285 $2.85 \times 10^{-1}$
f. $0.00000000000000000033 \times 10^{-19}$
11. Convert the following numbers in scientific notation to standard notation:
a. $6.437 \times 10^{8}$
643,700,000
b. $4.7 \times 10^{12}$
4,700,000,000,000
c. $6 \times 10^{14}$
600,000,000,000,000
d. $5.3 \times 10^{-4}$
0.00053
e. $7.9 \times 10^{-6}$
0.0000079
f. $8.3 \times 10^{-12}$
0.0000000000083
12. Ingenuity:

Consider the following sequence: $1,3,7,15, \ldots$
a. what are the next three terms? $1,3,7,15$, $\qquad$ 31,63, 127
b. What is a formula for the nth term? $\mathrm{a}_{\mathrm{n}}=$ $2^{n}-1$

## Section 7.5 - Unique Prime Factorization

## Big Idea:

Developing the understanding that any integer greater than 1 is either prime or can be written as a unique product of prime numbers. (Unique Prime Factorization)

## Key Objectives:

- Use a factor tree to factor an integer.
- Use exponents to write a prime factorization.
- Understand that multiplication is commutative.
- Understand that the number 1 is neither prime nor composite.


## 7th Grade Strategy:

Review the Fundamental Theorem of Arithmetic on the second page of the section.
Do Exercise 1 (parts e, g, h, and i), 4, and 5

## Materials:

## Calculator

## Pedagogical/Orchestration:

- Introduce the lesson with the Factor Chain activity. While the Fundamental Theorem of Arithmetic may seem complicated to middle school students, it provides the reason why prime factorizations are written in order from least to greatest.
- This section relies on number sense and can develop a greater appreciation and facility for it. There are some students and teachers who remain confused about the fact that 1 is not a prime number. The Fundamental Theorem of Arithmetic can demonstrate the reason 1 is not a prime. If 1 were a prime, there would never be a unique prime factorization, because you could have any number of factors of 1 and the Fundamental Theorem of Arithmetic would not hold.


## Activity:

"Doughnut Races" from the end of the section and on the CD

## Exercises:

Exercises 2 and 3 build new vocabulary words: perfect squares and cubes. Exercises 4 and 5 can be challenging for students with weak number sense ability or poor recall of multiplication facts.

## Vocabulary:

unique prime factorization, prime factorization, tree diagram, perfect square, perfect cube, factor tree

## TEKS:

6.1 (D); 7.1(C);
7.14(A); $7.15(\mathrm{~A}, \mathrm{~B}) ;$
8.1(C);
8.15(A);
8.16(A);

## WARM-UPS for Section 7.5

1. Which of the following is equivalent to -4 ?
a. $4\left[3^{2}-4(3)\right]-11$
b. $\quad 2\left[5^{2}-4(3)\right]-22$
c. $7\left[4^{2}-5(3)\right]-11$
d. $(-3)\left[4^{2}+3(-3)\right]+16$

## Ans: (c)

## Launch for Section 7.5:

## Factor Chains

In groups, instruct students to represent the number 360 as the longest chain of factors they can find without using a 1 as a factor. Give students about 5 minutes to come up with a response.

Have a member from each group come to the board to report the longest "chain" they found for the number.
Once all groups have reported, conduct a class discussion to observe patterns found. Some groups will have only factor pairs. Others will have longer "factor chains." Discuss what those groups did to expand the "chain."

Example: $\quad 360=10 \cdot 36 \quad$ (group 1)
$360=2 \cdot 5 \cdot 36 \quad$ (group 2)
$360=2 \cdot 5 \cdot 4 \cdot 9 \quad($ group 3)
In many cases, groups will have the same factors but in different orders. This is an example of multiplication being commutative.

Focus on the longest "factor chain" and ask students to explain how it is related to shorter chains.
Ask students to explain why the original directions stated that you cannot use a 1 .

## SECTION 7.5 UNIQUE PRIME FACTORIZATION

One reason we are so interested in prime numbers is that they are the building blocks of the integers. In the previous section, we learned that a prime number is a positive integer greater than 1 that can be written as a product of two positive integers in only one way. For example,

$$
13=13 \cdot 1=1 \cdot 13 .
$$

We cannot write 13 as a product of two positive integers without using the number 13 itself. In this way the number 13 cannot be divided into smaller equal whole parts. We can, however use the number 13 , together with other prime numbers, to form many other numbers:

$$
\begin{aligned}
& 13 \cdot 2=26 \\
& 13 \cdot 3=39 \\
& 13 \cdot 5=65 \\
& 13 \cdot 7=91
\end{aligned}
$$

Primes are combined in various ways to form different positive integers. In some cases, you might use a certain prime factor more than once when building a number:

$$
\begin{aligned}
4=2 \cdot 2 & =2^{2} \\
8 & =2 \cdot 2 \cdot 2
\end{aligned}=2^{3},
$$

In a similar way, every positive integer greater than 1 that is not prime can be written as the product of prime factors. In other words, each positive integer can be identified by its prime factors and the number of times each of these factors occurs. For example, $n$ is a positive integer that is composed of 3 factors of 2,1 factor of 3 and 2 factors of 5 . What is the exact value of $n$ ? Does it matter if $n$ is $2 \cdot 3 \cdot 5 \cdot 2 \cdot 5 \cdot 2$ or $2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5$ ? Is there an accepted way to organize these factors as a product of $n$ ?

We can answer these questions with the following:

While the Fundamental Theorem of Arithmetic may seem complicated to students, it provides motivation to write the prime factorization with the factors ordered from least to greatest. This can be done in only one way. It is unique.

Make sure students notice that the even numbers always have at least one factor that is 2 in the first two steps.

## THEOREM 7.1: FUNDAMENTAL THEOREM OF ARITHMETIC

If $n$ is a positive integer, $n>1$, then $n$ is either prime or can be written as a product of primes

$$
n=p_{1} \cdot p_{2} \cdot \cdots \cdot p_{k^{\prime}}
$$

for some prime numbers $p_{1}, p_{2}, \ldots, p_{k}$ such that $p_{1} \leq p_{2} \leq \cdots \leq p_{k}$ In fact, there is only one way to write $n$ in this form.

The Fundamental Theorem of Arithmetic, or FTA, tells us that there is only one way to decompose a given integer into prime factors with the factors written in order. That is, if two different people correctly express a positive integer as a product of prime factors, their products always contain exactly the same prime factors, whether the order is the same or not. Using the FTA, however, the prime factors should be in increasing order.

The prime factorization of an integer gives us useful information about its factors. Factoring a number into primes usually takes some trial and error, but there is a technique that makes the process easier. Let's look at an example:

## EXAMPLE 1

Find the prime factors of 60 .

## SOLUTION

When looking for prime factors of a positive integer, it is useful to have a list of prime numbers. Look at the Sieve of Eratosthenes from the activity in Section 7.1 to confirm that the first few primes are

$$
2,3,5,7,11,13,17,19,23, \ldots
$$

Work your way through these primes to see if any of them are factors of 60 . Divide 60 by 2 , to get a quotient of 30 and a remainder of 0 . So 2 is a factor of 60 , with

$$
60=2 \cdot 30
$$

Reinforce the multiplication facts and the fact that any number ending in 5 has a factor of 5 .

It is tempting to go on to the next prime in our list, but remember that a prime might appear more than once in a prime factorization. So before you continue, notice that 30 is even and realize that 2 is a factor of 30 . Dividing, you will find that $30=2 \cdot 15$. So

$$
60=2 \cdot 2 \cdot 15
$$

Using the multiplication facts, divide 15 by 3 to find that $15=3 \cdot 5$. So

$$
60=2 \cdot 2 \cdot 3 \cdot 5
$$

Because each of the factors 2, 2, 3 and 5 is prime, you are through. You have written 60 as a product of prime factors. A useful way to write your result is to use exponents:

$$
60=2^{2} \cdot 3 \cdot 5
$$

The process of finding the prime factors of a number is called prime factorization. We also use the same term to describe the result of the process. For example, the prime factorization of 60 is $2^{2} \cdot 3 \cdot 5$.

## EXAMPLE 2

Find the prime factors of 672.

## SOLUTION

If you want, use the same step-by-step process you used in the previous example. But there is a faster way to write the information. You can track the prime factors using a tree diagram. You know that 2 is a prime factor of 672 because 672 is even, so start by dividing 672 by 2 :


Continue factoring by 2 until the remaining number is odd:


Remember that $7 \cdot 3=21$, so you know that the last 2 factors in the process are 7 and 3. The prime factorization of 672 is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7=2^{5} \cdot 3 \cdot 7$.

Notice that the processes used in Examples 1 and 2 are related. The second process is the same as the first, except that it organizes the factoring process more visually using a tree diagram, which can be helpful when working with larger numbers. Another useful tool in factoring is the Divisibility Table you came up with in Section 7.2, as shown below.

| A number is divisible by: | If |
| :---: | :---: |
| 2 | The last digit is $0,2,4,6$, or 8. |
| 3 | The sum of the digits is divisible by 3. |
| 5 | The last digit is 0 or 5. |
| 6 | The number is divisible by 2 or 3 |
| 9 | The sum of the digits is divisible by 9 |
| 10 | The last digit is 0. |

## EXERCISES

1. a. $6=2 \cdot 3$
e. $\quad 36=2^{2} \cdot 3^{2}$
b. $9=3^{2}$
f. $\quad 252=2^{2} \cdot 3^{2} \cdot 7$
i. $\quad 1225=5^{2} \cdot 7^{2}$
c. $16=2^{4}$
g. $897=3 \cdot 13 \cdot 23$
j. $\quad 3195=3^{2} \cdot 5 \cdot 71$
h. $819=3^{2} \cdot 7 \cdot 13$
k. $6292=2^{2} \cdot 11^{2} \cdot 13$
d. $23=23$
l. $\quad 6300=2^{2} \cdot 3^{2} \cdot 5^{2} \cdot 7$
2. a. $2 \cdot 5$
b. $2^{2} \cdot 5^{2}$
c. $2^{3} \cdot 5^{3}$
a. $2^{4} \cdot 5^{4}$
b. $2^{5} \cdot 5^{5}$
3. a. $3 \cdot 7$
b. $2 \cdot 3 \cdot 5 \cdot 7$
c. $2^{2} \cdot 3 \cdot 5^{2} \cdot 7$
a. $3 \cdot 7 \cdot 2^{4} \cdot 5^{4}$
b. $3 \cdot 7 \cdot 2^{5} \cdot 5^{5}$
4. A
5. C
6. Yes, $9,16,36,1225$. Perfect squares have prime factors with even powers.

## EXERCISES

1. Factor each of the following integers into primes. Write the integer as a product of its prime factors, using exponents when there are repeated prime factors. See TE.
a. 6
C. 16
e. 36
g. 897
i. 1,225 k. 6,292
b. 9
d. 23
f. 252
h. 819 j. 3,195 ।.
6,300
2. Factor each of thee following numbers into its prime factoriztion
a. 10
b. 100
c. 1,000

What pattern do you notice?
Use ths pattern to factor the following numbers:
a. 10,000
b. 100,000
3. Factor each of the following numbers into its prime factorization:
a. 21
b. 210
c. 2,100

What pattern do you notice?
Use this pattern to factor the following numbers:
a. 21,000
b. 210,000
4. Which of the following is the prime factorization of 350 ? Select the best choice and explain your answer.
a. $2 \cdot 5^{2} \cdot 7$
b. $2 \cdot 5 \cdot 35$
c. $2 \cdot 5 \cdot 7$
d. $\quad 2^{2} \cdot 5 \cdot 7$
5. Which of the following is the prime factorization of 440 ? Select the best choice and explain your answer.
a. $2 \cdot 5 \cdot 11$
b. $\quad 2^{2} \cdot 5 \cdot 11$
c. $\quad 2^{3} \cdot 5 \cdot 11$
d. $2^{2} \cdot 55$
6. The prime factorization of a number can be used to find a perfect square, which is the square of an integer. Look at the numbers given in Exercise 1. Are any of these perfect squares? If so, which ones? How can you use their prime factorization to determine whether each is a perfect square? See TE.
7. A perfect cube is an integer $n$ that can be written in the form $n=k^{3}$, where $k$ is an integer. Some examples of perfect cubes are

$$
0^{3}=0,1^{3}=1,2^{3}=8,3^{3}=27,4^{3}=64, \ldots
$$

How can you use the prime factors of a number to determine whether it is a perfect cube? The powers of every prime factor must be multiples of 3 .
9. Although the two numbers differ by 1 , their factorizations are very different. The prime factorization of 1224 is $2^{3} \cdot 3^{2} \cdot 17$ and 1225 is $5^{2} \cdot 7^{2}$. Thus, 1225 is a perfect square while 1224 is not.
14. $\quad 140=2^{2} \cdot 5 \cdot 7$
$141=3 \cdot 47$
$142=2 \cdot 71$
$143=143$
$144=2^{4} \cdot 3^{2}$
$145=5 \cdot 29$
$146=2 \cdot 73$
$147=147$
$148=2^{2} \cdot 37$
$149=149$
$150=2 \cdot 3 \cdot 5^{2}$
15. $333=3^{2} \cdot 37$
$444=2^{2} \cdot 3 \cdot 37$
$555=3 \cdot 5 \cdot 37$
$888=2^{3} \cdot 3 \cdot 37$
Each of these are multiples of 111 which has 37 as a factor.
8. What is the smallest positive integer that has four different prime factors? 210
9. Write the prime factorization for each of 1,224 and 1,225 . What do you notice about their prime factorizations?
10. $30=2 \cdot 3 \cdot 5$
a. List all the factor pairs of 30 .
b. Find the prime factorization of each of the factors of each factor pair. For example, $24=2 \cdot 12=2 \cdot\left(2^{2} \cdot 3\right)$. What do you notice?
11. Use the prime factorization, $40=2^{3} \cdot 5$, to find the number of rectangles you can make with integer side lengths and area equal to 30 square units.
12. Use the prime factorization, $2 \cdot 2 \cdot 3 \cdot 5=60$, to find the number of rectangles you can make with integer side lengths and area equal to 60 square units.
13. $p$ and $q$ are primes and $n=p^{3} q$
a. List all the factor pairs of $n$. (hint: look at exercise above)
b. How many factors does $n$ have?
14. Find the prime factorization for all whole numbers from 140 to 150 . Write the prime factorization in exponential notation. Explain any patterns you may have noticed.
15. Find the prime factorization for the integers $333,444,555$, and 888 . Write the prime factorization in exponential notation. Explain any patterns you may have noticed.
16. Ingenuity:

In general, if $p$ and $q$ are primes, how many factors are there for $p^{4}$ ? For $q^{3}$ ? For $p^{4} \cdot q^{3}$ ? What about $p^{m} \cdot q^{n}$ ?

## DOUGHNUT RACES

Objective: Students will practice and review multiplication, prime /composites, factors, multiples, prime factorization, fractions, decimals, percents

## Materials:

Doughnut Circles (for whole group game); or Doughnut worksheet (for individual or small group game)
Pencils if using student copy (for individual or small group play)
Chalk or Dry erase markers (if using board for whole group play )

## Activity Instructions:

Teacher calls out skill in random order:
multiplication (teacher rolls the dice, calls out the number and students multiply around the "dough nut" as fast as they can. First player to correctly complete their doughnut wins the round.
prime/composites (students write P or C around the donut)
factors (students list the factors of each number)
multiples (students list the first 4 multiples of each number)
prime factorization (students write the prime factors of each number and use exponential notation if needed.)
fractions (teacher rolls the die and calls out the numbe. The number rolled is the denominator and the numbers on the "doughnut" are the numerators. Students are to write each fraction and simplify them if needed. (reduce \& convert)
fractions decimals (repeat \#6 except convert fractions to decimals and/or to percents.)
fractions, decimals, percents (on the backside of the donut use the decimal forms to convert to fractions and/or to percents.)
Students work out the skill and the first person to complete their "doughnut" correctly earns a point. Team to earn 10 points wins the game.

Variation 1:
Students can play in 2 teams, using the giant doughnuts (made out of poster board, then laminated and taped on the chalkboard) Two players compete at a time and earn points for their team then rotate players. Winning team gets to eat doughnuts, other team gets doughnut holes. We play this game on the last Friday of the 6 weeks and 2 students volunteer to bring doughnuts and d-holes for the prize.

## Variation 2:

Use the multiple doughnuts worksheet to play many rounds at a time individually or in small groups. This allows for more participation and success.



Name $\qquad$ Date $\qquad$ Period
Student Copy
1.

4.
2.

3.


5.

6.

1.

| $g$ | $h$ |  |
| :--- | :--- | :--- |
| 9 | 117 | $9 \cdot 13=117$, so 9 is a factor of 117 |
| 7 | 48 | $7 \cdot 7=49$, so 7 is not a factor of 49 |
| 8 | 112 | $8 \cdot 14=112$, so 8 is a factor of 112 |
| 3 | 123 | $3 \cdot 41=123$, so 3 is a factor of 123 |

2. The first 12 multiples of 7 are: $7,14,21,28,35,42,49,56,63,70,77,84$

Therefore, 77 is the greatest multiple of 7 that is less than 80 .
3. The different group sizes you could have are: 1 group of 24,2 groups of 12,3 groups of 8,4 groups of 6,6 groups of 4,8 groups of 3,12 groups of 2 , and 24 groups of 1 .
4. $1 \times 72,2 \times 36,3 \times 24,4 \times 18,6 \times 12,8 \times 9$
5. a. 47 is prime
d. 71 is prime
b. 21 is composite, $3 \cdot 7$
e. 111 is composite, $3 \cdot 37$
c. 35 is composite, $5 \cdot 7$
6. a. $100=2 \cdot 2 \cdot 5 \cdot 5$, so the prime factors of 100 are 2 and 5 .
b. $22=2 \cdot 11$, so the prime factors of 22 are 2 and 11 .
c. $49=7 \cdot 7$, so the prime factor of 49 is 7 .
d. $60=2 \cdot 2 \cdot 3 \cdot 5$, so the prime factors of 60 are $2,3,5$.
e. $125=5 \cdot 5 \cdot 5$, so the prime factor of 125 is 5 .
7. a. $20=2 \cdot 2 \cdot 5=2^{2} \cdot 5$
d. $\quad 144=2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3=2^{4} \cdot 3^{2}$
b. $36=2 \cdot 2 \cdot 3 \cdot 3=2^{2} \cdot 3^{2}$
e. $504=2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7=2^{3} \cdot 3^{2} \cdot 7$
c. $50=2 \cdot 5 \cdot 5=2 \cdot 5^{2}$

The numbers in parts (b) and (d) are the only perfect square since they are a product of even powers of primes.
8. a. $2^{4}$
c. $3^{3} \cdot 5^{3}$
e. $x^{3} y^{4}$
b. $2^{3} \cdot 3$
d. $2^{2} \cdot 7^{7}$
9. a. $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5$
b. $7 \cdot 7 \cdot 11 \cdot 11 \cdot 11$
c. $2 \cdot 2 \cdot 7 \cdot 7 \cdot 7 \cdot 5$

## REVIEW PROBLEMS

1. Determine whether $g$ is a factor of $h$.

| $g$ | $h$ | Explain |
| :---: | :---: | :---: |
| 9 | 117 |  |
| 7 | 48 |  |
| 8 | 112 |  |
| 3 | 123 |  |

2. What is the greatest multiple of 7 that is less than 80 ?
3. If there are 24 students in your class, what are the different equal size groups that can be arranged? Explain.
4. What are the possible dimensions of all the possible rectangles with area 72 ?
5. Determine and explain whether the following numbers are prime or composite.
a. 47
b. 21
c. 35
d. 71
e. 111
6. For the following numbers, find all the prime factors of each number.
a. 100
b. 22
c. 49
d. 60
e. 125
7. Determine the prime factorization of each of the following integers, using exponents when necessary. Indicate if and why any of the integers are perfect squares.
a. 20
b. 36
c. 50
d. 144
e. 504
8. Write the following using exponents.
a. $2 \cdot 2 \cdot 2 \cdot 2$
b. $2 \cdot 2 \cdot 2 \cdot 3$
c. $3 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 5$
d. $2 \cdot 2 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$
e. $x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y$
9. Write the following in prime factorizations without exponents:
a. $2^{3} \cdot 3^{2} \cdot 5^{2}$
b. $7^{2} \cdot 11^{3}$
c. $2^{2} \cdot 5 \cdot 7^{3}$
d. $a^{4} \cdot b^{2}$
10. a. $9^{2}>4^{3}$
c. $11^{2}<5^{3}$
b. $2^{5}>3^{3}$
d. $8^{2}=4^{3}$
11. a. $12^{2}=12 \cdot 12=144$
c. $4^{3}=4 \cdot 4 \cdot 4=64$
b. $4^{4}=4 \cdot 4 \cdot 4 \cdot 4=256$
d. $5^{5}=5 \cdot 5 \cdot 5 \cdot 5 \cdot 5=3125$
12. a. $7+6 \cdot 8^{2}=7+6 \cdot 64=7+6 \cdot 64=7+384=391$
b. 7,623
c. $10+6 \cdot 82-12=10+492-12=10-12+492=-2+492=490$
d.
e. $-7+6 \cdot 64=-7+384=377$
f. 40,870
g. 9,408
13. a. $1.45 \times 10^{5}$
b. $3.4 \times 10^{10}$
c. $3.4 \times 10^{-3}$
d. $5.27 \times 10^{-6}$
14. a. $25,000,000$
b. 83,670
c. .00005421
d. 000004
15. Compare the following numbers using $>,<$ or $=$ :
a. $9^{2}$
$4^{3}$
c. $11^{2}$ $\square$ $5^{3}$
b. $2^{5} \square 3^{3}$
d. $8^{2} \square 4^{3}$
16. Compute:
a. $12^{2}$
b. $\quad 4^{4}$
c. $4^{3}$
d. $5^{5}$
17. Compute:
a. $7+6 \cdot 82$
b. $7 \cdot 10^{3}+6 \cdot 10^{2}+2 \cdot 10+3$
c. $10+6 \cdot 82-12$
d. $5-4^{3} \div(17-9) \cdot 2$
e. $-7+6 \cdot 8^{2}$
f. $4 \cdot 10^{4}+8 \cdot 10^{2}+7 \cdot 10$
g. $9 \cdot 10^{3}+4 \cdot 10^{2}+8$
18. Write the following numbers using scientific notation:
a. 145,000
c. . 0034
b. $34,000,000,000$
d. . 00000527
19. Write the following numbers using standard notation:
a. $2.5 \cdot 10^{7}$
b. $8.367 \cdot 10^{4}$
c. $5.421 \cdot 10^{-5}$
d. $4 \cdot 10^{-6}$
20. $9^{2}$
21. a. 17
b. 25
c. 14
d. 21
22. Write an expression using exponents that could be used to find the number of small squares.

23. Evaluate:
a. $\sqrt{289}$
b. $\sqrt{625}$
c. $\sqrt{196}$
d. $\sqrt{441}$

Section 7.1: 12, 14, and 21
Solution: Since , the ways of writing 168 as a product of three numbers are, $1 \cdot 1 \cdot 168$ (sum 170), 1•2•84 (87), $1 \cdot 3 \cdot 56(60), 1 \cdot 4 \cdot 42(47), 1 \cdot 6 \cdot 28(35), 1 \cdot 7 \cdot 24(32), 1 \cdot 8 \cdot 21(30), 1 \cdot 12 \cdot 14(27), 2 \cdot 2 \cdot 42(46), 2 \cdot 3 \cdot 28$ (33), $2 \cdot 4 \cdot 21(27), 2 \cdot 6 \cdot 14(22), 2 \cdot 7 \cdot 12(21), 3 \cdot 4 \cdot 14(21), 3 \cdot 7 \cdot 8(18)$, and $4 \cdot 6 \cdot 7(17)$. The only sums appearing more than once are 27 , with the oldest child 14 or 21 , and 21 , with the oldest child 12 or 14 .

Section 7.2: 193
Solution: Let $x=8 a$ and $y=8$, so we have $64\left(a^{2}-b^{2}\right)+1=p$. Now 65 is not prime; 129 is not prime and there is no integer solution to $a^{2}-b^{2}=2 ; 193$ is prime with $a=2$ and $b=1$.

## Section 7.3: 8

Solution: The fastest we can make the exponent grow is to multiply each power by itself, thus doubling the exponent. Thus we need more than six multiplications since that can only get us to an exponent of $2^{6}=64$. Three possible sequences are $x \rightarrow x^{2} \rightarrow x^{4} \rightarrow x^{8} \rightarrow x^{16} \rightarrow x^{32} \rightarrow x^{64} \rightarrow x^{96} \rightarrow x^{100}$ , $x \rightarrow x^{2} \rightarrow x^{6} \rightarrow x^{12} \rightarrow x^{24} \rightarrow x^{25} \rightarrow x^{50} \rightarrow x^{100}$, and
$x \rightarrow x^{2} \rightarrow x^{4} \rightarrow x^{8} \rightarrow x^{9} \rightarrow x^{16} \rightarrow x^{25} \rightarrow x^{50} \rightarrow x^{100}$. Note that all of the sequences, at different stages, double the exponent 6 times. Can you find other sequences of equal length?

Section 7.4: 1995
Solution: We must use each of the digits $1,3,5,7$, and 9 exactly once. Since 3,5 , and 7 are primes they can be factors, and 19 uses 1 and 9 for a product of 1995.

## CHALLENGE PROBLEMS

## Section 7.1:

A man has three children, whose ages he can't remember. He remembers that the product of their ages is 168 , and he remembers the sum of their ages, but he still can't figure out how old they are. What are all possible ages of his oldest child?

## Section 7.2:

Find the smallest prime number $p$ such that $x^{2}+y^{2}+1=p$ where $x$ and $y$ are both multiples of 8 .

## Section 7.3:

We can get $x^{10}$ from $x$ in 5 multiplications of previously generated powers, e.g. . What is the minimum number of such multiplications required to get $x^{100}$ from $x$ ?

## Section 7.4:

Find the smallest positive integer whose prime factorization uses each odd digit exactly once.

## Section 8.1-GCF and Equivalent Fractions

## Big Idea:

Finding common factors and greatest common factor
Recognizing and using fractional models to represent part of a group and part of a whole

## Key Objectives:

- Use unique factorization to find common factors, including GCF.
- Use Venn diagrams to find common factors.
- Understand definition of "relatively prime" and how it relates to simplifying fractions.
- Use linear models, area models, and discrete models to represent fractions.
- Understand how to find equivalent fractions.
- Understand the process of simplifying a fraction and be able to write a fraction in simplest form.


## Materials:

- Paper for folding (each sheet should be the same size to represent one whole)
- Grid paper


## Pedagogical/Orchestration:

- To help students better understand Example 3 and the prime factorization method; give other examples where the numbers are smaller such as 14 and 18.
- This section presents fractions as parts of groups and wholes with basic vocabulary.
- The Folding Paper Activity in the lesson is an excellent way for students to see a visual representation of equivalent fractions.


## Activities:

"Clock Worksheet"; "Fraction Friend"; "Connect Three" found at the end of the section and on the CD.

## Exercises:

Be sure to emphasize what happens when two numbers have no common prime factors before the students attempt Exercise 1. This situation will come up when they get to Exercise 1c and 1d, and the students need to be aware that in these situations the GCF is 1 .

Your students may find that Exercise 19 is time consuming, but it is a very important problem. Allow them to work together to find all of the values of $d$, and encourage them to work strategically to be sure that they have found all of the possible values.

## Vocabulary:

common factor, greatest common factor, relatively prime, simplest form, (See CD: like fractions, unlike fractions), area model, numerator, denominator, equivalent, equivalent fractions, simplifying, simplest form

TEKS:
$6.1(A)(B)(C) ; \quad 6.12(A) ; \quad 6.13(B) ; \quad 7.2(A)(B) ; \quad 7.13(D) ; \quad 7.14(A)(B) ; \quad 7.15(A)(B) ; \quad 8.1(B) ; \quad 8.2(B) ; \quad 8.14(B) ;$ 8.15(A); 8.16(A)

## WARM-UPS for Section 8.1

1. Which of the following numbers does not have 128 as a multiple?
a. 8
b. 16
C. 24
d. 32

Ans: (c) because 128 is a power of 2 and only 24 is not a power of 2 in the possible answers.
2. Find the GCF for each pair integers:
a. 8 and 10 Ans: Both are even. They are each divisible by 2. But, they cannot be both a multiple of any number greater than 2 because they are only 2 units apart on the number line. So, the GCF is 2 .
b. 24 and 26 Ans: $\mathbf{G C F}=2$ for the same reason as part a.
c. 3464 and 3466 (Hint: There is a short way.) Ans: $\mathrm{GCF}=2$ for the same reason as part a.
d. 2565 and 2567 Ans: Both are odd. They are each not divisible by 2. But, they cannot be both a multiple of any number greater than 2 because they are only 2 units apart on the number line. So, the GCF is 1 .

## Launch for Section 8.1:

Refer to Exploration 1

## EXPLORATION 1

Remind the students that "so on" refers to going on sequentially.

1. Have students draw a number line from 0 to 60 . Use it to identify the frogs that land on both 24 and 36 . Solution: 2, 3, 4, 6, and 12 frogs.
2. The longest jumping frog that will land on both 24 and 36 is the 12 -frog. This is because 12 is the largest common factor of 24 and 36 .
3. 4 -frog
4. 1-frog

# FRACTIONS, <br> DECIMALS AND <br> P ERCENTS 

## SECTION 8.1 GCF AND EQUIVALENT FRACTIONS

We know that when we multiply 3 and 8 to obtain the product 24 , the numbers 3 and 8 are factors of 24 . Notice that $2 \cdot 12$ also equals 24 , so 2 and 12 are factors of 24. You have also discovered several other numbers that are factors of 24. In this section, we examine how we can use what we know about factors to determine factors common to two or more numbers. When is it necessary to find common factors? Let's examine the following situation to see.

## EXPLORATION 1

To prepare for a frog-jumping contest, Fernando decided to train a group of his fellow frogs. Each frog was trained to jump a certain length along a number line starting at 0 . He trained a 1 -frog to jump a distance of 1 unit in each hop. He also trained a 2 -frog to jump 2 units, a 3-frog to jump 3 units and so on. The frogs always start at the zero point on the number line. Now Fernando wants to know which frogs will land on certain locations on the number line.

1. Which of his frogs will land on both the locations 24 and 36 ?
2. Which is the longest jumping frog that will land on both 24 and 36? Explain why this answer makes sense.
3. What is the longest jumping frog that will land on both 20 and 32 ?
4. What is the longest jumping frog that will land on both 24 and 25 ?

If a frog lands on 24 , then the length of its jump is a factor of 24 . So, if a frog lands on both 24 and 36, then the length of its jump is a factor of 24 and 36. Another way to ask the question in part 2 above is: "What is the greatest number that is a factor of both 24 and 36?"

There is a problem in mathematics when it enters the symbolic stage. There are not enough unique symbols, in this case grouping symbols, to have unambiguous expressions. For instance, in function notation $f(x)$ means "the function of $x$," but in algebraic multiplication it means " $f$ times $x$. ." This is always a source of confusion and frustration for some students. Call your students' attention to the fact that both points " $(\mathrm{x}, \mathrm{y})$ " and " $\mathrm{GCF}(\mathrm{m}, \mathrm{n})$ " use parentheses with an ordered pair. The careful reader can, however, distinguish which way the parentheses are being used and avoid confusion. An example of this outside of mathematics is that many students have the same. name but are, in fact, very different individually.

## DEFINITION 8.1: COMMON FACTOR AND GCF

Suppose $m$ and $n$ are positive integers. An integer $d$ is a common factor of $m$ and $n$ if $d$ is a factor of both $m$ and $n$. The greatest common factor, or GCF, of $m$ and $n$ is the greatest positive integer that is a factor of both $m$ and $n$. We write the GCF of $m$ and $n$ as $\operatorname{GCF}(m, n)$.

You saw that in question 2, the GCF of 24 and 36 is 12 . From question 3 , the GCF of 20 and 32 is 4 , and the GCF of 24 and 25 is 1 .

There are several different ways to calculate the GCF of two numbers. Here is one way that reinforces the term Greatest Common Factor.

## EXAMPLE 1

Find the GCF of 30 and 36 .

## SOLUTION

Find the GCF of two numbers by first listing all the factors of each of the numbers in a chart. Find all the common factors of the two numbers, then choose the greatest.

| $\mathbf{3 0}$ |  |
| :---: | :---: |
| 1 | 30 |
| 2 | 15 |
| 3 | 10 |
| 5 | 6 |


| 36 |  |
| :---: | :---: |
| 1 | 36 |
| 2 | 18 |
| 3 | 12 |
| 4 | 9 |
| 6 | 6 |

So the common factors of 30 and 36 are 1,2,3, and 6, and the GCF of 30 and 36 is 6 because it is the largest of the four common factors.

Rule: The GCF of two prime numbers is 1 .

All the factors of 108 : $\quad(1,2,3,4,6,9,12,18,27,36,54,108)$
All the factors of $168: \quad(1,2,3,4,6,7,8,12,14,21,24,28,42,56,84,168)$

## EXAMPLE 2

Find the GCF of 27 and 32 .

## SOLUTION

Use the same method you used in the previous example. First, list the factors of each number.

| $\mathbf{2 7}$ |  |
| :---: | :---: |
| 1 | 27 |
| 3 | 9 |$\quad$| $\mathbf{3 2}$ |  |
| :---: | :---: |
|  | 2 |
| 2 | 16 |
|  |  |

In this case, there is only one common factor: 1. Therefore, the GCF of 27 and 32 , written also as $\operatorname{GCF}(27,32)$, is 1 . There is a term to describe a relationship between numbers whose GCF is 1 .

DEFINITION 8.2: RELATIVELY PRIME
Two integers $m$ and $n$ are relatively prime if the GCF of $m$ and $n$ is 1 .

Based on the definition above, the numbers 27 and 32 are relatively prime. Notice that neither 27 nor 32 are prime numbers. If we consider two prime numbers like 3 and 7, what is their GCF? Check a few more examples. Make a generalization about the GCF of any two prime numbers.

Find the GCF of a pair of larger numbers like 108 and 168 using the process of first finding factors, then the common factors, and eventually the greatest common factor.

First, list all the factors of 108 in order. Then, list all the factors of 168 in order. There are many factors to find. If you do not have 12 factors for 108 and 16 factors for 168 , go back and find them all.

## EXAMPLE 3

If the number is even, factor out 2 and continue to look for even numbers. After that, look for multiples of three and so forth. Encourage students to be systematic and use number sense to factor.

Determine all the factors the two numbers have in common. You should find that the common factors are $1,2,3,4,6,12$. From this list, you can see that the greatest common factor of both 108 and 168 is 12. As you discovered, this method for finding the GCF works well. However, the more factors the numbers have, the more time it takes to make the list of factors for each number. Fortunately, prime factorization makes finding the GCF of two numbers easier.

Here is an example of the efficiency of prime factorization.

## EXAMPLE 3

Find the GCF of 108 and 168 using prime factorization.

## SOLUTION

Start by finding the prime factors of 108 and 168.
Use factor tree diagrams to do this:


PROBLEM 1
a. 12
b. 2
c. $x^{2} y$

Recall that the prime numbers are the building blocks of the integers. If you want to find the GCF of two integers, find the building blocks, or prime factors, the two numbers have in common. Remember, when doing this, it is helpful to write out the prime factors of each number in an organized way using exponents.

$$
\begin{aligned}
& 108=2 \cdot 2 \cdot 3 \cdot 3 \cdot 3=2^{2} \cdot 3^{3} \\
& 168=2 \cdot 2 \cdot 2 \cdot 3 \cdot 7=2^{3} \cdot 3 \cdot 7
\end{aligned}
$$

You have written the prime factors of 108 and 168 with and without exponents. It is clear which factors the two numbers have in common:

$$
\begin{aligned}
& 108=2^{-2} \cdot 2 \cdot \cdot 3 \cdot 3 \cdot 3=2^{2} \cdot 3^{3} \\
& 168=2 \cdot 2 \cdot 2 \cdot 3: \quad \cdot 7=2^{3} \cdot 3 \cdot 7
\end{aligned}
$$

The prime factors have two 2 's and one 3 in common. So $2 \cdot 2 \cdot 3=12$ is the greatest common factor of 108 and 168 . It is also possible to check the common factors using long division. If there were a larger common factor, the quotients would have even more common prime factors, or higher powers of common prime factors, or both. However, the quotient after dividing 108 by 12 is 9 and the quotient after dividing 168 by 12 is 14 . It is easy to check that 9 and 14 have no common factor other than 1 , and therefore are relatively prime.

## PROBLEM 1

Use the prime decomposition (factorization) to find the GCF for each of the following pairs of integers:
a. 72 and 84
b. 48 and 58
c. $x^{2} y^{3}$ and $x^{3} y z$

A visual representation called a Venn Diagram might help you to see how the GCF of numbers like 108 and 168 is constructed from the prime factorization. To make the Venn Diagram, draw a circle for each number you are considering. Inside each circle write all its prime factors. If there are common prime factors, then the circles will intersect and the common prime factors will be in the area common to both, or in the intersection of the circles. If the circles have no common prime factors,

## PROBLEM 2

196 $=2 \cdot 2 \quad \cdot 7 \cdot 7=2^{2} 7^{2}$
$210=2 \cdot 3 \cdot 5 \cdot 7=2 \cdot 3 \cdot 5 \cdot 7$
GCF $=2(7)=14$

then you conclude that the only common factor is 1 . Remember 1 is always a common factor for any pair of integers, and it is the greatest common factor if there are no other common factors.

The Venn Diagram will look like this.


$$
\mathrm{GCF}=2(2)(3)=12
$$

## PROBLEM 2

Compute the GCF of 196 and 210. Use a Venn Diagram and the method from Example 3, and decide which you prefer.

## EXAMPLE 4

Compute the GCF of 8,12 , and 18 .

## SOLUTION

We shall solve this problem using the prime factorization method. The factor trees for these 3 numbers are shown below.


Thus, it is clear which factors the three numbers have in common:

Ask students, "What do you know about a fraction?" Keep an ongoing list or chart of vocabulary and different representations of fractions.

## Activity: Folding Paper

Watch to see if students use the same size sheet to represent the whole if they compare the parts or fractions of these wholes with each other. The issue is discussed on the next page in the text.

Equivalence in fractions is an essential concept in understanding how fractions work. Don't assume all your students understand equivalence until they can demonstrate it with Activity: Paper Folding in this section.
Step 1: Yes, folding lengthwise or diagonally will also represent one half.
Step 2: We fold the paper twice to make fourths. All four parts on the sheet are equal to $\frac{1}{4}$. The fraction that represents 3 of these parts is $\frac{3}{4}$. There are two possible fractions that represent 2 parts of the paper: $\frac{2}{4}$ or $\frac{1}{2}$ :
Let your students record how many folds it takes to half a sheet, fold it into fourths, eighths and sixteenths. Then ask them to compare the number of times they folded the sheet to the number of parts they produced with that fold. Students may notice that as you fold the paper in half each time that the number of pieces increases as a power of two: $2,4,8,16$, etc. Later your students will discover they were producing fractional equivalents to the negative powers of two.

$$
\begin{aligned}
8 & =: 2 \cdot 2 \cdot 2=2^{3} \\
12 & =2: 2 \cdot 3=2^{2} \cdot 3 \\
18 & =2 \cdot 3 \cdot 3=2 \cdot 3^{2}
\end{aligned}
$$

Therefore, the greatest common factor of 8,12 , and 18 is 2 .

Now that we know how to find the GCF, let us look at how to use it. T-charts, prime factorization, and Venn Diagrams will be used later to simplify fractions.

## Review of Fractions

We use two numbers in writing a fraction, the numerator and the denominator.
The numerator is above the denominator. The denominator tells into how many parts the whole is divided. "Denominator" comes from the same root as "name." It names the parts into which the whole is divided, like halves or fourths. "Numerator" comes from the same root as "number." It counts the number of parts.

## ACTIVITY: FOLDING PAPER

Materials: You will need several sheets of paper for this activity. Each sheet represents one whole.
Step 1: Fold the paper to represent the number $\frac{1}{2}$. Write $\frac{1}{2}$ on each of the two parts of the folded paper. Is there more than one way to represent $\frac{1}{2}$ ?
Step 2: Use a new sheet to create $\frac{1}{4}$. How many parts equal to $\frac{1}{4}$ are there in the whole sheet of paper? What fraction represents three of these parts? What represents two parts of the paper?

Step 3: Use a new sheet of paper to make a folded piece that has eight equal parts. Identify and make a list of as many fractions involving the denominator 8 as you can. Which of these fractions represent the same fractional part as the fractions in Step 1 and 2 with different denominators?

Step 4: Fold this same sheet of paper once more to make sixteenths. How many times did you fold the paper?

Discuss with the students that the AREA MODEL is not restricted to just rectangles. For example, we can look at other geometric shapes like circles that we see in pizzas. When this book refers to the AREA MODEL, we mean looking at the area of the shape but not restricting ourselves to any particular shape. With that said, we often use the rectangle as a convenient shape.

Cutting horizontally foreshadows the area model for multiplying fractions.

They are both 2 parts out of 4 parts. The difference is the size of the whole or "1." Ask your students for another example of 2 parts out of 4 parts that looks different.

Use the area model to draw the fractions below. Draw the whole as a rectangle because it is easier to divide into equal pieces. For each fraction, identify the numerator and denominator. Then write the fraction mathematically.
a. Three-fourths
c. One-fifth
b. Two-fifths
d. Three-tenths

It is possible for two fractions with different numerators and denominators to represent the same amount. Consider the following example:


If we divide a whole into 4 equal parts, 2 of the 4 parts can be written as the fraction


Are these both a representation of $\frac{2}{4}$ ? They do not look the same. So what is the difference?

Shading 2 equal parts out of 4 is equivalent to shading 1 part out of 2 . This means the fractions $\frac{1}{2}$ and $\frac{2}{4}$ are equivalent. Another way to show that the fraction $\frac{1}{2}$ is equivalent to the fraction $\frac{2}{4}$ is to take the picture representing $\frac{1}{2}$ and draw a horizontal slice as shown below:


The horizontal slice doubles the numerator and also doubles the number of parts into which the whole is divided, the denominator.

Suppose we make three horizontal cuts in the original rectangular model for $\frac{1}{2}$ to form equal sized pieces. What fraction is shaded?

The whole class has 12 students. The fraction of the class that is girls is $\frac{8}{12}$, and the fraction of the class that is boys is $\frac{4}{12}$. Ask if these fractions are equivalent to simpler fractions, that is, a fraction with a smaller denominator. Yes, $\frac{8}{12}=\frac{4}{6}=\frac{2}{3}$ and $\frac{4}{12}=\frac{2}{6}=\frac{1}{3}$. Add these to the class chart.

## Example 5

This discussion can easily be one you lead orally.


Like the example above, the picture represents both $\frac{1}{2}$ and $\frac{4}{8}$. These fractions are equivalent. We write $\frac{1}{2}=\frac{2}{4}=\frac{4}{8}$ because the fractions represent the same part of a whole.

Remember, the "whole" is not just a geometric shape that represents one whole, like a circle or a rectangle, subdivided into equal parts. For example, the whole might be a class that has 8 girls and 8 boys. What fraction of the class is female? Male?

## EXAMPLE 5

A class of 12 consists of 4 boys and 8 girls. We know that the fraction of the class that is male can be written $\frac{4}{12}$, and the fraction of the class that is female can be written $\frac{8}{12}$. Write another way to express the fraction of the class that is boys as well as the fraction of the class that is girls, using equivalent fractions.

## SOLUTION

If you said $\frac{1}{3}$ as another way to express the fraction of the class that represents the boys, then you were correct. In the original fraction, the denominator 12 represented the number of people in the class, and the numerator 4 represented the number of boys in the class.

Remember, two fractions that represent the same part of a whole are called equivalent fractions.

In general, we can find equivalent fractions by multiplying the numerator and denominator by the same number. For example,

$$
\frac{1}{4}=\frac{(2)(1)}{(2)(4)}=\frac{(1)(2)}{(4)(2)}=\frac{2}{8}
$$

Again, a basic concept in understanding fractions is the ability to simplify them so that the numerator and denominator are relatively prime.

Sometimes we say that a fraction is in simplest form when you cannot simplify it any more. However, this statement is not precise. How do you know you cannot simplify it any more? When factoring the numerator and denominator into their prime factorizations, if they have no factors in common, then the fraction is in simplest form.

In simplifying a fraction, you can think of it two ways. One is to divide the numerator and denominator by the same factor, repeatedly if necessary. For example: $\frac{12}{6}=\frac{12 \div 2}{36 \div 2}=\frac{6}{18}=\frac{6 \div 2}{18 \div 2}=\frac{3}{9}=\frac{3 \div 3}{9 \div 3}=\frac{1}{3}$. However, a more straightforward way of using Property 8.1 is as follows: $\frac{12}{36}=\frac{6 \cdot 2}{18 \cdot 2}=\frac{6}{18}=\frac{3 \cdot 2}{9 \cdot 2}=\frac{3}{9}=\frac{1 \cdot 3}{3 \cdot 3}=\frac{1}{3}$

Pictorially,


Multiplying the numerator and denominator by 2 has the effect of doubling the number of slices:


Multiplying the numerator and denominator by the same number changes the number of shaded parts and the total number of parts by the same factor, yielding an equivalent fraction.

## PROPERTY 8.1: EQUIVALENT FRACTION PROPERTY

For any number $a$ and nonzero numbers $k$ and $b$

$$
\frac{a}{b}=\frac{k \cdot a}{k \cdot b}=\frac{a \cdot k}{b \cdot k}=\frac{a k}{b k}
$$

We have generated equivalent fractions using the area model by dividing a given representation into smaller equal pieces, as seen in the diagram, by converting 1 part out of 4 parts into 2 parts out of 8 parts. Notice that the new denominator was always a multiple of the original denominator. But many times we will want to find an equivalent fraction with a smaller denominator, if possible. We call this process simplifying a fraction. We will do this by using Property 8.1 in reverse or by using the GCF. For example, to simplify the fraction $\frac{6}{10}$, we first recognize that $\frac{6}{10}=\frac{2(3)}{2(5)}$. The numerator 6 and the denominator 10 have a greatest common factor, 2. Dividing both the numerator and the denominator by the common factor of 2 produces an equivalent fraction. Using the Equivalent Fractions Property, we see that $\frac{6}{10}$ is equivalent to $\frac{3}{5}$. So, we have simplified $\frac{6}{10}$ to the form $\frac{3}{5}$. A fraction is in simplest form if the numerator and denominator have no common factors except 1.

## Exercises

1. Have students share the different methods they use for doing these. Students could use prime factorization, listing, and Venn Diagrams.
a. 1,$2 ;$ CGF is 2 f. $1,2,3 ;$ GCF is 4
b. 1,3 ; GCF is 3 g. $1,3,9$; GCF is 9
c. 1 ; GCF is 1 h. 1,$5 ;$ GCF is 5
d. 1 ; GCF is 1 i. $1,2,3,6$; GCF is 6
e. $1,3,4,6,12 ;$ GCF is 12 j. $1,3,5 ;$ GCF is 5
2. 4 boxes where each box contains 3 jars of strawberry jam, 4 jars of grape jam, and 6 jars of pineapple jam.
3. 6 bags where each bag contains 4 lollipops, 2 candy bars, and 7 pieces of gum.
4. $b$
5. d, because although 3 is a factor, it is not the greatest common factor of 9,27 , and 36.18 and 24 are not factors of all 3 numbers.

## EXERCISES

1. Find the GCF of each pair of integers using one of the following strategies: T-chart, prime factorization, or Venn Diagram.
a. 12 and 10
f. 44 and 84
b. 12 and 15
g. 45 and 81
c. $\quad 15$ and 22
h. 65 and 90
d. 80 and 81
i. 66 and 90
e. 24 and 36
j. 120 and 195
2. John has 12 jars of strawberry jam, 16 jars of grape jam, and 24 jars of pineapple jam. He wants to place the jars into the greatest possible number of boxes so that each box has the same number of jars of each kind of jam. How many boxes does he need, if every jar of jam is used?
3. Sarah is making candy bags for her birthday party. She has 24 lollipops, 12 candy bars, and 42 pieces of gum. She wants each bag to have the same number of each kind of candy. What is the greatest number of bags she can make if all the candy is used? How many pieces of each kind of candy will be in each bag?
4. Mrs. Blackburn wrote the following riddle on the board for her mathematics class. We are 2 -digit numbers. Our greatest common factor is 14 . Our difference is 42 . Our sum is 98 . What are the 2 numbers of the riddle?
a. 14 and 42 because their greatest common factor is 14 .
b. 28 and 70 because their difference is 42 , their greatest common factor is 14 , and their sum is 98 .
c. 14 and 56 because their difference is 42 , and their greatest common factor is 14 .
d. 42 and 84 because their difference is 42 .
5. Which of the following is the greatest common factor of 9,27 , and 36 ? Select the best choice and explain your answer.
a. 3
b. 24
C. 18
d. 9
6. a, common factors are $1,2,3,6$. So 6 is the GCF of $\{12,18$, and 36$\}$.
7. It is not possible to tile a $5 \times 7$ grid with square tiles larger than $1 \times 1$. You can tile a $9 \times 15$ grid with tiles larger than $1 \times 1$ tiles, more specifically with $3 \times 3$ tiles. This exercise involves common divisors because the square tiles must evenly divide both the length and width of the grid, so you must find square tiles with sides that evenly divide both numbers - they must be common divisors of each number. The largest square tiles that can cover a $24 \times 36$ square grid are $12 \times 12$ tiles.
8. 1 because the only factors of any prime numbers are 1 and the number itself.
9. Have students share different ways they find equivalent fractions. Answers may vary. Some answers may include:
a. $\frac{4}{5}, \frac{16}{20}, \frac{24}{30}, \frac{32}{40}$
b. $\frac{2}{1^{4}}, \frac{3}{21}, \frac{4}{48}, \frac{5}{35}$
d. $\frac{4}{5}, \frac{8}{10}, \frac{24}{30}, \frac{36}{45}$
10. Some of these fractions have several simpler equivalent fractions. Watch to see if your students understand when a fraction is in its simplest form. If they don't, discuss what criteria they might use to determine whether a fraction is in its simplest form. (cf = common factor; gcf = greatest common factor)
a. $\frac{4}{5}$;
b. $\frac{2}{3}$;
c. $\frac{2}{3}$;
d. $\frac{2}{3}$;
e. $\frac{5}{7}$;
f. $\frac{17}{24}$;
11. a. $\frac{27}{40}$
f. No, 210 and 221 are relatively prime.
b. Simplest, 27 and 32 are relatively prime.
g. No, 59 and 83 are relatively prime.
c. $\frac{8}{25}$
h. No, 124 and 125 are relatively prime.
d. $\frac{20}{39}$
e. $\frac{76}{177}$
$\begin{array}{ll}\text { i. } & \frac{p}{q} \\ \text { j. } & \frac{1}{w x y^{2}}\end{array}$
12. Which of the following is the greatest common factor of 12,18 , and 36 ? Select the best choice and explain your answer.
a. 6
b. 1
c. 12
d. 24
13. Draw a $5 \times 7$ grid. Is it possible to tile this grid with squares larger than $1 \times 1$ ? What about tiling a $9 \times 15$ grid with larger square tiles? How does this exercise involve common divisors? Without drawing the grid, determine the dimensions of the largest size square that tiles a $24 \times 36$ grid. See TE.
14. What is the GCF of two prime numbers $p$ and $q$ ? Explain your reasoning.
15. Find three equivalent fractions for each of the fractions below. You may use paper folding or any other model to determine the equivalent fractions.
a. $\frac{8}{10}$
b. $\frac{1}{7}$
c. $\frac{3}{6}$
d. $\frac{4}{8}$ e. $\frac{12}{15}$
f. $\frac{1}{x}$
16. For each of the following fractions, find a common factor in the numerator and denominator. Then, simplify the fraction. See TE.
a. $\frac{24}{30}$
C. $\frac{24}{36}$
b. $\frac{14}{21}$
d. $\frac{32}{48}$
e. $\frac{25}{35}$
f. $\frac{51}{72}$
17. Use common factors in both numerator and denominator to rewrite each of the following fractions in simplest form. If the fraction is already in its simplest form, explain why it is. See TE
a. $\frac{108}{160}$
b. $\frac{27}{32}$
c. $\frac{200}{625}$
d. $\frac{340}{663}$
e. $\frac{380}{885}$ f. $\frac{210}{221}$
g. $\frac{59}{83}$
h. $\frac{124}{125}$
i. $\frac{p^{2} q}{p q^{2}} \quad$ j. $\frac{x y^{2}}{w x^{2} y^{4}}$
18. a. $\frac{1}{2}$
c. $\frac{6}{18}$ or $\frac{1}{3}$
d. $\frac{4}{18}$ or $\frac{2}{9}$
19. A represents $\frac{1}{4}$
B represents $\frac{1}{1,6}$
C represents $\frac{1}{32}$
D represents $\frac{1}{8}$
E represents $\frac{1}{16}$
F represents $\frac{1}{8}$
Grepresents $\frac{1}{16}$
20. Use Exercise 3 to talk to your students about the best shape for the whole when working the exercises. For instance, when dealing with time, a circle for the analog clock is best. Then talk about how many parts to divide it into - twelve. Jeremy practiced juggling for $\frac{40}{60}$ or $\frac{2}{3}$ of an hour. Amy practiced $\frac{45}{60}=\frac{3}{4}$ hour.
21. $\frac{12}{30}=\frac{2}{5}$ of the class is male. $\frac{18}{30}=\frac{3}{5}$ of the class is female.
22. In half an hour it will be 3:20 P.M., and in $\frac{1}{4}$ of an hour it will be 3:05 P.M.; have students use an analog clock to explain their answer.
23. Answers may vary for equivalent fractions to $\frac{8}{12}$ that have larger denominators. Some answers may include: $\frac{16}{24}, \frac{24}{36}, \frac{10}{15}$. But there are only 3 equivalent fractions for $\frac{8}{12}$ with smaller denominators: $\frac{2}{3}, \frac{4}{6}$, and $\frac{9}{12}$.
24. What fraction is represented by the shaded portion of each figure below?
a.

C.

b.

d.

e. Simplify each of your fractions in a -d , if possible.
25. Determine the fraction that represents each of the labeled regions assuming the large square represents the whole or 1 .

26. Jeremy practiced juggling for 40 minutes and Amy practiced for 45 minutes. For what fraction of an hour did each practice? Simplify each fraction.
27. A class has 12 boys and 18 girls. What fraction of the class is male? What fraction of the class is female? Simplify each of these fractions.
28. If the time is $2: 50$ p.M., what time will it be in half an hour? What time will it be in $\frac{1}{4}$ of an hour?
29. Find an equivalent fraction for $\frac{8}{12}$ that has a larger denominator. Find 2 equivalent fractions for $\frac{8}{12}$ that have smaller denominators.
30. a. yes, $\frac{5}{6}$
b. no, 53 is prime.
c. no, 12 and 25 have no prime factors in common.
d. no, 23 is prime.
e. Knowing the factors of the numerators and denominators or whether or not the numerator or denominator was prime. We want students to articulate how they know if a fraction is in simplest form.
31. Make sure that your students start out with large scaled versions of the rectangles rather than trying to figure out what's going on using tiny diagrams. Start with a square that is at least $16 \times 16$.

Students should notice that the new square has dimensions half of the original and each new area is one-fourth of the original.

18. Determine whether each of the following fractions has an equivalent fraction with a denominator that is less than the denominator in the given fraction. That is, determine whether you can simplify the fractions. If you can simplify, do so. See TE.
a. $\frac{15}{18}$
b. $\frac{24}{53}$
c. $\frac{12}{25}$
d. $\frac{18}{23}$
e. What properties of the numerator or denominator helped you to answer parts a-d?
19. Ingenuity:

A positive integer $d$ has the following properties:

- When 100 is divided by $d$, the remainder is 2 .
- When 145 is divided by $d$, the remainder is 5 .
a. What are all the possible values of $d$ ? 7,14
b. What is the greatest possible value of $d$ ? 14


## 20. Investigation:

Using grid paper, draw a large $1 \times 1$ square. Shade an area of the square that is the same shape and $\frac{1}{4}$ of the original area. What are the dimensions of the smaller square? Repeat this process for several other rectangles, each time shading a smaller rectangle that is the same shape and $\frac{1}{4}$ of the original rectangle's area. What do you notice?

## FRACTION FRIEND

Objective: The students will use models to find equivalent fractions.

## Materials Needed:

Construction paper ( 5 sheets of different color per student)
Scissors
Glue stick
Markers or crayons
Yarn (optional)

## Activity Instructions:

1) Choose a color and discuss that it will represent the body or one whole. Label it as 1 whole in big print.
2) Choose another color. Fold it into "fourths" and cut them out. These parts will represent the arms and the legs. Label each part $1 / 4$ in large print then cut and glue them to the whole (body).
3) Choose another color. Fold it into "eighths" by starting with $1 / 2$, then $1 / 4$, then $1 / 8$. Label each part $1 / 8$ in large print. These will be the calves and forearms. Cut and glue 2- "eighths" to each "fourth" (arms and legs).
4) Choose another color. Fold it into "sixteenths." Start with folding the paper in half and repeat this four times to make sixteenths ( $1 / 2,1 / 4,1 / 8$, then $1 / 16$ ). Label each part $1 / 16$, then cut and glue to eighths (fingers and toes).
5) Using a gallon ice cream lid, have students trace the circle, cut it out, and decorate a face for their fraction friend using only math symbols (+ = - / \% \# symbol for pi, etc...). They can use yarn to add hair, mustache, or eyelashes. Some students have used scraps to make bows, ties, hats, etc. (using the math symbol criteria).
You can relate this activity to the "Gallon Guy" since most students are familiar with it.

1 gallon = 1 whole
1 quart $=1 / 4$ of a gallon
1 pint $=1 / 8$ of a gallon
1 cup $=1 / 16$ of a gallon
EXTENSION: Students can write the decimal and percent equivalent form for each part.

## Clock Worksheet Practice Exercises

1. Jeremy practiced juggling for 40 minutes. For what fraction of an hour did Jeremy practice? - $\frac{2}{3}$
2. Daniel practiced trombone for 25 minutes. For what fraction of an hour did Daniel practice? - $\frac{5}{12}$
3. Mike played the piano for $11 / 2$ hours. For how many minutes did Mike play? - 90 minutes
4. Ann spent $3 / 4$ of an hour making dinner. After dinner, Nathan spent $1 / 3$ of an hour cleaning up. How long did the two of them take? Can you give your answer in minutes as well as in fraction? - 65 minutes


## Section 8.2 - Unit Fractions and Mixed Numbers

## Big Idea:

Comparing and converting improper fractions and mixed numbers

## Key Objectives:

- Understand the relationship between improper fractions and mixed numbers.
- Learn the definition of unit and proper fractions.
- Learn the definition of reciprocal.
- Use fractions within the context of real-life situations.


## This entire section is a review for 7th grade.

## Materials:

Grid paper

## Pedagogical/Orchestration:

This section involves connecting improper fractions to their mixed number equivalents through division. The computational practice in this section is important to understanding and being able to compute using fractions. The practice with the multiplicative inverse and real-life situations is also important.

- Beginning of lesson foreshadows multiplication of fractions.
- Improper fractions are greater than 1 , and proper fractions are less than 1.
- In this section, there should be some direct instructions on how to convert from mixed number to improper fraction and vice versa.
- Connection to Measurement: ask students to find $21 / 4$ inches on a ruler and so forth. See "Ruler Activity" on CD.


## Internet Resource:

Jeopardy Game: Improper/Proper Fractions- http://www.quia.com/cb/186132.htm|

## Activities:

"The Best Sugar Cookies" on CD
"Ruler Activity" on CD

## Exercises:

Foreshadowing mixed number division in Exercises 12,13.

## Vocabulary:

multiplicative inverse, reciprocal, unit fraction, proper fraction, improper fraction, mixed number

## TEKS:

$6.1(\mathrm{~B}) ; \quad 7.1(\mathrm{~A}) ; \quad 7.2(\mathrm{G}) ; \quad 7.13(\mathrm{C}) ; \quad 7.14(\mathrm{~A})(\mathrm{B}) ; \quad 8.2(\mathrm{C}) ; \quad 8.14(\mathrm{~B}) ; \quad 8.15(\mathrm{~A})$;

## WARM-UPS for Section 8.2

1. a. Sonia spent $3 \frac{1}{2}$ years in college. How many months was she in college? Ans: 42 months
b. Fred worked on a project for the space program for 21 months. How many years did he work on the project? Ans: $1 \frac{9}{12}=1 \frac{3}{4}$ years
2. Maria worked on cleaning up an oil spill in the Gulf for 100 days. Which of the following best describes the number of weeks she worked?
a. between 13 and 14 weeks
b. between 14 and 15 weeks
c. between 15 and 16 weeks
d. between 16 and 17 weeks

Ans: (b) because $100 \div 7=14 \frac{2}{7}$
3. What are the next three members of the following list:
a. $1 \frac{1}{2}, 3,4 \frac{1}{2}, 6,7 \frac{1}{2},-,-\quad-$
Answer: $9,10 \frac{1}{2}, 12$
b. $2,2 \frac{3}{4}, 3 \frac{1}{2}, 4 \frac{1}{4}$, $\qquad$ Answer: $5,5 \frac{3}{4}, 6 \frac{1}{4}$

## Launch for Section 8.2:

In this Launch, the teacher can elect to visually demonstrate the concept by bringing in sugar, a $\frac{1}{4}$ measuring cup and a 2 cup liquid measuring cup. Tell the class, "You are going to bake cookies and the recipe calls for $1 \frac{3}{4}$ cups of sugar. You look all through the kitchen and the only measuring cup you can find is a $\frac{1}{4}$ measuring cup. How could you use that cup to measure the required amount of sugar? Let students discuss and explain to others in the class that 4 of the $\frac{1}{4}$ measuring cups will give one whole cup, and 3 of thg $\frac{1}{4}$ cups will give the $\frac{3}{4}$ cups. This is 7 all together of the $\frac{4}{4}$ cups. Let students know that this is actually called $\frac{7}{4}$ of a cup and is known as an improper fraction because it is larger than 1 whole. Visually demonstrate by scooping 7 of the $\frac{1}{\Phi}$ cups into the sugar and pouring them into the 2 cup liquid measuring cup. It will fill to the $1 \frac{3}{4}$ line. Therefore $1 \frac{8}{4}=\frac{7}{4}$. Tell your students, "The focus for today will be converting between improper fractions and mixed numbers just as we did with the amount of sugar in the cookie recipe."

Foreshadows multiplication of fractions.

## SECTION 8.2 UNIT FRACTIONS AND MIXED NUMBERS

In Chapter 4, we developed a model for multiplication based on repeated addition. Apply this model to fractions like $\frac{1}{3}$, which are called unit fractions. Using the frog model, if the length of each jump is $\frac{1}{3}$ and the frog takes 3 jumps, it will land on 1 . In other words, $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=\frac{1}{3} \cdot 3=1$.

If the length of each jump is $\frac{1}{4}$ and the frog takes 4 jumps, it will land on 1 . We write this as $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{1}{4} \cdot 4=1$. In a similar way, we can show that

$$
\begin{aligned}
& \frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{1}{5} \cdot 5=1 \text {, and } \\
& \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{8} \cdot 8=1 .
\end{aligned}
$$

## DEFINITION 8.3: MULTIPLICATIVE INVERSE (RECIPROCAL)

The number $x$ is called the multiplicative inverse or reciprocal of the positive integer $n$ if $x \cdot n=1$.

For example, you say $\frac{1}{7}$ is the multiplicative inverse of 7 because $\frac{1}{7} \cdot 7=1$ and 6 is the multiplicative inverse of $\frac{1}{6}$ because $6 \cdot \frac{1}{6}=1$.

## THEOREM 8.1: UNIT FRACTION

For any positive integer $n$, the multiplicative inverse or reciprocal of $n$ is the unit fraction $\frac{1}{n}$.

A unit fraction always has 1 in the numerator. The denominator is a positive integer. For example, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, and so on are all examples of unit fractions.

Students should be reminded of the customary conversions of mass and capacity.

Now extend the addition of unit fractions to make another connection to multiplication. You have seen several models that represent the fraction $\frac{3}{5}$. In the area model, $\frac{3}{5}$ represents three $\frac{1}{5}$ 's of a whole. This means $\frac{3}{5}$ is the sum $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$. In the frog model, this is the same as taking 3 jumps of length $\frac{1}{5}$. That is, $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{1}{5} \cdot 3=\frac{3}{5}$. This understanding can be extended to all fractions. For example, the fraction $\frac{5}{9}$ is the same as the sum of 5 copies of $\frac{1}{9}$ :

$$
\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\frac{1}{9} \cdot 5=\frac{5}{9}
$$

Write the sum of 8 copies of $\frac{1}{5}$. Using the frog model, what is the result of 8 jumps of length $\frac{1}{5}$ each?
Using math with recipes can be fun not only for learning but for eating, too! Usually recipes involve quantities of ingredients and cooking directions. Here is a chocolate chip cookie recipe:

- $2 \frac{1}{4}$ cups flour - 1 tsp vanilla
- $\frac{3}{4}$ cup sugar
- 1 tsp salt
- $\frac{3}{4}$ cup brown sugar
- 1 tsp baking soda
- 12 oz chocolate chips

This makes approximately 6 dozen cookies.
Each of the sugar quantities is $\frac{3}{4}$ of a cup. This quantity is called a proper fraction because it is less than 1 .

Notice that one easy way to check if a fraction is proper is to make sure the numerator is less than the denominator. A fraction that is not proper is called improper. An improper fraction is a fraction that is greater than or equal to 1 . For a fraction $\frac{\mathrm{a}}{\mathrm{b}}$ to be improper, the numerator must be greater than the denominator, or $a>b$.

Look at the first ingredient in our recipe. Did you think "two and one-fourth cups?" Remember, "and" means adding two cups to one-fourth of a cup. Quantities like $2 \frac{1}{4}$ are called mixed numbers because they consist of an integer like 2 , in addition to a fractional part that is less than a whole, like $\frac{1}{4}$. It is customary

Have a discussion of the word "proper."

## PROBLEM 1

Have a discussion with many examples to illustrate the differences between proper and improper fractions. You could have discussion start in small groups and then share with the whole class.
to write the fractional part in simplified form. The mixed number $2 \frac{1}{4}$ is actually the sum $2+\frac{1}{4}$. The rest of the recipe contains both fractional parts of cups or teaspoons and numbers of ounces and dozens.
Look at the mixed number $2 \frac{1}{4}$. If you have only a quarter-cup measure, describe how you can measure the correct amount with the quarter cup.
Did you find $2 \frac{1}{4}$ equivalent to $\frac{9}{4}$ ? In fact, what you have found are two ways to write the same quantity: as a mixed number, $\frac{1}{4}$, and as an improper fraction, $\frac{9}{4}$. How would you describe improper fractions? Why do you think they are called improper?

## PROBLEM 1

State the difference between proper and improper fractions. What is the advantage of using an improper fraction or using a mixed number?

If $a$ is positive $(a>0)$ what is the value of $a x$ ?
There are three possibilities:

1. $x<1$. For example, $x=\frac{1}{3}$. Multiplying $\frac{1}{3} \bullet a$ is the same as taking $\frac{1}{3}$ of $a$, which is less than $a$.
2. $x=1$. Then $1 \bullet a=a$.
3. $x>1$. For example $x=3$. Multiplying $3 \bullet a$ is greater than $a$.

If $x \bullet a>a$, what is the value of $x$ ? There are 3 possibilities:

$$
x>1, x=1 \text {, or } x<1 \text {. }
$$

The value $x$ cannot be 1 or $x$ less than 1 because in both cases, $x \bullet a$ is not greater than $a$.

So if $x \cdot a>a$, then $x>1$.
Similarly, if $x \cdot a=a$, then $x=1$, and if $x \bullet a<a$, then $x<1$.

Problem 2:

| a. | $\frac{7}{5}$ |
| :--- | :--- |
| b. | $\frac{7}{3}$ |
| c. | $\frac{15}{4}$ |

A ruler is a type of number line people use daily. Determine where $2 \frac{1}{4}$ inches is located on a ruler. Use the ruler to explain why $\frac{9}{4}$ is an equivalent fraction to $2 \frac{1}{4}$. What does the 4 represent in the fraction?
A recipe for pancakes calls for $1 \frac{3}{4}$ cups of flour. Locate this point on the number line. Describe the equivalent, improper form for the mixed number and use the number line to explain the value in the numerator.

Jack has three identical pans of brownies and decides to divide each pan into 12 equal pieces. How many brownie pieces does he have in all? Because Jack was very hungry, he ate 2 of the pieces. If you assume each brownie pan represents 1 or a whole, express the amount of brownies that remain in terms of the whole and pieces.

If he takes half of the uneaten brownies to a party, what quantity will he take? Using the area model, draw the brownie quantities he will take and leave. Be sure to include the fact that each pan is divided into 12 pieces.

## PROBLEM 2

Write each of the following mixed numbers as improper fractions:
a. $1 \frac{2}{5}$
b. $3 \frac{3}{4}$
c. $\quad 2 \frac{1}{3}$

## PROBLEM 3

Write each of the following mixed numbers as improper fractions:
a. $\frac{5}{2}$
b. $\frac{7}{4}$
c. $\quad \frac{11}{3}$

1. a. $\left(\frac{1}{5}\right) 5=(0.2) 5=1.0=1$
c. $\left(\frac{1}{25}\right) 25=(0.04) 25=1.0=1$
b. $\left(\frac{1}{8}\right) 8=(0.125) 8=1.0=1$
d. $\left(\frac{1}{40}\right) 40=(0.025) 40=1.0=1$.

Encourage your students to use number sense or a, shortened form of long division when converting $\frac{1}{40}$ to decimal form. First they might realize that $\frac{1}{40}$ is $\frac{1}{4}$ of $\frac{1}{10}$. Because $\frac{1}{10}=0.10, \frac{1}{4}$ of $\frac{1}{10}$ is $\frac{1}{4}$ of 0.10 or 0.025 , just as 2.5 is $\frac{1}{4}$ of 10 . Also, they can divide 40 into 1 by seeing that taking 40 into 1 is like taking 4 into 100 after a conversion that involves dividing by 1000. The quotient 25 divided by $1000=0.025$.
2. a. The row of 4 chips is $\frac{20}{32}$ or $\frac{5}{8}$ inch long.
b. A,total of 6 computer chips can fit in one row.
c. $\frac{2}{32}$ or $\frac{1}{16}$ inch is left.
3. a. $\frac{5}{3}=1 \frac{2}{3}$
c. $\frac{6}{5}=1 \frac{1}{5}$
b. $\frac{7}{6}=1 \frac{1}{6}$
d. $\frac{9}{4}=2 \frac{1}{4}$
4. $\frac{7}{4}=1 \frac{3}{4}, \frac{19}{4}=4 \frac{3}{4}, \frac{10}{4}=2 \frac{2}{4}=2 \frac{1}{2}, \frac{14}{4}=3 \frac{2}{4}=3 \frac{1}{2}, \frac{17}{4}=4 \frac{1}{4}, \frac{11}{4}=2 \frac{3}{4}, \frac{12}{4}=3$.

5. $\frac{12}{5}=2 \frac{2}{5}$. Students may use the number line model as in exercise 4 or they may use the fact that every fraction is a division problem and ask themselves - how many times does 5 go into 12? 2 times with a remainder of 2 and thus $2 \frac{2}{5}$.

7. $2 \frac{5}{6}=\frac{17}{6}$. Answers may vary as to what model students use. They may use the number line model or a modified multiplication model that reverses the process they used in Exercise 5. Using a modified multiplication model students will compute common denوminators and add fractions. Multiplying 2 and 6 to get 12 and then adding 5 which is 17 , the numerator of $\frac{17}{6}$.
8. $2 \frac{4}{5}=\frac{14}{5}, 3 \frac{4}{7}=\frac{25}{7}, 4 \frac{1}{3}=\frac{13}{3}, 2 \frac{7}{15}=\frac{37}{15}, 3 \frac{1}{7}=\frac{22}{7}, 8 \frac{1}{2}=\frac{17}{2}, 8 \frac{1}{6}=\frac{49}{6}, 3 \frac{5}{16}=\frac{53}{16}$.

## EXERCISES

1. Verify that the multiplicative inverse rule is true by substituting the equivalent decimal for each of the $\mu$ nit fractions below to perform the multiplication indicated. For example, $\frac{1}{2} \cdot 2=(0.5) 2=1.0=1$.
a. $\frac{1}{5} \cdot 5=1$
b. $\frac{1}{8} \cdot 8=1$
c. $\left(\frac{1}{25}\right) 25=1$
d. $\left(\frac{1}{40}\right) 40=1$
2. A type of computer chip is $\frac{5}{32}$ inch long. In assembling the motherboard of a new computer, a technician lines up 4 of the chips.
a. How long is this row of chips with no gap between them?
b. If this part of the motherboard has only 1 inch of space, how many chips can fit in one row?
c. How much space is left?
3. Rewrite these sums as an improper fraction and as a mixed number:
a. $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=$
b. $\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=$
c. $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=$
d. $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=$
4. Convert each of these improper fractions to a mixed number. Sketch a number line from 0 to 5 with fourths marked and locate each mixed number.

$$
\frac{7}{4}, \frac{19}{4}, \frac{10}{4}, \frac{14}{4}, \frac{17}{4}, \frac{11}{4}, \frac{12}{4}
$$

5. Write the improper fraction $\frac{12}{5}$ as a mixed number. Explain your process
6. Convert each of these improper fractions to a mixed number.

Then order the fractions from least to greatest.

$$
\frac{5}{3}, \frac{8}{3}, \frac{11}{3}, \frac{17}{5}, \frac{23}{7}, \frac{14}{8}, \frac{18}{4}, \frac{28}{8}, \frac{36}{10}, \frac{30}{20}, \frac{24}{9}
$$

7. Convert the mixed number $2 \frac{5}{6}$ to an improper fraction. Explain your process
8. Convert each of these mixed numbers to an improper fraction.
9. There are 14 one-eighth cups of flour in $1 \frac{3}{4}$ cups of flour.
10. $2 \frac{2}{3}>\frac{15}{6}$. Student explanations may vary.
11. $\frac{23}{7}>3 \frac{1}{4}$. Student explanations may vary.
12. Uncle Jim can serve shrimp to 10 people. Foreshadows division of a mixed number by a fraction. Students will probably use the linear model and skip count to compute the number of servings.
13. Each piece of candy cane is $1 \frac{7}{8}$ inches long. This problem may lead some students to discover that dividing by 2 is equivalent to multiplying by $\frac{1}{2}$.

$$
2 \frac{4}{5}, 3 \frac{4}{7}, 4 \frac{1}{3}, 2 \frac{7}{15}, 3 \frac{1}{7}, 8 \frac{1}{2}, 8 \frac{1}{6}, 3 \frac{5}{16}
$$

9. How many one-eighth cups of flour are in $1 \frac{3}{4}$ cups of flour?
10. Which is greater: $2 \frac{2}{3}$ or $\frac{15}{6}$ ? Explain your reasoning.
11. Which is greater: $\frac{23}{7}$ or $3 \frac{1}{4}$ ? Explain.
12. Uncle Jim brings home $3 \frac{1}{3}$ pounds of shrimp. He estimates he will need an average of $\frac{1}{3}$ pound of shrimp per serving. How many people can he serve?
13. Johnny has a candy cane that is $3 \frac{3}{4}$ inches long. He cuts it in two equal pieces and gives one to his sister. How long is each piece?
14. Draw a number line with 0 and $\frac{1}{x}$ as shown below. Plot $\frac{2}{x}, \frac{3}{x}$, and $\frac{4}{x}$ on your number line.

15. For each of the following questions, make a copy of the picture below and use it as a linear model for a fraction bar.


For each scenario below, determine the numbers that each of the other labled points represent. Write the values below each point
a. If the point $A$ represents the number 1
b. If the point $B$ represents the number 1
c. If point $C$ represents the number 1
d. If point $D$ represents the number 1
e. If point $E$ represents the number 1
f. If point $F$ represents the number 1

## Ingenuity

16. The knowledge of how mixed fractions are converted into improper fractions, which students learn by using variables, not numbers, is important to computation in higher math.
a. $\frac{18+\mathrm{x}}{9}$
b. $\frac{8 \mathrm{x}+}{8}$
c. Start with a positive $n$ and plot the fractions on a number line.
d. Answers may vary. Some students will justify their answers using common denominators, others will use number lines.
e. If $n>0, \frac{n}{3}$; if $n<0, \frac{n}{4}$.

## Investigation

17. In this exercise, "copy" need not be taken literally. To work this exercise, use grid paper, setting each unit square equal to the unit fraction that corresponds with each section For instance, in part (a), set each unit square equal to $\frac{1}{3}$. Then draw the first rectangle using 2 squares for $\frac{2}{3}$. Because $1=\frac{3}{3}$, extend the rectangle another square to get $\frac{3}{3}=1$. Shade the whole extended rectangle. The idea is that the numerator represents the number of divisions present, so in $\frac{2}{3}$, divide the rectangle into halves. Then find 3 of those divisions. To do this, you will have to make another half.

18. Let $\mathrm{a}=12$. Compute ax for each value of x below, and determine if the value is greater than, equal to, or less than a. Explain how to do this without computation.
a. $x=2$
b. $x=3$
c. $x=1 / 2$
d. $x=1 / 3$

## 17. Ingenuity:

Suppose $x$ is a positive integer. Convert the following mixed numbers to improper fractions.
a. $2 \frac{x}{9}$
b. $\times \frac{7}{8}$

Assume that $n$ is a nonzero integer.
c. First, suppose $n$ is positive. Plot $n, \frac{n}{2}, \frac{n}{3}, \frac{n}{4}$ and $\frac{n}{5}$ on a number line. Now make another number line representing the case where $n$ is negative
d. Find a fraction between $\frac{n}{3}$ and $\frac{n}{4}$ and then another between $\frac{-n}{3}$ and 4.
e. The number $n$ is a nonzero integer. Which fraction is greater, $\frac{n}{3}$ or $\frac{n}{4}$ ? Explain your answer.

## 18. Investigation:

Copy the rectangle to the right for each problem and suppose that its area is given in each of the problems below. Determine and shade an area
 of 1 square unit for each of the rectangles.
a. $1 \frac{2}{3}$
b. $2 \frac{2}{3}$
c. $\frac{5}{4}$
d. $\frac{2}{3}$

## FACTORIZE WITH ARRAYS

Objective: The students will practice the skills learned in sections 8.3 and 8.4 to divide or factorize numbers into two factors and build arrays to represent each factorization.

## Materials:

Computer with internet
Notebook paper

## Activity Instructions:

Direct students to the following website and have them complete the activity several times. This activity will work best if you can provide one computer per student. If, however, you don't have enough computers for all of your students, you can do this as a class activity if you have a way to project your computer screen (or Active Board) to the whole class. If you are doing the whole class activity, you can break the students into groups and have one group at a time come up and work a problem. If they get it correct, that team gets a point. It is important, however, that all students get a chance to work at least one problem on their own.

## http://illuminations.nctm.org/Activities.aspx?grade=3\&srchstr=factors

Once your students are at the correct website, direct them to the button labeled ACTIVITES; in the next screen, check off the level 6-8 and type in the key word box:
FACTORIZE and click on SEARCH. Next, click on the activity FACTORIZE; in the next screen, students will be given step-by-step instructions to follow.

It would be best if the students kept a written log of the original numbers and their final answers. Decide on a set number of problems for them to complete in a certain amount of time. The students can turn in their written work to show you what they have accomplished.

## Section 8.3 - Common Multiples and the LCM

## Big Idea:

Finding common multiples and least common multiple

## Key Objectives:

- Formalize that skip counting produces multiples, and that multiple lists are infinitely long.
- Review listing method to find LCM.
- Use prime factorization to find LCM.
- Use Venn diagrams to find LCM.


## Materials:

No extra materials needed.

## Pedagogical/Orchestration:

- The prime factorization method for finding LCM can be introduced to 6th grade, but most students are more comfortable using the listing method.
- 6th Grade: The Exploration after Example 4 may be challenging for 6th grade but would be good as an extension.
- Use word problems, such as the hot dog problem, as application of LCM in real life situations.


## Activity:

"Factoring with Venn Diagrams" at the end of the section and on CD (needs computer and internet)

The word problems may require guidance from the teacher for set up.

## Vocabulary:

multiple, common multiple, least common multiple (LCM)
TEKS:
6.1(F);
6.12(A);
6:13(B); 7.13(A)(D);
$7.14(A)$
7.15(A)(B);
8.14(A);
8.15(A);
8.16(A)

## WARM-UPS for Section 8.3

1. Two numbers are relatively prime if their greatest common factor, GCF, is 1 . Which of the following integer pairs are not relatively prime? Explain.
a. $(17,19)$
b. $(25,27)$
C. $(84,87)$
d. $(127,128)$

Ans: (c) because both 84 and 87 are divisible by 3 .
2. Spot's Barkery specializes in homemade organic dog biscuits. They also create gift baskets for their furry friends. They have 48 luscious liver biscuits, 36 chewy cheese biscuits and 24 tasty turkey biscuits.
a. What is the maximum number of gift baskets that can be made if each basket has the same number of treats and all of the treats get used? Ans: 12 baskets
b. How many of each flavor is in each basket? Ans: $\mathbf{2}$ turkey, $\mathbf{3}$ cheese, and 4 liver

## Launch for Section 8.3:

Ask students if any of them have ever volunteered at an animal shelter. If none have, ask if any would like to volunteer. Select two students as your volunteers and have them sit in chairs in front of the room facing the class. Use the students' own names: the names Jose and Kerry will be used for this example. During the summer, Jose goes to the animal shelter every three days, and Kerry goes every five days. If they just saw each other today (day 0), when is the next time the two volunteers will meet at the shelter? Tell the volunteers that you will be counting days and they will need to stand every time it is his or her turn to go to the shelter. Tell the class to keep track of the days each volunteer goes to the shelter by making a list. Start counting slowly, and at 3, Jose should stand and then sit down. At 5, Kerry should stand and then sit down. This should continue until at 15, for the first time, Jose and Kerry stand together. Make a note of this to the class and continue counting at least until 30. Ask a student from the class to write on the board, the days that Jose visited the shelter, and another student to write the days that Kerry visited the shelter. Ask the class if they know a name for all the numbers in Jose's list. If they do not know, tell them the numbers in Jose's list are the multiples of 3 . Then ask about the numbers in Kerry's list, and they should be able to tell you those are multiples of 5 . Circle the 15 that is in both lists, and ask what the number 15 would represent. Students may remember that this is called a common multiple. Ask students what other multiples the two lists would have in common and if they can see a pattern. Students may notice that these are the multiples of 15. Remind students that the numbers 3 and 5 are called factors of 15 . Tell students that 15 , which is the smallest of the common multiples, is a special number we call the Least Common Multiple, and that it is used in working with fractions. Tell students, "Today we will be building on what we know about multiples."

## SECTION 8.3 COMMON MULTIPLES AND THE LCM

Have you noticed that hot dogs often come in packages of eight, and hot dog buns come in packages of twelve? When people plan to cook hot dogs, they tend to buy one package of hot dogs and one package of buns. But if they do this, they are left with four extra buns.


Some people who pay for the extra hot dog buns don't want to waste them. What can they do? They could buy another package of eight hot dogs:


But now there are four extra hot dogs without buns. If they buy more buns:


There are eight buns without hot dogs, even more extra buns than the first time. Will this process ever end? Try buying one more package of hot dogs:


Aha! We have finally reached a point where we have exactly the same number of hot dogs and buns. Of course, in order to get there, the consumers had to buy two packages of buns and three packages of hot dogs.

What happened mathematically with the hot dogs and buns? One way to organize the number of hot dogs and the number of buns is

$$
\begin{array}{lcccccccccc}
\text { Hot dogs: } & 0 & 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 \ldots \\
\text { Buns: } & 0 & 12 & 24 & 36 & 48 & 60 & 72 \ldots
\end{array}
$$

Look for some positive integer $N$ where anyone could buy exactly $N$ hot dogs and $N$ buns. Finding the number gives the consumer the number of hot dogs and the number of buns to buy so that they come out even.

It is easy to see that 24 is the smallest positive integer that is in both lists. So, the smallest value of $N$ for both items is 24 . Notice that the numbers in the first list are the multiples of 8 . This makes sense, because it is only possible to buy 8 hot dogs at a time. The numbers in the second list are the multiples of 12 , because buns come only in packages of 12 . Thus, 24 is the smallest positive integer that is a multiple of both 8 and 12. Mathematicians have a term for this:

## DEFINITION 8.4: COMMON MULTIPLE AND LCM

Integers $a$ and $b$ are positive. An integer $m$ is a common multiple of $a$ and $b$ if $m$ is a multiple of both $a$ and $b$. The least common multiple, or LCM, of $a$ and $b$ is the smallest integer that is a common multiple of $a$ and $b$. We write the LCM of $a$ and $b$ as LCM $(a, b)$.

Notice that 48 and 72 are also common multiples of 8 and 12 but not the least.

As we found with the GCF, there are several ways to find the LCM of two numbers. Try some examples:

## EXAMPLE 1

Find the LCM of 5 and 7.

## EXAMPLE 1

Remind students that multiples are infinitely long, so if a common multiple does not appear with the first 10 multiples, the lists should be expanded.

## SOLUTION

One way to find the least common multiple of two numbers is to list the positive multiples of each number in increasing order until you find an integer that is in both lists. For example, start by writing the first fourteen positive multiples of 5 and the first 10 positive multiples of 7 :

Multiples of 5 :

$$
5,10,15,20,25,30,35,40,45,50,55,60,65,70, \ldots
$$

Multiples of 7:

$$
7,14,21,28,35,42,49,56,63,70, \ldots
$$

Notice that 35 is in both lists. 70 is also on both lists. In fact, if we continued listing the multiples, we would find other common multiples of 5 and 7 . However, the smallest positive integer that is a multiple of both 5 and 7 is 35 . That means 35 is the LCM of 5 and 7.

As you have seen, with very large numbers, it might be necessary to write many multiples to find their LCM. This can be very time-consuming, so it would be better to have a more efficient method for computing the LCM of two numbers, just as you did when looking for the GCF. Thankfully, prime factorization comes to the rescue again.

## EXAMPLE 2

Find the LCM of 54 and 63.

## EXAMPLE 2

The prime factorization method for finding the LCM is a more efficient method, but most students are more comfortable using the listing method.

## SOLUTION

First, factor 54 and 63, using prime factorization:


The prime factorizations are $54=2 \cdot 3^{3}$, and $63=3^{2} \cdot 7$. Again, it is useful to express the prime factors with the exponents, aligning like factors:

$$
\begin{aligned}
& 54=2 \cdot 3^{3}=2 \cdot 3 \cdot 3 \cdot 3 \\
& 63=3^{2} \cdot 7=3 \cdot 3 \cdot 7
\end{aligned}
$$

To find a number that is a multiple of both 54 and 63 , the multiple must have the prime building blocks that both 54 and 63 have. Before continuing, consider how you might be able to construct a multiple of 54 and 63 using the prime factors for each of the numbers. How can you construct the smallest such multiple?

Now, look at the method for finding the LCM of numbers using their prime factorization. By examining the two sets of prime factors, you can see that a common multiple must include 2,3 , and 7 . However, $2 \cdot 3 \cdot 7=42$ is not a multiple of 54 or 63 . Because 54 has three factors of 3 and 63 has two factors of 3 , a common multiple must have three factors of 3 . Why won't two factors of 3 be enough? Now, multiply the factors $2,3^{3}$, and 7 .

$$
2 \cdot 3^{3} \cdot 7=378
$$

## EXPLORATION

Have your students rehearse the rule for finding the GCF from Section 8.2 and compare it to the rule for finding the LCM using prime factorization.
$\operatorname{LCM}(m, n)=2^{6} \cdot 3^{5} \cdot 5^{4} \cdot 7^{2} \cdot 11^{3} \cdot 13^{2} \cdot 17^{2}=(23 \cdot 35 \cdot 5 \cdot 72 \cdot 113 \cdot 172)(23 \cdot 53 \cdot 132)=(\mathrm{m})(23 \cdot 53 \cdot 132)$
(a multiple of m)
and
$26 \cdot 35 \cdot 54 \cdot 72 \cdot 113 \cdot 132 \cdot 172=(26 \cdot 3 \cdot 54 \cdot 7 \cdot 132 \cdot 17)(34 \cdot 7 \cdot 113 \cdot 17)=(n)(34 \cdot 7 \cdot 113 \cdot 17)$
(a multiple of $n$ )
The GCF is the product of all the common primes raised to the lowest in either factorization. The LCM is the product of all the primes that occur, raised to the highest exponent of each factor. This is given as a rule at the end of the section. The Investigation in Exercise 12 will show that the product $m \cdot n=\operatorname{LCM}(m, n) \cdot \operatorname{GCF}(m, n)$. Be prepared if a student notices this pattern.

Notice that $378=2 \cdot 3 \cdot 3 \cdot 3 \cdot 7=(2 \cdot 3 \cdot 3 \cdot 3) \cdot 7=54 \cdot 7$ which is a multiple of 54 . Also, $378=2 \cdot 3 \cdot 3 \cdot 3 \cdot 7=(2 \cdot 3) \cdot(3 \cdot 3 \cdot 7)=6 \cdot 63$ which is a multiple of 63 .
This is the smallest integer that contains all of the building blocks or prime factors in both sets of prime factors. Therefore, 378 is the LCM of 54 and 63.

## EXPLORATION 1

Describe a method for finding the LCM of two numbers using their prime decompositions. Use it to find the LCM for $m$ and $n$ without computing the two numbers.

$$
\begin{aligned}
m & =2^{3} \cdot 3^{5} \cdot 5 \cdot 7^{2} \cdot 11^{3} \cdot 17^{2} \\
n & =2^{6} \cdot 3 \cdot 5^{4} \cdot 7 \cdot 13^{2} \cdot 17
\end{aligned}
$$

Another approach to solving for the LCM is to look at the Venn diagram, as you did in Section 8.1, when you worked with GCFs. Examine the prime factors of 6 and 8 . The Venn diagram includes the prime factors for each number in the respective circles. Note the common factors in the overlapping part of the circles. This is the GCF of 6 and 8 . Note that the product of $6 \cdot 8=(2 \cdot 3)(2 \cdot 2 \cdot 2)$ is a common multiple of 6 and 8 .


But it is the least common multiple. A multiple of 6 requires a factor of 2 and 3 ; a multiple of 8 requires 2,2 , and 2 . So, a multiple of 6 and 8 requires one 3 and three 2 s as factors. The shortest list of factors that will produce a multiple of both is $2 \cdot 2 \cdot 2 \cdot 3$. The greatest common factor is 2 and we avoid double counting it in finding the LCM. To compute the LCM $\{6,8\}$, you compute the product of all the prime factors in the Venn Diagram, $3 \cdot 2 \cdot 2 \cdot 2$, which avoids using factors in the overlapping part twice. Thus, to get the LCM of 6 and 8 , take the product of the highest power of all the factors that occur in either number, that is $2^{3} \cdot 3$, to get the LCM of 24 . Remember to use the highest power of any prime for the LCM just as you used the lowest power of any common prime for the GCF.

Have your students rehearse the rule for finding the GCF from Section 8.1 and compare it to the rule for finding the LCM using prime factorizations. Namely, for the GCF: the product of all the common primes raised to the lowest exponent that appears in either factorization. LCM: the product of all the primes that occur, raised to the highest exponent that appears in either factorization. The Investigation in Exercise 15 will show the very nice relationship that the product $m \cdot n=\operatorname{LCM}(m, n) \cdot G C F(m, n)$. Don't try to get them to see this yet, but be prepared if a student notices this pattern.

You will find that the Venn diagram is more practical when the numbers are larger and there are three numbers or more. For example, to find the least common multiple of 70,36 , and 60 , write all the prime factors and notice which factors are common.

$$
\begin{aligned}
& 70=5 \cdot 7 \cdot 2 \\
& 36=2^{2} \cdot 3^{2} \\
& 60=2^{2} \cdot 3 \cdot 5
\end{aligned}
$$

We can represent this information as a Venn diagram:


Both 70 and 60 have a common factor of 5 , so 5 is the intersection of the 70 circle and the 60 circle, but not the 36 circle. The factors $2^{2}$ and 3 are in both 60 and 36 , so $2^{2} \cdot 3$ is the intersection of the 60 circle and the 36 circle. Factor common to 70 and 36 is 2

After you have separated the factors into the different regions in the Venn diagram by multiplying all the numbers in the circles and their intersections, you will have the LCM of 70,36 , and 60 . Did you get 1260 ? If so, you are correct.

To summarize these investigations, the LCM of two integers $a$ and $b$ is equal to the product of all the primes that occur in the prime factorizations, raised to the highest exponent that appears in either factorization.

## EXERCISES

1. Make sure your students show enough work.
2. Make sure your students show enough work.
3. Yes, the pairs of integers in the previous exercises that are relatively prime: $3 \& 11$ and $13 \& 7$. Answers will vary as to how far you go using this method to find whether or not two integers are relatively prime. But, the idea is using prime factorization is more efficient. If $p$ and $q$ are relatively prime, then their LCM is their product, pq.

The GCF is useful in finding the simplest equivalent fraction. The LCM can also be useful when working with fractions. For example, consider the fractions $\frac{1}{6}$ and $\frac{3}{8}$. Find equivalent fractions for both fractions that have the same denominator. How might this information help you to find equivalent fractions with a common denominator? The LCM is useful in adding and subtracting fractions.

## EXERCISES

1. Find the LCM of the following pairs of numbers by listing multiples of the two numbers until you find the first multiple common to both lists.
a. 4 and 612
b. 5 and 1010
c. 5 and 630
d. 3 and 1133
e. $\quad 10$ and $14 \quad 70$
f. $\quad 13$ and 791
2. Find the LCM of the given numbers either by listing multiples of the two numbers until you find the first multiple common to both lists or by using the prime factorization of each number.
a. 15 and $18 \quad 90$
d. 30 and $20 \quad 60$
b. 32 and 20160
e. 120 and 16240
c. 5, 6, and 7
f. 8,9 , and 12
3. Were there any pairs of integers in the previous exercise where the integers were relatively prime? Explain whether you have to check every prime factor of both integers to find that the two integers are relatively prime. Based on the evidence in the previous exercises, if $p$ and $q$ are different prime numbers, what is the LCM of $p$ and $q$ ? Use what you know about the common factor of two relatively prime numbers to explain why your previous answer makes sense. See TE.

If you have not discussed the word conjecture in class yet, this would be a perfect opportunity to do so.
4. e. When one number is a multiple of the other, the larger is the LCM.
5. a. $\mathrm{GCF}=36$ and $\mathrm{LCM}=540$
c. $\mathrm{GCF}=10$ and $\mathrm{LCM}=880$
b. $\mathrm{GCF}=1$ and $\mathrm{LCM}=5544$
d. $\mathrm{GCF}=125$ and $\mathrm{LCM}=5000$
6. They will ride together again 35 days after the initial Sunday morning they start (since 35 is divisible by 7 , the next time they bike together will also be a Sunday, but this is extra information and not part of the original question - yet an interesting question for students to ponder).
7. If Teresa and Vanessa carpool on June 18th, the next two days they will carpool are July 6th and July 24th. If Vanessa waited ten days instead of nine, then the next time they would go swimming is July 18th.
This is more challenging because your students will need to use a calendar. You might remind them that working with a calendar involves a variation of clock arithmetic, only the clock changes depending on the month of the year.
8. You can buy 12 juice boxes and breakfast bars. You will have to buy 2 packs of juice boxes and 3 packs of breakfast bars in order to do this.
9. She can reward 40 students without having any pencils or erasers left over. She will buy 4 packs of pencils and 5 packs of erasers.
4. Find the LCM for each of the following pairs of numbers:
a. $\quad 12$ and $60 \quad 60$
b. 6 and 7272
c. 45 and 1545
d. $p q$ and $p q^{2} \quad p q^{2}$
e. Look for a pattern in computing the LCM for a-c. Make a conjecture that explains this pattern. See TE.
5. For each pair of integers below, use prime factorization to find the GCF and LCM. Check student work for prime factorizations.
a. 108 and 180
b. 77 and 72
c. $\quad 80$ and 110
d. $\quad 1000$ and 625
6. Terry rides his bike every 5 days, and Max rides his bike every 7 days. If they start at the same time and same place on a Sunday morning, how many days will it be before they ride together again?
7. Teresa and Vanessa like to go swimming every few days to keep in shape. Each time Teresa goes swimming, she waits exactly six days before swimming again. For example, if she goes swimming on a Monday, the next time she goes swimming is Sunday. Each time Vanessa goes swimming, she waits exactly nine days before swimming again. Each time the two go swimming on the same day, they carpool to the swimming pool. If Teresa and Vanessa carpool on June 18 , what will be the next two days they carpool? What date will they next carpool if Vanessa waits ten, rather than nine, days every time she goes swimming? See TE.
8. At Happy Days Day Care, the director will give one juice box and one blueberry breakfast bar as a morning snack to each child. Juice comes in packs of 6 and breakfast bars in packs of 4. If she wants to buy enough so that there are the same number of juice boxes and breakfast bars with none left over, what is the fewest number of each she will have to buy? What is the number of packs of each she will have to buy?
9. Mrs. Tolento wants to give pencils and erasers as gifts to her students. Pencils come in packs of 10 and erasers in packs of 8 . How many students can she reward with exactly one pencil and one eraser with none left over? How many packages of each will she need to buy?
10. Encourage your students to be careful when setting up this problem. They need to read carefully and notice that Randy counts as one of the "eaters" in this problem. This is a good time to make a connection between the concept of LCM and Exercise 8.1.7.

## Investigation

11. $\operatorname{GCF}(15,24,36)=3$

$$
\operatorname{LCM}(15,24,36)=2^{3} \cdot 3^{2} \cdot 5
$$


12. $\operatorname{LCM}$ of $\{(23 \times(1,000,000), 37 \times(1,000,000)\}$ is $(23)(37)(1000000)$
13. You must buy 4 packs of invitations and 2 packages of party favors in order to have 48 of each. This is the least number you can buy to have none left over and only one for each person.
14. What we hope here is that your students will discover that $\mathrm{GCF} \cdot \mathrm{LCM}=m \cdot n$.
10. Randy is baking cookies for a family get-together at his house. He wants to bake enough cookies so that each person at the party, including himself, gets the same number of cookies. Randy knows that there are seven family members who will definitely come to the party. In addition, Randy has an aunt, an uncle, and two cousins who might or might not come. Because they are in the same family, either all four of these people will show up, or none of them will. What is the smallest number of cookies that Randy can cook if he wants to guarantee that everyone gets the same number of cookies, with no leftovers? 24 cookies
11. Use a Venn diagram to find the GCF and LCM for 15,24 , and 36 .
12. What is the LCM of $23,000,000$ and $37,000,000$ ? $851,000,000$
13. You are planning a party and want to invite 48 people. You need to buy invitations and party favors. Invitations come in packages of 12, and party favors come in packages of 24 . What is the least number of packs of invitations and party favors you should buy to have one for each person and none left over?

## 14. Investigation:

The table below lists two variables $m$ and $n$. Copy and fill out this table. In the column labeled $\operatorname{GCF}(m, n)$, write the GCF of $m$ and $n$. In the column labeled LCM $(m, n)$, write the LCM of $m$ and $n$. As you fill out the table, what do you notice?

| $m$ | $n$ | GCF $(m, n)$ | LCM $(m, n)$ |
| :---: | :---: | :---: | :---: |
| 3 | 7 | 1 | 21 |
| 4 | 6 | 2 | 12 |
| 5 | 15 | 5 | 15 |
| 8 | 15 | 1 | 120 |
| 10 | 36 | 2 | 180 |
| 20 | 48 | 4 | 240 |
| 30 | 50 | 10 | 150 |
| 30 | 65 | 5 | 390 |
| 77 | 81 | 1 | 6237 |
| 96 | 100 | 4 | 2400 |

15. a. Answers will vary among any of the following pairs of numbers: $(1,4),(2,4),(4,4)$. There are 3 such ordered pairs.
b. Answers will vary among any of the following pairs of numbers: $(1,12),(2,12),(3,12),(4,12),(6,12)$, $(12,12),(3,4),(4,6)$. There are 8 such ordered pairs.
c. Answers will vary among any of the following pairs of numbers: $(1,120),(2,120),(3,120),(4,120),(5,120)$, $(6,120),(8,120),(10,120),(12,120),(15,120),(20,120),(24,120),(30,120),(40,120),(60,120),(120$, $120),(3,40),(6,40),(10,24),(12,40),(8,15),(8,30),(8,60),(5,24),(40,60),(30,40),(15,40),(15,24)$, $(20,24),(24,30),(24,40),(24,60)$. There are 32 such ordered pairs.

An example of how to approach part (b):
Students may approach this problem by trying to make an exhaustive list, but this can often lead to missing answers. Encourage students to use prime factorization and to choose which factors to include in the pair of numbers to make a LCM of 4,12 , or 120. Let us look more closely at part (b). We know that the prime factorization for 12 is $2^{2} 3$, so one could think about factors of 12 in the following way:
Each factor can be represented as such: _ - _ where the first slot may or may not contain one or two 2 s and the third slot may or may not contain a 3 (since we are looking for prime factors of 12 we break down into $2 s$ and 3 s , but if we are looking for another number we would have other slots, like 15 is composed of 3 and 5 as factors, so our slots would have $3 s$ and $5 s$ in them). All of the $(n, 12)$ pairs can be generated in this way, where $n=$ $\qquad$ -._.
Since each of the slots can have a particular factor or it cannot have a particular factor, you can think about the slots as lights that turn off and on, where the first slot has 3 different settings: high, low, and off. What are the different combinations of ways for the lights to be on? You can have both lights off (this represents $n=1$ ), the first light can be on low and the second off ( $n=2$ ), or the first light can be on high and the second off ( $n=4$ ). You can see the results in the following table:
After students have discovered all of the pairs of the form ( $n, 12$ ), they can use a similar method with what they know about LCMs to find all of the other pairs.

| fir i s t <br> light | second <br> light | \# of 2s | \# of 3s | $n$ |
| :--- | :--- | :--- | :--- | :--- |
| off | off | 0 | 0 | 1 |
| low | off | 1 | 0 | 2 |
| high | off | 2 | 0 | 4 |
| off | on | 0 | 1 | 3 |
| low | on | 1 | 1 | 6 |
| high | on | 2 | 1 | 12 |

15. Ingenuity:
a. Find a pair of numbers $m$ and $n$ for which $\operatorname{LCM}(m, n)$ is 4 . How many such pairs $(m, n)$ are there? See TE.
b. Find a pair of numbers $m$ and $n$ for which $\operatorname{LCM}(m, n)$ is 12 . How many such pairs $(m, n)$ are there? See TE.
c. Find a pair of numbers $m$ and $n$ for which $\operatorname{LCM}(m, n)$ is 120 . How many such pairs $(m, n)$ are there? See TE.

## FACTORING WITH VENN DIAGRAMS

Objective: The students will practice the skills learned in Sections 8.2 and 8.3 to find the prime factorization of numbers, categorize these factors in a Venn diagram, and use this Venn diagram to find LCMs and GCFs.

## Materials:

Computer with internet
Notebook paper

## Activity Instructions:

Direct students to the following website and have them play several rounds of the activity. This activity will work best if you can provide one computer per student. If, however, you don't have enough computers for all of your students, you can do this as a class activity if you have a way to project your computer screen to the whole class. If you are doing the whole class activity, you can break the students into groups and have one group at a time come up and work a problem. If they get it correct, that team gets a point. It is important, however, that all students get a chance to work at least one problem on their own.

## nlvm.usu.edu/en/nav/frames_asid_202_g_3_t_1.html

Once your students are at the correct website, direct them to the bottom of the page and ask them to click on "two" for the number of trees and "computer" for the problems. One problem will be in blue and the other will be in yellow. They find prime factors for each of these numbers until the factor trees are complete. Once both trees are complete and correct, the program will automatically display a Venn diagram. The students then click and drag the prime factors from the trees into the appropriate areas of the Venn diagram.

After the Venn diagram is complete and correct, the program prompts the students to fill in the LCM and GCF of the two numbers. Once they think they have found them, they should click the button to check. The computer lets them know if they are correct. If the student is incorrect, the computer lets them know where to look for their mistake.

It would be best if the students kept a written log of the original numbers and their final answers. Decide on a set number of problems for them to complete in a certain amount of time. The students can turn in their written work to show you what they have accomplished.

## Section 8.4 - Addition and Subtraction of Fractions

## Big Idea:

Developing addition and subtraction of fractions with common and uncommon denominators.

## Key Objectives:

- Use area and linear models to visualize addition and subtraction of fractions and to develop a method for adding and subtracting fractions.
- Discover the advantage of using the LCM for the common denominator.
- Solve real-life application problems using addition and subtraction of fractions.


## Materials:

Grid paper, Picture of ruler for demonstration

## Pedagogical/Orchestration:

- This section reviews adding and subtracting fractions using the area model. Some of your students may find grid paper to be useful. For instance, in Example 2, a 2 by 3 rectangle on a grid is chosen as the whole and used to help students add $\frac{1}{2}$ and $\frac{1}{3}$.
- Do not shy away from the fractions that have variables in the denominators found throughout the section. Dealing with variables in the denominator is actually easier in many cases than dealing with constants, and can help the students develop a strategy for finding the least common denominator.


## Activity:

"Spinning for LCD" and "Pattern Block Fun" at the end of the section and on CD to reinforce addition and subtraction with fractions at end of Section 9.1.

If pattern blocks are not available, use blackline master copy of pattern blocks.

## Exercises:

Exercise 9 connects to measurement.

## Vocabulary:

common denominator, least common denominator

## TEKS:

6.2(A); 7.2(A)(B)(F);
7.13(A)(C);
7.15(A)(B);
8.2(A,B);
8.14(A);
8.16(A)

## WARM-UPS for Section 8.4

1. A square has perimeter 17.2 cms . Which of the following best describes the area of the square?
a. between 17 and 17.5 square cms
b. between 17.5 and 18 square cms
c. between 18 and 18.5 square cms
d. between 18.5 and 19 square cms

## Ans: (c) The area is (4.3)(4.3) $=18.49 \mathrm{sq} \mathrm{cms}$

2. Mrs. Jones had baked a cake. Each of her children cut and ate a piece of cake. Jack says he ate $\frac{1}{3}$ of the cake. Marissa says she ate $\frac{1}{4}$ of the cake. Alex says he ate $\frac{1}{6}$ of the cake. Mrs. Jones finds that $\frac{1}{8}$ of the cake is remaining. If she knows nobody else ate a piece, are her children telling the truth? Explain your answer.
Ans: the sum $\frac{1}{3}+\frac{1}{4}+\frac{1}{6}=\frac{4}{12}+\frac{3}{12}+\frac{2}{12}=\frac{3}{4}$. So there should be $1-\frac{3}{4}=\frac{1}{4}$ cake left. Someone is not telling the truth.

## Launch for Section 8.4:

Tell your students that Greg and Martha are baking cookies for their class, and need 8 eggs to do so. Greg has $\frac{1}{4}$ of a carton of eggs and Martha has $\frac{1}{3}$ of a carton of eggs. A carton holds a dozen eggs. Ask the students, "Do Greg and Martha have enough eggs to bake the cookies?" Have students draw a picture that will demonstrate this problem. Ask them, "How can you use fractions to solve this problem?" Let students share their various strategies and make sure to highlight any strategies that involve drawing a picture so that $\frac{1}{4}$ of a carton is represented as 3 out of 12 eggs, and $\frac{1}{3}$ of a carton is represented as 4 out of 12 eggs. Greg and Martha need 8 eggs but only have 7. Ask students to think about what strategies they used that help make adding $\frac{1}{4}$ to $\frac{1}{3}$ easier. Tell them these strategies will be used throughout this lesson in learning how to add and subtract fractions.

## PROBLEM 1

Students may use the number line or compute $\frac{3}{8}+\frac{2}{8}=\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{8}+\frac{1}{8}\right)=\frac{5}{8}$.

Notice that we don't warn kids to think about whether the denominators have to be the same. Make sure that they discover that you can only directly add the numerators when the denominators are the same.

## SECTION 8.4 ADDITION AND SUBTRACTION OF FRACTIONS

Adding 1 foot to 2 feet equals 3 feet. Combining 1 apple with 2 apples gives 3 apples. In each case, both numbers and units are important. Given these two examples, it seems reasonable to say that the sum of 1 fifth and 2 fifths is 3 fifths. More precisely, in Section 8.2, the linear skip counting model demonstrated that $\frac{3}{5}$ is $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$. Using skip counting, it is easy to see that

$$
\frac{2}{5}+\frac{1}{5}=\left(\frac{1}{5}+\frac{1}{5}\right)+\frac{1}{5}=\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{3}{5} .
$$

In general, for each positive integer $m$ and $n$, the fraction $\frac{m}{n}$ is the sum of $m$ unit fractions in the form $\frac{1}{n}$. For the rest of this chapter, assume all possible denominators are positive integers.

## PROBLEM 1

Compute the sum of $\frac{3}{8}$ and $\frac{2}{8}$. Explain your answer.

Now look at the area model. How is the sum $\frac{1}{5}+\frac{2}{5}$ computed using the area model? Use a candy bar model. Betsy had $\frac{1}{5}$ of a candy bar, and her friend had $\frac{2}{5}$ of a candy bar like Betsy's.


Together, they have $\frac{3}{5}$ of a candy bar. Express this as $\frac{1}{5}+\frac{2}{5}=\frac{3}{5}$.
Write rules to generalize the previous discussion of adding fractions.

## PROBLEM 2

$\frac{(7-4)}{9}=\frac{3}{9}=\frac{1}{3}$

## EXAMPLE 1

The problem can be modeled by subtraction as $1-\frac{2}{3}=\frac{1}{3}$.

## RULE 8.1: SUMS WITH LIKE DENOMINATORS

The sum of two fractions with like denominators, $\frac{a}{n}$ and $\frac{b}{n}$ , is given by

$$
\frac{a}{n}+\frac{b}{n}=\frac{a+b}{n}
$$

The same principle applies when subtracting fractions.

## PROBLEM 2

Compute $\frac{7}{9}-\frac{4}{9}$ and explain how to obtain the answer.

Describe how to subtract fractions with like denominators. What is the difference $\frac{a}{n}-\frac{b}{n}$ ? How does your method compare to the addition rule above?

## EXAMPLE 1

If you eat $\frac{2}{3}$ of a candy bar, how much of the candy bar is left? How can you use subtraction of fractions to answer this question?

## SOLUTION

Using mathematical fractions, the problem looks like this: $1-\frac{2}{3}$. In order to perform this calculation, begin by drawing a picture of a candy bar and divide it into three pieces. First convert 1 into the fraction $\frac{3}{3}$ :

$$
1=\frac{3}{3}=\begin{array}{|l|l|l|}
\hline & & \\
\hline
\end{array}
$$

## PROBLEM 3

Students might struggle with the 2 . The 2 can be written as $1+\frac{4}{4}$ or $\frac{8}{4}$. The first way will lead to the answer $1 \frac{3}{4}$ and the second to the answer $\frac{7}{4}$.

What happens when you subtract $\frac{2}{3}$ of the candy bar? Shading the portions that are subtracted, the difference is

$$
1-\frac{2}{3}=\frac{1}{3}=\square \begin{array}{|l|l|}
\hline & \\
\hline
\end{array}
$$

Write this as $1-\frac{2}{3}=\frac{3}{3}-\frac{2}{3}=\frac{1}{3}$.

Another way to think of this subtraction uses the linear model. Like the model for subtracting integers, move 1 unit to the right and then back up a distance of $\frac{2}{3}$ to land on the number $\frac{1}{3}$. This model represents $1-\frac{2}{3}$.


For integers, the subtraction problem $n-m$ is equivalent to the addition problem $n+(-m)$. Similarly, the subtraction problem $1-\frac{2}{3}$ is equivalent to the addition problem $1+\left(-\frac{2}{3}\right)$. What is the car model picture for this addition problem?

## PROBLEM 3

Compute the difference $2-\frac{1}{4}$ and illustrate the process with either the area model or the linear model.

Now that you've explored adding and subtracting fractions with like denominators, explore finding the sum of an integer and a fraction, like the addition problem $2+\frac{3}{5}$. Applying a similar process for this sum as you did for the last sum, there are several ways to combine these two quantities:

$$
\begin{aligned}
& 2+\frac{3}{5}=1+1+\frac{3}{5}=1+\frac{5}{5}+\frac{3}{5}=1+\frac{8}{5}, \text { or } \\
& 2+\frac{3}{5}=\frac{10}{5}+\frac{3}{5}=\frac{13}{5} \text { or } \\
& 2+\frac{3}{5}=2 \frac{3}{5} .
\end{aligned}
$$

The first method trades one addition problem for another addition problem. The second results in an answer that is an improper fraction. The third way results in the mixed number $2 \frac{3}{5}$. In Section 8.2 , you learned how to convert mixed numbers to improper fractions and vice versa. Now you see that a mixed number can also be thought of as a sum.

## EXAMPLE 2

Explore how to use the ideas just learned to compute the sum of two fractions when the denominators are not the same.
Use the area model to compute the sum $\frac{1}{2}+\frac{1}{3}$.

## SOLUTION

Begin by looking at a visual representation.


Notice that each outer rectangle is the same size because they represent the whole or 1 . Also notice that one fraction is modeled with a horizontal cut and the other with a vertical cut. Why is this helpful?

## EXPLORATION

The computation should be: $\frac{1}{3}+\frac{1}{4}=\frac{1(4)}{3(4)}+\frac{1(3)}{4(3)}=\frac{4}{12}+\frac{3}{12}=\frac{7}{12}$.
The general process: $\frac{1}{a}+\frac{1}{b}=\frac{1(b)}{a(b)}+\frac{1(a)}{b(a)}=\frac{b}{a b}+\frac{a}{a b}=\frac{b+a}{a b}$.
Teacher, encourage your students to derive a general process for adding all fractions, a general $\frac{a}{b}+\frac{c}{d}$.
The process: $\frac{a}{b}+\frac{c}{d}=\frac{a(d)}{b(d)}+\frac{c(b)}{d(b)}=\frac{a d}{b d}+\frac{c b}{b d}=\frac{a d+c b}{b d}$.
The rule is $\frac{a}{b}+\frac{c}{d}=\frac{a d+c b}{b d}$.

Is it possible to combine the shaded amounts? Modify the picture above to display equivalent divisions of the whole.


To do this, divide the first model horizontally to represent $\frac{1}{2}$ as 3 parts out of 6 parts. Then, divide the second model vertically to represent $\frac{1}{3}$ as 2 parts out of 6 parts. It is easy to see from the model that $\frac{1}{2}=\frac{3}{6}$ and $\frac{1}{3}=\frac{2}{6}$ It is also easy to see how to add the two fractions in their equivalent forms. Using the rule for adding fractions with like denominators, the sum is

$$
\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}
$$

In order to add the fractions, find a common-sized piece so that the two fractions can be written with the same or common denominator.

Using the equivalent fractions property, transform the two fractions to fractions with the same denominator.

$$
\frac{1}{2}=\frac{1 \cdot 3}{2 \cdot 3}=\frac{3}{6} \text { and } \frac{1}{3}=\frac{1 \cdot 2}{3 \cdot 2}=\frac{2}{6}
$$

The most important thing to remember when adding fractions is to ensure that you have a common denominator.

## EXPLORATION

Compute the sum $\frac{1}{3}+\frac{1}{4}$ by first using the area model and then the equivalent fractions property to convert the fractions into equivalent fractions with like denominators.
Find the pattern to add the fractions $\frac{1}{\mathrm{a}}$ and $\frac{1}{\mathrm{~b}}$ and show the process.

## PROBLEM 4

Some possible common denominators are 24,36 and 12 . They are all common multiples of 6 and 4 . We could use $4 \cdot 6=24$ as the common denominator: $\frac{1}{6}+\frac{1}{4}=\frac{1(4)}{6(4)}+\frac{1(6)}{4(6)}=\frac{4}{24}+\frac{6}{24}=\frac{10}{24}$. Watch for any student who sees a better strategy than just multiplying the two denominators to get a "common denominator." By using 12, we get $\frac{1}{6}+\frac{1}{4}=\frac{1(2)}{6(2)}+\frac{1(3)}{4(3)}=\frac{2}{12}+\frac{3}{12}=\frac{5}{12}$, which is equivalent to $\frac{10}{24}$. Don't bring up the idea of LCD or LCM yet, but members of the class might make the connection to LCM. Let them work the next examples before formalizing the idea of LCD.

Teachers, don't worry if kids don't see this right away. Do not force the issue about least common denominators and adding fractions. This is covered more thoroughly in the text later.

## PROBLEM 5

Have the class members share their work. Reflect with the class about each of the common multiples or common denominators they used. Students who make lists of multiples of denominators will use the LCM while others will just multiply the two denominators. Compare the two methods on the board and talk about the advantages or disadvantages for each. Using the LCM usually leads to an answer that is already in simplified form. The other method, often leads to an answer that needs to be simplified. After students see the advantage of using the LCM, define Least Common Denominator.
a. $\frac{7}{36}$
b. $\frac{19}{24}$
c. $\frac{31}{36}$

An example where using the LCD does not yield a simplified form is $\frac{1}{6}+\frac{1}{3}=\frac{1}{6}+\frac{2}{6}=\frac{3}{6}=\frac{1}{2}$.

## PROBLEM 4

Find three common denominators for the fractions $\frac{1}{6}$ and $\frac{1}{4}$. Write each fraction in equivalent forms using the three denominators. What do you notice about these common denominators? Which denominator would be the best choice for computing the sum $\frac{1}{6}+\frac{1}{4}$ ? Why?

Look at all the denominators you created in Problem 4. What is the relationship between each of the common denominators and the original denominators? For which choice of denominator did you not have to simplify? Now combine the discoveries about common denominators in Problem 4 with those about common multiples from Section 8.3.

## PROBLEM 5

For each of the following sums: (1) find a common multiple for both denominators, (2) use it to find equivalent fractions for each fraction, (3) compute their sum and (4) simplify your answer, if necessary.
a. $\frac{1}{9}+\frac{1}{12}$
b. $\frac{3}{8}+\frac{5}{12}$
C. $\frac{7}{12}-\frac{5}{18}$

## DEFINITION 8.5: LEAST COMMON DENOMINATOR

The least common denominator of the fractions $\frac{p}{n}$ and $\frac{k}{m}$ is the least common multiple of $n$ and $m$.

In adding or subtracting fractions, the LCM of the denominators produces the least common denominator or LCD.

Although it does not appear to be so, the placement of the negative sign in the first fraction is actually aligned with the fraction bar. While in the second fraction, the negative sign accompanies the numerator.

In Example 1, you saw that subtracting a fraction is equivalent to adding the negative of that fraction. What is the connection between $-\frac{2}{3}$ and $\frac{-2}{3}$ ? The number $-\frac{2}{3}$ is the opposite of $\frac{2}{3}$, that is, $\frac{2}{3}+-\frac{2}{3}=0$ and $-\frac{2}{3}$ is located the same distance from 0 and on the opposite side from $\frac{2}{3}$ :


What is $\frac{-2}{3}$ equivalent to and where is it located on the number line? First, think of $\frac{-2}{3}$ as the quotient $-2 \div 3$. Using the missing factor model, $\frac{-2}{3}=x$ means $3 \cdot x=-2$. The number $x$ must be negative because the product of 3 and $x$ is -2 . Using the frog method, if $x$ is the length of each jump and 3 is the number of jumps, the frog lands on -2 . To do that, the frog must divide -2 into 3 equal jumps:


As you can see from the number line above, the directed length of each jump is $-\frac{2}{3}$ . So by the missing factor model, the quotient $\frac{-2}{3}=-2 \div 3$ is the fraction $-\frac{2}{3}$. In general, if each of $a$ and $b$ is a positive number, then $-\frac{a}{b}=\frac{-a}{b}=\frac{a}{-b}$.
Now that you can plot rational numbers on a number line, you can also locate points in the coordinate plane with rational coordinates.

## PROBLEM 3

Locate the points below in the coordinate plane:
A. $\left(\frac{2}{3}, \frac{1}{2}\right)$
B. $\left(-\frac{2}{3}, \frac{1}{2}\right)$
C. $(5,0)$
D. $\left(-4,-10 \frac{1}{3}\right)$
E. $\left(3 \frac{1}{2},-3 \frac{1}{2}\right)$
F. $\left(-\frac{2}{3}, \frac{1}{4}\right)$

## EXERCISES

7th grade teachers should assign the following exercises:

1. a-h
2. a-i, n
3. 
4. $a, b, c, e$
5. 
6. 
7. 
8. a, d
9. h-i
10. 
11. Where appropriate, give the answer as a mixed number.
a. $\frac{6}{9}=\frac{2}{3}$
b. $\frac{4}{8}=\frac{1}{2}$
c. $\frac{10}{7}=1 \frac{3}{7}$
d. $\frac{5}{a}$
e. $\frac{3}{7}$
f. $\frac{8}{8}=1$
g. $2 \frac{4}{7}$
h. $\frac{5}{y}$
i. $\frac{6}{3}=2$
j. $\frac{15}{3}=5$
k. $\frac{2}{x}$
I. 0

Observe how students deal with parts (d), (h), (k) and (I) to check if they understand the process. You might need to review what a variable is when working these problems.

## EXERCISES

1. Locate the following numbers on the number line:

$$
\frac{2}{3}, \frac{1}{5}, 0-2, \frac{1}{2},-5 \frac{1}{3}
$$

2. Graph the points below.

$$
\begin{aligned}
& A=\left(2 \frac{1}{2}, 2 \frac{1}{2}\right) \\
& B=\left(-3 \frac{1}{3}, 5 \frac{1}{4}\right) \\
& C=\left(-3 \frac{1}{3},-5 \frac{1}{4}\right) \\
& D=\left(-2 \frac{1}{2}, 0\right) \\
& E=\left(4 \frac{1}{5}, 3 \frac{2}{5}\right)
\end{aligned}
$$

3. In the graph below, what are the coordinates of the points $A, B, C, D, E$.

4. Add or subtract the following fractions. Write your answers in simplest form.
a. $\frac{4}{9}+\frac{2}{9}$
b. $\frac{3}{8}+\frac{1}{8}$
c. $\frac{4}{7}+\frac{6}{7}$
d. $\frac{2}{a}+\frac{3}{a}$
e. $\quad 1-\frac{4}{7}$
f. $\frac{3}{8}+\frac{5}{8}$
g. $3-\frac{3}{7}$
h. $\frac{8}{y}-\frac{3}{y}$
i. $\frac{8}{3}-\frac{2}{3}$
j. $\frac{10}{3}+\frac{5}{3}$
k. $\frac{1}{x}+\frac{1}{x}$
5. $\frac{3}{2}-\frac{3}{2}$
6. a. $\frac{31}{35}$
f. $\frac{1}{42}$
k. $\frac{8}{15}$
b. $\frac{7}{8}$
g. $\frac{19}{24}$
l. $\frac{14}{24}=\frac{7}{12}$
C. $\frac{9}{20}$
h. $\frac{7}{42}=\frac{1}{6}$
m. $\frac{49}{90}$
d. $\frac{11}{14}$
i. $\frac{1}{24}$
n. $\frac{y+x}{x y}$
e. $\frac{13}{18}$
j. $\frac{43}{72}$
7. $\frac{2 b+3 a}{a b}$

Note that in parts ( n ) and ( 0 ) the LCM or LCD is the product of the denominators. Why?
3. a. $\frac{3 \cdot 7+2^{2} \cdot 5}{2^{3} \cdot 3^{2} \cdot 5 \cdot 7}=\frac{21+20}{2520}=\frac{41}{2520}$
b. $\frac{4 \cdot 5+8 \cdot 3 \cdot 7}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7}=\frac{20+168}{11225}=\frac{188}{11225}$
c. $\frac{7}{2 \cdot 3 \cdot 3 \cdot 5}+\frac{25}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 7}=\frac{7 \cdot 2^{2} \cdot 7+25 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7}=\frac{571}{2520}$
d. $\frac{99}{7 \cdot 100}-\frac{77}{45 \cdot 100}=\frac{99 \cdot 45-77 \cdot 7}{7 \cdot 5 \cdot 9 \cdot 100}=\frac{4455-539}{31500}=\frac{3916}{31500}=\frac{979}{7875}$
4. $\frac{2}{3}+\frac{1}{8}=\frac{19}{24}$ gallons of chocolate milk.
5.
a. $\frac{7}{3 x}$
b. $\frac{3}{4 a}$
c. $\frac{5 y+4 x}{20 x y}$
d. $\frac{7}{6 n}$
e. $\frac{1}{2 x}$
f. $\frac{3 y-2 x}{12 x y}=\frac{9 y-4 x}{24 x y}$
g. $\frac{3 x+2 x}{24}=\frac{5 x}{24}$
h. $\frac{4 \mathrm{~A}}{10}=\frac{2 \mathrm{~A}}{5}$
i. $\frac{4 a+3 b}{48}$
j. $\frac{3 a+c}{3 b}$
k. $\frac{6 x+25 x}{30}=\frac{31 x}{30}$
l. $\quad \frac{3 a-2}{12 b}=\frac{6 a-4}{24 b}$
m. $\frac{x^{2}+y^{2}}{x y}$
n. $\frac{y+3}{x y}$
0. $\frac{7+3 a}{a b}$

Notice the difference in how students find a LCD in Exercise 4, Part (0) and Exercise 2, Part (0). Ask students how each of these parts are different.
6. $\frac{1}{3}+\frac{1}{2}=\frac{5}{6}$ part that she gives away. The part she has left is $1-\frac{5}{6}=\frac{1}{6}$.
5. Compute the sums or differences. Write your answers in simplest form.
a. $\frac{2}{7}+\frac{3}{5}$
b. $\frac{1}{4}+\frac{5}{8}$
d. $\frac{5}{14}+\frac{3}{7}$
e. $\frac{8}{9}-\frac{1}{6}$
f. $\frac{1}{6}-\frac{1}{7}$
g. $\frac{3}{8}+\frac{5}{12}$
i. $\frac{1}{12}-\frac{1}{24}$
k. $\frac{7}{10}-\frac{1}{6}$
C. $\frac{1}{4}+\frac{1}{5}$
I. $\frac{5}{24}+\frac{3}{8}$
h. $\frac{8}{21}-\frac{3}{14}$
j. $\frac{2}{9}+\frac{3}{8}$
m. $\frac{5}{18}+\frac{4}{15}$
n. $\frac{1}{x}+\frac{1}{y}$
0. $\frac{2}{a}+\frac{3}{b}$
6. Compute the following sums or differences. Write your answers in simplest form.
a. $\frac{1}{2^{3} \cdot 3 \cdot 5}+\frac{1}{2 \cdot 3^{2} \cdot 7}$
b. $\frac{4}{3^{2} \cdot 5 \cdot 7^{2}}+\frac{8}{3 \cdot 5^{2} \cdot 7}$
c. $\frac{7}{90}+\frac{25}{168}$
d. $\frac{99}{700}-\frac{77}{4500}$
7. Jennifer mixes $\frac{2}{3}$ of a gallon of milk with $\frac{1}{8}$ of a gallon of chocolate syrup. How much chocolate milk will she make?
8. Find the sum or difference of the following algebraic fractions. Simplify your answer, if necessary.
a. $\frac{2}{x}+\frac{1}{3 x}$
b. $\frac{1}{4 a}+\frac{1}{2 a}$
c. $\frac{1}{4 x}+\frac{1}{5 y}$
d. $\frac{2}{3 n}+\frac{1}{2 n}$
e. $\frac{5}{6 x}-\frac{1}{3 x}$
f. $\frac{1}{4 x}-\frac{1}{6 y}$
g. $\frac{x}{8}+\frac{x}{12}$
h. $\frac{A}{2}-\frac{A}{10}$
k. $\frac{2 x}{10}+\frac{5 x}{6}$
I. $\frac{a}{4 b}-\frac{1}{6 b}$
i. $\frac{a}{12}+\frac{b}{16}$
j. $\frac{a}{b}+\frac{c}{3 b}$
m. $\frac{x}{y}+\frac{y}{x}$
n. $\frac{1}{x}+\frac{3}{x y}$
0. $\frac{7}{a b}+\frac{3}{b}$
9. Julie bakes a carrot cake. She plans to take half of it to her grandmother and serve $\frac{1}{3}$ of the cake to her two brothers for a snack. How much of the cake will be left?
7. Earl ate $\frac{1}{x}$ part of the first candy bar and $\frac{1}{2 x}$ part of the second bar. Together, he ate $\frac{1}{x}+\frac{1}{2 x}=\frac{3}{2 x}$ part of a candy bar.
8. Fred is off by $\frac{2}{3}-\frac{3}{5}=\frac{10}{15}-\frac{9}{15}=\frac{1}{15}$. Rene is off by $\frac{7}{10}-\frac{2}{3}=\frac{21}{30}-\frac{20}{30}=\frac{1}{30}$. Since $\frac{1}{30}$ is less than $\frac{1}{15}$, then Rene's estimate is closer than Fred's estimate.
9. a. $\frac{5}{16}-\frac{1}{8}=\frac{3}{16}$
b. $\frac{1}{2}-\frac{1}{8}=\frac{3}{8}$
C. $\frac{3}{4}-\frac{1}{8}=\frac{5}{8}$
d. $\frac{1}{2}-\frac{5}{16}=\frac{3}{16}$
e. $\frac{3}{4}-\frac{1}{2}=\frac{1}{4}$

Observe whether students use subtraction or a strategy of counting from one point to the next. Reflect on both methods.
10. Express as a mixed fraction, if needed:
a. $\frac{4+6+9}{18}=\frac{19}{18}=1 \frac{1}{18}$
b. $\frac{2+4+1}{8}=\frac{7}{8}$
c. $\frac{5+4+6}{20}=\frac{15}{20}=\frac{3}{4}$
d. $\frac{8+3+10}{24}=\frac{21}{24}=\frac{7}{8}$
e. $\frac{1+8+12}{24}=\frac{21}{24}=\frac{7}{8}$
f. $\frac{30+24+15}{40}=\frac{69}{40}=1 \frac{19}{40}$
g. $\frac{20+15+12}{60}=\frac{47}{60}$
h. $\frac{12+3+2}{24}=\frac{17}{24}$
i. $\frac{45+16+42}{72}=\frac{103}{72}=1 \frac{31}{72}$
11. a. $\frac{4+9}{6}=\frac{13}{6}=2 \frac{1}{6}$, which is $\frac{2^{2}+3^{2}}{2 \cdot 3}$ b. $\frac{16+25}{20}=\frac{41}{20}=2 \frac{1}{20}$, which is $\frac{4^{2}+5^{2}}{4 \cdot 5}$
c. $\frac{9+49}{21}=\frac{58}{21}=2 \frac{16}{21}$, which is $\frac{3^{2} \cdot 7^{2}}{3 \cdot 7}$
12. a. The LCD of $\frac{1}{p^{3} q}$ and $\frac{1}{p q^{2}}$ is $p^{3} q^{2}$. In order to rewrite each fraction with this common denominator, we see that the first fraction needs a factor of $q$ and the second fraction needs a factor of $p^{2}$. We multiply each fraction by the appropriate big "ONE" and get the following: $\frac{1}{p^{3} q}+\frac{1}{p q^{2}}=\frac{q}{q} \cdot \frac{1}{p^{3} q}+\frac{p^{2}}{p^{2}} \cdot \frac{1}{p q^{2}}$ which is equal to $\frac{q}{p^{3} q^{2}}+\frac{p^{2}}{p^{3} q^{2}}=\frac{q+p^{2}}{p^{3} q^{2}}$.
b. The LCD of $\frac{1}{p^{2} q^{3}}$ and $\frac{1}{p^{4} q^{2}}$ is $p^{4} q^{3}$. ${ }^{3} q^{2}$ order to rewrite each fraction with this common denominator, we see that the first fraction needs a factor of $p^{2}$ and the second fraction needs a factor of $q$. We multiply each fraction by the appropriate big "ONE" and get the following: $\frac{1}{p^{2} q^{3}}=\frac{1}{p^{4} q^{2}}=\frac{1}{p^{2} q^{3}} \cdot \frac{p^{2}}{p^{2}}+\frac{1}{p^{4} q^{2}}$ $\cdot \frac{q}{q}$ which is equal to $\frac{p^{2}}{p^{4} q^{3}}+\frac{q}{p^{4} q^{2}}=\frac{p^{2}+q}{p^{4} q^{2}}$.
10. There are two identical bars of chocolate. Bar $A$ is cut into $x$ equal-sized pieces. Bar $B$ is cut into twice as many equal-sized pieces. If Earl eats one piece from each chocolate bar, what fraction of a chocolate bar has he eaten, in terms of $x$ ?
11. Jill has a jar of marbles. Fred estimates that $\frac{3}{5}$ of the marbles are blue and Rene estimates that $\frac{7}{10}$ of them are blue. Jill knows that exactly $\frac{2}{3}$ of her marbles are blue. Use subtraction to determine whose estimate was closest to the correct fraction, and by how much.
12. On a one-inch ruler, the $\frac{1}{8}$ mark is labeled point $A$, the $\frac{5}{16}$ mark is labeled point $B$, the $\frac{1}{2}$ mark is labeled point $C$ and the $\frac{3}{4}$ mark is labeled point $D$. Show how subtraction can be used to calculate these distances.
a. What is the distance from point $A$ to point $B$ ?
b. What is the distance from point $A$ to point $C$ ?
c. What is the distance from point $A$ to point $D$ ?
d. What is the distance from point $B$ to point $C$ ?
e. What is the distance from point $C$ to point $D$ ?
13. Compute and simplify.
a. $\frac{2}{9}+\frac{1}{3}+\frac{1}{2}$
b. $\frac{1}{4}+\frac{1}{2}+\frac{1}{8}$
d. $\frac{1}{3}+\frac{1}{8}+\frac{5}{12}$
e. $\frac{1}{24}+\frac{1}{3}+\frac{1}{2}$
f. $\frac{3}{4}+\frac{3}{5}+\frac{3}{8}$
g. $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$
h. $\frac{1}{2}+\frac{1}{8}+\frac{1}{12}$
C. $\frac{1}{4}+\frac{1}{5}+\frac{3}{10}$
i. $\frac{5}{8}+\frac{2}{9}+\frac{1}{12}$
14. Compute the following sums. Express each as a simplified mixed fraction.
a. $\frac{2}{3}+\frac{3}{2}$
b. $\frac{4}{5}+\frac{5}{4}$
C. $\frac{3}{7}+\frac{7}{3}$
d. Check to see that your general answer from Exercise 4, part $\mathbf{m}$ agrees with the answers using arithmetic in parts $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.
15. Both $p$ and $q$ are prime numbers. For each of the following sums, find the LCD of each pair of fractions and use it to compute the sum:
a. $\frac{1}{p^{3} q}+\frac{1}{p q^{2}}$
b. $\frac{1}{p^{2} q^{3}}+\frac{1}{p^{4} q^{2}}$
$\begin{array}{ll}\text { 13. a. } & \frac{x(x+1)}{x}=x+1 \\ \text { c. } & \frac{x^{2} y^{2}\left(1+x y^{2}\right)}{x^{4} y}=\frac{x^{2} y}{x^{2} y} \cdot \frac{y\left(1+x y^{2}\right)}{x^{2}}=\frac{y\left(1+x y^{2}\right)}{x^{2}}\end{array}$
You might need to review the distributive property. This problem is crucial to seeing the advantage of using the LCD in the next section.
14. For the fractions $\frac{1}{a^{2} b}$ and $\frac{1}{a b}$, a common denominator is the product of the two denominators, $a^{3} b^{2}$. Using this common denominator, the sum is $\frac{1}{a^{2} b}+\frac{1}{a b}=\frac{a b}{a b} \cdot \frac{1}{a^{2} b}+\frac{a^{2} b}{a^{2} b} \cdot \frac{1}{a b}=\frac{a b}{a^{3} b^{2}}+\frac{a^{2} b}{a^{3} b^{2}}=\frac{a b+a^{2} b}{a^{3} b^{2}}$ - However, this answer is not in simplest form. The numerator and denominator have a common factor of $a b$. Thus, you get the following: $\frac{a b+a^{2} b}{a^{3} b^{2}}=\frac{a b(1+a)}{a^{3} b^{2}}=\frac{a b}{a b} \cdot \frac{(1+a)}{a^{2} b}=\frac{(1+a)}{a^{2} b}$. Other possible common denominators are $a 3 b 2, a 2 b 2$ and $a 2 b$, which is the LCD. Each of these denominators requires multiplying by a big "ONE", but the LCD requires the fewest factors to use.
Using the LCD on this sum, you get $\frac{1}{a^{2} b}+\frac{1}{a b}=\frac{1}{a^{2} b}+\frac{a}{a} \cdot \frac{1}{a b}=\frac{1}{a^{2} b}+\frac{a}{a^{2} b}=\frac{1+a}{a^{2} b}$. Notice that the answer is in simplest form.

## 16. Ingenuity:

Write the following fractions in simplest form:
a. $\frac{x^{2}+x}{x}$
b. $\frac{a^{2} b+a b}{a b}$
c. $\frac{x^{2} y^{2}+x^{3} y^{4}}{x^{4} y}$
17. Investigation:

Ifonefraction is $\frac{1}{a^{2} b}$ and another fraction is $\frac{1}{a b}$, what common denominator(s) do the fractions have? Explain how to decide which denominator to use when adding the fractions. Then compute the addition problem. Perform the addition using another common denominator and check to see how it compares with your first answer.

## PATTERN BLOCK FUN



Objective: Students will use visuals with pattern blocks to reinforce addition and subtraction with fractions.

## Materials:

Pattern Block Fun worksheet
Pattern Blocks (optional)

## Activity Instructions:

A brief explanation about the four pattern block shapes may be necessary before the students begin this activity. If you have pattern blocks in your classroom, pull them out and let the students hold them, play with them and discuss the relationships between the shapes. If you don't have pattern blocks in your classroom, you can find examples of the blocks on the internet and copy them for your students to see. After a brief discussion about the pattern blocks, your students are ready to begin the activity.

Make a copy of the Pattern Block Fun worksheet for each student. Ask your students to follow the directions on the worksheet to complete the activity. This activity will work well if the students are grouped together, but it can also be done individually.

## Answers:

1) $1 / 6$
2) $1 / 3$
3) $1 / 2$
4) $2 / 3$
5) $1 / 9$
6) $1 / 4$
7) 4
8) $3 / 7$
9) 1
10) 

Name $\qquad$

## Pattern Block Fun!

Directions: Look at the diagrams below and see if you can figure out the fractional patterns to answer each question.

1) If

$\qquad$ .

## 2)


$\qquad$ .
3) If
 $=1$, then

$\qquad$ .
4) If

$\qquad$ .
5) If

$\qquad$
6) If
 $=1$, what is A $\qquad$
7) If


$\qquad$
8) If

$\qquad$
9) If

$?$ $\qquad$
10) If
 $=2 / 3$, what is $1 ?$
Draw the shape here

## SPINNING FOR LCD

Objective: Students will be able to reinforce their flexibility with numbers by figuring out LCD's of two or more denominators.

## Materials:

Number spinner (1 to 9)
Challenge spinner
Paper clip any size

## Activity Instructions:

Each student spins a number spinner as many times as needed to find 2 or 3 different denominators. Students will start with one-digit numbers. As students become more confident, they may use the challenge spinner. Students then, write these numbers down.

After the denominators are generated, each student finds the LCD of them. To figure out LCD, students may list common multiples \& circle the LCM or LCD; use prime factorization (take the product of each prime raised to its larger exponent; Ex. the LCD is LCM $(6,9)=2 \times 32=18)$; or any other strategy that makes sense to them.

## NUMBER SPINNER



## CHALLENGE SPINNER



## Section 8.5 - Common Denominators and Mixed Numbers

## Big Idea:

Adding and subtracting mixed numbers

## Key Objectives:

- Use models to add or subtract mixed numbers.
- Develop an algorithm for computing least common denominator.
- Use least common denominator to add or subtract mixed numbers.


## Materials:

No extra materials needed

## Pedagogical/Orchestration:

This section carefully shows the difference between common denominator and LEAST common denominator. Although this is a review of 6th grade TEKS, we go deeper into the process of addition of fractions.

## Activity:

"Subtracting Fractions with Regrouping" at the end of the section and on CD

## Vocabulary:

least common denominator (LCD), improper fractions, vertical addition, like parts

## TEKS:

6.2(B); 6.11(B); 7.2(B)(F); 7.13(A)(C)(D); 7.14(A); 8.2(B,C); 8.14(A); 8.15(A);

## WARM-UP for Section 8.5

1. Which of the following sums is NOT between $\frac{1}{2}$ and $\frac{2}{3}$ ?
a. $\frac{1}{4}+\frac{1}{3}$
C. $\frac{2}{5}+\frac{3}{10}$
b. $\frac{1}{10}+\frac{1}{2}$
d. $\frac{1}{6}+\frac{2}{5}$

Ans: (c) because $\frac{2}{5}+\frac{3}{10}=\frac{7}{10}>\frac{2}{3}$

## Launch for Section 8.5:

Write $1 \frac{1}{a}+1 \frac{1}{b}$ on the board. Tell your students that we do not know what $a$ and $b$ are exactly but we know that they are positive integers greater than 1 . Tell your students that the values to be added are called mixed numbers because they represent the sum of a whole number and a fraction. In fact the mixed numbers can be rewritten like this: $1+\frac{1}{a}+1+\frac{1}{b}$. Ask students to estimate what $1+\frac{1}{a}+1+\frac{1}{b}$ equals. What two whole numbers would the sum be between? Let students think about this for awhile. If they struggle, suggest replacing $a$ and $b$ with integers such as 2 and 4 , etc. The answer is that the sum is greater than 2 but less than or equal to 3 . The students may even be able to tell you that the sum equals $2+\frac{a+b}{a b}$. Go as far as you can with the students and then let them know that today they will be adding and subtracting mixed numbers and developing strategies for finding the least common denominator.

## EXAMPLE 1

Even though the second problem has variables rather than known integers, it is easier to see how the process of developing the LCD works.

## SECTION 8.5 COMMON DENOMINATORS AND MIXED NUMBERS

In Section 8.4, the discussion of adding two fractions involved finding a common denominator. In many problems, you probably used the least common denominator (LCD), the LCM of the given denominators.

In the Exploration in Section 8.4 you discovered a rule for adding two fractions with unknown and unlike denominators:

$$
\frac{1}{a}+\frac{1}{b}=\frac{1 \cdot b}{a \cdot b}+\frac{1 \cdot a}{b \cdot a}=\frac{b}{a \cdot b}+\frac{a}{a \cdot b}=\frac{b+a}{a \cdot b}
$$

## EXAMPLE 1

Assume $p$ and $q$ are primes. Compute the following two sums using the pattern from above. Notice that the first sum is a special case of the second, where $p=2$ and $q=3$.
a. $\frac{1}{6}+\frac{1}{9}$
b. $\frac{1}{p q}+\frac{1}{q^{2}}$

## SOLUTION

## Common Denominator Method:

As in the rule above, you can create a common denominator for each sum by multiplying the two given denominators:

$$
\begin{aligned}
\frac{1}{6}+\frac{1}{9} & =\frac{1 \cdot 9}{6 \cdot 9}+\frac{1 \cdot 6}{9 \cdot 6} & \frac{1}{p q}+\frac{1}{q^{2}} & =\frac{1 \cdot q^{2}}{p q \cdot q^{2}}+\frac{1 \cdot p q}{q^{2} \cdot p q} \\
& =\frac{9+6}{6 \cdot 9}=\frac{15}{54} & & =\frac{q^{2}+p q}{p q^{3}}
\end{aligned}
$$

Remember Exercise 12 in Section 9.1.

Repeat with the class the way the LCD method builds the LCD starting with either denominator.

## PROBLEM 1

Make sure your students use the prime factorization for the first few examples and some of the exercises. The main point is to choose the right factors in multiplying the numerator and denominator so that they get an equivalent fraction with the LCD. The process is shown for part (a) and is similar for parts (b) and (c).
a. $\quad \frac{1}{40}+\frac{1}{50}=\frac{1}{2^{2} 5}+\frac{1}{2 \cdot 5^{2}}=\frac{1(5)}{2^{3} \cdot 5(5)}+\frac{1\left(2^{2}\right)}{2 \cdot 5^{2}\left(2^{2}\right)}=\frac{5}{(40)(5)}+\frac{4}{50(4)}$

$$
=\frac{5}{2^{3} \cdot 5^{2}}+\frac{2^{2}}{2^{3} \cdot 5^{2}}=\frac{5}{200}+\frac{4}{200}=\frac{5+2^{2}}{2^{3} \cdot 5^{2}}=\frac{5+4}{200}=\frac{9}{200}
$$

b. $\quad \frac{3}{8}+\frac{5}{12}=\frac{3}{2 \cdot 2 \cdot 2}+\frac{5}{2 \cdot 2 \cdot 3}=\frac{3 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 3}+\frac{5 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 3}=\frac{9+10}{2 \cdot 2 \cdot 2 \cdot 3}=\frac{19}{24}$
c. $\quad \frac{7}{10}+\frac{4}{9}=\frac{7}{2 \cdot 5}+\frac{4}{3 \cdot 3}=\frac{7 \cdot 3 \cdot 3}{2 \cdot 5 \cdot 3 \cdot 3}+\frac{4 \cdot 2 \cdot 5}{2 \cdot 5 \cdot 3 \cdot 3}=\frac{63+40}{2 \cdot 3 \cdot 3 \cdot 5}=\frac{103}{90}$

Are these fractions simplified? Do the numerator and denominator have any common factors? The numerator can be factored using the distributive property:

$$
\frac{15}{54}=\frac{3 \cdot 5}{2 \cdot 3 \cdot 3 \cdot 3}=\frac{5}{18} \quad \frac{q^{2}+p q}{p q^{3}}=\frac{q(q+p)}{p q^{3}}=\frac{q+p}{p q^{2}}
$$

Notice that this approach did not involve finding the LCD.

## LCD Method:

Another approach is to first find the LCD of the fractions in each sum or, equivalently, the LCM of the denominators. Look at the prime factorizations of the denominators of the fractions: $6=2 \cdot 3,9=3^{2}$, and because $p$ and $q$ are primes, $p q$ and $q^{2}$ are their own prime factorizations. Remember the rule for finding the LCM of two numbers from their prime factorizations: take the product of each prime raised to its larger exponent. So, the $\operatorname{LCDs}$ are $\operatorname{LCM}(6,9)=2 \cdot 3^{2}$ and $\operatorname{LCM}\left(p q, q^{2}\right)=p q^{2}$. Now, in computing the sums $\frac{1}{6}+\frac{1}{9}$ and $\frac{1}{\mathrm{pq}}+\frac{1}{\mathrm{q}^{2}}$, multiply the numerator and denominator of each fraction by a factor that will make the denominator the LCD:

$$
\begin{aligned}
\frac{1}{6}+\frac{1}{9} & =\frac{1}{2 \cdot 3}+\frac{1}{3^{2}} & \frac{1}{p q}+\frac{1}{q^{2}} & =\frac{1}{p q}+\frac{1}{q^{2}} \\
& =\frac{1 \cdot 3}{(2 \cdot 3) \cdot 3}+\frac{1 \cdot 2}{3^{2} \cdot 2} & & =\frac{1 \cdot q}{p q \cdot q}+\frac{1 \cdot p}{q^{2} \cdot p} \\
& =\frac{3+2}{2 \cdot 3^{2}}=\frac{5}{18} & & =\frac{q+p}{p q^{2}}
\end{aligned}
$$

Notice that the final answer in each sum is already simplified.

## PROBLEM 1

Use the process just developed to compute the LCD for the fractions and then compute the sum:
a. $\frac{1}{40}+\frac{1}{50}$
b. $\frac{3}{8}+\frac{5}{12}$
C. $\frac{7}{10}+\frac{4}{9}$

Teachers, formulating the written procedure is a group activity that acts as a final check for understanding and a way for students to help each other and themselves when they discuss the procedure.

## EXPLORATION 1

Have the class work on this problem and look for 2 approaches: (1) Coverting the mixed fractions to improper fractions, add them and then convert them to a mixed fraction or (2) Add $3+5$ and $\frac{1}{6}+\frac{1}{4}$. Then combine to form a mixed fraction. Reflect on the advantage of both methods. In the class discussion, look for a general approach for adding mixed fractions.

The general rule for the correct form in an answer is to leave the answer in its original form, in this case a mixed fraction.

Make sure that your students discover that when adding mixed numbers, adding just the whole number gives a good estimate of the result. Silvia has at least $3+5=8$ pounds of sugar, and no more than $4+6=10$ pounds.

Formulate a written procedure that describes the process of:

- Finding the LCD for any two fractions
- Rewriting fractions equivalently using the LCD
- Computing sums and differences of two fractions.


## EXPLORATION 1

Silvia is baking six sheet cakes for a party. The recipe she is using calls for $3 \frac{1}{6}$ pounds of refined sugar and $5 \frac{1}{4}$ pounds of unrefined sugar. First, use the linear model to give an estimate of how much sugar Silvia needs. Then, compute how many pounds of sugar Silvia needs. Explain your process for both the estimation and the calculation. Can you use the same process to add other mixed numbers?

## EXAMPLE 2

Compute the sum $6 \frac{3}{5}+3 \frac{5}{7}$.

## SOLUTION

There are at least three ways to compute this sum.

## 1. Improper Fractions:

One approach is to treat this as an ordinary fraction addition problem by converting from mixed numbers to improper fractions and back again.
First, convert the mixed numbers to improper fractions:

$$
\begin{gathered}
6 \frac{3}{5}=6+\frac{3}{5}=\frac{6 \cdot 5}{1 \cdot 5}+\frac{3}{5}=\frac{33}{5} \\
\text { and } \\
3 \frac{5}{7}=3+\frac{5}{7}=\frac{3 \cdot 7}{1 \cdot 7}+\frac{5}{7}=\frac{26}{7}
\end{gathered}
$$

You may wish to remind students what the Commutative and Associative Properties of Addition are.

Then, find the LCD and compute the sum. Note that in this case the denominators are relatively prime, so the LCD is their product.

$$
\begin{aligned}
\frac{33}{5}+\frac{26}{7} & =\frac{33 \cdot 7}{5 \cdot 7}+\frac{26 \cdot 5}{7 \cdot 5} \\
& =\frac{231}{35}+\frac{130}{35} \\
& =\frac{361}{35}
\end{aligned}
$$

Finally, convert the improper fraction to a mixed number and simplify. Because the largest multiple of 35 less than 361 is 350 , convert 361 to $35 \cdot 10+11=350+11$ or $361 \div 35$ is 10 with a remainder of 11 .

$$
\begin{aligned}
\frac{361}{35} & =\frac{350+11}{35} \\
& =\frac{350}{35}+\frac{11}{35} \\
& =10+\frac{11}{35} \\
& =10 \frac{11}{35}
\end{aligned}
$$

## 2. Combining Like Parts:

The improper fractions approach can be cumbersome because it involves working with relatively large numbers. Another approach is to consider each mixed number as the sum of an integer and a proper fraction and regroup, using the Commutative and Associative Properties of Addition:

$$
6 \frac{3}{5}+3 \frac{5}{7}=\left(6+\frac{3}{5}\right)+\left(3+\frac{5}{7}\right)=(6+3)+\left(\frac{3}{5}+\frac{5}{7}\right)
$$

The general rule for the correct form in an answer is to leave the answer in its original form: in this case, a mixed fraction.

Note that in each step, the whole problem is written in the column. The process of finding the LCD and rewriting the fractions in equivalent form with the LCD as their denominators is not included below. This work should be shown somewhere else.

After going through the example, reflect with the class on advantages and disadvantages of the horizontal and stacking methods.

Finding the difference uses the same techniques, so its method is the same unless it is impossible to subtract the fractional parts without altering the mixed numbers, as we will demonstrate.

This leads to the sum of proper fractions:

$$
\begin{aligned}
\frac{3}{5}+\frac{5}{7} & =\frac{3 \cdot 7}{5 \cdot 7}+\frac{5 \cdot 5}{7 \cdot 5} \\
& =\frac{21}{35}+\frac{25}{35} \\
& =\frac{46}{35} \\
& =1 \frac{11}{35}
\end{aligned}
$$

Combining these results, the original sum is

$$
\begin{aligned}
6 \frac{3}{5}+3 \frac{5}{7} & =\left(6+\frac{3}{5}\right)+\left(3+\frac{5}{7}\right) \\
& =(6+3)+\left(1+\frac{11}{35}\right) \\
& =10+\frac{11}{35} \\
& =10 \frac{11}{35}
\end{aligned}
$$

As you can see, in computing the sum of mixed numbers, it is often easier to regroup the mixed numbers as whole parts and fractional parts, add each group and then combine these two partial sums.

## 3. Vertical Addition:

There is another way to organize and write this same process vertically:

$$
\begin{aligned}
& 6 \frac{3}{5} \\
&+3 \frac{5}{7} \longrightarrow 6 \begin{array}{c}
\frac{3}{5} \\
+\underline{3}+\frac{5}{7}
\end{array}+\begin{array}{r}
\frac{21}{35} \\
+\frac{3}{9}+\frac{25}{35} \\
\frac{46}{35}
\end{array}=9+\left(1+\frac{11}{35}\right) \\
&=10+\frac{11}{35} \\
&=10 \frac{11}{35}
\end{aligned}
$$

How would finding the difference between two mixed numbers be different?

When subtracting $5 \frac{3}{10}$ in the stacking method, you must distribute the subtraction sign.

## EXAMPLE 3

Compute the following differences:
a. $8 \frac{4}{5}-5 \frac{3}{10}$
b. $6 \frac{3}{5}-3 \frac{5}{7}$

## SOLUTION

a. Use the vertical method from the previous example:

$$
\begin{aligned}
8 \frac{4}{5} \rightarrow 8-\frac{4}{5} \rightarrow 8 & \frac{8}{10} \\
\underline{-5 \frac{3}{10}}-\underline{-5}-\frac{3}{10} \quad \frac{-5}{3} \frac{-\frac{3}{10}}{\frac{5}{10}} & =3+\frac{5}{10} \\
& =3+\frac{1 \cdot 5}{2 \cdot 5} \\
& =3 \frac{1}{2}
\end{aligned}
$$

Notice that the fraction $\frac{5}{10}$ in the solution is simplified to its equivalent fraction $\frac{1}{2}$.
b. Again, use the vertical method. However, a complication arises when attempting to subtract $\frac{5}{7}$ from $\frac{3}{5} \cdot \frac{5}{7}$ is greater than $\frac{3}{5}$.

To avoid the negative fraction, rename 6 as $5+1$ and associate 1 , or $\frac{35}{35}$ , with the fraction $\frac{21}{35}$.

$$
\begin{aligned}
& =2 \frac{31}{35}
\end{aligned}
$$

## EXERCISES

Assign the following parts of exercises:
Ex. 1: a, e, i
Do: Ex. 5, 6, 7, 8, 9, 10, 11
Ex. 2: b, e, f
Ex. 3: g, c, d
Ex.4: Skip

1. a. $\frac{3}{2 \cdot 2 \cdot 7}+\frac{5}{2 \cdot 2 \cdot 3}=\frac{3 \cdot 3+5 \cdot 7}{2 \cdot 2 \cdot 3 \cdot 7}=\frac{44}{2 \cdot 2 \cdot 3 \cdot 7}=\frac{2 \cdot 2 \cdot 11}{2 \cdot 2 \cdot 7}=\frac{11}{21}$
b. $\frac{3}{2 \cdot 2}+\frac{5}{2 \cdot 2 \cdot 2 \cdot 2}=\frac{3 \cdot 2 \cdot 2+5}{2 \cdot 2 \cdot 2 \cdot 2}=\frac{17}{2 \cdot 2 \cdot 2 \cdot 2}=\frac{17}{16}=1 \frac{1}{16}$
c. $\frac{7}{2 \cdot 2 \cdot 2 \cdot 3}+\frac{5}{2 \cdot 2 \cdot 3 \cdot 3}=\frac{7 \cdot 3+5 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3}=\frac{31}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3}=\frac{31}{72}$
d. $\frac{8}{3 \cdot 5}+\frac{3}{2 \cdot 2 \cdot 7}=\frac{8 \cdot 2 \cdot 2 \cdot 7+3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 3 \cdot 5 \cdot 7}=\frac{244+45}{2 \cdot 2 \cdot 3 \cdot 5 \cdot 7}=\frac{269}{420}$
e. $\frac{11}{2 \cdot 2 \cdot 3 \cdot 5}-\frac{1}{2 \cdot 2 \cdot 3 \cdot 5}=\frac{11 \cdot 3 \cdot 7+1 \cdot 2 \cdot 5}{2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7}=\frac{241}{2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7}=\frac{241}{1260}$
f. $\frac{1}{2 \cdot 2 \cdot 5}-\frac{1}{3 \cdot 5 \cdot 5}=\frac{1 \cdot 3 \cdot 5 \cdot 1 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 3 \cdot 5 \cdot 5}=\frac{11}{2 \cdot 2 \cdot 3 \cdot 5 \cdot 5}=\frac{11}{300}$
g. $\frac{7}{2 \cdot 2 \cdot 3 \cdot 3 \cdot 3}+\frac{5}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3}=\frac{7 \cdot 2+5 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3}=\frac{29}{216}$
h. $\frac{19}{3 \cdot 7}-\frac{71}{2 \cdot 2 \cdot 3 \cdot 7}=\frac{19 \cdot 2 \cdot 2 \cdot 71}{2 \cdot 2 \cdot 3 \cdot 7}=\frac{5}{2 \cdot 2 \cdot 3 \cdot 7}=\frac{5}{84}$
i. $\frac{1}{3 \cdot 3 \cdot 3}-\frac{1}{2 \cdot 2 \cdot 7}=\frac{1 \cdot 2 \cdot 2 \cdot 7 \cdot 1 \cdot 3 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 7}=\frac{28-27}{2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 7}=\frac{1}{756}$
j. $\frac{2}{3 \cdot 3 \cdot 5}+\frac{3}{2 \cdot 2 \cdot 2 \cdot 5}=\frac{2 \cdot 2 \cdot 2 \cdot 2+3 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5}=\frac{16+27}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5}=\frac{43}{360}$
k. $\frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3+3 \cdot 5 \cdot 5}{2 \cdot 2 \cdot 5 \cdot 5}=\frac{123}{100}=1 \frac{23}{100}$. $\quad \frac{3 \cdot 7+2 \cdot 7}{2 \cdot 2 \cdot 2 \cdot 3}=\frac{5 \cdot 7}{2 \cdot 2 \cdot 2 \cdot 3}=1 \frac{11}{24}$

2
a. $4 \frac{6}{8}=4 \frac{3}{4}$
d. $9+5+\frac{8}{15}+\frac{11}{12}=14+\frac{32+55}{60}=14+\frac{87}{60}=15 \frac{27}{60}=15 \frac{9}{20}$
b. $6+8+\frac{2}{5}+\frac{3}{7}=8+\frac{14+15}{35}=8+\frac{29}{35}=8 \frac{29}{35}$
c. $7 \frac{3}{3}$
e. $3+5+\frac{5}{6}+\frac{3}{4}=8+\frac{10+9}{12}=8+\frac{19}{12}=9+\frac{7}{12}=9 \frac{7}{12}$
f. $6+2+\frac{3}{8}+\frac{11}{12}=8+\frac{9+22}{24}=8+\frac{31}{24}=8+1+\frac{7}{24}=9+\frac{7}{24}=9 \frac{7}{24}$
3.
a. $\quad 1 \frac{4}{8}=1 \frac{1}{2}$
$\begin{array}{lll}\text { b. } & 2 \frac{2}{3} & \text { e } \\ \text { c. } & 3 \frac{4}{7} & f\end{array}$
d. $4+\left(\frac{32}{60}-\frac{55}{60}\right)=3+\frac{92-55}{60}=\frac{37}{60}$
$\begin{array}{ll}\text { g. } & 4 \frac{1}{72} \\ \text { h. } & 2 \frac{1}{80}\end{array}$
e. $4+\left(\frac{1}{5}-\frac{1}{6}\right)=4+\frac{6-5}{30}=4 \frac{1}{30}$
f. $\quad 4 \frac{1}{180}$
4. a. $\frac{1}{x^{4} y^{3}}+\frac{1}{x^{2} y^{5}}=\frac{y^{2}}{x^{4} y^{5}}+\frac{x^{2}}{x^{4} y^{5}}=\frac{y^{2}+x^{2}}{x^{4} y^{5}}$
d. $\frac{2 x}{x}+\frac{1}{x}=\frac{2 x+1}{x}$
b. $\frac{1}{x y^{2} z}+\frac{1}{x z^{2}}=\frac{z}{x y^{2} z^{2}}+\frac{y^{2}}{x y^{2} z^{2}}=\frac{z+y^{2}}{x y^{2} z^{2}}$
e. $\quad \frac{3 x y}{x y}+\frac{a}{x y}=\frac{3 x y+a}{x y}$
c. $\frac{1}{p^{2} q r^{3}}+\frac{1}{p^{3} q^{4} r^{2}}=\frac{p q^{\frac{1}{3}}}{p^{3} q^{4} r^{2}}+\frac{r}{p^{3} q^{4} r^{2}}=\frac{p q^{3}+r}{p^{3} q^{4} r^{2}}$
f. $\quad \frac{3 a}{x z} \frac{y}{y}+\frac{4 b}{x z} \frac{x}{x}=\frac{3 a y+4 b x}{x y z}$

Explain to your students that because we know nothing about a variable's true value, we must assume that each variable in each sum is a prime number. You might mention that this is only one of the many times that variables, no matter how scary at first, are easier to work with than constants are.
5. $3 \frac{3}{8}+6 \frac{1}{5}=3+6+\frac{3}{8}+\frac{1}{5}=9+\frac{15+8}{40}=9+\frac{23}{40}=9 \frac{23}{40}$ gallons of honey.
6. Miguel ran $5 \frac{1}{4}-2 \frac{5}{8}$ miles further. Subtracting, you get $5 \frac{1}{4}-2 \frac{5}{8}=3+\left(\frac{1}{4}-\frac{5}{8}\right)=2+\left(\frac{5}{4}-\frac{5}{8}\right)=2+\left(\frac{10-5}{8}\right)=2 \frac{5}{8}$
7. The amount of milk left in the container is $4-2 \frac{1}{3}=3 \frac{3}{3}-2 \frac{1}{3}=1 \frac{2}{3}$ cups.

## EXERCISES

1. Compute the following sums and differences by using the prime factorization of each denominator to find the LCD. Show all the steps in the process.
a. $\frac{3}{28}+\frac{5}{12}$
e. $\frac{11}{60}-\frac{1}{1,26}$
i. $\frac{1}{27}-\frac{1}{28}$
b. $\frac{3}{4}+\frac{5}{16}$
f. $\frac{1}{20}-\frac{1}{75}$
g. $\frac{1}{108}+\frac{5}{72}$
h. $\frac{19}{21}-\frac{71}{84}$
$\begin{array}{ll}\text { j. } & \frac{2}{45}+\frac{3}{40} \\ \text { k. } & \frac{12}{35}+\frac{3}{4} \\ \text { l. } & \frac{7}{8}+\frac{7}{12}\end{array}$
2. Compute the following sums of mixed numbers using either the horizontal or vertical method. Show all the steps in the process.
a. $3 \frac{1}{8}+1 \frac{5}{8}$
b. $6 \frac{2}{5}+2 \frac{3}{7}$
c. $5 \frac{1}{3}+2 \frac{2}{3}$
d. $\quad 9 \frac{8}{15}+5 \frac{11}{12}$
e. $3 \frac{5}{6}+5 \frac{3}{4}$
f. $6 \frac{3}{8}+2 \frac{11}{12}$
3. Compute the following differences of mixed numbers using the stacking method. Show all the steps in the process.
a. $3 \frac{1}{8}-1 \frac{5}{8}$
b. $\quad 5 \frac{1}{3}-2 \frac{2}{3}$
c. $4-\frac{3}{7}$
d. $\quad 9 \frac{8}{15}-5 \frac{11}{12}$
e. $6 \frac{1}{5}-2 \frac{1}{6}$
f. $7 \frac{1}{30}-3 \frac{1}{36}$
g. $5 \frac{5}{24}-1 \frac{7}{36}$
h. $4 \frac{1}{30}-2 \frac{1}{48}$
4. Assume that each variable in this problem is a prime number. For each of the following sums, find the LCD of the denominators and use it to compute the sum. Show all the steps in the process.
a. $\frac{1}{x^{4} y^{3}}+\frac{1}{x_{1}^{2} y^{5}}$
b. $\frac{1}{x y^{2} z}+\frac{1}{x z^{2}}$
c. $\frac{1}{p^{2} q r^{3}}+\frac{1}{p^{3} q^{4} r^{2}}$
d. $2+\frac{1}{x}$
e. $\quad 3+\frac{a}{x y}$
f. $\frac{3 a}{x z}+\frac{4 b}{y z}$
5. Susie has $3 \frac{3}{8}$ gallons of honey. Her friend John has $6 \frac{1}{5}$ gallons of honey. How much honey do they have together?
6. Gabriel ran $2 \frac{5}{8}$ miles while Miguel ran $5 \frac{1}{4}$ miles. How much further did Miguel run?
7. Sophia is making a cake that requires $2 \frac{1}{3}$ cups of milk. If she pours her milk out of a four-cup milk container, how much will she have left in the container?
8. $5 \frac{1}{4}-\frac{2}{3}=5+\left(\frac{1}{4}-\frac{2}{3}\right)=4+\left(\frac{5}{4}-\frac{2}{3}\right)=4+\left(\frac{15-8}{12}\right)=4+\frac{7}{12}=4 \frac{7}{12}$ feet of wire.
9. $3 \frac{2}{3}+2 \frac{3}{5}=5+\frac{2}{3}+\frac{3}{5}=5+\frac{10+9}{15}=5+\frac{19}{15}=6 \frac{4}{15}$ cups of liquid.
10. $2 \frac{1}{4}+1 \frac{2}{3}+3 \frac{2}{5}=6+\frac{1}{4}+\frac{2}{3}+\frac{2}{5}=6+\frac{15+40+24}{60}=6+\frac{79}{60}=7 \frac{19}{60}$ inches of rain.
11. Class $A$ has 20 students of which $\frac{1}{4}$ are female. Since $\frac{1}{4}=\frac{5}{20}$, the number of females in class $A$ is 5 . Class B has 32 students of which $\frac{3}{4}$ are female. Since $\frac{3}{4}=\frac{24}{32}$, the number of females in class B is 24 . The total number of females in both classes is 29. The total number of students in both classes is 52 . The fraction of females in the PE class is $\frac{29}{52}$. This is not a classic addition of fractions problem. The key issue is what is the whole. The fraction of girls in Class A is $\frac{5}{20}$ and the fraction of girls in Class B is $\frac{24}{32}$. The fraction of the PE class is $\frac{5+24}{20+32}=\frac{29}{52}$. This is a subtle problem and is often missed by high school students.

## Ingenuity

12. The amount of a house that Jack and Jill can paint together in a day is $\frac{1}{8}+\frac{1}{6}=\frac{7}{24}$ of a house. In 3 days, they can paint $\frac{7}{24}+\frac{7}{24}+\frac{7}{24}=\frac{21}{24}$ of a house. In 4 days they can paint $\frac{28}{24}=1 \frac{4}{24}=1 \frac{1}{6}$ of a house. So they will take between 3 and 4 days to paint a house; they will take exactly $3 \frac{3}{7}$ days.

## 13. Investigation

You may work these problems using a visual area model (rectangles). The following is an algebraic method for computing these fractions. In part $b$, we deftly avoid the issue of division of fractions by using the fact that $\frac{3 x}{2}=\frac{3}{2} x=1.5 x$.
a. Class $C$ has 3 times as many students as Class D. Let $x=$ the number of students in Class D. So, Class C has $3 x$ number of students. Together, the two classes have $4 x$ number of students. The fraction of sixth grade students in Class $C$ is $\frac{3 x}{4 x}=\frac{3}{4}$. Class C has $\frac{3}{4}$ and Class D has $\frac{1}{4}$ of the sixth graders.
b. One third of Class $C$ are girls, so the number of girls in Class $C$ is $\frac{3 x}{3}=x$. One half of the students in Class $D$ are girls, so the number of girls in Class $D$ is $\frac{x}{2}$. So together, there are $x+\frac{x}{2}=\frac{3}{2} x=1.5 x$ number of girls. The total number of students is $4 x$. So the fraction of sixth grade students that are girls is $\frac{1.5 \mathrm{x}}{4 \mathrm{x}}=\frac{1.5}{4}=\frac{1.5}{4} \cdot \frac{2}{2}=\frac{3}{8}$.
An area model is helpful. This foreshadows multiplication of fractions. The girls in Class C are $\frac{1}{3}$ of $\frac{3}{4}=\frac{1}{4}$ of all sixth graders. The girls in Class D are $\frac{1}{2}$ of $\frac{1}{4}=\frac{1}{8}$ of all sixth graders. Together, $\frac{1}{4}+\frac{1}{8}=\frac{3}{8}$ of all sixth graders are girls. You can use algebraic equations to show this process. Some students will pick a particular number of students in Class D and triple it to get the number of students in Class C and then compute the fractions with these numbers. That is a valid non-algebraic solution.

## Investigation

14. $6 \frac{3}{5}-3 \frac{5}{7}=\left(6+\frac{3}{5}\right)-\left(3+\frac{5}{7}\right)=6+\frac{3}{5}-3-\frac{5}{7}=(6-3)+\left(\frac{3}{5}-\frac{5}{7}\right)=3+\left(\frac{3}{5}-\frac{5}{7}\right)$. Notice that the subtraction of fractions will give you a negative numerator. So we take 1 from the 3, leaving 2, and add it to the first fraction to get $1+\frac{3}{5}=\frac{8}{5}$ so as to be able to continue the subtraction. This gives us $2+\left(\frac{8}{5}-\frac{5}{7}\right)=2+\left(\frac{56}{35}-\frac{25}{35}\right)=2+\frac{31}{35}=2 \frac{31}{35}$
15. Adam cut $\frac{2}{3}$ foot off of a $5 \frac{1}{4}$ foot wire. How much wire was left?
16. Lydia drinks $3 \frac{2}{3}$ cups of milk, and $2 \frac{3}{5}$ cups of juice. How much liquid does she consume?
17. On Monday it rained $2 \frac{1}{4}$ inches. On Tuesday it rained $1 \frac{2}{3}$ inches. On Wednesday it rained $3 \frac{2}{5}$ inches. What was the total rainfall for these three days?
18. There are two fifth grade classes in a school. Class A is one-fourth female and Class B is three-fourths female. Both classes have PE together. Class A has 20 students and Class B has 32 students. What fraction of the combined PE class is female?
19. Ingenuity:

Jack and Jill are house painters. Jack can paint $\frac{1}{8}$ of a standard-sized house in a day. Jill can paint $\frac{1}{6}$ of a standard-sized house in one day. Estimate about how long it might take Jack and Jill to paint a standard house working together.
13. Investigation:
a. The students in the sixth grade are all in two classes, Class C and Class D . If Class $C$ has three times as many students as Class $D$, what fraction of the sixth grade students are in Class C? Draw a picture.
b. In the same sixth grade class, Class C is one-third girls and Class D is one-half girls. What fractional part of the sixth grade is made up of girls?
14. Investigation:

In Example 2, the process of adding two mixed numbers was written horizontally. Formulate a similar horizontal process for computing the difference $6 \frac{3}{5}-3 \frac{5}{7}$.

## Subtracting Fractions with Regrouping

To rename a whole number into a mixed number you borrow "one" from the whole number and rename it as a fraction.

## Example:

Step 1: $\quad$ Borrow $\quad 7-1=6$
Step 2: $\quad$ Rename $\quad 6+1 \quad$ (" 1 " is renamed to $\frac{4}{4}$ )
Step 3: $\quad$ Regroup $\quad 6+\frac{4}{4}$

$$
=6 \frac{4}{4}
$$

Practice Renaming:

1) $9=8 \overline{3}$
2) $4=3-$
3) $7=6 \overline{8}$

Practice Subtracting Mixed Numbers from Whole Numbers:
4) $9-2 \frac{5}{8}$
5) $4-1 \frac{2}{3}$
6) $7-6 \frac{3}{4}$

Practice Subtracting Mixed Numbers w/like Denominators:
7) $9 \frac{3}{8}-2 \frac{5}{8}$
8) $4 \frac{1}{3}-1 \frac{2}{3}$
9) $7 \frac{1}{4}-6 \frac{3}{4}$

Practice Subtracting Mixed Number w/unlike Denominators:
10) $8 \frac{1}{9}-2 \frac{5}{6}$
11) $9 \frac{3}{10}-4 \frac{1}{2}$
12) $4 \frac{5}{6}-3 \frac{8}{9}$

1. a. $\operatorname{GCF}(20,25)=5$
c. $\operatorname{GCF}(12,36)=12$
e. $G C F=12$
b. $\quad \operatorname{GCF}(45,65)=5$
d. $\operatorname{GCF}(16,17)=1$
f. $\mathrm{GCF}=42$
g. $\operatorname{GCF}(14,35,56)=7$
h. $\operatorname{GCF}(105,120,135)=3$
i. $\mathrm{GCF}=x y^{3}$
2. Answers may vary, but some equivalent fractions include:
a. $\frac{4}{10}$
C. $\frac{4}{6}$
b. $\frac{3}{5}$
d. $\frac{3}{12}$
e. $\frac{2}{3}$
3. Ordered from least to greatest:
e. $\frac{2}{8}=\frac{1}{4}$
c. $\frac{2}{6}=\frac{1}{3}$
b. $\frac{3}{8}$
d. $\frac{4}{10}=\frac{2}{5}$
a. $\frac{5}{10}=\frac{1}{2}$
f. $\frac{4}{6}=\frac{2}{3}$
4. Isabel worked on her homework for $\frac{20}{60}$ of an hour $=\frac{1}{3}$ of an hour.
5. a. $\frac{4}{5}$
c. $\frac{1}{3}$
e. $\frac{7}{8}$
b. $\frac{9}{13}$
d. $\frac{16}{17}$
f. $\frac{21}{44}$

## REVIEW PROBLEMS

1. Find the greatest common factor of each pair of integers. Show your work with the Venn Diagram method or the unique factorization method.
a. 20 and 25
b. 45 and 65
c. 12 and 36
d. 16 and 17
e. 60 and 84
f. 378 and 420
g. 14,35 , and 56
h. 105,120 , and 135
i. $x^{2} y^{3}$ and $x y^{5}$
2. Find an equivalent fraction for each of the following:
a. $\frac{2}{5}$
C. $\frac{2}{3}$
b. $\frac{6}{10}$
d. $\frac{1}{4}$
e. $\frac{8}{12}$
3. Label the shaded fractional parts of the following and order from least to greatest:
a.

d.

b.

e.

f.

c.

4. If Isabel worked on her math homework for 20 minutes, what fraction of an hour did she work on her homework? Write this fraction in its simplest form.
5. Rewrite the following fractions in simplest form.
a. $\frac{20}{25}$
b. $\frac{45}{65}$
c. $\frac{12}{36}$
d. $\frac{16}{17}$
e. $\frac{63}{72}$
f. $\frac{126}{264}$
6. . $\frac{7}{9}$
b. $\frac{3}{4}$
c. $\frac{3}{5}$
d. $\frac{77}{81}$
e. $\frac{15}{22}$
7. I ate the most pizza: $\frac{1}{2}=\frac{3}{6}$. My friend ate $\frac{1}{3}=\frac{2}{6}$. There is $\frac{1}{6}$ of the pizza left over.
8. She can invite 23 people plus herself so that nothing is left over. Cookies: $6,12,18,24 ;$ Juice: $8,16,24$. There can only be a total of 24 people, including Dora.
9. There are 6 people playing in all. Each player has 2 water guns and 3 water balloons.
10. a. $\frac{8}{5}=1 \frac{3}{5}$
b. $\frac{20}{8}=2 \frac{4}{8}=2 \frac{1}{2}$
c. $\frac{25}{10}=2 \frac{5}{10}=2 \frac{1}{2}$
11. Simplify each of the following fractions:
a. $\frac{56}{72}$
b. $\frac{90}{120}$
c. $\frac{126}{210}$
d. $\frac{77}{81}$
e. $\frac{225}{330}$
12. Your friend eats $\frac{1}{3}$ of a pizza and you eat $\frac{1}{2}$ of the same pizza. Who ate the most pizza? How much of the pizza is left over?
13. Dora is planning to have a party. Cookies are sold in packages of six. Juice drinks are sold in packages of eight. She is trying to figure out how many friends she is able to invite so that each friend gets exactly one cookie and one drink with nothing left over. How many friends can Dora invite to her party?
14. Christopher is having friends over for a fun day of water games. He has 12 water guns and 18 water balloons. If all the water guns and water balloons are handed out and each person has the same number of water guns and water balloons, how many people were playing? How many water guns and water balloons did each person have?
15. Write the following shaded areas as an improper fraction and a mixed number.
a.

b.

c.

16. a. $1 \frac{5}{7}$
c. $9 \frac{1}{6}$
e. $6 \frac{7}{12}$
b. $5 \frac{3}{4}$
d. $3 \frac{1}{3}$
17. a. $\frac{11}{3}$
c. $\frac{65}{8}$
d. $\frac{77}{6}$
18. a. $\operatorname{LCM}(8,12)=24$
c. $\operatorname{LCM}(14,21)=42$
e. $\operatorname{LCM}(15,20)=60$
b. $\operatorname{LCM}(10,12)=60$
d. $\operatorname{LCM}(15,8)=120$
f. $\operatorname{LCM}(25,36)=900$
g. $\operatorname{LCM}(4,16,8)=16$
h. $\operatorname{LCM}(8,12,20)=120$
i. $\operatorname{LCM}\left(a^{2} b\right.$ and $\left.x y^{5}\right)=a^{2} b x y^{5}$
19. a. 16 and $32, \mathrm{GCF}=16, \mathrm{LCM}=32$
d. $\quad G C F=12, L C M=240$
b. 24 and $30, \mathrm{GCF}=6, \mathrm{LCM}=120$
e. $\mathrm{GCF}=1 . \mathrm{LCM}=240$
c. 60 and $12, \mathrm{GCF}=12, \mathrm{LCM}=60$
f. $G C F=60, L C M=1,260$
20. a. $\frac{7}{10}$
c. $\frac{7}{10}$
b. $\frac{8}{9}$
d. $\frac{13}{20}$
21. Marina's cookie dough will have $\frac{17}{15}$ or $1 \frac{2}{15}$ cups of the three ingredients.
22. Convert each improper fraction to a mixed number.
a. $\frac{12}{7}$
b. $\frac{23}{4}$
c. $\frac{55}{6}$
d. $\frac{10}{3}$
e. $\frac{79}{12}$
23. Convert each mixed number to an improper fraction.
a. $3 \frac{2}{3}$
b. $5 \frac{4}{5}$
c. $8 \frac{1}{8}$
d. $12 \frac{5}{6}$
24. Find the least common multiple of each pair of integers. Show your work with the Venn Diagram method or the unique factorization method.
a. 8 and 12
b. 10 and 12
c. 14 and 21
d. 15 and 8
e. 15 and 20
f. 25 and 36
g. 4,16 , and 8
h. 8,12 , and 20
i. $\quad a^{2} b$ and $x y^{5}$
25. Find the greatest common factor and least common multiple for each of the following. Show your work with the Venn Diagram method or the unique factorization method.
a. $2 \cdot 2 \cdot 2 \cdot 2$ and $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
b. $2 \cdot 2 \cdot 2 \cdot 3$ and $2 \cdot 3 \cdot 5$
c. $2 \cdot 2 \cdot 3 \cdot 5$ and $2 \cdot 2 \cdot 3$
d. 48 and 60
e. 16 and 15
f. $\quad 180$ and 420
26. Add or subtract the following fractions. Write your answers in simplest form.
a. $\frac{4}{7}+\frac{2}{7}$
b. $\frac{5}{9}+\frac{1}{9}$
c. $\frac{5}{6}-\frac{2}{6}$
d. $\frac{8}{12}-\frac{3}{12}$
27. Compute the sums or differences. Write your answers in simplest form.
a. $\frac{3}{10}+\frac{2}{5}$
b. $\frac{5}{9}+\frac{1}{3}$
c. $\frac{5}{6}-\frac{2}{15}$
d. $\frac{1}{4}+\frac{2}{5}$
e. $\frac{5}{8}+\frac{1}{6}$
f. $\frac{7}{8}+\frac{3}{10}$
28. Marina is mixing cookie dough. She would like the dough to include $\frac{1}{3}$ cup chocolate chips, $\frac{3}{5}$ cup peanut butter and $\frac{1}{5}$ cup pecans. How many total cups of these ingredients will the cookie dough have?
29. There will be $\frac{1}{12}$ more white buttons than green buttons. There will also be $\frac{1}{24}$ more green buttons than pink buttons.
30. a. $\frac{16}{12}=1 \frac{4}{12}=1 \frac{1}{3}$
c. $8 \frac{11}{8}=9 \frac{3}{8}$
e. $2 \frac{1}{6}$
b. $\frac{21}{16}=1 \frac{5}{16}$
d. $3 \frac{2}{3}$
f. $4 \frac{7}{12}$
g. $\frac{31}{60}$
h. $\frac{17}{48}$
31. Abigail and Brianna had $5 \frac{3}{3}=6$ feet of ribbon altogether. Brianna has $\frac{2}{3}$ feet more ribbon than Abigail.
32. Nicole spent $7 \frac{1}{2}$ hours babysitting her cousins.
33. Nancy and Leslie are sewing buttons to a quilt. Nancy would like $\frac{1}{8}$ of the buttons to be pink while Leslie would like $\frac{1}{6}$ of the buttons to be green. Both Nancy and Leslie agree that $\frac{1}{4}$ of the buttons should be white. How many more white buttons will be on the quilt than green buttons? How many more green buttons will there be than pink buttons?
34. Compute the following. Write your answer in simplest form.
a. $\frac{3}{4}+\frac{7}{10}$
b. $\frac{3}{8}+\frac{15}{16}$
c. $5 \frac{7}{8}+3 \frac{1}{2}$
d. $\quad 10 \frac{1}{3}-6 \frac{2}{3} \quad$ f. $7 \frac{1}{4}-2 \frac{2}{3}$
e. $8 \frac{2}{3}-6 \frac{1}{2}$
g. $\frac{23}{30}-\frac{3}{12}$
h. $\frac{9}{16}-\frac{5}{24}$
35. Abigail has $2 \frac{2}{3}$ feet of ribbon. Brianna has $3 \frac{1}{3}$ feet of ribbon. How much ribbon do they have all together? How much more ribbon does Brianna have than Abigail?
36. Nicole babysat her younger cousins for $2 \frac{1}{3}$ hours on Monday. She babysat for $1 \frac{1}{2}$ hours on Wednesday and for $3 \frac{3}{4}$ hours on Friday. How much time did she spend babysitting her cousins?

## Section 8.3: 2

Solution: The LCM will have to be divisible by $25=5^{2}$ but not by $5^{3}=125$, and by $64=2^{6}$, so we get two factors of $10=2 \cdot 5$, making the number end in 2 zeros.

Section 8.4: 11, 33, and 99
Solution: The fraction is $\frac{a b}{99}$, so the denominator must divide 99. That leaves $1,3,9,11,33$, and 99 . We can't get 1 , since the numerator would have to be 99 but a and $b$ can't both be 9 . Similarly 3 is out since the numerator can't be 33 or 66 . Also 9 comes only from $11,22,44,55,77$, or 88 . We can get $0.0909 \ldots=\frac{9}{99}=\frac{1}{11}$ (or $0.1818 \ldots=\frac{18}{99}=\frac{2}{11}$, etc. $), 0.0303 \ldots=\frac{3}{99}=\frac{1}{33}$, and $0.0101 \ldots=\frac{1}{99}$.

Section 8.5: 20
Solution: Since the length is multiplied by $1.25=\frac{5}{4}$, the width must be multiplied by $\frac{4}{5}=0.8$, a decrease of $20 \%$.

Section 8.6: 0.222
Solution: Note $0 . a b c=\frac{a}{10}+\frac{b}{100}+\frac{c}{100}$, so the numerator is
$\frac{2 a+2 b+2 c}{10}+\frac{2 a+2 b+2 c}{100}+\frac{2 a+2 b+2 c}{1000}$. Dividing by $a+b+c$ leaves $\frac{2}{10}+\frac{2}{100}+\frac{2}{1000}=0.222$.

## CHALLENGE PROBLEMS

## Section 8.3:

The LCM of $1,2,3, \ldots, 98,99$ ends in how many zeros?

## Section 8.4:

If $0 . a b a b . .$. is a repeating decimal with digits $\mathrm{a} \neq \mathrm{b}$, what are the possible values of the denominator of the reduced fraction?

## Section 8.5:

If the length of a rectangle is increased by $25 \%$, by what percentage must the width be decreased to keep the area from changing?

## Section 8.6:

Let $a, b$, and $c$ denote single digits, not all 0 . Find all possible values of

$$
\frac{0 \cdot a b c+0 \cdot a c b+0 \cdot b a c+0 \cdot b c a+0 \cdot c a b+0 \cdot c b a}{a+b+c}
$$

## Section 9.1 - Multiplication of Fractions

## Big Idea:

Multiplying fractions
Key Objectives:

- Use the linear and area models to multiply fractions.
- Develop a method for multiplying fractions.
- Find the reciprocal of non-zero integers and non-zero fractions.


## Materials:

- Number lines
- Grid paper


## Pedagogical/Orchestration:

Although the linear model for multiplying fractions is understandable, the area model is far superior. Encourage your students to have a repertoire of models, and other mathematical structures they feel comfortable with, when they are introduced to new mathematical concepts. Some, like the number line and the area model, are adaptable to many situations through algebra and calculus.

## Activity:

"Reciprocal Concentration" on CD.

## Vocabulary:

multiplicative inverse, reciprocal

## TEKS:

7.2(A)(E)(F)(G); 7.13(A)(C); 7.14(B); 7.15(A)(B); 8.2(A,C); 8.14(A,B); 8.16(A)

## WARM-UPS for Section 9.1

1. Four fifths of a class likes peanut butter and jelly. If there are 35 students in the class, how many students like peanut butter and jelly?
a. 25 students
b. 28 students
c. 27 students
d. 30 students

Ans: (b) because $\frac{4}{5}=\frac{28}{35}$.
2. Simplify each of the following improper fractions and write the answer as a mixed number:
a. $\frac{22}{4}$
b. $\frac{46}{6}$
C. $\frac{48}{9}$
d. $\frac{72}{16}$

Answers: a. $5 \frac{1}{2}$, b. $7 \frac{2}{3}$, c. $5 \frac{1}{3}$, d. $4 \frac{1}{2}$
3. Nama has $6 \frac{1}{3}$ cups of sugar and gives $2 \frac{2}{3}$ cups to Terry.
a. How much sugar does Nama have left? Ans: $3 \frac{2}{3}$ cups
b. What if Nama started out with $7 \frac{1}{4}$ cups of sugar? Ans: $4 \frac{7}{12}$

## Launch for Section 9.1:

We use the linear model to understand addition and subtraction of fractions. Initially, the linear model is also useful in understanding multiplication of fractions. We will begin by reviewing the linear model.

What would $\frac{1}{3}$ of 6 be? In other words, what is the product $\frac{1}{3} \cdot 6$ ? Illustrate the process on the number line below and represent the product in words.

$\begin{array}{lllllll}0 & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} \\ & & 1 & & & 2\end{array}$

## PROBLEM 1

Another way to write this product is $6 \cdot \frac{1}{3}$. So, $\frac{1}{3} \cdot 6=2$. One third of a jump 6 units long is 2 units long. Revisit the number line with fractions and partitioning fractionally as in Chapter 8 .

Do Example 1 together as a class. It is to show how awkward linear model is for $\frac{1}{2} \cdot \frac{1}{3}$. Transition to area model.

# M U LTIPLYING AND DIVIDING FRACTIONS 

SECTION 9.1 MULTIPLICATION OF FRACTIONS

We used the linear model to understand addition and subtraction of fractions. Initially, the linear model is also useful in understanding multiplication of fractions. We will begin by reviewing the linear model.

## PROBLEM 1

What would $\frac{1}{3}$ of 6 be? In other words, what is the product $\frac{1}{3} \cdot 6$ ? Illustrate the process on the number line below and represent the product in words.


## EXAMPLE 1

Jane has $\frac{1}{2}$ a yard of ribbon and needs to cut $\frac{1}{3}$ of its length. To do this, she finds out what $\frac{1}{3}$ of $\frac{1}{2}$ is. Use a number line to show how much ribbon she cuts.

Teachers, you might point out that "arithmetic" the adjective is not pronounced like "arithmetic" the noun. Encourage students to look up the word in the dictionary.

This is $\frac{1}{3}$ of $\frac{1}{2}$. The area is $\frac{1}{6}$ because it is one out of six equal pieces of the whole.

## SOLUTION

To find how much ribbon Jane cuts, use the linear model to calculate $\frac{1}{3} \cdot \frac{1}{2}$ :


The arithmetic statement is $\frac{1}{3} \cdot \frac{1}{4}=\frac{1}{6}$. If each jump is $\frac{1}{3}$ yard and the frog makes a $\frac{1}{2}$ of a jump, it travels $\frac{4}{6}$ of a yard.

With the linear model it is important to be very exact when drawing the picture. To see the advantage of the area model, look at the problem above: $\frac{1}{3} \cdot \frac{1}{2}$. To begin the process using the area model, draw $\frac{1}{3}$ as a shaded part of the whole rectangle with area 1.


One way to represent $\frac{1}{2}$ of the shaded area is to cut the rectangle representing $\frac{1}{3}$ in half by cutting vertically. But this is the same process as the linear model. Instead, we cut the rectangle into 2 equal pieces by cutting horizontally.


One of the 2 pieces from the second cut is shaded to represent $\frac{1}{2}$ of the original $\frac{1}{3}$ rectangle. What part of the whole rectangle is the double-shaded area?


EXPLORATION 1: Hand out template with blank rectangles (found at the end of this section).

Have groups do one of the following problems and present their work to the rest of the class.
a. The answer is $\frac{1}{5} \cdot \frac{1}{2}=\frac{1}{10}$.
b. Translate into the product: $\frac{1}{3} \cdot \frac{1}{4}=\frac{1}{12}$.
c. The answer is $\frac{1}{42}$.
d. Have students put up representations of these products. Reflect on whether they used the linear or area model. The area model is more transparent and easier to see. One of the best ways to show part (d) is to simply rotate the rectangle so the vertical and horizontal divisions are switched.

## PROBLEM 2

$\frac{3}{8}$ is the area that is shaded and cross-hatched.
Have students explore these problems individually and in groups. Also, make sure they know why $b$ and $d$ cannot be zero.
a. $\frac{1}{3}$ Use the area model of rectangles and have $\frac{2}{3}$ vertically and $\frac{1}{2}$ horizontally. Cross hatching should occur
in $\frac{1}{3}$ of the rectangle.
b. $\frac{4}{15}$ Use the area model of rectangles and have $\frac{2}{5}$ vertically and $\frac{2}{3}$ horizontally. Cross hatching should occur in $\frac{4}{15}$ of the rectangle.

c. $\frac{12}{35}$ Use the area model of rectangles and have $\frac{3}{5}$ vertically and $\frac{4}{7}$ horizontally. Cross hatching should occur in $\frac{12}{35}$ of the rectangle.
d. The answer is $\frac{15}{28}$.

The product is less than both of the factors.

## EXPLORATION 1

a. Translate $\frac{1}{2}$ of $\frac{1}{5}$ into a multiplication problem and draw the corresponding picture to find the product.
b. Translate $\frac{1}{4}$ of $\frac{1}{3}$ into a multiplication problem and draw the corresponding picture to find the product.
c. Predict what is $\frac{1}{6}$ of $\frac{1}{7}$ without drawing a model.
d. Explain why multiplication of unit fractions is commutative, that is

$$
\frac{1}{m} \cdot \frac{1}{n}=\frac{1}{n} \cdot \frac{1}{m}
$$

## PROBLEM 2

What is $\frac{1}{2}$ of $\frac{3}{4}$ ?


## EXPLORATION 2

What is the product of the fractions $\frac{a}{b}$ and $\frac{c}{d}$, where $a, b, c$, and $d$ are positive integers with $b$ and $d$ not zero? Use the area model to compute the following products:
a. What is $\frac{1}{2}$ of $\frac{2}{3}$ ? Use the area model to illustrate and find the answer.
b. What is $\frac{2}{5}$ of $\frac{2}{3}$ ? Use the area model to illustrate and find the answer.
c. What is $\frac{3}{5}$ of $\frac{4}{7}$ ? Use the area model to illustrate and find the answer.
d. Predict what $\frac{3}{4}$ of $\frac{5}{7}$ is without drawing a model.

What do you notice about the product of two proper fractions?

PROBLEM 3
a. $\frac{3}{8}$
b. $\frac{6}{35}$
c. 1
d. 1
e. $\frac{5}{12}$

Summarizing this pattern of multiplying fractions,

## RULE 9.1: MULTIPLYING FRACTIONS

The product of two fractions $\frac{a}{b}$ and $\frac{c}{d}$, where $b$ and $d$ are non-zero, is

$$
\frac{a}{b} \cdot \frac{c}{d}=\frac{a \cdot c}{b \cdot d}
$$

## PROBLEM 3

Multiply the following fractions:
a. $\frac{1}{2} \cdot \frac{3}{4}$
b. $\frac{3}{5} \cdot \frac{2}{7}$
c. $\frac{3}{5} \cdot \frac{5}{3}$
d. $\frac{3}{4} \cdot \frac{4}{3}$
e. $\frac{5}{9} \cdot \frac{3}{4}$

In the linear model, we see that:

$$
\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{1}{5} \cdot 5=1 .
$$

This leads us to the following definition:

RULE 9.2: RECIPROCAL OF AN INTEGER
If $n$ is a number, other than 0 , then the multiplicative inverse, or reciprocal, of $n$ is the fraction $\frac{1}{n}$. The product $\frac{1}{n} \cdot n$ is

$$
\frac{1}{n} \cdot n=1 .
$$

## EXPLORATION 3

This is a great opportunity to see if your students try to apply the Rule for Multiplying Fractions and see that the numerator must equal the denominator, so $\left(\frac{a}{b}\right)\left(\frac{b}{a}\right)=1$ implies $\left(\frac{2}{3}\right)\left(\frac{3}{2}\right)=1$. If they try the linear or area model, they might say, "What jump length will yield 1 after $\frac{2}{3}$ of a jump?" Or, they might try to solve $\left(\frac{2}{3}\right)(?)=$ 1, which asks, "How many jumps of length $\frac{2}{3}$ will reach 1 unit?" You may point out that this involves solving the equation $\left(\frac{2}{3}\right) \cdot x=1$.
Using the number line helps to visualize that $\frac{4}{3}$ of $\frac{3}{4}$ equals can be understood as $1 \cdot \frac{3}{4}+\frac{1}{3}$ of $\frac{3}{4}=\frac{3}{4}+\frac{1}{4}$, which is 1 . So, $1+\frac{1}{3}=\frac{4}{3}$, the reciprocal of $\frac{3}{4}$
Remind students of parts c and d in the Problem above.

Have your students discuss the patterns they have seen in previous Explorations. The reciprocal of the fraction $\frac{a}{b}$ is the fraction $\frac{b}{a}$ where neither $a$ nor $b$ is zero.

What is the relationship between the rule for Reciprocals of an Integer and the rule for Multiplying Fractions? Each positive integer $n$ can be written as the fraction $\frac{n}{1}$. Substituting $\frac{n}{1}$ for $n$ in the Rule for Reciprocals and using the fact that $n \cdot 1=n$, the equation becomes

$$
\frac{1}{n} \cdot n=\frac{1}{n} \cdot \frac{n}{1}=\frac{1 \cdot n}{n \cdot 1}=\frac{n}{n}=1
$$

## EXPLORATION 3

What fraction can be multiplied by $\frac{2}{3}$ to get 1 ? In other words, what times $\frac{2}{3}$ equals 1? Explain your answer.

Remember that the reciprocal of a number is the number that, when multiplied by the original number, equals 1 . What is the reciprocal of $\frac{3}{4}$ ? Verify that the product of $\frac{3}{4}$ and its reciprocal is 1 .
Make a conjecture about the reciprocal of any fraction $\frac{\mathrm{a}}{\mathrm{b}}$.
For example, the reciprocal of $\frac{x}{y}$ is $\frac{y}{x}$. This makes sense because $\frac{x}{y} \cdot \frac{y}{x}=\frac{x y}{y x}=\frac{x y}{x y}=1$.
You found that the product of $\frac{2}{3}$ and $\frac{3}{2}$ equals 1 , and the fractions are reciprocals of each other. Notice that $\frac{3}{2}$ is a fraction larger than 1 . In general, if a positive number is less than 1 , then its reciprocal is greater than 1 .

## RULE 9.3: RECIPROCAL OF A FRACTION

In general, the multiplicative inverse or reciprocal of $\frac{x}{y}$ is the fraction $\frac{y}{x}$ since

$$
\frac{x}{y} \cdot \frac{y}{x}=\frac{x y}{y x}=\frac{x y}{x y}=1 .
$$

## EXAMPLE 2

Discuss the different ways to represent that 8 is $\frac{1}{3}$ of 24 .

PROBLEM 4
a. 3
b. 4
c. $\frac{4}{15}$
d. 15

## EXAMPLE 2

Lisa has 24 books in her library, one third of which are hardback books.
a. How many of her library books are hardback?
b. How many of her books are not hardback?

## SOLUTION

a. Take $\frac{1}{3}$ of $24: 24 \cdot \frac{1}{3}=\frac{24}{1} \cdot \frac{1}{3}=\frac{24}{3}=24 \div 3=8$ books.
b. Take $\frac{2}{3}$ of 24: $24 \cdot \frac{2}{3}=\frac{24}{1} \cdot \frac{2}{3}=\frac{48}{3}=48 \div 3=16$ books.

Or, you can subtract 8 from 24 to get 16 books.

## PROBLEM 4

Compute each of the following, and simplify as needed.
a. $12 \cdot \frac{1}{4}$
b. $12 \cdot \frac{1}{3}$
c. $\frac{4}{5} \cdot \frac{1}{3}$
d. $\frac{3}{7} \cdot 35$

## EXAMPLE 3

Compute the product $\frac{21}{32} \cdot \frac{16}{35}$.

## SOLUTION

Using the Rule for Multiplying Fractions,

$$
\frac{21}{32} \cdot \frac{16}{35}=\frac{21 \cdot 16}{32 \cdot 35}=\frac{336}{1120}
$$

## PROBLEM 5

a. 3
b. $\quad \frac{5}{34}$
c. $\quad \frac{156}{245}$
d. $\frac{6}{15}$

The answer certainly does not look simplified. To simplify this fraction, before you multiply the numerators and denominators, write the numerator and denominator in their prime factorization form. Completing the factoring process,

$$
\begin{aligned}
& 21 \cdot 16=(3 \cdot 7) \cdot(2 \cdot 2 \cdot 2 \cdot 2)=2^{4} \cdot 3 \cdot 7 \\
& 32 \cdot 35=(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) \cdot(5 \cdot 7)=2^{5} \cdot 5 \cdot 7
\end{aligned}
$$

The answer can be simplified by looking for an equivalent fraction as follows:

$$
\frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 7}=\frac{2^{4} \cdot 3 \cdot 7}{2^{5} \cdot 5 \cdot 7}=\frac{3}{2 \cdot 5}=\frac{3}{10}
$$

In the next section, we will explore division of fractions more carefully.

## PROBLEM 5

Compute each of the following products and simplify as needed:
a. $\quad \frac{5}{1} \cdot \frac{3}{5}$
b. $\frac{12}{17} \cdot \frac{5}{24}$
c. $\quad \frac{36}{49} \cdot \frac{13}{15}$
d. $\frac{24}{25} \cdot \frac{15}{36}$

Consider the problem of computing the product $2 \frac{1}{3} \cdot 3 \frac{1}{2}$. Recall from Section 8.2 that $2 \frac{1}{3}$ is also the sum $2+\frac{1}{3}$ and that $3 \frac{1}{2}$ is $3+\frac{4}{2}$. To multiply two mixed numbers, convert the product of mixed numbers $2 \frac{1}{3} \cdot 3 \frac{1}{2}$ to the product of sums:

$$
\left(2+\frac{1}{3}\right)\left(3+\frac{1}{2}\right)=2 \cdot 3+2 \cdot \frac{1}{2}+3 \cdot \frac{1}{3}+\frac{1}{3} \cdot \frac{1}{2}=6+1+1+\frac{1}{6}=8 \frac{1}{6} .
$$

Below is a visual model of this product using the area model.

Have a discussion, and use the area model, to see the 2 by 3 rectangle inside, and outline a 3 by 4 rectangle on the outside of the model above.

PROBLEM 6
a. $16 \frac{1}{2}$
b. $4 \frac{8}{15}$
c. $12 \frac{5}{6}$
d. $27 \frac{13}{15}$


The area model illustrates the use of the distributive property that was introduced previously in Section 4.2. One advantage of this method is that you can estimate that the product will be greater than $2 \cdot 3$ and less than $3 \cdot 4$. That is, the product will be between 6 and 12. Explain why this is true.

Remember in Section 8.5 Example 2, you can add two mixed numbers by first rewriting the fractions as improper fractions. Similarly, you can multiply two mixed numbers by first rewriting them as improper fractions. For example, the product $2 \frac{1}{3} \cdot 3 \frac{1}{2}$ could be computed as

$$
2 \frac{1}{3} \cdot 3 \frac{1}{2}=\frac{7}{3} \cdot \frac{7}{2}=\frac{49}{6} .
$$

Check to see if the answer is in simplest form. The answer can be left as an improper fraction or converted back to a mixed number, whichever is more useful for the original problem.

## PROBLEM 6

Find the products of each of the following problems. Write your answer as an improper fraction in simplest form and as a simplified mixed number.
a. $4 \frac{1}{2} \cdot 3 \frac{2}{3}$
b. $3 \frac{2}{5} \cdot 1 \frac{1}{3}$
c. $2 \frac{3}{4} \cdot 4 \frac{2}{3}$
d. $3 \frac{4}{5} \cdot 7 \frac{1}{3}$
1.
a. $\frac{4}{9}$
b. $\quad \frac{3}{10}$
c. $\quad \frac{1}{10}$
d. $\quad \frac{5}{21}$
e. $\frac{5}{24}$
2.
a. $\frac{5}{3}$
b. $\quad \frac{21}{16}$
c. 4
d. $\frac{91}{118}$
e. $\frac{y}{x}$
3. It should be relatively easy to convince your students that simplifying before multiplication is easier and simpler than multiplying first.
a. $\frac{1}{8}$
$\begin{array}{ll}\text { e. } & \frac{12}{35} \\ \text { f. } & \frac{1}{1500} \\ \text { g. } & \frac{9}{64}\end{array}$
h. $\frac{1}{6}$
b. 1
c. $\frac{1}{72}$
i. 18
d. 2
4. Always use units in the answer of a word problem, if appropriate, even if it is not specifically asked for.
$\frac{4}{5} \cdot \frac{1}{3}=\frac{4}{15}$ mile
5. $\frac{3}{8} \cdot 24=9$ students
6. (C) multiply 345.50 by $\frac{2}{5}$

## EXERCISES

1. Compute the following products and simplify if needed:
a. $\frac{2}{3} \cdot \frac{2}{3}$
b. $\frac{1}{2} \cdot \frac{3}{5}$
c. $\quad \frac{2}{5} \cdot \frac{1}{4}$
d. $\frac{2}{7} \cdot \frac{5}{6}$
e. $\frac{5}{8} \cdot \frac{1}{3}$
2. Find the reciprocal of the following:
a. $\frac{3}{5}$
C. $\frac{7}{28}$
b. $\frac{16}{21}$
d. $\frac{118}{91}$
e. $\frac{x}{y}$
f. $\frac{3 a}{5 b}$
3. Compute the following products and simplify if needed:
a. $\frac{5}{12} \cdot \frac{3}{10}$
b. $\frac{7}{5} \cdot \frac{5}{7}$
c. $\frac{1}{6} \cdot \frac{1}{12}$
d. $\frac{1}{6} \cdot 12$
e. $\frac{10}{21} \cdot \frac{18}{25}$
f. $\frac{1}{20} \cdot \frac{1}{75}$
g. $\frac{25}{48} \cdot \frac{27}{100}$
h. $\frac{19}{21} \cdot \frac{7}{38}$
i. $27 \cdot \frac{2}{3}$
j. $\frac{8}{45} \cdot \frac{9}{40}$
4. Beth lives $\frac{4}{5}$ of a mile from her school. Walking to school, $\frac{1}{3}$ of the trip is downhill. How long is the trip downhill?
5. Three-eighths of Ms. Chen's class have a birthday in the summer. If her class has 24 students, how many of them have birthdays in the summer?
6. Mr. Alexander earned $\$ 345.50$ commission this week. He earned $\frac{2}{5}$ of his commission on Tuesday. Which equation can be used to determine the amount of commission, $m$, Mr. Alexander earned on Tuesday? Select the best choice and explain your answer.
a. $345.50 \div \frac{2}{5}$
b. $345.50-\frac{2}{5}$
7. $\frac{3}{4} \cdot \frac{2}{5}=\frac{3}{10}$ square miles
8. D
9. The answer is $\frac{3}{8} \div 3$, which is the same as $\frac{3}{8} \cdot \frac{1}{3}=\frac{1}{8}$ of a pan of brownies to each sister.
10. a. This is an addition and subtraction problem: $1-\left(\frac{1}{3}+\frac{2}{5}\right)=1-\frac{11}{15}=\frac{4}{15}$. The students who prefer juice or soft drinks is $\frac{2}{5}+\frac{1}{3}=\frac{11}{15}$ of the student body. The fraction of those who prefer water is $1-\frac{11}{15}=$ $\frac{4}{15}$ of the student body.
b. The fraction of the student body that prefers sugar-free soft drinks is $\frac{3}{4} \cdot \frac{1}{3}=\frac{1}{4}$.
c. $\frac{2}{5} \cdot 300=120$ prefer juice, $\frac{1}{3} \cdot 300=100$ prefer soft drinks, and $\frac{4^{4}}{15} \cdot 300=80$ prefer water. (Note: of those who prefer soft drinks, 75 prefer sugar-free, and 25 prefer regular soft drinks.)
11. There are $30 \cdot \frac{3}{5}=18$ students with dogs, and $18 \cdot \frac{1}{3}=6$ students with Chihuahuas.
12. $\left(2 \frac{1}{2}\right) \cdot\left(4 \frac{2}{3}\right)=\left(\frac{5}{2}\right) \cdot\left(\frac{14}{3}\right)=\frac{35}{3}=11 \frac{2}{3}$ square meters of cloth.
c. $345.50 \times \frac{2}{5}$
d. $345.50 \div \frac{2}{5}$
13. Mr. Rodriguez has a large farm that is in the shape of a rectangle. It is $\frac{3}{4}$ of a mile long and $\frac{2}{5}$ of a mile wide. What is his farm's area?
14. Which rule can be used to find the value of the $n$th term in the sequence below, where $n$ represents the position of the term? Explain your answer.

| Position | Value of Term |
| :---: | :---: |
| 1 | 5 |
| 2 | 8 |
| 3 | 11 |
| 4 | 14 |
| 5 | 17 |
| $n$ |  |

a. $5 n$
b. $3 n+1$
c. $2 n+1$
d. $3 \mathrm{n}+2$
9. Jack has $\frac{3}{8}$ of a pan of brownies left over from a party. He wants to give what is left equally to his 3 younger sisters. How much will each sister receive?
10. In a school survey, $\frac{2}{5}$ of the students preferred juice, $\frac{1}{3}$ preferred soft drinks, and the rest preferred water.
a. What fraction of the student body preferred water?
b. Of those students who preferred soft drinks, $\frac{3}{4}$ of them preferred sugarfree to regular soft drinks. What fraction of the student body prefers sugar-free soft drinks?
c. There are 300 students in the school. How many students prefer each kind of beverage?
11. Mr. Johnson took a survey of his class. He found that $\frac{3}{5}$ of his students have dogs and that $\frac{1}{3}$ of these students have a Chihuahua. If there are 30 students in his class, how many have a Chihuahua?
12. Ramon is making a bulletin board that is $2 \frac{1}{2}$ meters wide and $4 \frac{2}{3}$ meter long. He plans to cover the board with cloth. How much cloth does he need?
13. al $\left.\frac{4}{5}\right) \cdot(1200)=\left(\frac{9}{5}\right) \cdot 1200 .=\frac{5 a c}{b^{0}} 0 \cdot d \frac{b c}{\frac{6}{a}} a^{2}=\frac{5 c^{2}}{7 a b}$
b. $\frac{3}{x y} \cdot \frac{y}{6 x}=\frac{1}{2 x^{2}}$
e. $\frac{4 a^{2}}{3 b^{2} c^{3}} \cdot \frac{b^{3} c^{2}}{13 a^{3}}=\frac{4 b}{39 a c}$
c. $\frac{4 x}{y^{2}} \cdot \frac{y}{10 x^{3}}=\frac{2}{5 x^{2} y} \quad$ f. $\quad \frac{2 y x}{5 x^{2}} \cdot \frac{15 x}{4 z^{3}}=\frac{3 y}{2 x z^{2}}$
g. $\quad \frac{6 a x}{b^{2} y} \cdot \frac{b y^{3}}{10 a x^{3}}=\frac{3 y^{2}}{5 x^{2} b}$
h. $\frac{4 b x}{3 a y} \cdot \frac{3 a y}{4 b x}=\frac{1}{1}=1$
i. $\quad \frac{4 x}{y^{2}} \cdot \frac{2 x^{2}}{5 y^{3}}=\frac{8 x^{3}}{5 y^{5}}$

## Ingenuity

15. The number of servings is the number of cakes multiplied by the number of servings in each cake. $5 \frac{1}{4}$ cakes $=$ $\frac{21}{4}$. So $\frac{21}{4}(16)=84$ servings.

## Investigation

16. a. $1-x^{2}$
b. $1-x^{3}$
c. $1-x^{4}$
d. $1-x^{16}$
17. In 1960, a large ranch in south Texas had a population of about 1200 deer. By 1995 , the deer population on the ranch was $1 \frac{4}{5}$ times as large as in 1960 . What was the deer population on the ranch in 1995?
18. Compute each of the following products. Assume that each variable is a non-zero number. Simplify the answer if needed.
a. $\frac{x}{y} \cdot \frac{y}{x}$
b. $\frac{3}{x y} \cdot \frac{y}{6 x}$
c. $\frac{4 x}{y^{2}} \cdot \frac{y}{10 x^{2}}$
d. $\frac{5 a c}{b^{2}} \cdot \frac{b c}{7 a^{2}}$
e. $\frac{4 a^{2}}{3 b^{2} c^{3}} \cdot \frac{b^{3} c^{2}}{13 a^{3}}$
f. $\frac{2 y z}{5 x^{2}} \cdot \frac{15 x}{4 z^{3}}$
g. $\frac{6 a x}{b^{2} y} \cdot \frac{b y^{3}}{10 a x^{3}}$
h. $\frac{4 b x}{3 a y} \cdot \frac{3 a y}{4 b x}$
i. $\frac{4 x}{y^{2}} \cdot \frac{2 x^{2}}{5 y^{3}}$
19. Ingenuity:

At a wedding party, the cook made a number of equal size cakes. Each person is served $\frac{1}{16}$ of a piece of a cake. The server counts that there are 5 $\frac{1}{4}$ cakes left. How many more servings can she cut?
16. Investigation:

Compute each of these products:
a. $(1+x)(1-x)$
b. $\left(1+x+x^{2}\right)(1-x)$
c. $\left(1+x+x^{2}+x^{3}\right)(1-x)$
d. Guess what the following product is: $\left(1+x+x^{2}+x^{3}+\ldots+x^{14}+x^{15}\right)(1-x)$

## Section 9.2 - Division of Fractions

## Big Idea:

Dividing fractions

## Key Objectives:

- Use linear and area models to understand the meaning of division of fractions.
- Understand division by a fraction and division of a fraction.
- Develop a method for dividing fractions.
- Understand the relationship between fractions and division.
- Simplify complex fractions that have fractions in both the numerator and denominator.
- Use fractions in daily situations.


## Materials:

Number lines, Grid paper, Ribbon and Scissors for Launch

## Pedagogical/Orchestration:

This section explains in several diverse ways the division of fractions. Another possibly helpful idea is to go back to one of the basic ideas of multiplication. The product of 2 and 3 can involve taking 3 groups of 2 . In a similar way, dividing 4 by $\frac{1}{2}$ involves dividing 4 into a number of halves. So dividing 4 by $\frac{1}{2}$ answers the question, "How many halves are there in 4 ?" Using the area model, there are 8 halves in 4 , so 4 divided by $\frac{1}{2}=8$.

## Activities:

Launch Ribbon Cutting activity connects to measurement.
"Tic Tac Frac" at the end of the section and on CD

## Exercises:

Make sure your students understand that the complex fractions in Exercises 2 and 13 are a different way of representing division problems.

## Vocabulary:

no new vocabulary

## TEKS:

7.2(A)(B)(D)(E)(F)(G); 7.5(A); 7.13(C); 8.2(A,B,C,D)

## WARM-UPS for Section 9.2

1. Elena's hair grows $\frac{1}{4}$ of an inch of a week. How long does it take her to grow 8 more inches of hair?
a. 26 weeks
b. 28 weeks
c. 30 weeks
d. 32 weeks

Ans: Each inch is 4 weeks, so 8 inches takes 32 weeks.
2. At 10:20 a.m., Janet looks at two jobs she needs to finish. One task will take half an hour and the second task will take a third of an hour. If she takes a fifteen-minute break between the two tasks, when will she finish the second task? Ans: 11:25a.m.

## Launch for Section 9.2:

Tell students, "The big idea for today will be dividing with fractions. Let's go back to the idea of division with whole numbers. What is one way to think about 24 divided by 4?" Students will probably just give the answer, 6 , but remind them that $24 \div 4$ is another way of asking, "How many groups of 4 are in 24 ?" (The first page of this section discusses the models of division.) Tell your students, "Now let's relate this idea to fractions. If you are asked to work a problem such as $4 \div \frac{1}{2}$, how can you relate this to the meaning of division? In other words, how many half units are there in 4 units? We will do an activity that will help us make sense of this." Put the students in their groups and hand each group a length of ribbon 4 feet long. Tell them it takes a certain length of ribbon to make a bracelet or necklace. Assign each group a different fractional length so that the ribbon is cut into pieces of that size. For instance, the groups could be assigned $\frac{1}{2}$ foot lengths, $\frac{1}{4}$ foot lengths, $\frac{3}{4}$ foot lengths and so forth. The question is how many bracelets or necklaces can they make of given length from the 4 foot ribbon they were given. Also ask them what division problem they are modeling. For instance, the group assigned $\frac{1}{2}$ foot lengths is modeling $4 \div \frac{1}{2}$. Have a class discussion on the results and write the results for each group on the board. Inform your students, "As you work through today's lesson, think about the meaning of division to help you come up with methods for dividing fractions, and make sure you notice patterns that will make the process easier."

## PROBLEM 1

Using the linear model, count by jumps of length $3 / 4$ to find that it takes 4 jumps. So $3 \div \frac{3}{4}=4$. Checking, we see that $\frac{3}{4} \cdot 4=3$ is true.

## EXPLORATION 2

Have students draw the area model. We want them to discover that dividing by 5 is the same as multiplying by $\frac{1}{5} \ldots$


## PROBLEM 2

Each person will receive $\frac{3}{12}$ or $\frac{1}{4}$ of a pizza.


## SECTION 9.2 DIVISION OF FRACTIONS

## EXPLORATION 1

Rene has 6 pounds of jellybeans. She plans to make little party bags containing $\frac{3}{8}$ pound of jellybeans. How many party bags can she make?

## PROBLEM 1

Compute the following quotients:
What is 2 divided by $\frac{1}{3}$, that is, $2 \div \frac{1}{3}$ ?
What is 3 divided by $\frac{1}{5}$, that is, $3 \div \frac{1}{5}$ ?
What is 3 divided by $\frac{3}{4}$, that is, $3 \div \frac{3}{4}$ ?

## EXPLORATION 2

Madison has two-thirds of a pan of brownies and shares it evenly among her five friends. How much does each friend receive? Using the area model, each friend gets $\frac{2}{15}$ of Madison's brownie pan or

$$
\frac{1}{5} \cdot \frac{2}{3}=\frac{2}{3} \cdot \frac{1}{5}=\frac{2}{15} .
$$

## PROBLEM 2

Maria has $\frac{3}{4}$ pizza in the refrigerator. She wants to share this equally with 3 friends. Use an area model to determine how much of a pizza each person will get.

In Exploration 2, dividing by 5 produces the same result as multiplying by $\frac{1}{5}$. You can verify that $\frac{2}{3} \div 5=\frac{2}{15}$ by checking to see that

$$
5 \cdot \frac{2}{15}=\frac{10}{15}=\frac{2}{3} .
$$

Use the area model to check that this relationship between division by a whole and multiplication by a fraction works for each of the following:
a. $\frac{1}{4} \div 2$
b. $\frac{3}{5} \div 6$
c. $\frac{2}{5} \div 4$
d. $\frac{5}{6} \div 3$

You've discovered that dividing by $n$ is the same as multiplying by $\frac{1}{n}$. The following sequence of calculations shows the connection between division and multiplication with fractions:

$$
\begin{array}{ll}
1 \div 5=1 \cdot \frac{1}{5}=\frac{1}{5} & 4 \div 10=4 \cdot \frac{1}{10}=\frac{4}{10} \\
2 \div 3=2 \cdot \frac{1}{3}=\frac{2}{3} & 5 \div 5=5 \cdot \frac{1}{5}=\frac{5}{5}=1 \\
3 \div 8=3 \cdot \frac{1}{8}=\frac{3}{8} &
\end{array}
$$

In general, a fraction $\frac{m}{1^{n}}$ is another way to write the division problem $m \div n$. Also, as above, $\frac{m}{n}=m \cdot \frac{1}{n}$.

Look at a similar problem but with fractional quantities: How many $\frac{1}{4}$-pound bags does it take to pack $\frac{3}{4}$ pounds of sand? In other words, what is $\frac{3}{4} \div \frac{1}{4}$ ?

The number of $\frac{1}{4}$-pound bags.

You may use measuring cups and beans as a visual aid to demonstrate $\frac{1}{8} \div \frac{1}{4}$. And ask the question "How many one-fourth's are in $\frac{1}{8}$ ?

Using a repeated subtraction model, make 3 equal parts. With the first $\frac{1}{4}$-pound bag, $\frac{3}{4}-\frac{1}{4}=\frac{1}{2}$ pounds are left. The second $\frac{1}{4}$-pound leaves $\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$, so the third $\frac{1}{4}$-pound bag leaves no sand.


Writing this as a division problem, $\frac{3}{4} \div \frac{1}{4}=3$. At first, it might be surprising that when dividing two fractions, the answer is an integer, especially when the integer is large compared to the fractions. What does 3 represent in this case?

Another way to think about this problem uses the missing factor method. What number times $\frac{1}{4}$ equals $\frac{3}{4}$ ? Starting at 0 on the number line, 3 jumps of length $\frac{1}{4}$ equals $\frac{3}{4}$. So, $\frac{3}{4} \div \frac{1^{4}}{4}=3$.


Notice in the earlier example with bags of sand, the quantity of sand exceeded the bag size. A $\frac{3}{4}$-pound bag was being separated into smaller $\frac{1}{4}$-pound bags. The number of bags was $\frac{3}{4} \div \frac{1}{4}=3$ bags. What if the initial quantity is less than the bag size, like having $\frac{1}{8}$ pound of sand and a bag that holds $\frac{1}{4}$ of a pound? What is $\frac{1}{8} \div \frac{1}{4}$ ?


It is impossible to use a "repeated-subtraction" model, because there is no way to fill even one $\frac{1}{4}$-pound bag with only $\frac{1}{8}$ pound of sand. You can see that with the $\frac{1}{8}$ pound, only $\frac{1}{2}$ of the bag is filled. Therefore, $\frac{1}{8} \div \frac{1}{4}=\frac{1}{2}$.

PROBLEM 3
a. $\quad \frac{\frac{1}{2}}{\frac{1}{3}}=\frac{\frac{1}{2}}{\frac{1}{3}} \cdot \frac{\frac{3}{1}}{\frac{1}{3}}=\frac{\frac{1}{2} \cdot \frac{3}{1}}{\frac{1}{3} \cdot \frac{3}{1}}=\frac{\frac{3}{2}}{1}=\frac{3}{2}$
b. $\quad \frac{1}{3} \div \frac{1}{2}=\frac{\frac{1}{3}}{\frac{1}{2}}=\frac{\frac{1}{3} \cdot \frac{2}{1}}{\frac{1}{2} \cdot \frac{2}{1}}=\frac{\frac{2}{3}}{1}=\frac{2}{3}$

Using the relationship between fractions and division, that $m \div n$ is the same as $\frac{\mathrm{m}}{\mathrm{n}}$, rewrite $\frac{1}{8} \div \frac{1}{4}$ as a big fraction, that is $\frac{\frac{1}{8}}{\frac{1}{8}}$. Now this looks pretty complicated, but luckily it can be simplified. $\frac{1}{4}$

The first problem with the big fraction is that the denominator is a fraction. Recall that simplifying a fraction requires rewriting the fraction as an equivalent fraction. But instead of factoring the numerator and denominator, create an equivalent fraction by multiplying both the numerator and denominator by a number that will convert the denominator to a very friendly product. To get a friendly product, multiply the denominator by its own reciprocal. What happens to the denominator when it is multiplied by its reciprocal? To produce an equivalent fraction, the numerator must also be multiplied by the reciprocal of the denominator.

$$
\frac{1}{8} \div \frac{1}{4}=\frac{\frac{1}{8}}{\frac{1}{4}}=\frac{\frac{1}{8}}{\frac{1}{4}} \cdot \frac{\frac{4}{1}}{\frac{4}{1}}=\frac{\frac{4}{8}}{\frac{4}{4}}=\frac{\frac{4}{8}}{1}=\frac{4}{8}=\frac{1}{2}
$$

In general, when the denominator of a fraction is a fraction, multiplying both the numerator and denominator by the reciprocal of the denominator produces a simpler fraction. Another way to simplify complicated fractions uses the pattern that $m \div n=\frac{m}{n}=m\left(\frac{1}{n}\right)$. Because of this, dividing by $n$ is the same as multiplying by the reciprocal of $n$. Using this pattern, dividing by $\left(\frac{1}{n}\right)$ is the same as multiplying by the reciprocal of $\frac{1}{n}$, which is $n$.
Using this pattern, rewrite $\frac{\frac{1}{8}}{\frac{1}{4}}$ as $\frac{1}{4} \cdot \frac{4}{1}$, because the reciprocal of $\frac{1}{4}$ is $\frac{4}{1}$. Then multiply to find the answer: ${ }^{\frac{1}{8}} \frac{1}{2} \cdot \frac{4^{8}}{1}=\frac{4^{\prime}}{8}=\frac{1}{2}$.

## PROBLEM 3

Compute the following division of fractions using the stacking method from above.
a. $\frac{1}{2} \div \frac{1}{3}$
b. $\frac{1}{3} \div \frac{1}{2}$

## PROBLEM 4

$\frac{1}{6} \div \frac{2}{3}=\frac{\frac{1}{6}}{\frac{2}{3}}=\frac{\frac{1}{6}}{\frac{2}{3}} \cdot \frac{\frac{3}{2}}{\frac{3}{2}}=\frac{\frac{3}{12}}{\frac{6}{6}}=\frac{\frac{3}{12}}{1}=\frac{3}{12}=\frac{1}{4}$

## PROBLEM 5

Encourage students to do the division of fractions using the stacking method as shown below for the first 10 problems, before letting them use the "short method" of multiplying by the reciprocal of the divisor.

Visual models are great but quickly get too complicated to draw. The computation showing the steps in detail looks like this:

$$
\frac{7}{10} \div \frac{2}{5}=\frac{\frac{7}{10}}{\frac{2}{5}}=\frac{\frac{7}{10} \cdot \frac{5}{2}}{\frac{2}{5} \cdot \frac{5}{2}}=\frac{\frac{7}{10} \cdot \frac{5}{2}}{1}=\frac{7}{10} \cdot \frac{5}{2}=\frac{7 \cdot 5}{10 \cdot 2}=\frac{7 \cdot 5}{2 \cdot 5 \cdot 2}=\frac{7}{4}=1 \frac{3}{4}
$$

A straightforward way is to use the approach above that $\frac{m}{n}=m \cdot \frac{1}{n}$. We then have $\frac{7}{10} \div \frac{2}{5}=\frac{7}{10} \cdot \frac{5}{2}=\frac{7}{2}$ - $\frac{1}{2}=\frac{7}{4}=1 \frac{3}{4}$.

Note on Division by Zero:
Division by zero is an undefined operation. Why is this so? We know that $6 \div 2=3$, means $2 \cdot 3=6$.
What could $2 \div 0$ mean? If $2 \div 0$ is some number $n$ then $0 \cdot n=2$. What could $n$ be? Nothing. No number times 0 is equal to 2 . Try some possible values for $n$ if your students seem doubtful.

Students might also ask about $0 \div 0 . \frac{0}{0}$ could be equal to 1 because any number divided by itself is 1 . Or $\frac{0}{0}$ could be equal to 0 because 0 divided by any number is 0 . But no number can be equal to both 1 and 0 . Therefore, this division by zero is undefined.

Always make sure your students know why $\mathrm{b}, \mathrm{c}$ and d cannot be zero. If n is zero, then we have $\mathrm{m} \div 0$. But this equals no number because if there were a number, call it q , then $\mathrm{q} \cdot 0$ must equal m . If m is not 0 , there is a problem because $\mathrm{q} \cdot 0=0$. If $\mathrm{m}=0$ then q can be anything. However, by the definition of division, the quotient must be a unique number. Therefore, fractions like $\frac{m}{n}$ must always have denominators that are nonzero.

## PROBLEM 4

Christina's bird feeder holds $\frac{1}{6}$ of a cup of birdseed. Christina is filling the bird feeder with a scoop that holds $\frac{2}{3}$ of a cup. How many scoops of birdseed will Christina put into the feeder? Use the numerical technique from above. Write your answer in simplest form.

## PROBLEM 5

Compute the following quotients.
a. $\frac{7}{10} \div \frac{2}{5}$
b. $\frac{2}{5} \div \frac{7}{10}$
c. $\frac{7}{8} \div \frac{1}{6}$
d. $\frac{3}{11} \div \frac{6}{33}$

## RULE 9.4: RECIPROCAL OF AN INTEGER

If $n$ is a positive integer, then the multiplicative inverse or reciprocal of $n$ is the unit fraction $\frac{1}{n}$. The product $n \cdot \frac{1}{n}$ is

$$
n \cdot \frac{1}{n}=1
$$

Note that, by the commutative property of multiplication:
$n \cdot \frac{1}{n}=\frac{1}{n} \cdot n=1$

To summarize, when dividing by a fraction, or simplifying a fraction whose denominator is a fraction, use one of the two following techniques:
Method 1: Write the division problem as a fraction and multiply the numerator and denominator of this fraction by the reciprocal of the denominator. Notice that multiplying both the numerator and denominator of a fraction by the same number, does not change the value of the fraction, because $\frac{x}{x}=1$, so the fraction is multiplied by 1 . This results in an equivalent fraction with denominator 1 :

$$
\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{\frac{a}{b} \cdot \frac{d}{c}}{\frac{c}{d} \cdot \frac{d}{c}}=\frac{\frac{a}{b} \cdot \frac{d}{c}}{1}=\frac{a}{b} \cdot \frac{d}{c}=\frac{a d}{b c}
$$

See Note on Division by Zero on the previous page..

Method 2: Or, because division is equivalent to multiplication by the reciprocal, rewrite the division as multiplication:

$$
\frac{a}{b} \div \frac{c}{d}=\frac{a}{b} \cdot \frac{d}{c}
$$

Either approach will find the quotient. The major point is division transforms into multiplication, not magically, but from a well-motivated reason using a deep understanding of fractions and how they work.

Remember that in $m \div n$, $n$ cannot be zero. See $T E$.

## EXAMPLE 1

A $3 \frac{1}{4} \mathrm{ft}$. long party sub is being cut into pieces that are $\frac{1}{2} \mathrm{ft}$. pieces. How many servings can be cut?

## SOLUTION

Let $x=$ the number of $\frac{1}{2}$ ft. pieces in $3 \frac{1}{4} \mathrm{ft}$. of sub. $\frac{1}{2} \cdot x=3 \frac{1}{4}$. Solving for $x$, we say

$$
x=3 \frac{1}{4} \div \frac{1}{2}=\frac{\frac{13}{4}}{\frac{1}{2}}=\frac{\frac{13}{4}}{\frac{1}{2}} \cdot \frac{\frac{2}{1}}{\frac{2}{1}}=\frac{\frac{26}{4}}{\frac{2}{2}}=\frac{\frac{26}{4}}{1}=\frac{26}{4}=6 \frac{1}{2}
$$

We can also use the multiplicative inverse property along with other properties previously studied to solve equations.

## EXAMPLE 2

Solve the equation $3(x+2)=7$ algebraically, indicating the properties you use at each step.

## SOLUTION

We simplify the equation by writing:

$$
3(x+2)=7
$$

## EXERCISES

1. a. $\frac{3}{2}$
C. 4
e. $\frac{3}{2}=1 \frac{1}{2}$ g. 5
i. $\frac{5}{2}$
b. 7
d. $\frac{1}{4}$
f. $\frac{2}{3}$
h. $\frac{1}{5}$
j. $\frac{2}{5}$

$$
\begin{array}{ll}
3 x+6=7 & \text { by the distributive property } \\
(3 x+6)-6=7-6 & \text { by the subtraction property } \\
3 x+(6-6)=7-6 & \text { by the associative property } \\
3 x+0=7-6 & \text { by the additive inverse property } \\
3 x=7-6 & \text { by the additive identity property } \\
3 x=1 & \text { subtraction identity } \\
\frac{1}{3} \bullet(3 x)=\frac{1}{3} \bullet 1 & \begin{array}{l}
\text { multiplication property of equality } \\
\text { (just like addition and subtraction) }
\end{array} \\
\left(\frac{1}{3} \bullet 3\right) \bullet x=\frac{1}{3} \bullet 1 & \text { associative property of multiplication } \\
\left(\frac{1}{3} \cdot 3\right) \cdot x=\left(3 \cdot \frac{1}{3}\right) \cdot x & \text { by commutative property of multiplication } \\
1 \bullet x=\frac{1}{3} \bullet 1 & \text { Multiplicative inverse property } \\
x=\frac{1}{3} & \text { Multiplicative identity property of } 1 .
\end{array}
$$

The key idea is that if you have an expression like $3 x$, then when you multiply by $\frac{1}{3}$, you will get $x$ back again, using associativity of multiplication, commutativy of multiplication, the multiplicative inverse, and multiplicative identity properties.

## EXERCISES

1. Use a visual model to compute the following quotients. Check your work by using the product of the quotient and the divisor.
a. $1 \div \frac{2}{3}$
b. $\frac{7}{5} \div \frac{1}{5}$
c. $6 \div \frac{3}{2}$
d. $\frac{3}{2} \div 6$
e. $\frac{1}{2} \div \frac{1}{3}$
f. $\frac{1}{3} \div \frac{1}{2}$
g. $2 \div \frac{2}{5}$
h. $\frac{2}{5} \div 2$
i. $\frac{1}{2} \div \frac{1}{5}$
j. $\frac{1}{5} \div \frac{1}{2}$
2. a. $\frac{\frac{5}{3}}{\frac{3}{2}}=\frac{\frac{5}{3} \times \frac{2}{3}}{\frac{3}{2} \times \frac{2}{3}}=\frac{\frac{5}{3} \times \frac{2}{3}}{1}=\frac{5}{3} \times \frac{2}{3}=\frac{10}{9}=1 \frac{1}{9}$
b. $\quad \frac{\frac{7}{5}}{\frac{5}{7}}=\frac{\frac{7}{5} \times \frac{7}{5}}{\frac{5}{7} \times \frac{7}{5}}=\frac{\frac{7}{5} \times \frac{7}{5}}{1}=\frac{7}{5} \times \frac{7}{5}=\frac{49}{25}=1 \frac{24}{25}$
C. $\frac{\frac{1}{6}}{\frac{1}{12}}=\frac{\frac{1}{6} \times \frac{12}{1}}{\frac{1}{12} \times \frac{12}{1}}=\frac{\frac{1}{6} \times \frac{12}{1}}{1}=\frac{1}{6} \times \frac{12}{1}=2$
d. $\frac{\frac{1}{6}}{12}=\frac{\frac{1}{6} \times \frac{1}{12}}{\frac{12}{1} \times \frac{1}{12}}=\frac{\frac{1}{6} \times \frac{1}{12}}{1}=\frac{1}{6} \times \frac{1}{12}=\frac{1}{72}$
g. $\quad \frac{\frac{5}{24}}{\frac{5}{18}}=\frac{\frac{5}{24} \times \frac{18}{5}}{\frac{5}{18} \times \frac{18}{5}}=\frac{\frac{5}{24} \times \frac{18}{5}}{1}=\frac{5}{24} \times \frac{18}{5}=\frac{3}{4}$
h. $\frac{\frac{9}{10}}{\frac{3}{8}}=\frac{\frac{9}{10} \times \frac{8}{3}}{\frac{3}{8} \times \frac{8}{3}}=\frac{\frac{9}{10} \times \frac{8}{3}}{1}=\frac{9}{10} \times \frac{8}{3}=\frac{72}{30}=\frac{12}{5}=2 \frac{2}{5}$
i. $\frac{\frac{5}{7}}{\frac{15}{14}}=\frac{\frac{5}{7} \times \frac{14}{15}}{\frac{15}{14} \times \frac{14}{15}}=\frac{\frac{5}{7} \times \frac{14}{15}}{1}=\frac{5}{7} \times \frac{14}{15}=\frac{2}{3}$
j. $\quad \frac{\frac{8}{27}}{\frac{16}{9}}=\frac{\frac{8}{27} \times \frac{9}{16}}{\frac{16}{9} \times \frac{9}{16}}=\frac{\frac{8}{27} \times \frac{9}{16}}{1}=\frac{8}{27} \times \frac{9}{16}=\frac{1}{6}$
e. $\frac{\frac{3}{4}}{\frac{9}{16}}=\frac{\frac{3}{4} \times \frac{16}{9}}{\frac{9}{16} \times \frac{16}{9}}=\frac{\frac{3}{4} \times \frac{16}{9}}{1}=\frac{\frac{3}{4} \times \frac{4}{3} \times \frac{4}{3}}{1}=\frac{1 \times \frac{4}{3}}{1}=1 \times \frac{4}{3}=\frac{4}{3}=1 \frac{1}{3}$
f. $\frac{\frac{1}{20}}{\frac{1}{75}}=\frac{\frac{1}{20} \times \frac{75}{1}}{\frac{1}{75} \times \frac{75}{1}}=\frac{\frac{1}{20} \times \frac{75}{1}}{1}=\frac{1}{20} \times \frac{75}{1}=\frac{1}{4 \times 5} \times \frac{3 \times 5 \times 5}{1}=\frac{3 \times 5}{4}=\frac{15}{4}=3 \frac{3}{4}$
3. $21 \div \frac{3}{5}=\frac{21}{\frac{3}{5}}=\frac{21 \times \frac{5}{3}}{\frac{3}{5} \times \frac{5}{3}}=\frac{21 \times \frac{5}{3}}{1}=7 \times 5=35$ decorations.
4. Using $d=r t$, the distance is 12 miles and the rate is $3 \frac{2}{5}=\frac{17}{5} \frac{\text { miles }}{\text { hour }}$. So the time is computed by dividing the distance by the rate: $12 \div \frac{17}{5}=12 \times \frac{5}{17}=\frac{60}{17}=3 \frac{9}{17}$ which is approximately 3.53 hours.
5. Using $d=r t$, the distance is 9 miles and the time is $\frac{3}{4}$ hours. So the rate is the distance divided by the time: $9 \div \frac{3}{4}=9 \cdot \frac{4}{3}=12$ miles per hour.
6. You can make $\frac{5}{6}$ or $83.3333 \ldots . \%$ of the recipe.
7. $\frac{4}{5} \div \frac{3}{25}=\frac{4}{5} \cdot \frac{25}{3}=\frac{20}{3}=6 \frac{2}{3}$ adapters. In reality, one would expect to be able to purchase an integer number, which in this case would be 6 adapters. However, the leftover of $\frac{2}{3}$ adapter translates to $\frac{2}{3}$ of the $\frac{3}{25}$ meters needed for one whole adapter. Multiplying $\frac{2}{3} \cdot \frac{3}{25}$ we get $\frac{2}{25}$ meters left over.
8. You can use the number line to see how many jumps of length $\frac{3}{5}$ hour it takes to get to 9 . Another method is to divide 9 by $\frac{3}{5}$ to get $9 \div \frac{3}{5}=9 \cdot \frac{5}{3}=15$ number of boxes.
9. Shelly can't make any costumes.
10. a. You can solve this equation several ways. You can divide both sides of the equation by 3 to get:
$(3 x) \div 3=\frac{3}{2} \div 3$ and so $x=\frac{3}{2} \cdot \frac{1}{3}=\frac{1}{2}$. You can also solve this equation by multiplying each side of the equation by $\frac{1}{3}$ to get $c$.
b. $4 \cdot \frac{1}{4} x=x=4 \cdot \frac{3}{4}=3$
c. $\frac{3}{2} \cdot \frac{2}{3} x=x=\frac{3}{2} \cdot \frac{1}{6}=\frac{3}{12}=\frac{1}{4}$.
d. $\frac{1}{5} 5 x=x=\frac{1}{5} \cdot \frac{3}{4}=\frac{3}{20}$
11. Compute the following quotients by using the process developed in Method 1. Check by using Method 2 . Simplify if needed.
a. $\frac{\frac{5}{3}}{\frac{3}{2}}$
C. $\frac{\frac{1}{6}}{\frac{1}{12}}$
b. $\frac{\frac{7}{5}}{\frac{5}{7}}$
d. $\frac{\frac{1}{6}}{12}$
e. $\frac{\frac{3}{4}}{\frac{9}{16}}$
f. $\frac{\frac{1}{20}}{\frac{1}{75}}$
g. $\frac{\frac{5}{24}}{\frac{5}{18}}$
h. $\frac{\frac{9}{10}}{\frac{3}{8}}$
i. $\frac{\frac{5}{7}}{\frac{15}{14}}$
j. $\frac{\frac{8}{27}}{\frac{16}{9}}$
12. Sylvia has 21 meters of cloth on a roll. She needs a piece of material $\frac{3}{5}$ of a meter long for each decoration she is making. How many decorations can she make with the cloth she has?
13. A duck walks at a rate of $\frac{3}{4} \mathrm{mph}$. What fraction of an hour does it take the duck to walk $\frac{1}{8}$ of a mile?
14. Henry walks at a speed of $3 \frac{2}{5}$ miles per hour. How long will it take him to walk 12 miles?
15. Linda biked 9 miles in $\frac{3}{4}$ of an hour. What was her average speed?
16. A recipe calls for $\frac{2}{3}$ cup of sugar and you have $\frac{1}{2}$ cup of sugar. What part of the recipe can you make?
17. Robert has $\frac{4}{5}$ of a meter of electrical cord. He needs pieces $\frac{3}{25}$ meters long to make adapters for computers. How many adapters can he make with the $\frac{4}{5}$ meter of cord? How much cord will be left over?
18. James can build a wooden box in $\frac{3}{5}$ of an hour. How many boxes can he build in 9 hours?
19. Shelly needs a costume that requires $\frac{3}{4}$ of a yard of material. The seamstress has $\frac{1}{2}$ yard of material. How may costume(s) can she make?
20. Solve each of the following equations for $x$. Describe how you solved a below:
a. $3 \cdot x=\frac{3}{2}$
b. $\frac{1}{4} \cdot x=\frac{3}{4}$
c. $\frac{2}{3} \cdot x=\frac{1}{6}$
d. $5 \cdot x=\frac{3}{4}$
21. Solve each of the following equations for $x$.
a. $3 \cdot x=\frac{3}{2}$
b. $\quad 5 \cdot x=\frac{3}{4}$
c. $\frac{1}{4} \cdot x=\frac{3}{4}$
d. $\frac{x}{4}=\frac{3}{4}$
e. $\frac{2 x}{3}=\frac{1}{6}$
f. $\quad \frac{2}{3} \cdot x=\frac{1}{6}$
22. a. $x=\frac{1}{3}$
c. $x=3$
e. $x=\frac{1}{4}$
b. $x=\frac{3}{20}$
d. $x=3$
f. $x=\frac{1}{4}$
23. a. $\frac{\frac{15}{28 a}}{\frac{10}{7 a}}=\frac{\frac{15}{28 a} \times \frac{7 a}{10}}{\frac{10}{7 a} \times \frac{7 a}{10}}=\frac{\frac{15}{28 a} \times \frac{7 a}{10}}{1}=\frac{\frac{3}{4} \times \frac{1}{2}}{1}=\frac{\frac{3}{8}}{1}=\frac{3}{8}$
d. $\frac{24 x^{3} y^{3}}{z^{4}} \times \frac{16 z^{4}}{x^{3} y^{2}}=384 y$
b. $\frac{\frac{12 x^{2}}{y z^{2}}}{\frac{z^{3}}{10 x^{2} y}}=\frac{12 x^{2}}{y z^{2}} \times \frac{10 x^{2} y}{z^{3}}=\frac{120 x^{4}}{z^{5}}$
e. $\frac{15 a c^{4}}{b^{2}} \times \frac{a^{2} b}{c^{10}}=\frac{15 a^{3}}{b c^{6}}$
c. $\frac{24 b^{3} c^{2}}{a^{6}} \div \frac{36 c^{6}}{25 a^{2} b}=\frac{24 b^{3} c^{2}}{a^{6}} \times \frac{25 a^{2} b}{36 c^{6}}=\frac{50 b^{4}}{3 a^{4} c^{4}}$
f. $\frac{x^{6} z^{3}}{y^{4}} \times \frac{x^{2} y z}{1}=\frac{x^{8} z^{4}}{y^{3}}$
24. a. 4
c. -15
e. -27
b. 6
d. -10
25. a. $\frac{35}{6}=5 \frac{5}{6}$
C. $\frac{8}{15}$
e. $\frac{39}{6}=6 \frac{3}{6}$
b. $\frac{17}{6}=2 \frac{5}{6}$
d. $\frac{15}{8}=1 \frac{7}{8}$
f. $\frac{42}{12}=3 \frac{6}{12}$

Ingenuity
16. The key is to realize that 4 girls change the proportion of girls in Ms. Adams' class to from $\frac{2}{3}$ to $\frac{1}{2}$. That makes 4 girls $=\frac{1}{6}$ of the class. So the class has 24 students.
13. Compute each of the following quotients. Assume that each variable is a non-zero number. Simplify the answer if needed.
a. $\frac{\frac{15}{28 a}}{\frac{10}{7 a}}$
b. $\frac{\frac{12 x}{y z}}{\frac{z}{10 x y}}$
c. $\frac{\frac{24 b c}{a}}{\frac{36 c}{25 a b}}$
d. $\frac{\frac{24 x y}{z}}{\frac{x y}{16 z}}$
e. $\frac{\frac{15 a c}{b}}{\frac{c}{a b}}$
f. $\frac{\frac{x z}{y}}{\frac{1}{x y z}}$
14. Solve the following equations for x . Show your work and indicate what properties you are using at each step. Check your answers.
a. $2 x+4=12$
b. $\frac{1}{5}(x-2)=20$
c. $3(x+4)=2 x-3$
d. $-2 x=20$
e. $-2(x-3)=15$
15. Compute the following quotients.
a. $2 \frac{1}{3} \div \frac{2}{5}$
C. $\frac{4}{5} \div 1 \frac{1}{2}$
b. $3 \frac{2}{5} \div 1 \frac{1}{5}$
d. $1 \frac{1}{2} \div \frac{4}{5}$
e. $4 \frac{1}{3} \div \frac{2}{3}$
f. $5 \frac{1}{4} \div 1 \frac{1}{2}$
16. Ingenuity:

Two-thirds of Ms. Adams' fifth-grade students are girls. To make the number of boys and girls equal, 4 girls go to the other fifth-grade class, and 4 boys come from that class into Ms. Adams' class. Now one-half of her students are boys. How many students are in Ms. Adams' class?

## Investigation

17. Many students find it more comfortable to give a variable point a reasonable numerical value. You can work the problem this way, but if you have your students work in groups, several of the reasonable numerical values should be different, which can lead to some productive generalizations about rational numbers less than 0 , between 0 and 1 , and greater than 1 .
a. The point $B$ because $C$ and $D$ are each less than 1 so their product will be less than both $C$ and $D$ and still positive.
b. The point E because $\mathrm{D}>\mathrm{C}$ so $\mathrm{D} \div \mathrm{C}$ will be greater than 1 .
c. Since $B<C$, then $B \div C$ is less than 1 . Since $D \div C$ is greater than 1 , then $B \div C$ is less than $D \div C$.
d. This is tricky. Assume that $0<\frac{C}{D}<B$, then $\frac{C}{B}<D$. But $D<1$. Then $\frac{C}{B}<1$, which implies that $C<B$, which is not true. So $0<\frac{C}{D}<B$ cannot be true. Since $C<D$, then we know that $\frac{C}{D}<1$ and not greater than 1. Assume that $\frac{C}{D}$ is between $B$ and $C$, that is $B<\frac{C}{D}<C$. But this implies that $1=\frac{C}{C}<D$ and 1 $<\mathrm{D}$, which is not true. The remaining two possibilities can both occur.
Case 1: Let $C=\frac{1}{4}$ and $D=\frac{3}{4}$ and $\frac{C}{D}=\frac{1}{3}$. So $\frac{C}{89}$ is between $C$ and $D$.
Case 2: Let $C=\frac{89}{100}$ and $D \stackrel{90}{=} 100$ and $\frac{C}{D}=\frac{89}{90}=0.9888 \ldots>0.9=D$. So $\frac{C}{D}$ is between $D$ and 1 . If you try to answer this question based on the points in the picture, it looks like C is about $\frac{1}{2}$ and that C is little more than half the length of $D$. So that $\frac{C}{D}$ would fall between $C$ and $D$.

## 17. Investigation:

Consider the following number line.

a. What point best represents the product of the fractions $C$ and $D$ ? Why?
b. What point best represents the quotient the quotient $D \div C$ ? Explain.
c. Is the quotient $B \div C$ greater than or less than $D \div C$ ? Explain.
d. In which interval would you most likely find $C \div D$ : between 0 and $B$, between $B$ and $C$, between $C$ and $D$, between $D$ and 1 , or greater than 1? Explain why you think so.

## TIC-TAC-FRAC



Objective: The students will play Tic-Tac-Frac to reinforce the skill of dividing fractions. This game will include whole numbers divided by fractions, fractions divided by fractions, and fractions divided by whole numbers.

## Materials:

Integer Chips (or two-color counters)
Colored Tape (to make a tic-tac-toe grid on the floor of your room)
Dry Erase Boards \& Markers (optional, but work very well for this game if you have them)

## Activity Instructions:

Before playing this game, you will need use the colored tape to make a fairly large tic-tac-toe grid on the floor in the center of your room. Arrange your students' desks around this grid, either in a circle, oval, or big square. If this is not possible, you can just have your students sit in a big circle on the floor, as long as they have a way to work out their math problems (either on a dry erase board or using a book as a hard surface while they use their own notebook paper.)

Using the sample division problems on the attached page, choose one problem at a time to share with your students. Students will individually work on this problem and hold up their answer for you to check when they are finished. If a student has successfully divided the fractions, hand them one chip. Once you have checked everyone's answers, the fun begins. Everyone with a chip in their hand gets to toss their chip in the direction of the tic-tac-frac grid on the floor, hoping to land in one of the square spaces. As the game continues, the students will flip more and more chips onto the grid. The first student to get three chips in a row (horizontally, vertically, or diagonally) wins the game. Once you have a winner, pick up all of the chips and start the game over, if time allows.

## Helpful Hints:

Things can get a little confusing when flipping the chips onto the grid, as chips will roll, flop, bounce etc. One suggestion that may help with this confusion is to have the students use a dry erase marker to write their initials on one side of their chip before they flip it onto the grid. This helps everyone keep track of their own chips on the floor, and it is also very easy to clean up. A tissue will easily wipe the marker off of your integer chips when the game is over.

Another suggestion to help keep the game fair and fun is to set parameters on the allowable spaces for chip placement. It may be helpful to tell the students that any chip that lands "touching" the tape in any way is considered "OUT". Chips must land INSIDE the area of the square.

One last suggestion, tell students that they must remain seated (bottoms on the floor or in their chair) during the toss process. If they are allowed to lean in, step forward, or move their bodies in any way, taller students may have an advantage over shorter students.

It is estimated that you may need to play around 10 rounds of the game before somebody gets tic-tac-toe. Although it is possible that a student could win after just three rounds, this is very unlikely. The further you position your students away from the grid on the floor, the better your chances of having them practice more division problems before somebody wins.

## Sample Fraction Division Problems for Tic-Tac-Frac

$$
\begin{array}{lll}
\frac{3}{5} \div \frac{9}{10}=\frac{2}{3} & \frac{7}{9} \div \frac{12}{5}=\frac{45}{108} & \frac{4}{9} \div \frac{11}{12}=\frac{16}{33} \\
\frac{3}{4} \div \frac{9}{8}=\frac{2}{3} & \frac{8}{14} \div \frac{8}{5}=\frac{5}{14} & 2 \div \frac{7}{4}=\frac{8}{7} \\
\frac{11}{12} \div \frac{5}{6}=\frac{11}{10} & \frac{8}{9} \div 3=\frac{8}{27} & \frac{15}{20} \div \frac{3}{4}=1 \\
\frac{21}{22} \div \frac{3}{2}=\frac{7}{11} & \frac{4}{17} \div \frac{4}{3}=\frac{3}{17} & 5 \div \frac{7}{3}=\frac{15}{7}
\end{array}
$$

$$
\frac{4}{6} \div \frac{3}{4}=\frac{8}{8}
$$

$$
\frac{5}{9} \div \frac{14}{6}=\frac{5}{21}
$$

$$
7 \div \frac{2}{3}=\frac{21}{2}
$$

$$
\frac{4}{11} \div \frac{1}{22}=8
$$

$$
\frac{1}{6} \div \frac{5}{11}=\frac{11}{30}
$$

$$
\frac{3}{4} \div 6=\frac{1}{8}
$$

# Sample Fraction Division Problems for Tic-Tac-Frac 

$$
\frac{3}{5} \div \frac{9}{10}=\frac{2}{3}
$$

$$
\frac{7}{9} \div \frac{12}{5}=\frac{35}{108}
$$

$$
\frac{4}{9} \div \frac{11}{12}=\frac{16}{33}
$$

$$
\frac{3}{4} \div \frac{9}{8}=\frac{2}{3}
$$

$\frac{8}{14} \div \frac{8}{5}=\frac{5}{14}$
$2 \div \frac{7}{4}=\frac{8}{7}$
$\frac{11}{12} \div \frac{5}{6}=\frac{11}{10}$
$\frac{8}{9} \div 3=\frac{8}{27}$
$\frac{15}{20} \div \frac{3}{4}=1$
$\frac{21}{22} \div \frac{3}{2}=\frac{7}{11}$
$\frac{4}{17} \div \frac{4}{3}=\frac{3}{17}$
$5 \div \frac{7}{3}=\frac{15}{7}$
$\frac{4}{6} \div \frac{3}{4}=\frac{8}{8}$
$\frac{5}{9} \div \frac{14}{6}=\frac{5}{21}$
$7 \div \frac{2}{3}=\frac{21}{2}$
$\frac{4}{11} \div \frac{1}{22}=8$

$$
\frac{1}{6} \div \frac{5}{11}=\frac{11}{30}
$$

$$
\frac{3}{4} \div 6=\frac{1}{8}
$$

## Section 9.3 - Fraction, Decimal, \& Percent Equivalents

## Big Idea:

Comparing and ordering fractions, and converting them to decimal form

## Key Objectives:

- Convert fractions to decimal form.
- Convert decimals to fractions.
- Understand the relationship between fractions of dollars and decimals representing cents.
- Become familiar with repeating decimals and their notations.
- Compare and order fractions using an area model, linear model and no decimals.
- Compare and order fractions using a linear model.


## Materials:

- Paper for folding (sentence strips or adding machine tape) Grid paper, Fraction Chart from the end of the section or the CD


## Pedagogical/Orchestration:

- This section involves connecting fractions to their decimal equivalents through division. Students also get practice using the linear model for fractions through paper folding, and using the area model for fractions by shading regions on a grid. Students can be shown the Fraction Chart from the CD as it is an extension of the linear model that makes ordering common proper fractions visually very simple.
- Refer to Section 6.1 to review place value. The Place Value Chart will help students make a connection of decimals to fractions. Reinforce to students that when you say the name of a decimal, you are saying the fraction to which it is equal. For example: $0.751=\frac{751}{1000} ; 0.75=\frac{75}{100} ; 0.7=\frac{7}{10}$.


## Internet Resource:

Jeopardy Game: Fractions, Decimals, and Percent Conversions- http://www.quia.com/cb/34887.htm|

## Activities:

"Equivalent Fractions," "Concentration Game;" and "Factorize with Arrays" from the end of the section and the CD

## Exercises:

Exercise 6: Remind students about repeating decimals from fractions covered in Section 6.4. Allow students to use guess and check with calculators to find the repeating decimal values, and remind them to look for patterns.

## Vocabulary:

fraction, decimal, equivalent, rounding

## TEKS:

6.1(A)(B);
6.3(A);
7.1(A)(B);
$7.13(\mathrm{D}$
7.14(A)
$7.15(A)(B) ;$
8.15(A);
8.16(A)

## WARM-UPS for Section 9.3

1. Julian is painting a room and is $45 \%$ done. Which of the following fractions is equivalent to the part of the room that has been painted? Explain.
a. $\frac{11}{20}$
b. $\frac{9}{20}$
c. $\frac{4}{10}$
d. $\frac{1}{2}$
Ans: (b) because $\frac{45}{100}=\frac{9}{20}$
2. Aunt Mary baked a pie and served $\frac{7}{16}$ of it to her nephews. She sent one third of what was left home with her nephews. How much pie does she have left?
a. $\frac{3}{16}$
b. $\frac{1}{4}$
c. $\frac{3}{8}$
d. $\frac{9}{16}$

Ans: (c) because there was $\frac{9}{16}$ left and $\left(\frac{1}{3}\right)\left(\frac{9}{16}\right)=\frac{3}{16}$ was sent home so $\frac{9}{16}-\frac{3}{16}=\frac{6}{16}$ was left over and $\frac{6}{16}=\frac{3}{8}$.
3. Put the following numbers in order from least to greatest:
a. $0.5,0.37,0.66, \frac{1}{3}, 23 \%, 69 \%, 43 \%$
Ans: $23 \%, \frac{1}{3}, 0.37,43 \%, .5,0.66,69 \%$
b. $11 \%, \frac{9}{10}, \frac{3}{4}, 30 \%, 0.01,42 \%, 0.43$
Ans: $0.01,11 \%, 30 \%, 42 \%, 0.43, \frac{3}{4}, \frac{9}{10}$

## Launch for Section 9.3:

Tell your class there are two situations that you want them to discuss in their groups.
Situation 1: The other day you asked your grandmother what time it was, and she said it was a quarter past 2 . What did she mean by that? What time was it? What time would a quarter till 2 be?

Situation 2: You are at the store trying to buy a soda, and the clerk tells you that you are a quarter short. What does he mean by that? How many cents is he talking about? How is this different than the quarter your grandmother referred to in Situation 1?
Let the groups discuss both situations bringing out the importance of knowing what the whole is. A quarter ( $\frac{1}{4}$ ) of an hour is different than a quarter ( $\frac{1}{4}$ ) of a dollar. $\frac{1}{4}$ of an hour is 15 minutes, whereas $\frac{1}{4}$ of a dollar is 25 cents, etc. Let students know that in today's lesson, they will be thinking about money in order to understand the connection between fractions and decimal numbers.

Refer to Chapter 6 for discussion of how repeating decimals arise in division, i.e. $\frac{1}{3}=0.333333 \ldots$

## SECTION 9.3 FRACTION, DECIMAL, \& PERCENT EQUIVALENTS

In the past, you have probably referred to one-half of a dollar as $\$ 0.50$, or 50 cents. One half is a fraction that is equal to 0.50 , a decimal. We say that $\frac{1}{2}$ is equivalent to 0.50 . In this section, we will review how a fraction can be represented as a decimal number and how some decimals can be represented as fractions.

If you buy four apples for a dollar, how much does each apple cost? In Chapter 6, we found that each apple costs $\$ 1 \div 4=\$ 0.25$, or 0.25 dollars. We can also say each apple costs a quarter, or $\frac{1}{4}$ of a dollar. So, $\frac{1}{4}$ and 0.25 are equal, or equivalent. Does the fraction $\frac{1}{5}$ have a decimal form? If we buy 5 bananas for $\$ 1$, we know that each banana costs $\$ 1 \div 5=\$ 0.20$, or 20 cents. In other words, each banana costs $\frac{1}{5}$ dollar because it takes $5(\$ 0.20)$ to make a whole dollar. So, $\frac{1}{5}=1 \div 5=0.20$, or 20 hundredths. But the decimal 0.20 , or twenty hundredths, has the same name as the fraction $\frac{20}{100}$. Does this mean the fraction $\frac{1}{5}$ is equal to $\frac{20}{100}$ ? These are equivalent fractions, so the decimal 0.20 and the fractions $\frac{1}{5}$ and $\frac{20}{100}$ are equal.

## PROPERTY 9.1: FRACTIONS AND DIVISION

For any number $m$ and nonzero number $n$ the fraction $\frac{m}{n}$ is equivalent to the quotient $n \longdiv { m }$.

Now we ask, "What decimal is equivalent to $\frac{1}{3}$ ?" We could also ask, "What is $\frac{1}{3}$ of a dollar?" We can use our new rule to see that $\frac{1}{3}$ is equivalent to the quotient $1 \div 3$. In Section 6.4 , we discovered that this quotient is $1 \div 3=0.3333 \ldots=0 . \overline{3}$ . Thus, $\frac{1}{3}$ of a dollar is $\$ 0.33 \overline{33}$, and we cannot practically divide $\$ 1$ into 3 equal parts with our present set of coins. There are other fractions that equal repeating decimals:

$$
\begin{aligned}
& 2 \div 3=0.6666 \ldots=0.66 \overline{6}=0 . \overline{6} \\
& 1 \div 6=0.1666 \ldots=0.166 \overline{6}=0.1 \overline{6}
\end{aligned}
$$

Indeed, we learned in the Repeating Decimal Game from Section 6.4 that there are many fractions that have repeating decimals.

| a. $\quad 0.25$ | b. | 0.2 | c. | 0.125 | d. | $0.333 \ldots$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0.5 |  | 0.4 |  | 0.25 |  | $0.033 \ldots$ |
|  | 0.75 |  | 0.6 |  | 0.375 | $0.111 \ldots$ |
|  |  |  | 0.8 |  | 0.625 | $0.0111 \ldots$ |

## PROBLEM 1

a. 0.8
b. 0.55
c. 0.98
d. 0.375

## EXPLORATION

Convert the following fractions to decimal form. You may verify your answer by dividing with a calculator, if necessary.
a. $\frac{\frac{1}{4}}{\frac{2}{4}}$
b. $\frac{1}{5}$
C. $\begin{aligned} & \frac{1}{8} \\ & \frac{2}{8}\end{aligned}$
d. $\frac{1}{3}$
$\frac{4}{5}$
$\frac{5}{8}$ $\frac{1}{9}$
$\frac{1}{90}$

In converting a decimal to a fraction, we take advantage of the fact that we use the base ten system to write each decimal number. For example, the number 0.3 is called three-tenths and so is equivalent to $\frac{3}{10}$. The number 0.35 is read as 35 hundredths and so is the same as $\frac{35}{100}$.

Some fractions can be converted to an equivalent fraction with a denominator that is a power of 10 , such as 10,100 , or 1000 . In this case, the new equivalent fraction can be written as a decimal quite easily. For example, $\frac{3}{25}$ is equivalent to $\frac{3 \cdot 4}{25 \cdot 4}=\frac{12}{100}=0.12$. We used the factor 4 because 4 times 25 gives us the product 100 . Compare this to the process of dividing 3 by $25: \frac{0.12}{2 5 \longdiv { 3 . 0 0 }}$.

## PROBLEM 1

Convert the following fractions to decimal form by using equivalent fractions:
a. $\frac{4}{5}$
b. $\frac{11}{20}$
c. $\frac{49}{50}$
d. $\frac{3}{8}$

You have learned that fractions such as $\frac{3}{4}$ can be written as the equivalent fraction $\frac{75}{100}$. This equivalent fraction can also be represented by the decimal 0.75 . In some instances, this number can then be converted to 75 percent, $75 \%$. The word percent means "in a hundred" in Latin.

## Activity 1:

Have the class conduct a survey and make a chart with data, fractions and the percentage of the class for each category of the survey. Sample Questions:

1. How many students prefer cats to dogs? Dogs to cats? Neither?
2. How many students prefer some chocolate in their ice cream?
3. How many students are the oldest children in their families?

3 How many students are wearing sneakers today?
To convert a percent to a decimal, as we did at the beginning of this section, reverse the process and divide by 100 . For example, $35 \%$ is equivalent to $35 \div 100=35 / 100=0.35$. If the percent includes a decimal part, simply divide by 100 to get its decimal equivalent. For example, $64.8 \%$ is equivalent to $64.8 \div 100=0.648$.

| Fraction | Fraction in <br> Hundredths | Decimal | Percent |
| :---: | :---: | :---: | :---: |
| $\frac{3}{4}$ | $\frac{75}{100}$ | 0.75 | $(0.75)(100)=75 \%$ |
| $\frac{12}{25}$ | $\frac{48}{100}$ | 0.48 | $(0.48)(100)=48 \%$ |
| $\frac{4}{5}$ |  | 0.8 | $80 \%$ |
| $\frac{9}{15}$ |  | 0.6 | $60 \%$ |
| $\frac{4}{32}$ |  | 0.125 | $12.5 \%$ |
| $\frac{7}{10}$ |  | 0.7 | $70 \%$ |
| $\frac{1}{20}$ |  | 0.05 | $5 \%$ |
| $\frac{5}{8}$ |  | 0.625 | $62.5 \%$ |

## Activity 2 :

Use the activity below to reinforce students' understanding of the connection between the three forms of fraction, decimal, and percent, and as a way to practice comparing and ordering within the three representations.

In small groups, write 4 or 5 numbers in any form (fraction, decimal, or percent). Practice placing the numbers in order from least to greatest. Switch cards with other groups for additional practice.

It is useful to convert some fractions into decimal form by finding the equivalent fraction with the denominator of 100. In converting decimals to percents, you can multiply the decimal by 100 to get the percent. For instance $(0.75)(100)=75 \%$. We illustrate this using the chart below. Test your skills by completing it.

| Fraction | Fraction in <br> Hundredths | Decimal | Percent |
| :---: | :---: | :---: | :---: |
| $\frac{3}{4}$ | $\frac{75}{100}$ | 0.75 | $(0.75)(100)=75 \%$ |
| $\frac{12}{25}$ | $\frac{48}{100}$ | 0.48 | $(0.48)(100)=48 \%$ |
| $\frac{4}{5}$ |  |  |  |
| $\frac{9}{15}$ |  |  |  |
| $\frac{4}{32}$ |  |  |  |
| $\frac{7}{10}$ |  |  |  |
| $\frac{1}{20}$ |  |  |  |
| $\frac{5}{8}$ |  |  |  |

Similarly, you can reverse the pattern of converting percents to decimals by dividing the percent by 100 . For example, $75 \%$ is equivalent to $75 \div 100=\frac{75}{100}=0.75$. Even if the percent includes a decimal part, simply divide by 100 to get its decimal equivalent. For example, $6.5 \%$ is equivalent to $6.5 \div 100=0.065$.

| Fraction | Decimal | Percent |
| :---: | :---: | :---: |
|  |  | $1 \%$ |
|  |  | $12.5 \%$ |
|  |  | $0.25 \%$ |
|  |  | $24 \frac{1}{2} \%$ |


| Fraction | Decimal | Percent |
| :---: | :---: | :---: |
| $\frac{1}{100}$ | 0.01 | $1 \%$ |
| $\frac{12.5}{100}$ | 0.125 | $12.5 \%$ |
| $\frac{0.25}{100}$ | 0.0025 | $0.25 \%$ |
| $\frac{0.245}{100}$ | 0.245 | $24 \frac{1}{2} \%$ |

Make sure your students know that neither 0.3 nor 0.33 is the same as $\frac{1}{3}$. It is close, just as 3.14 is close to the value of $\pi$. But the exact value is not the approximate value.

A mixed fraction is a sum of an integer and a proper fraction, or a fraction between 0 and 1 . The notation of a mixed fraction can be confusing, but it was discussed in Section 8.6.

While the process used in the table is useful for many fractions, it only works for "friendly fractions." A friendly fraction is one that can easily be converted to a fraction whose denominator is a power of ten.

## EXAMPLE 1

How do you convert a fraction like $\frac{5}{16}$ into a decimal and a percent?

## SOLUTION

When a fraction is not "friendly," you can use the division process to convert it to a decimal. For example, $\frac{5}{16}$ is equivalent to $5 \div 16=0.3125=31.25 \%$.

The trickiest conversion from a fraction to a percent involves fractions with repeating decimals. For example, $\frac{1}{3}=0.333 \ldots$ To convert from a decimal to a percent, multiply $(0.333 \ldots)(100)=33.333 \ldots \%$, which can also be written as the mixed fraction $33 \frac{1}{3} \%$. The idea of mixed fractions was discussed in more detail in Section 8.6.

## EXAMPLE 2

In a random survey to find people's favorite car color, $60 \%$ of the people surveyed liked red, $10 \%$ liked blue, and $30 \%$ liked black.
a. Represent this data with a pie graph.
b. If 12 people in the survey liked red, how many people total were surveyed? Solve numerically, and also by using your pie graph.
c. Based on this survey, in a group of 1000 people, about how many might be expected to like red?

## SOLUTION

a.


PROBLEM 2
a. $0 . \overline{190476} ; 19 . \overline{047619}$ b. $0 . \overline{28125} ; 28.125 \%$ c. $0.8 \overline{3} ; 83 \frac{1}{3} \%$
b. Let $T$ is the total number of people. Setting up a proportion of red to total, $\frac{12}{\mathrm{~T}}=\frac{60}{100}$.
Multiplying both sides of the equation by $100 T$

$$
\begin{gathered}
\frac{12}{T} \cdot 100 T=\frac{60}{100} \cdot 100 T \\
12 \cdot 100=60 T \\
1200=60 T \\
T=20
\end{gathered}
$$

c. In 1000 people, approximately $60 \%$ of the total should prefer red.

$$
60 \% \text { of } 1000=0.60 \bullet 1000=600 \text {. }
$$

## PROBLEM 2

Convert each of the following fractions to a decimal and a percent:
a. $\frac{4}{21}$
b. $\frac{9}{32}$
c. $\frac{5}{6}$

## EXAMPLE 3

A class of 25 students has 12 girls. What percent of the class are girls?

## SOLUTION

Notice that this problem does not give the percent. You are, in fact, asked to find the percent of girls given that of the 25 students, 12 are girls.

If 12 of the 25 students are girls, the fraction of girls to the total number of students is $\frac{12}{25}$. We can convert this fraction to a decimal and then to a percent. Or, we can find an equivalent fraction for $\frac{12}{25}$ that has a denominator of 100 . The fraction $\frac{48}{100}$ is equivalent to $\frac{12}{25}$ and to the decimal 0.48 . It is good practice to check your work by checking that the product of the percent of the class that is girls times the class total, ( 0.48 )(25), is equal to $12=$ the number of girls in class.

Even though many times a fraction is not so easily transformed into hundredths, some fractions have a surprisingly nice conversion into hundredths. Remind students of the exercises in Section 7.2.

## PROBLEM 3

The fraction of shots made to shots taken is 9 to 15 or $\frac{9}{15}$. To convert this fraction to a decimal, divide 9 by 15 to get $\frac{9}{15}=0.6=0.60=\frac{60}{100}=60 \%$, the percent of baskets made. It is a little surprising that the fraction $\frac{9}{15}$ is so nice. Many times a fraction is not so easily transformed into hundredths. Again, it is best to check that Amy made (0.60)(15) $=9$ baskets.

$$
\frac{9}{15}=0.60=60 \%
$$

## PROBLEM 4

Using an algebraic equation to solve this problem, we let $x=$ percent of the mixed candy that is lemon in decimal form. Then $x \cdot 32=4$. To solve for $x$, we divide both sides by $32: x=\frac{4}{32}$. The fraction of lemon to the whole bag is $\frac{4}{32}=\frac{1}{8}$ As you might recall, the decimal for $\frac{1}{8}$ is 0.125 . How can you convert this to a percent? What kind of answer do you expect? Convert 0.12 and 0.13 to percents $12 \%$ and $13 \%$ respectively. Because 0.125 is the midpoint between 0.12 and 0.13 on the number line, 0.125 is equivalent to $12.5 \%$, the percent of lemon candy in the bag.

Our numeration system is often called the base-ten system or the Hindu-Arabic system. The base-ten refers to the place-values being powers of ten. The numerals evolved from the Asian and Arabic cultures.

## LINEAR MODEL FOR FRACTIONS ACTIVITY

You can simultaneously make a large number line model on the board or on the floor using string that does not stretch and masking tape. But this is no substitute for a students' own number line model. Place the fractions above the line and decimals below the line. As you build the model, we want students to use their knowledge about converting fractions to decimals and decimals to fractions to build the number line.
4. Some possible comparisons the students might notice include:

- The ruler and yard stick are divided into fractional amounts, but their number line has labels for each fractional amount.
- The yardstick, ruler, and their own number lines are using different scales, i.e., half of a yard is different than half of a foot or half of their own units.

5. Talk about how to estimate the decimal form from what you know. If it is valuable, have a student with a calculator check the class' guesses. If necessary, remind students of the fraction-dollar relationship.
a. 0.05
0.35
b. $\approx 0.08$
$\approx 0.4$
c. $\approx 0.06$
d. 0.04
$\approx 0.3$
0.08

## PROBLEM 3

Amy was shooting hoops in her backyard. She made 9 of 15 baskets. What percent of her shots did she make?

## PROBLEM 4

In a small bag of 32 pieces of mixed candy, there are 4 pieces of lemon candy. What percent is lemon? 12.5\%

## LINEAR MODEL FOR FRACTIONS ACTIVITY

On a number line, each integer corresponds to a point. Recall that there are many other points between each pair of integers on the number line, and each of these points also corresponds to a number. We will now fill out a table based on a number line. We will locate points and label them with fractions (above) and decimals (below).

Materials: You will need a long strip of paper like a sentence strip or an 18 -inch piece of adding machine paper and $10 \times 10$ grid paper.

1. Locate and label the point representing $\frac{1}{2}=0.5$. Color each of these strips the same color.
2. Do the same for $\frac{1}{3}$ and $\frac{1}{4}$ in groups. Check that the whole class is on target.
3. Fill in the remaining strips.
4. Compare the number line with a typical foot ruler or yardstick.
5. Use your new number line to estimate the decimal and percent form of the following fractions:
a. $\frac{1}{20}$
b. $\frac{1}{12}$
C. $\frac{1}{16}$
d. $\frac{1}{25}$
$\frac{7}{20}$
$\frac{5}{12}$
$\frac{5}{16}$
$\frac{2}{25}$
6. Use your $10 \times 10$ grid paper to find the decimal and percent form for:
a. $\frac{1}{20}$
b. $\frac{1}{12}$
c. $\frac{1}{16}$
d. $\frac{1}{25}$

Students can compare the numerators of equivalent fractions with the same denominators or decimal representations to decide which fraction is greater. We will focus on using two visual models first, then discuss the advantage of finding equivalent fractions with the same denominator. We also verify order of fractions with decimals.

## COMPARING AND ORDERING FRACTIONS: THREE METHODS

I. Linear Model:

We want students to locate these fractions on the master number line or the Fraction Chart at the end of the section.
A. $\frac{3}{5}$ is greater.

This is not easy to do with a number line and is a weakness of the linear model. If they recognize that each has an equivalent fraction in forty-fifths, they can find the difference. Don't suggest subtraction yet.
Don't get bogged down by the second question. It is good for them to think about distance without resolving the issue yet.
The area model will enable students to answer this question.
B. $\frac{4}{5}$ is greater.
C. From smallest to largest: $\frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}$. Looking at the number line, we see what fraction is farthest left (least) then build from there rightward.
D. From largest to smallest: $\frac{7}{10}, \frac{2}{3}, \frac{5}{8}, \frac{3}{5}, \frac{5}{9}$. Looking at the number line, we see what fraction is farthest right (greatest) then build from there leftward.

## II. Area Model:

Teachers, you might want to review the Area Model and Fudge Model to help your students remember the concept. If they draw the two fractions separately, they must overlay one on the other to compare them. This is foreshadowing common denominator. If your students don't use the term or concept, don't mention it yet.
A. With vertical cuts for fourths and horizontal cuts for thirds, the area model has produced twelfths. We see that $\frac{1}{4}=\frac{3}{12}$ and $\frac{1}{3}=\frac{4}{12}$, so that $\frac{1}{3}$ is greater by $\frac{1}{\mathrm{~d}_{2}}$.
B. The area model tells us that $\frac{2}{3}=\frac{3}{12}$ and $\frac{3}{4}=\frac{\mathrm{g}^{2}}{12}$, so that $\frac{3}{4}$ is greater by $\frac{1}{12}$.
C. The area model tells us that $\frac{2}{5}=\frac{14}{35}$ and $\frac{\frac{3}{7}}{7}=\frac{15}{35}$, so $\frac{3}{7}$ is greater by $\frac{1}{35}$.

Reflect on how useful it is to rewrite fractions with a common denominator in order to compare them. Contrast this to their decimal representations. For instance, which decimal is greater, 0.37 or 0.38 , and by how much?

Getting your students who have a hard time with fractions to use and understand the area model will be a great service to them.
7. Use the number line chart to determine which fraction is greater, $\frac{2}{5}$ or $\frac{3}{7}$.

Given two fractions, how can you determine which of them is greater? We can now locate fractions on the number line. The fraction that is to the right of the other is the greater. What problems might arise from this method? For one, its accuracy depends on the quality of the comparative number lines. It becomes harder as the fractions get closer to the same value.

Use the master number line that you constructed or the Fraction Chart to decide which fraction is greater, $\frac{2}{3}$ or $\frac{3}{4}$. Explain your answer. Is there another way to explain which is greater?

## COMPARING AND ORDERING FRACTIONS: THREE METHODS

## I. Linear Model:

Use the number line from 0 to 1 to answer the following. Verify your answer by comparing the decimal form for each pair of fractions.
A. Which fraction is greater: $\frac{3}{5}$ or $\frac{4}{9}$ ? Can you tell how much greater using only the number line?
B. Which fraction is greater, $\frac{3}{4}$ or $\frac{4}{5}$ ?
C. Use the number line to order the following fractions from least to greatest. Justify your choices. $\frac{2}{5}, \frac{3}{7}, \frac{3}{8}, \frac{4}{9}$.
D. Write the following fractions in order from greatest to least:
$\frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{7}{10}, \frac{5}{9}$.

## II. Area Model:

Use the area model to compare the following pairs of fractions. Use vertical cuts for one fraction and horizontal cuts for the second fraction.
A. Which fraction is greater, $\frac{1}{4}$ or $\frac{1}{3}$ ? How much greater?

III. A. $0.75,0.7,0.666$ (repeating) C. $0.4,0.428571$ (repeating), $0.41666 \ldots$
B. $0.375,0.444$ (repeating), 0.5 D. 0.615384 (repeating), $0.6,0.625$
B. Which fraction is greater, $\frac{2}{3}$ or $\frac{3}{4}$ ? How much greater?
C. Which fraction is greater, $\frac{2}{5}$ or $\frac{3}{7}$ ? How much greater?

## III. Conversion to Decimal

Use the method of converting fractions to their deciaml equivalents to place each group of fractions in order.
A. $\frac{3}{4}, \frac{7}{10}, \frac{2}{3}$
B. $\frac{3}{8}, \frac{4}{9}, \frac{1}{2}$
C. $\frac{2}{5}, \frac{3}{7}, \frac{5}{12}$
D. $\frac{8}{13}, \frac{3}{5}, \frac{5}{8}$

Comparing fractions can be useful in real-life problems as well.

## EXAMPLE 4

Sara has a 16-ounce cup that has ten ounces of water in it. Mary has a 12-ounce cup that t has eight ounces of water in it.
a. They look about equal. Are they? Which cup is fuller?
b. Sara and Mary each drink threes ounces of water from their glasses. Which cup is fuller now?

## SOLUTION 4

a. Calculate the percentage of water in Sara's cup. The ratio is 10 ounces water: 16 ounces total, or $\frac{10}{16}$, so Sara's cup is $\frac{5}{8}$ full. Converting to a percentage, Sara's glass is $62.5 \%$ full.

The ratio of water to the capacity for Mary's cup is 8 ounces water: 12 ounces total, or $\frac{8}{12}$, so Mary's cup is $\frac{2}{3}$ full. Converting to a percentage, Mary's cup is 66.6\% full.
b. When Sara drinks three ounces, there will be only 7 ounces of water remaining. Her glass is now $\frac{7}{16}$ or $43.75 \%$ full.
When Mary drinks three ounces of water, her glass is $\frac{5}{12}$ or $41.67 \%$ full.
Mary's cup is not as full as Sara's because she drank $\frac{3}{12}=\frac{1}{4}$ or $25 \%$ of her cup's capacity.

Sara only drank $\frac{3}{16}$ or $18.75 \%$ of her cup's capacity. So three ounces makes a bigger difference to Mary's amount of water left than to Sara's.

Thinking of the situation above in terms of fractions, originally Sara's water to cup ratio versus Mary's was $\frac{10}{16}<\frac{8}{12}$.
Subtracting 3 from each denominator, yielded the fractions $\frac{7}{16}$ and $\frac{5}{12}$, but $\frac{7}{16}>\frac{5}{12}$.

## PROBLEM 5

Finding percents from picture models.
All 750 students at Miller Middle School in San Marcos were asked whether they played a musical instrument and whether they played on a sports team at the school. The Venn Diagram shows the results of the survey.
a. What percent of the students played a musical instrument?
b. What percent of the students played on a sports team?
c. What percent of the students played a musical instrument and played on a sports team?
d. What percent did neither?

## EXERCISES

1. First, monitor to see whether your students use the unit fraction's decimal equivalent or whether they use division each time. Encourage best practices here. Look for simplified forms or denominators that convert to powers of 10. As a last resort, use division.
a. 0.2
b. $\quad 0.35$
0.1
0.35
0.05
0.44
0.025
0.44
c. 0.33
d. $\quad 0.75$
0.166...
0.75
0.0833...
0.75
0.0125
0.04166...
0.8
0.020833...
2. a. $30 \%=0.30=\frac{3}{10}$
e. $\quad 85 \%=0.85=\frac{17}{20} \quad$ i. $\quad 0.6 \%=0.006=\frac{3}{500}$
b. $25 \%=0.25=\frac{1}{4}$
c. $40 \%=0.40=\frac{2}{5}$
d. $2 \%=0.02=\frac{1}{50}$
f. $\quad 0.1 \% 0.001=\frac{1}{1000} \mathrm{j} . \quad 20.6 \%=0.206=\frac{103}{500}$
g. $2.4 \% \quad 0.024=\frac{3}{125}$ k. $\quad 2.08 \%=0.0208=\frac{13}{625}$
h. $24 \%=0.24=\frac{6}{25}$
3. Have students reflect on any patterns they see, e.g., they are decreasing, some are multiples of previous numbers. There is no general pattern between all of these fractions.

| $\frac{1}{2}=0.5$ | $\frac{1}{3}=0.333 \ldots$ | $\frac{1}{4}=0.25$ | $\frac{1}{5}=0.2$ |
| :--- | :--- | :--- | :--- |
| $\frac{1}{6}=0.1666 \ldots$ | $\frac{1}{7}=0.142857142857 \ldots$ | $\frac{1}{8}=0.125$ | $\frac{1}{9}=0.111 \ldots$ |
| $\frac{1}{10}=0.1$ | $\frac{1}{11}=0.0909 \ldots$ | $\frac{1}{12}=0.08333 \ldots$ | $\frac{1}{13}=0.076923076923 \ldots$ |
| $\frac{1}{14}=0.071428571428 \ldots$ | $\frac{1}{15}=0.0666 \ldots$ | $\frac{1}{16}=0.0625$ | $\frac{1}{17}=0.058823529 \ldots$ |
| $\frac{1}{18}=0.0555 \ldots$ | $\frac{1}{19}=0.05263157894 \ldots$ | $\frac{1}{20}=0.05$ |  |

## EXERCISES

1. Convert each fraction to its equivalent decimal and order them least to greatest.
a. $\frac{1}{5}$
$\frac{1}{10}$
$\frac{1}{20}$
$\frac{1}{40}$
$\frac{1}{80}$
b. $\frac{7}{20}$
$\frac{11}{25}$
c. $\frac{1}{3}$
d. $\frac{12}{16}$
100
$\frac{44}{100}$
$\frac{27}{36}$
$\frac{15}{20}$

$\frac{16}{20}$
2. Convert each percent to a decimal and then to a fraction in simplest form.
a. $30 \%$
c. $40 \%$
e. $85 \%$
g. $2.4 \%$
i. $0.6 \%$
k. 2.08\%
b. $25 \%$
d. $2 \%$
f. $0.1 \%$
h. $24 \%$
j. 20.6\%
3. Complete the table below by converting each unit fraction to an equivalent decimal. Use calculators as necessary.

| $\frac{1}{2}=$ | $\frac{1}{3}=$ | $\frac{1}{4}=$ | $\frac{1}{5}=$ |
| :--- | :--- | :--- | :--- |
| $\frac{1}{6}=$ | $\frac{1}{7}=$ | $\frac{1}{8}=$ | $\frac{1}{9}=$ |
| $\frac{1}{10}=$ | $\frac{1}{11}=$ | $\frac{1}{12}=$ | $\frac{1}{13}=$ |
| $\frac{1}{14}=$ | $\frac{1}{15}=$ | $\frac{1}{16}=$ | $\frac{1}{17}=$ |
| $\frac{1}{18}=$ | $\frac{1}{19}=$ | $\frac{1}{20}=$ |  |

4. Use the number line, paper strip, and $10 \times 10$ grid paper to find the following in decimal and percent form:
(a) $3 / 10$
(b) $3 / 25$
(c) $1 / 8$
5. Encourage students to make connections between a fraction in hundredths and the decimal form of a number. Students can also extend what they know about fractions to find an equivalent form in tenths and thousandths.

| Simplest Form | Fraction in Hundredths | Decimal Form |
| :---: | :--- | :--- |
| $\frac{3}{5}$ | $\frac{60}{100}$ | 0.6 |
| $\frac{7}{10}$ | $\frac{70}{100}$ | 0.7 |
| $\frac{9}{20}$ | $\frac{45}{100}$ | 0.45 |
| $\frac{13}{40}$ | $\frac{12 \frac{1}{2}}{100}$ or $\frac{12.5}{100}$ | 0.125 |
| $\frac{7}{25}$ | $\frac{28}{100}$ | 0.28 |

5. Have your students read the correct name for each decimal. For instance, 0.375 is "three hundred seventy-five thousandths," not "point three seven five." Correctly naming a decimal reveals its fractional equivalent of $375 / 1000$. Also encourage the following insights: 0.375 is 0.125 less than $0.5 ; 0.375$ is half way between 0.25 and 0.5 ; and 0.375 is 0.125 more than 0.25 .
$\begin{aligned} \text { 5. a. } 0.4 & =\frac{2}{59} & \text { b. } \quad 0.15=\frac{3}{20} & \\ 0.9 & =\frac{\text { c. }}{10} & & 0.125=\frac{1}{30} \\ 0.04 & =\frac{9}{25} & 0.12=\frac{3}{25} & 0.98=\frac{49}{50} \\ 0.05 & =\frac{3}{20} & 0.65=\frac{3}{8} & 0.075=\frac{3}{40}\end{aligned}$
6. $\frac{21}{24}=0.875=87.5 \%$
7. For each fraction in simplest form, write an equivalent fraction with a denominator of 100, or in hundredths. Then convert each fraction to its decimal form.

| Simplest Form | Fractions in Hundredths <br> or Thousandths | Decimal Form | Percent |
| :---: | :---: | :---: | :---: |
| $\frac{3}{5}$ |  |  |  |
| $\frac{7}{10}$ |  |  |  |
| $\frac{9}{20}$ |  |  |  |
| $\frac{13}{40}$ |  |  |  |
| $\frac{7}{25}$ |  |  |  |

6. For each of the following decimals, find a fraction that is equivalent.
a. 0.4
0.9
b. 0.15
c. 0.375
0.125
0.98
0.04
0.12
0.075
0.05
0.65
0.48
7. Ms. Johnson's class of 20 students has 9 boys.
a. The boys make up what percent of the class?
b. Draw a pie graphs representing the students in the school, showing the percentage of boys and percentage of girls based in the class.
c. The percentage of boys is the same in Mr. Johnson's class as in the whole school, and there are 360 boys in the school. How many students are in the school? How many girls are in the school?
8. In Mr. Henry's math class, 25 students out of 29 passed their test. In Ms. Wiley's class, 21 students out of 25 passed their test. Which class had a higher passing percentage?
9. In a survey of 24 children, 21 preferred chocolate to vanilla ice cream. What percent of those surveyed preferred chocolate ice cream?
10. A biologist collected 40 turtles of a certain species at the beach. Fifteen of them had spots on their shells.
11. 

a. $\quad \frac{4}{\frac{10}{10}}=\frac{2}{5}$
b.
$\frac{125}{1000}=\frac{15}{Y 00}=\frac{3}{20}$
$\frac{98}{100}=\frac{\frac{375}{4900}}{50}=\frac{3}{8}$
$\frac{4}{100}=\frac{1}{25}$
$\frac{12}{100}=\frac{3}{25}$
$\frac{75}{1000}=\frac{3}{40}$
$\frac{5}{100}=\frac{1}{20}$
$\frac{65}{100}=\frac{13}{20}$
$\frac{48}{100}=\frac{12}{25}$
a. What percent of the turtles had spots on their shells?
b. If a random sample of turtles of this species had 25 spotted turtles, about how many turtles were in the sample?
11. For each fraction in the answers to Exercise 5, find the simplest equivalent fraction.
12. In a survey, 745 students were asked what to choose their favorite subject between Math, English, PE, and music. The results were the following:

| Class | Boys | Girls |
| :---: | :---: | :---: |
| Math | 105 | 164 |
| English | 125 | 114 |
| PE | 54 | 65 |
| Music | 85 | 103 |
| Total | 369 | 446 |

a. Make a pie graph of the results for the boys.
b. Make a pie graph of the results for the girls.
c. Make a pie graph of both boys and girls.
d. Use the pie graphs to find the percentage of boys, the percentage of girls, and the percentage of both who preferred each subject. List your results in a table.
e. Is the percentage of both boys and girls the sum of the other percentages? Why or why not?
f. Use the pie graph to find the percentage of girls who preferred either math or music.
10. a. $\frac{1}{3}$
$\frac{1}{30}$ $\frac{1}{300}$ $\frac{1}{15}$
b. $\frac{1}{9}$
$\frac{4}{9}$
$\frac{2}{45}$
$\frac{2}{450}$
c. $\frac{1}{6}$
$\frac{1}{60}$ $\frac{1}{600}$

Remind students of the Repeat Decimals from friendly fractions covered in Section 6.4. Expect students to see patterns and guess many of the fractions. Check with a calculator.
11. Expect your students to use common denominators and equivalent fractions to justify their choices. They might also use decimals and convert them to fractions.
12. $\frac{6}{7}$ is greater than $\frac{4}{5}$ by $\frac{2}{35}$.
13. Uncle Jim and Aunt Peggy ate $\frac{5}{12}$ of the pie. Only $\frac{7}{12}$ of the pie is left over. Answers will vary as to what fraction model students used to answer the questions. Foreshadows subtraction and comparison of fractions.
14. For some reason, the skill and habit of simplifying fractions has disappeared. Many times using the simplified form is easier and reveals relationships hidden by larger numbers. Unsimplified fractions encourage students to use calculators when number sense is easier and faster.
15. Explanations will vary.
a. $\quad 14.25$ dollars
b. $\quad \frac{1}{39} \mathrm{lb}$.
c. $\frac{-9}{12}$ of the pizza.
13. For each of the following decimals, find a fraction that is equivalent. Check your answer with a calculator, if necessary.
a. $0.333 \ldots$
0.0333...
0.0033...
b. $0.111 \ldots$
c. $0.166 \ldots$
0.4444...
0.0166...
0.06666...
0.0444...
0.00166...
0.00444...
14. Use the number line to discover and represent three fractions that are greater than $\frac{1}{4}$ and less than $\frac{1}{2}$. Explain why each fraction is between $\frac{1}{4}$ and $\frac{1}{2}$.
15. Which fraction is greater, $\frac{4}{5}$ or $\frac{6}{7}$ ? By how much? Explain your reasoning.
16. Your Uncle Jim ate $\frac{1}{4}$ of a cherry pie. Your Aunt Peggy ate $\frac{1}{6}$ of the same pie. How much pie was eaten? How much pie is left over? Which fraction model did you use to answer these questions?
17. Why is it important to know how to simplify fractions?
18. Determine whether a decimal or fraction representation is more appropriate in the following situation. Explain your answer.
a. Renee orders a meal at a restaurant. What best represents the cost of the meal, $\$ 14.25$ or $\$ 14 \frac{1}{4}$ ?
b. Billy orders a hamburger at a fast food restaurant. What best represents the weight of the hamburger, $\frac{1}{3} \mathrm{lb}$ or $0.333 \ldots \mathrm{lb}$ ?
c. A large pizza is divided into 12 slices. Danny and Marie eat 9 slices from a large supreme pizza. What best represents the portion of the pizza eaten, 0.75 or $\frac{9}{12}$ ?

## Equivalent Fractions



## Objective:

The purpose of these two activities is to give the students extra practice with finding equivalent fractions and simplifying fractions. These activities will give the students immediate feedback to help them strengthen these very important skills.

## Materials:

Computer, with printer

## Activity Instructions:

Direct the students to these two websites:
www. $321 \mathrm{know} . c o m / f r a 42 \mathrm{ax} 2 . \mathrm{htm}$
www. $321 \mathrm{know} . c o m / f r a 66 \mathrm{hx} 2 . \mathrm{htm}$
Both of these websites supply a mini-lesson and an interactive game for finding equivalent fractions and simplifying fractions.

Each game is timed and keeps a record of the percent of problems the student gets correct. You might have each child work on these skills for approximately 3 to 5 minutes, depending on their individual needs. Students can refer back to these websites when necessary, or until mastery is shown.

Linear Model for Fraction Activity (9.3)


## CONCENTRATION GAME

Objective: Students will reinforce their understanding of concepts in any lesson throughout the book, in this case, concepts in Section 9.3.

## Materials:

10 index cards (or more) and write a fraction on each
Corresponding decimal representation on the other 10 cards

## Activity Instructions:

1) Turn the cards face down and take turns trying to match the fraction with the decimal value.
2) The winner is the player who matches the most number of pairs correctly.
*Note: Teacher may want to duplicate this activity and use it to reinforce other concepts and skills throughout the book as needed.

## Section 9.4 - Fractions and Alternatives

## Big Idea:

Selecting and using fractions and their equivalent decimals and percents to solve equations and real-life application problems

## Key Objectives:

- Use the equivalent forms for fractions, decimals, and percents.
- Discover the advantage of the various forms of rational parts, given different contexts.
- Solve problems using fractions, decimals, and percents.


## Materials:

No extra materials needed.

## Pedagogical/Orchestration:

- This section reviews much of the skills and knowledge involved in adding, subtracting, multiplying and dividing fractions, decimals, and percents. It also contains several fine examples that use the different forms of rational parts.
- Contrast the process of solving equivalent equations $0.75 x=9$ and $\frac{3}{4} x=9$ and $\frac{3 x}{4}=9$.


## Activity:

"Fraction Equation Match" at the end of the section and on CD

## Vocabulary:

no new vocabulary

## TEKS:

$7.1(A)(B) ; \quad 7.2(B)(E)(F) ; \quad 7.3(A) ; \quad 7.5(A) ; \quad 7.13(A)(B) ; \quad 7.15(B) ; \quad 8.2(A, B) ; \quad 8.3(B) ; \quad 8.5(A) ; \quad 8.14(A) ;$ 8.16(A)

## WARM-UPS for Section 9.4

1. In a group of 26 students, 19 picked math as their favorite subject. Which of the following best describes what percent of the students preferred math?
a. between $73 \%$ and $74 \%$
b. between $74 \%$ and $75 \%$
c. between $75 \% 76 \%$
d. between $72 \%$ and $73 \%$

Ans: (a)
2. Which of the following fractions is less than $80 \%$ ?
a. $\quad \frac{80}{99}$
b. $\frac{81}{100}$
c. $\frac{79}{100}$
d. $\frac{81}{101}$

Ans: (c)

## Launch for Section 9.4:

This Launch can be used in conjunction with the questions at the beginning of the lesson in the first 3 paragraphs. Put the following on the table on the board and ask students to fill in the blanks.

| Fraction | Decimal | Percent |
| :---: | :---: | :---: |
| $\frac{1}{2}$ |  |  |
|  |  | $25 \%$ |
|  | 0.1 |  |

With the students, go through the questions and discussion found in the first 3 paragraphs of Section 9.4. End the launch by saying, "There are times when one fractional representation is easier to use than another. Today we will be looking at situations when being able to convert one form to another and using the alternate form is very useful."

EXPLORATION 1 \& 2 are review problems.

## EXPLORATION 1

A table might help them compare the form of each number. Be especially vigilant about the difference between $0.33,0.3$ and $\frac{1}{3}, \frac{1}{3}$, and $\frac{1}{3} \%$. Ask the class what percent is equivalent to $\frac{1}{3}$. Observe closely to see if they are able to choose $33 \frac{1}{3} \%$.

PROBLEM 1
9 students.

## SECTION 9.4 FRACTIONS AND ALTERNATIVES

In examining quantities like "half of a page," "a tenth of a block," or " $25 \%$ of the class," it is easy to see that fractional parts are represented in more than one way. For example, from the previous sections in Chapters 8 and 9 , we wrote "half of a page" numerically as $\frac{1}{2}$ of a page. Similarly, "a tenth of a block" can be written as 0.1 of a block. In other words, we use fractions, decimals, and percents to indicate parts of a whole.
We could have written $\frac{1}{2}$ of a page as 0.5 of a page, or $50 \%$ of a page. Express 0.1 of a block in fractional terms. How is 0.1 expressed as a percent? Is $\frac{1}{10}$ or 0.10 better? When is $25 \%$ better than its decimal equivalent? When is it better than its equivalent fraction?

There are times when one fractional representation is easier to use than another. In such situations, being able to convert one form to another is very useful.

## EXPLORATION 1

Order the numbers below from least to greatest and identify which of the numbers are equal to each other. Explain your reasoning.

$$
\frac{1}{3}, 30 \%, 0.33,0.3, \frac{3}{10}, \frac{1}{3} \%
$$

## PROBLEM 1

In Terry's class there are 36 students, and $25 \%$ of the class are absent due to a terrible flu epidemic. How many students are absent?

## EXPLORATION 2

This is a good opportunity to have different groups of students try to answer the different questions and then share answers and methods.

The problem leads to the question, "What is $25 \%$ of 36 ?" Earlier you learned that $25 \%$ is equivalent to the decimal 0.25 and to the fraction $\frac{1}{4}$. So, we can rephrase the question in terms of fractions: "What is $\frac{1}{4}$ of 36 ?" Why are these the same? You have two options for finding the answer. One option is to use what you learned about fractions in Section 9.3 to multiply. The other is to use the decimal equivalent and multiply. Which do you prefer?

## EXPLORATION 2

Roger needs $75 \%$ of a 160 -centimeter rope. Decide which of the following equivalent representations would be the easiest for him to use: $75 \%, 0.75$, or $\frac{3}{4}$ . What if he has 200 centimeters and wants to use $\frac{1}{5}$ of it? Decide which of the following equivalent representations would be easiest for him to use: $20 \%, 0.20$, or $\frac{1}{5}$. If he wanted $\frac{1}{6}$ of the 200 -centimeter rope, which method, fraction or decimal, would be better? In the problems above, what determines which method is better?

Next, revisit equations using an understanding of fractional and decimal multiplication and division. Notice how useful one form of a fractional representation might be over another.

In Chapter 4, you learned how to solve equations like $4 x=7$. Remember, to solve this equation, divide both sides of the equation by 4 .

## PROBLEM 2

Do the following questions as a class. Don't use too much time. Allow students to use calculators.
Encourage different groups to show how they solved each of these. This is an opportunity to review reciprocals. For instance, in part (a), they can multiply both sides by 4 or divide both sides by $\frac{1}{4}$, or 0.25 . Reflect with them the advantages and disadvantages of each method in each problem. In part (b), they can multiply by 6 , divide by $\frac{1}{6}$ or divide by $0.1666 \ldots$. In part (c), they can multiply by $\frac{5}{4}$, or 1.25 , or divide by $\frac{4}{5}$, or 0.8 .

## EXAMPLE 2

Students can solve the equation by dividing by $\frac{2}{3}$. The method shown is a simplified way of showing division.
Remind your students of the four-step process for solving equations from Chapter 3. Assure them that this is a good habit to get into for solving contextual algebraic problems.

Recall the four-step process as: (1) Define your variable, (2) Translate the problem into an equation, (3) Solve for the unknown variable, and (4) Check your answer.

Because dividing by 4 is equivalent to multiplying by its multiplicative inverse $\frac{1}{4}$ , the solution to the equation $4 x=7$ is

$$
\begin{gathered}
4 x \div 4=7 \div 4 \\
\text { or } \\
4 x \cdot \frac{1}{4}=7 \cdot \frac{1}{4} \\
\frac{4 x}{4}=\frac{7}{4} \\
x=\frac{7}{4}=1.75
\end{gathered}
$$

To check the answer, find that $4 \cdot \frac{7}{4}=7$, and $4 \cdot(1.75)=7$.

## PROBLEM 2

Solve each of the following pairs of equivalent equations. Choose the most convenient form of the equation and method for solving each equation.
a. $\frac{1}{4} x=3 \quad$ and $\quad 0.25 x=3$
b. $\frac{1}{6} x=4$ and $\quad(0.1666 \ldots) x=4$
c. $\frac{4}{5} x=7$ and $0.8 x=7$

## EXAMPLE 1

There are 18 green marbles in Julia's bag. Two-thirds of the marbles in her bag are green. How many marbles does she have in the bag? Justify each step.

We want students to compare Examples 2 and 3 and their equations and see that $\left(\frac{2}{3}\right) x=18$ and $\frac{2 \mathrm{~A}}{3}=18$ are essentially the same equations.

Make sure students see that the end of the solution for Example 3 shows different ways to set up and solve an equation.

If the solution is not obvious to your students, have them set up the four-step process to help them identify the relevant variables; set up the situation algebraically; and solve the equations. Notice that the solution in the book goes through exactly this process, but in a more abbreviated form. It's always good practice to check your work.

## SOLUTION

Let $x$ be the number of marbles in the bag. Then, $\frac{2}{3} x$ is the number of green marbles in the bag, so $\frac{2}{3} x=18$. Now solve the equation:

$$
\begin{aligned}
\frac{2}{3} x & =18 \\
\frac{3}{2} \cdot \frac{2}{3} x & =\frac{3}{2} \cdot 18 \\
1 x & =27
\end{aligned}
$$

Therefore, Julia's bag contains $x=27$ marbles.

## EXAMPLE 2

Amy has some pens in her desk. Nathan starts out with twice as many pens as Amy. Nathan then gives $\frac{1}{3}$ of his pens to his sister, Lisa. Lisa received 18 pens from Nathan. How many pens does Amy have?

## SOLUTION

Let $A$ be the number of pens Amy has. How many pens does Nathan have? He has $2 A$ pens. How many pens does Nathan give to Lisa? He gives $\frac{1}{3}$ of $2 A$ pens to Lisa, that is, he gives $\frac{1}{3} \cdot 2 A=\frac{2 A}{3}$ pens to Lisa. Lisa has 18 pens, so $\frac{2 A}{3}=18$. Does this equation look familiar? It has the same form as the one above. So, just as above, the solution is Amy has $A=27$ pens.

As you have seen, there are many equivalent ways to write a product involving fractions. For example, the following expressions are equivalent:

$$
\frac{3 x}{4}=3 \cdot \frac{x}{4}=\frac{3}{4} x=3 x \cdot \frac{1}{4}=0.75 x
$$

In Sections 8.1, you discovered that $\frac{a}{b}=a \cdot \frac{1}{b}=\frac{1}{b} \cdot a$. Extend this equivalence to see that

$$
\frac{a x}{b}=a x \cdot \frac{1}{b}=\frac{1}{b} \cdot a x=a \cdot \frac{x}{b}=x \cdot \frac{a}{b} .
$$

Encourage your students to share their solutions in small groups and then have the class discuss the best solution. Usually students prefer decimal answers. To encourage them to understand the value of fractional answers in some situations, ask them to solve the equation $\left(\frac{3}{8}\right) x=2$. They might see that this is easier than dividing 2 by 0.375 , which yields 5.33333 .

## EXERCISES

1. a. $\frac{1}{3} \cdot 780=260$ students
b. $\frac{7}{20} \cdot 780=273$
c. $\frac{12}{12} \cdot 780=65$
d. $\left(\frac{3}{4}\right) \cdot 780=585$
e. $\frac{2}{3} \cdot 780=520$
2. Students may use a calculator if they ask.
a. $16.666 \ldots \%$ or $16 \frac{2}{3} \%$ d.
8.333... \% or $8 \frac{1}{3} \%$ g. $\quad 0.5 \%$ or $\frac{1}{2} \%$
b. $55 \%$
e. $27.4 \%$
c. $62.5 \%$ or $62 \frac{1}{2} \%$ f. $2.8 \%$
3. a. Method 1: $\frac{4}{3} \cdot \frac{3 x}{4}=\frac{4}{3} \cdot 9=12,50, x=12$. Method 2: $4 \cdot \frac{3 x}{4}=4 \cdot 9=36,50,3 x=36$. Multiplying by $\frac{1}{3}$ , you get $x=\frac{3 x^{3}}{3}=\frac{36^{4}}{3}=12$.
b. $x=\frac{4}{3} \cdot \frac{3 x}{4}=\frac{4}{3} \cdot 2=\frac{8}{3}$
h. $x=\frac{10}{3} \cdot \frac{3 x}{10}=\frac{10}{3} \cdot 4=\frac{40}{3}=13 \frac{1}{3}$
C. $X=\frac{4}{3} \cdot \frac{3 x}{4}=\frac{4}{3} \cdot 1=\frac{4}{3}$
i. $x=\frac{0.3 x}{0.3}=\frac{2}{0.3}=\frac{2 \cdot 10}{0.3 \cdot 10}=\frac{20}{3}$
d. $x=\frac{5}{2} \cdot \frac{2 x}{5}=\frac{5}{2} \cdot 6=15$
j. $x=\frac{20}{3} \cdot \frac{3 x}{20}=\frac{20}{3} \cdot 9=60$
e. $x=\frac{5}{2} \cdot \frac{2 x}{5}=\frac{5}{2} \cdot 3=\frac{15}{2}=7 \frac{1}{2}$
k. $x=\frac{20}{3} \cdot \frac{3 x}{20}=\frac{20}{3} \cdot 4=\frac{80}{3}=26 \frac{2}{3}$
f. $x=\frac{5}{2} \cdot \frac{2 x}{5}=\frac{5}{2} \cdot 1=\frac{5}{2}$
I. $x=\frac{0.15 x}{0.15}=\frac{2}{0.15}=\frac{2 \cdot 100}{0.15 \cdot 100}=\frac{200}{15}=13.33 \overline{3}$
g. $x=\frac{10}{3} \cdot \frac{3 x}{10}=\frac{10}{3} \cdot 6=20$

If you substitute numbers for the variables, are the expressions still equivalent?

For the following equivalent equations, which form leads to the simplest way to compute a solution? Compute the solution to these equations as many ways as possible and compare the different approaches.

$$
\frac{3 x}{8}=12 \text { or } \frac{3}{8} x=12 \text { or } 0.375 x=12
$$

## EXERCISES

1. Jeffrey's school has 780 students. Find the number of students in each group below. Determine which form is the most convenient to use.
a. $\frac{1}{3}$ of the students live within 2 miles of the school.
b. $\frac{7}{2,0}$ of the students bring a sack lunch to school.
c. $\frac{1}{12}$ of the students prefer salad to pizza for lunch.
d. $75 \%$ of the students prefer ice cream to pie.
e. 2 out of every 3 students take the bus to school.
2. Convert each fraction or decimal into a percent:
a. $\frac{1}{6}$
b. $\frac{11}{20}$
$\begin{array}{ll}\text { C. } \frac{5}{8} \\ \text { d. } & \frac{1}{12}\end{array}$
e. 0.274
g. 0.005
f. 0.028
3. Solve each of the following equations in as many ways as you can. Compare the different methods you used.
a. $\frac{3 x}{4}=9$
b. $\frac{3 x}{4}=2$
c. $\frac{3 x}{4}=1$
d. $\frac{2 x}{5}=6$
e. $\frac{2 x}{5}=3$
f. $\frac{2 x}{5}=1$
g. $\frac{3 x}{10}=6$
h. $\frac{3}{10} x=4$
i. $\quad 0.3 x=2$
j. $\frac{3 x}{20}=9$
k. $\frac{3}{20} x=4$
I. $0.15 x=2$
4. a. $\quad R=\frac{1}{4} x=\frac{x}{4}$ is the number of cups of uncooked rice to feed $x$ people.
b. $R=\frac{1}{4} \cdot 20=5$ cups of rice to feed 20 people; $R(30)=\frac{1}{4} \cdot 30=7 \frac{1}{2}$ cups of rice to feed 30 people.
c. $R=\frac{1}{4} x=30$ cups of rice. Solving for $x$, you get $x=4 \cdot 30=120=$ the number of people 30 cups of uncooked rice will feed.
5. You can define the function $B(x)=\frac{1}{2} x=$ the number of bushels of peaches from $x$ trees.
a. To get 75 bushels of peaches, you set $B(x)=\frac{1}{2} x=75$. Solving for $x$, we get $\frac{1}{2} x=75$, and $x=2 \cdot 75=150=$ the number of trees expected to yield 75 bushels of peaches.
b. If the yield is actually $\frac{2}{3} \frac{\text { bushels }}{\text { tree }}$ then $B(x)=\frac{2}{3} x$. Then to fill the order of 75 bushels, you set the

$$
B=\frac{2}{3} x=75 \text { and solve: } \frac{3}{2} \frac{2}{3} x=\frac{3}{2} \cdot 75=\frac{150}{2}=112 \frac{1}{2} \text { trees. }
$$

6. The number of tables Mary can make in 20 days is $20 \div 2 \frac{1}{3}=20 \div \frac{7}{3}=20 \cdot \frac{3}{7}=\frac{60}{7}=8 \frac{4}{7}$, and the number of tables Mary can make in 21 days is $21 \div 2 \frac{1}{3}=21 \div \frac{7}{3}=21 \cdot \frac{3}{7}=9$ tables.
7. a. Since $C=\frac{2}{5} x=0.4 x$, then $C=0.4(12)=\frac{2}{5}(12)=\frac{24}{2}=4 \frac{4}{5}=4.80$.
b. $C=\frac{2}{5} x=0.4 x=8$, so we solve $\frac{2}{5} x=8$ to get $\frac{5}{2} \frac{2}{5} x=x=\frac{5}{2} \cdot 8=20$ pencils for $\$ 8$.
c. Setting $\frac{2}{5} x=0.4 x=15$, we solve to get $x=\frac{5}{2} \frac{2}{5} x=\frac{5}{2} \cdot 15=\frac{75}{2}=37 \frac{1}{2}$ pencils for $\$ 15$. Discuss with students how reasonable this answer is. The answer should be an integer since it doesn't make sense to buy half a pencil.
8. a. We let $C=\frac{8}{5} x=1.60 x$. Nine pens cost $C=\frac{8}{5}(9)=\frac{72}{5}=14 \frac{4}{5}=14.40$ dollars.
b. Setting $\frac{8}{5} x=1.6 \cdot x=200$, we get $x=\frac{5}{8} \frac{8}{8} x=\frac{5}{8} \cdot 200=5 \cdot 25=125$ pens for $\$ 200$.
c. Setting $\frac{8}{5} x=1.6 \cdot x=350$, we get $x=\frac{8}{8} \frac{8}{5} x=\frac{8}{8} \cdot 350=218.75$ pens for $\$ 350$.
9. a. We let $T=0.08 x=$ the tax paid on $\$ x$ of purchase.
b. The tax on a purchase of $\$ 54.50$ is $T(54.50)=0.08(54.50)=4.36$ dollars.
c. Setting $0.08 x=6.64$, we get $x=\frac{0.08 x}{0.08}=\frac{6.64}{0.08}=83$ dollars of purchase.
d. Note: Difficult problem

Let $x=$ the amount of purchase. Then $0.08 x=$ the tax paid on $\$ x$ of purchase.
The sum $x+0.08 x=75.06$, which is equivalent to $1.08 x=75.06$.
Solving for $x$, we get $x=\frac{1.08 x}{1.08}=\frac{75.06}{1.08}=69.50=$ amount of purchase.
This problem will most likely be worked by using an equation where your students solve for the cost of the goods, but a student might set up a proportion to work this problem. You might want to share the method with the class if one student does this.
4. It takes $\frac{1}{4}$ cup of uncooked rice for a meal for one person. Let $R$ be the amount of rice it takes to feed the guests at a banquet.
a. What is a formula for $R$ ?
b. How much uncooked rice will it take to prepare a meal for 20 people? 30 people?
c. How many people can you serve with 30 cups of uncooked rice?
5. A farmer grew peach trees and expected a yield of $\frac{1}{2}$ bushel of peaches per tree.
a. How many trees does she need to pick to fill an order of 75 bushels of peaches for a market?
b. The actual yield later that year was on average $\frac{2}{3}$ bushel per tree. How many trees did she end up needing to fill the order in part $\mathbf{a}$ ?
6. It takes Mary $2 \frac{1}{3}$ days to build a table. How many tables can she make in 20 days? 21 days?
7. A store sells pencils for $\$ 0.40$ each. The cost of pencils is given by the formula $C=0.4 x=\frac{2}{5} x$. Use the formula to answer the following questions:
a. What is the cost of 12 pencils?
b. How many pencils can you buy with $\$ 8$ ?
c. How many pencils can you buy with $\$ 15$ ?
8. A wholesale company sells pens for $\$ 1.60$ each. The cost of pens is given by the formula $C=1.6 x=\frac{8}{5} x$. Use the formulas to answer the following questions:
a. What is the cost of 9 pens?
b. How many pens can you buy with $\$ 200$ ?
c. How many pens can you buy with $\$ 350$ ?
9. The tax rate in Pottsboro is $8 \%$.
a. Write a formula for the tax paid for buying $x$ dollars of goods.
b. What is the tax on a $\$ 54.50$ purchase?
c. How much was spent if the tax was $\$ 6.64$ ?
d. What is the amount of tax paid if the total cost was $\$ 75.06$ ?
10. a. Let $S=0.12 x=$ the cups of syrup in $x$ cups of fruit punch.
b. In 16 cups of fruit punch, there are $S=0.12(16)=1.92$ cups of syrup.
c. If there are 2.6 cups of syrup, the amount of punch is $x=\frac{0.12 x}{0.12}=\frac{2.6}{0.12}=\frac{260}{12} \approx 21 . \overline{66}$.
11. a. $x=\frac{3}{2} \cdot \frac{2 x}{3}=\frac{3}{3} \cdot \frac{3}{4}=\frac{9}{8}$
b. $x=\frac{3}{4} \cdot \frac{4 x}{3}=\frac{3}{4} \cdot \frac{8}{5}=\frac{6}{5}$ c. $x=\frac{12}{7} \cdot \frac{7 x}{12}=\frac{12}{7} \cdot \frac{3}{16}=\frac{9}{28}$
d. First we add $\frac{3}{7}$ to both sides of the equation to get $\frac{2 x}{5}=\frac{8}{7}$. Multiplying both sides of this equation by $\frac{5}{2}$, we get $x=\frac{5}{2} \cdot \frac{2 x}{5}=\frac{5}{2} \cdot \frac{8}{7}=\frac{40}{14}=\frac{20}{7}=2 \frac{6}{7}$.
e. First we subtract $\frac{5}{2}$ from both sides of the equation to get $\frac{3 x}{10}=3-\frac{5}{2}=\frac{1}{2}$. Multiplying both sides of this equation by $\frac{10}{33}$, we get $x=\frac{10}{3} \cdot \frac{3 x}{10}=\frac{10}{3} \cdot \frac{1}{2}=\frac{5}{3}$.
f. First we add $\frac{3}{20}$ to both sides of the equation to get $\frac{7 x}{12}=\frac{3}{20}+\frac{11}{18}$. Simplifying the right side, we get $\frac{7 x}{12}=\frac{3}{20}+\frac{11}{18}=\frac{3}{2 \cdot 2 \cdot 5}+\frac{11}{2 \cdot 3 \cdot 3}=\frac{3 \cdot 3^{2}}{2^{2} \cdot 3^{2} \cdot 5}+\frac{11 \cdot 2 \cdot 5}{2^{2} \cdot 3^{2} \cdot 5}=\frac{27+110}{2^{2} \cdot 3^{2} \cdot 5}=\frac{137}{180}$.
Multiplying both sides of this equation by $\frac{12}{7}$, we get $\mathrm{x}=\frac{12}{7} \cdot \frac{7 \mathrm{x}}{12}=\frac{12}{7} \cdot \frac{137}{180}=\frac{137}{7 \cdot 15}=\frac{137}{105}$.

## Ingenuity

12. Remember that $104!=104 \cdot 103 \cdot 102!\cdot \frac{104!}{102!}=\frac{104 \cdot 103 \cdot 102 \cdot 101 \cdot 100 \cdot \ldots \cdot 3 \cdot 2 \cdot 1}{102 \cdot 101 \cdot 100 \cdots 3 \cdot 2 \cdot 1}=\frac{104!103!102!}{102!}=104 \cdot 103$ $=10,712$.

## Investigation

13. a. $\frac{5}{10}=\frac{1}{2}$
b. $\frac{5+5+5+5}{10+10+10+10}=\frac{20}{40}=\frac{1}{2}$
c. $5 \frac{\text { marbles }}{\text { bag }} \cdot$ bags $=60$ marbles.
14. Frida's Fruit Punch is made from water and syrup. Twelve percent of every cup is syrup.
a. Write a formula for the number of cups of syrup needed to make $x$ cups of fruit punch
b. How many cups of syrup are in 16 cups of fruit punch?
c. How many cups of fruit punch can be made from 2.6 cups of syrup?
15. Solve each of the following equations:
a. $\frac{2 x}{3}=\frac{3}{4}$
b. $\frac{4 x}{3}=\frac{8}{5}$
c. $\frac{7 x}{12}=\frac{3}{16}$
d. $\frac{2 x}{5}-\frac{3^{16}}{7}=\frac{5}{7}$
e. $\frac{3 x}{10}+\frac{5}{3}=3$
f. $\frac{7 \mathrm{x}}{12}-\frac{3}{20}=\frac{11}{18}$
16. Ingenuity:

Review the definition of factorial in the ingenuity in Section 4.3. Find $\frac{104!}{102!}$.
13. Investigation:

Meg has a bag of marbles containing 5 red marbles and 5 green marbles.
a. What fraction of the marbles are red?
b. Meg buys 4 more identical bags of marbles. What fraction of the marbles are red?
c. How many red marbles will she have when she has 12 such bags of marbles?

## FRACTION EQUATION MATCH



Objective: Students will match equations to their verbal representation.

## Materials:

Copy of Fraction Equation Match Cards (Sets A \& B)

## Activity Instructions:

To begin, make copies of both Set A and Set B cards and cut them into individual pieces. Set A cards will be shuffled and placed in a stack "face down." Set B cards should be displayed "face up" in a place where all players can easily see the equations.

Choose a player to go first, and have them pick the top card from Set A. Player 1 will read this card aloud. Player 1 will then look through the displayed cards from Set $B$ and try to find one that matches. If Player 1 is able to successfully find the match, he gets one point. If not, the next player will have the opportunity to find the match for this card. This card continues to rotate through players until the correct match is found. The player that finds the match is the player that gets the point.

Once the match is found, the player that earned the point now has the opportunity to solve the equation for the given value at the bottom of the card. If the player is successful in solving the equation, he earns one additional point. If this player is unsuccessful in solving, he keeps his one point from the correct match, but the next player will have the opportunity to solve and steal the extra point away from him. The opportunity to solve the equation and earn the extra point continues to rotate through the players until the equation is correctly solved.

## Set A

| Tom has $2 / 3$ as many pets as Amy. | The diet soda has $1 / 5$ the calories as the regular <br> soda. |
| :---: | :---: |
| The old copier is $50 \%$ slower than the new copier. | If Allison triples the number of cookies she has now, <br> she can split them evenly into four packages. |
| Baseball gloves cost $\$ 20.50$ each. | Jeremy pays 40 cents per minute for his new cell <br> phone plan. |
| The pencils need to be split evenly between 10 |  |
| students. | April's share of the work is $30 \%$ of Paul's share. |
| 2/9 of the class has lunch detention. |  |

Set B

| $Y=2 \bullet \frac{x}{3}$ |  |  |  |
| :---: | :---: | :---: | :---: |
| $Y=\frac{x}{2}$ | $Y=0.2 x$ |  |  |
| $F(x)=\frac{41 x}{2}$ | $Y=3$ | $Y=0.75 x$ |  |
| $Y=\frac{1}{10} x$ | $F(x)=8$ |  |  |
| $Y=2 x \bullet \frac{1}{9}$ |  | $Y=\frac{2 x}{5}$ |  |

1. a. $\frac{10}{3}=3 \frac{1}{3}$
c. $\frac{20}{30}=\frac{2}{3}$
e. $\frac{15}{6}=2 \frac{3}{6}=2 \frac{1}{2}$
b. $\frac{6}{12}=\frac{1}{2}$
d. $\frac{20}{15}=1 \frac{5}{15}=1 \frac{1}{3} \quad$ f. $\quad \frac{63}{168}=\frac{3}{8}$
g. $\frac{2}{15}$
h. $\frac{5}{4}=1 \frac{1}{4}$
2. James ate 3 cupcakes. $\frac{1}{4} \cdot 12=3$.
3. Daisy will use 28 jewels for the necklace. $\frac{2}{3} \cdot 42=28$.
4. 42
5.Since $\frac{2}{5} \cdot \frac{2}{3}=\frac{4}{15}$, then $\frac{4}{15}$ will be drained. Thus, $\frac{2}{3}-\frac{4}{15}=\frac{10-4}{15}=\frac{6}{15}=\frac{2}{5}$ of the tank remains.
5. $\frac{25}{4} \cdot \frac{81}{8}=\frac{2025}{32}=63.28 \mathrm{sq}$ in. Pablo will cover 63.28 sq in of the book.
6. a. $\frac{8}{1}=8$
c. $\frac{10}{8}=1 \frac{2}{8}=1 \frac{1}{4}$
b. $\frac{12}{4}=3$
d. $\frac{33}{3}=11$
e. $\frac{5}{3}=1 \frac{2}{3}$
7. $5 \frac{3}{5} \div 3=\frac{28}{5} \cdot \frac{1}{3}=\frac{28}{15}=1 \frac{13}{15}$. It takes Daisy $1 \frac{13}{15}$ hours to make one piece of jewelry.
8. $30 \div 4 \frac{3}{4}=\frac{30}{1} \div \frac{19}{4}=\frac{30}{1} \cdot \frac{4}{19}=\frac{120}{9}=6 \frac{6}{19}$ or 6.315 . Nila can get six $4 \frac{3}{4}$ inch hair strings from a 30 inch piece of yarn.
9. $7 \frac{3}{7}$, so she needs to prepare 8

## REVIEW PROBLEMS

1. Compute the following. Write answers in simplest form.
a. $5 \cdot \frac{2}{3}$
b. $\frac{2}{3} \cdot \frac{3}{4}$
c. $\frac{4}{5} \cdot \frac{5}{6}$
d. $\frac{10}{3} \cdot \frac{2}{5}$
e. $3 \cdot \frac{5}{6}$
f. $\frac{9}{14} \cdot \frac{7}{12}$
g. $\frac{10}{27} \cdot \frac{9}{25}$
h. $\frac{36}{21} \cdot \frac{35}{48}$
2. Iris baked a dozen cupcakes. James ate $\frac{1}{4}$ of the cupcakes. How many cupcakes did James eat?
3. Daisy would like to use $\frac{2}{3}$ of her 42 turquoise jewels to make a necklace. How many jewels will she use?
4. A poll showed that $\frac{3}{4}$ of those surveyed prefer pizza to hamburgers. If there were 56 people surveyed, how many of them prefer pizza?
5. A tank was $\frac{2}{3}$ full of water. If you drain $\frac{2}{5}$ of the water presently in the tank, how much water will be left in the tank?
6. Pablo is covering the face of his math book that measures $6 \frac{1}{4}$ inches wide and $10 \frac{1}{8}$ inches tall. How many total square inches will Pablo be covering?
7. Compute the following. Write answers in simplest form.
a. $2 \div \frac{1}{4}$
b. $\frac{3}{4} \div \frac{1}{4}$
c. $\frac{5}{8} \div \frac{1}{2}$
d. $3 \frac{2}{3} \div \frac{1}{3}$
e. $\frac{2}{3} \div \frac{2}{5}$
8. Daisy is making some jewelry for her sister. It takes her $5 \frac{3}{5}$ hours to make 3 pieces of jewelry. How long does it take Daisy to make 1 pieces of jewelry?
9. Nila is stringing together some yarn to make the hair on her paper doll. She would like each string to measure $4 \frac{3}{4}$ inches long. She is starting with a string that is 30 inches long. How many yarn hair strings can she get from this 30 inch string?
10. Sandra is running a marathon of 26 miles. If she wants to drink a bottle of water every $3 \frac{1}{2}$ miles, how many bottles does she need to prepare for the entire race?
11. Answers may vary, but an example is: $\frac{7}{10}, \frac{3}{4}, \frac{4}{5}$
12. a. 0.4
c. 0.5
e. 0.7
b. 0.48
d. 0.75
13. Answers may vary, but some equivalent fractions include:
a. $\frac{25}{100}=\frac{1}{4}$
c. $\frac{7}{100}$
e. $\frac{125}{1000}=\frac{1}{8}$
b. $\frac{3}{10}$
d. $\frac{60}{100}=\frac{3}{5}$
14. a. $20 \%$
c. $60 \%$
e. $50 \%$
b. $75 \%$
d. $37.5 \%$
15. a. $22 \%$
c. $75 \%$
e. $5 \%$
b. $10 \%$
d. $12.5 \%$
16. a. $\frac{24}{100}=24 \%$
c. $\frac{3}{10}=30 \%$
e. $\frac{9}{100}=9 \%$
b. $\frac{35}{100}=35 \%$
d. $\frac{375}{1000}=37.5 \%$
17. The order from least to greatest is:
$0.07,20 \%, 0.25, \frac{1}{3}, \frac{2}{5}, 50 \%, 0.75,1$
18. 

a. 14
b. 24
c. 20
d. 35
11. Using the number line below, give three fractions that are greater than $\frac{1}{2}$ and less than $\frac{4}{5}$. Plot each on the number line to show how they are ordered.

12. Convert the following fractions to an equivalent decimal.
a. $\frac{2}{5}$
b. $\frac{12}{25}$
$\begin{array}{ll}\text { c. } & \frac{6}{12} \\ \text { d. } \frac{3}{4}\end{array}$
e. $\frac{7}{10}$
13. Convert the following decimals to an equivalent fraction.
a. 0.25
b. 0.3
c. 0.07
d. 0.60
e. 0.125
14. Convert each fraction to a percent.
a. $\frac{1}{5}$
$\begin{array}{ll}\text { c. } & \frac{6}{10} \\ \text { d. } \frac{3}{8}\end{array}$
e. $\frac{7}{14}$
15. Convert each decimal to a percent.
a. 0.22
b. 0.1
c. 0.75
d. 0.125
e. 0.05
16. Convert each decimal to a fraction then a percent.
a. 0.24
b. 0.35
c. 0.3
d. 0.375
e. 0.09
17. Using the number line below, compare and order the following:
$0.75, \frac{1}{3}, 20 \%, 0.07,50 \%, 0.25, \frac{2}{5}$

18. Solve these equations:
a. $\frac{1}{2} x=7$
b. $\frac{3}{4} x=18$
c. $\frac{3}{5} y=12$
d. $\frac{5}{7} y=25$

## Section 9.1: 6

Solution: We can just try each possible denominator for possible numerators, and we get the following list: $\frac{1}{4}<\frac{2}{7}<\frac{1}{3}<\frac{3}{8}<\frac{2}{5}<\frac{3}{7}<\frac{4}{9}<\frac{1}{2}$. Note that for each consecutive $\frac{a}{b}<\frac{c}{d}$ we have $\mathrm{ad}-\mathrm{bc}=1$. This is an indication that there are no intermediate fractions with a smaller denominator, and it continues to hold if we take out first $\frac{4}{9}$, then $\frac{3}{8}$, etc. The increasing sequence of reduced fractions from 0 to 1 with a specified maximum denominator is called a Farey sequence. If we'd left, say, $\frac{3}{7}$ off of our list, then we would notice that $5 \cdot 4-2 \cdot 9=2$, not 1 , and we could construct the missing term by adding the numerators and denominators of the adjacent terms: $\frac{2+4}{5+9}=\frac{6}{14}=\frac{3}{7}$.
Section 9.2: 5
Solution: We have $\frac{a+b}{a b}=\frac{1}{6}$, cross-multiplying gives $6 a+6 b=a b$, and $a(b-6)=6 b$ so $a=\frac{6 b}{b-6}=6+\frac{36}{b-6}$. Thus $b-6$ divides 36 , which has 9 positive divisors. In particular, $b$ can be $7,8,9,10,12,15,18,24$, or 42 , making a $42,24,18,15,12,10,9,8$, or 7 , respectively. We reject the four cases in which $a>b$; thus $(a, b)=(12,12)$, $(10,15),(9,18),(8,24)$, and $(7,42)$.

## Section 9.3: $\quad 2^{18}$

Solution: When multiplying, we leave the first two denominators alone, then the 6 in the third fraction reduces to 2 thanks to the 3 in the second. Similarly the 5 in the third fraction reduces 10 to 2,7 reduces 14 to 2 , and 9 reduces 18 to 2 . Thus we are left with $\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{8} \cdot \frac{1}{2} \cdot \frac{11}{12} \cdot \frac{13}{2} \cdot \frac{15}{16} \cdot \frac{17}{2} \cdot \frac{19}{20}$. The 15 reduces to 12 and 20 to 4 each, giving a denominator of $2 \cdot 2^{2} \cdot 2 \cdot 2^{3} \cdot 2 \cdot 2^{2} \cdot 2 \cdot 2^{4} \cdot 2 \cdot 2^{2}=2^{18}=262,144$.

For further exploration, will the denominator of such a product always be a power of 2 ?

## Section 9.4: $\quad \frac{8}{5}$

Solution: Starting at the bottom, we have $1+\frac{1}{1}=2$, then $1+\frac{1}{2}=\frac{3}{2}$, then $1+\frac{2}{3}=\frac{5}{3}$, then $1+\frac{3}{5}=\frac{8}{5}$. In general, if we added more ones we would always get a ratio of consecutive Fibonacci numbers (the sequence that starts with 0 , then 1 , then each subsequent term is the sum of the previous two), and the value would approach the golden ratio $\frac{1+\sqrt{5}}{2}$.

## Section 9.5: 22

Solution: Since the fractions add less than 1 each, we start by observing that $1+4+7+10+13+16+19+$ $22=92$, and 8 fractions will still keep us under 100 , while adding 25 will send us over.

## CHALLENGE PROBLEMS

## Section 9.1:

How many fractions (in simplest form) between $\frac{1}{4}$ and $\frac{1}{2}$ have a denominator less than 10 ?

## Section 9.2:

How many pairs of positive integers $(a, b)$ with $a \leq b$ satisfy $\frac{1}{a}+\frac{1}{b}=\frac{1}{6}$ ?

## Section 9.3:

Compute the denominator, after simplifying, of $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{19}{20}$.

## Section 9.4:

Simplify $1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}}$.

## Section 9.5:

If $1 \frac{2}{5}+4 \frac{5}{6}+7 \frac{8}{9}+\cdots+n \frac{n+1}{n+2}<100$, what is the largest possible value of $n$ ?

## Section 10.1 - Rates and Ratios

## Big Idea:

Understanding and using rates and ratios

## Key Objectives:

- Understand the definition of a ratio
- Understand the definition of a rate.
- Understand the difference between ratios and rates.
- Represent rates and find distances using the distance/rate formula.
- Note the importance of units in representing ratios and rates.
- Use ratios and rates in problem settings.


## Materials:

Calculators suggested for Exercise 7

## Pedagogical/Orchestration:

This section explains rates and ratios and the difference between them. It also provides rich and numerous ways to solve problems involving ratios and rates.

## Internet Resource:

Jeopardy Game: Rates, Ratios, and Proportions- http://www.quia.com/cb/158527.htm|

## Activity:

"Ratios in the World" on CD and at the end of the section —use after going through section 10.1.

## Exercises:

Exercise 8 connects to measurement conversions.
$10 \& 11$ are challenging.

## Vocabulary:

rate, unit rate, ratio, (On CD: customary system, metric system)

## TEKS:

$6.2(C) ; \quad 6.3(B)(C) ; \quad 6.4(A) ; \quad 7.2(D)(F) ; \quad 7.3(A)(B) ; \quad 7.4(A) ; \quad 7.13(C) ; \quad 7.15(A)(B) ; \quad 8.2(A, D) ; \quad 8.3(A, B) ;$
8.4(A); 8.16(A)

## WARM-UPS for Section 10.1

1. Millie plays piano $1 \frac{1}{2}$ hours on her regular practice day. If her chart shows that she has practiced for 21 hours this month, how many days has she practiced?
a. 12 days
b. 14 days
c. 16 days
d. 18 days

Ans: (b) because $21 \div 1 \frac{1}{2}=14$ or you can build a table for $\mathrm{y}=\frac{3}{2} \mathrm{x}$ and find the output of 21 requires an input of 14.
2. Diann drove at a constant speed across a long straight road. If she drove 340 miles in 5 hours, how far did she drive in 8 hours?
Ans: 340 miles $\div 5$ hours $=68$ miles/hour. So ( $68 \mathrm{mi} / \mathrm{hr}$ )( 8 hr ) $=544$ miles.

## Launch for Section 10.1:

Ask and write "Give me examples of rates that you have seen or heard of before". Teacher wll write these on the board. Examples could include:

20 miles/hour 12 bananas for $\$ 1.25 \quad 2$ for the price of $1 \quad 4$ pieces of pizza for every 2 students
Let students discuss their thoughts, and if need be rewrite all of the rates using the fraction bar, such as $\frac{10 \text { miles }}{1 \text { hour }}$ . Let students know that these are all special ratios called rates. The two values being compared in a rate have different units and describe how one value changes in relation to another. Tell students, "In this section, you will be learning numerous ways to solve problems involving ratios and rates."

Note: 10.1 is a review up to Example 1.

Don't be worried if your students cannot figure this out. Correct answers that cannot be true logically and ask your students to be patient.

Emphasize that there are not suddenly just 7 students who live within 3 miles and 3 who don't. That is, these numbers are part of a ratio, rather than absolute numbers.

## RATES, <br> RATIOS AND <br> PROPORTIONS



## SECTION 10.1 RATES AND RATIOS

Fractions are often used to compare quantities. For example, Miller Junior High School has 400 students, and 280 of them live within 3 miles of the campus. Simplifying, $\frac{280 \text { students }}{400 \text { students }}=\frac{7}{10}$. Notice that both units are "students" and they simplify. How can you interpret the meaning of the simplified fraction?

In Miller Junior High School, 7 out of every 10 students live within 3 miles of the school. Converting the fraction $\frac{7}{10}$ into a percent, $70 \%$ of the students live within 3 miles. The fractional form of this comparison is called a ratio. A ratio is a division comparison of two quantities with or without the same units.

Ratios can also be written in the form of first one quantity, then a colon, followed by a second quantity. Ratios can be written using the word "to" in place of the colon and in fraction form. In this example, the unit of measure is the number of students.

Because there are 280 students who live within 3 miles, write
280 students : $\qquad$
Compared to these 280 students within 3 miles, there are 400 total students. So the ratio of students who live within 3 miles to total number of students is

280 students who live within 3 miles : 400 total students
Ignoring for a moment the units and using only the numbers, write the ratio as $280: 400$. Just as with fractions, we can simplify this to $7: 10$, that is, $\frac{280}{400}=\frac{7}{10}$. Always remember what kinds of things are being compared. In this problem, what does the ratio 7:3 describe?

## EXPLORATION 1

Suppose a class has 12 boys and 18 girls. How many other ratios can you discover in this situation? Write the following comparisons as ratios:
a. the number of boys to the number of girls $12: 18$ or $2: 3\left(\frac{12}{18}=\frac{2}{3}\right)$
b. the number of girls to the number of students $18: 30$ or $3: 5\left(\frac{3}{5}=\frac{18}{30}\right)$
c. the number of students to the number of girls $30: 18$ or $5: 3\left(\frac{30}{18}=\frac{5}{3}\right)$
d. the number of boys to the number of students $12: 30$ or $2: 5\left(\frac{12}{30}=\frac{2}{5}\right.$ )
e. the number of students to the number of boys $30: 12$ or $5: 2\left(\frac{30}{12}=\frac{5}{2}\right.$ )

In computing the ratios above, notice that some of the ratios are between a part and the whole, such as girls to students, and some of them are between two parts, such as boys to girls.

Rates are special ratios that compare different units. Suppose, you earn 48 dollars for doing 6 hours of yard work and mowing the lawn. You know that $48 \div 6=8$ indicates how much money you earned per hour. Using fractions, this calculation looks like $\frac{48}{6}=8$. However, it is usually helpful to write this problem using the units that describe each quantity. So the calculation becomes

$$
\frac{48 \text { dollars }}{6 \text { hours }}=\frac{8 \text { dollars }}{1 \text { hour }}=8 \frac{\text { dollars }}{\text { hour }} .
$$

| Hours | 1 | 2 | 3 | 4 | 5 | 6 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Dollar | 8 | 16 | 24 | 32 | 40 | 48 |

You read " $8 \frac{\text { dollars }}{\text { hour }}$ "as " 8 dollars per hour." The answer explains exactly how many dollars you earned each hour. This quantity is an example of a rate. A rate is defined as a division comparison between two quantities, usually with two different units, like dollars and hours. The simplified fractional answer in the example is called a unit rate because it represents a number or quantity per 1 unit, or hour in this case. The units may be written in fractional form, like $\frac{\text { dollars }}{\text { hour }}$ , $\frac{\text { miles }}{\text { hour }}$ or $\frac{\text { miles }}{\text { gallon }}$.

## EXPLORATION 2

There are 6 possible rates.
$(150$ miles $) /(3$ hours $)=50$ miles/hour $=50$ miles per hour $=$ number of miles driven in 1 hour
$(3$ hours $) /(150$ miles $)=1 / 50$ hour $/ \mathrm{mi}=$ how long it takes to drive 1 mile $=$ a little more than 1 minute per mile (150 mi.)/(5 gal.) = $30 \mathrm{mi} / \mathrm{gal}=30$ miles per gallon = number of miles driven on 1 gallon of gasoline $(5 \mathrm{gal}) /(150 \mathrm{mi})=.1 / 30 \mathrm{gal} / \mathrm{mi}=$ number of gallons it took to drive 1 mile $=$ about 0.03 gallons per miles ( 3 hours)/(5 gal.) $=3 / 5 \mathrm{hr} /$ gal $=$ how long he drove on 1 gallon of gasoline $=$ more than half an hour per gallon $(5 \mathrm{gal}) /(3$ hours $)=5 / 3 \mathrm{gal} / \mathrm{hr}=$ no. of gallons used while driving 1 hour $=$ almost 2 gallons per hour

Students are to complete the table to identify the pattern.

| cups of flower | 4 | 8 | 12 | 16 | 20 | 24 | 28 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| cakes | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

The word rate is usually used to distinguish an even more specific kind of ratio: one involving change, such as $\frac{\text { dollars }}{\text { hour }}, \frac{\text { miles }}{\text { hour }}$ and $\frac{\text { miles }}{\text { gallon }}$. From Exploration 1, suppose you wish to compare the genders in a class with 12 boys and 18 girls. Comparing the number of girls to boys in this class gives the ratio $\frac{18 \mathrm{girls}}{12 \text { boys }}$. Simplifying this fraction, the ratio of girls to boys in the class is $\frac{3}{2}$. Although it is mathematically correct to say the rate of girls per boy in the class is $\frac{3}{2}$, we rarely use the word rate in that comparison because the units are different and the ratio does not involve change.

## EXPLORATION 2

Juan drove 150 miles in 3 hours and used 5 gallons of gasoline. Make as many rates using these quantities and their units as possible. Explain what each unit fraction means.

## EXAMPLE 1

Sandra's bakery uses 12 cups of flour to make 3 cakes when she bakes.. How many cups of flour will she use when she bakes 7 cakes for a customer? 28 cups

## SOLUTION

The unit rate for the amount of flour per cake is given as $\frac{4 \text { cups }}{1 \text { cake }}=4 \frac{\text { cups }}{\text { cake }}$.

| cups |  |  |  |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| cakes |  |  |  |  |  |  |  |  |

To find how much flour is used to bake 7 cakes, multiply the unit rate by the number of cakes:

$$
4 \frac{\text { cups }}{\text { cake }} \cdot(7 \text { cakes })=28 \frac{\text { cups }}{\text { cake }} \cdot \text { cake }=28 \text { cups }
$$

## PROBLEM 1

Rate is given by miles per hour in this case. We take the distance, 18 miles, and divide by the time, $2 \frac{1}{2}$ hours. The result is $20 \div 2 \frac{1}{2}=30 \div \frac{5}{2}=20 \cdot \frac{2}{5}=18$ miles per hour. In an hour and a half, Karla must have traveled 8 mph - $1 \frac{1}{2}$ hours $=8$ miles/hour $\cdot \frac{3}{2}$ hours $=12$ miles.

## EXPLORATION 3

Have students make a table. Let them figure out how to build it. They might discover the importance of computing the unit rate first. They could divide by 2 several times to get the amount earned for 18 hours, 9 hours, and so forth. They will see the limitations of this approach. It is also important for them to see that building the table with increasing numbers of hours helps reveal the pattern.

| Hours | 36 | 5 | 10 | 15 |
| :--- | :--- | :--- | :--- | :--- |
| Earnings | $\$ 612$ | $17 \times 5=\$ 85$ | $17 \times 10=\$ 170$ | $17 \times 15=\$ 255$ |
| Rate in $\$ /$ hour | $612 / 36=\$ 17 /$ hour |  |  |  |

PROBLEM 2
a. Karen: 45 meters/minute; Karla: 50 meters/minute; Karla jogs faster.
b. $(45$ meters/minute) $(12$ minutes $)=540$ meters
c. Use $d=r t$. So, $600=50 \bullet t .600$ meters $\div 50$ meters/minute $=12$ minutes

Notice that the units of "cake" simplify to one and the answer is in cups. This is similar to computing the product $\left(\frac{7}{4}\right)(4)=7$. This way of keeping track of the units is very useful in application problems, especially in science.

What happens when we compare two quantities with the same unit?

## PROBLEM 1

Karla rode her bike for $2 \frac{1}{2}$ hours and traveled 20 miles. What was her average rate, or speed? Approximately how far did she travel in the first hour and a half?

## EXPLORATION 3

Sally works as a computer consultant for the Lennox Company and earns $\$ 612$ for working 36 hours. If she charges the same amount per hour, how much will she earn working 5 hours? 10 hours? 15 hours? Use unit rate to solve and show your work in table form.

## PROBLEM 2

Karen and Karla both jog everyday. Karen jogs an average of 1,800 meters in 40 minutes and Karla jogs an average of 1,500 meters in 30 minutes.
a. Who jogs faster?
b. On average, how far does Karen jog in 12 minutes?
c. Maintaining her pace, how long does it take Karla to jog 600 meters?

Problem 2 is an example of the rate formula that assumes that something is traveling at a constant rate. The distance traveled is equal to the constant rate multiplied by the time traveled: $\boldsymbol{d}=\boldsymbol{r} \cdot \boldsymbol{t}$ or $\boldsymbol{d}=\boldsymbol{r} \boldsymbol{t}$. The constant rate is often written as the unit rate. Typical units for $r$ are:
miles per hour $\left(\frac{\mathrm{mi}}{\mathrm{hr}_{\mathrm{f}}}\right)=$ miles traveled in one hour
feet per second $\left(\frac{\mathrm{ft}}{\mathrm{sec}}\right)=$ feet traveled in one second
meters per minute $\left(\frac{\mathrm{m}}{\mathrm{min}}\right)=$ meters traveled in one minute

Problem 3
unit rate $=\frac{12 \text { miles }}{3 \text { hours }}=4$ miles per hour
$(4$ miles $/ \mathrm{hr})(5 \mathrm{hr})=20$ miles

Conversion factors do not change the value of a specific quantity, only the units. Why is this so? Because we know that $x / x=1$ if $x=y$ then $x / y$ or $y / x$ is also equal to 1 . So, 1 yard $/ 1$ yard $=1$ and 1 yard $/ 3$ feet or 3 feet $/ 1$ yard is also equal to 1 and when we multiply a quantity by this conversion factor the value of the quantity does not change, only the units will change.

## PROBLEM 3

Sandy went canoeing for 3 hours and traveled 12 miles down river. How fast was she traveling? At the same rate, how far will she travel in 5 hours?

Some typical equivalents with the corresponding conversion rates equal to 1 are

$$
\begin{array}{rlrl}
3 \text { feet }=1 \text { yard } & \frac{3 \mathrm{ft}}{1 \mathrm{yd}} & =1=\frac{1 \mathrm{yd}}{3 \mathrm{ft}} \\
9 \mathrm{sq} \mathrm{ft}=1 \mathrm{sq} \mathrm{yd} & \frac{9 \mathrm{sqft}}{1 \mathrm{sqyd}} & =1=\frac{1 \mathrm{sqyd}}{9 \mathrm{sqft}} \\
60 \text { minutes } & =1 \text { hour } & \frac{60 \mathrm{~min}}{1 \mathrm{hour}} & =1=\frac{1 \text { hour }}{60 \mathrm{~min}} \\
8 \text { pints } & =1 \text { gallon } & \frac{8 \mathrm{p} \mathrm{ints}}{1 \mathrm{gal}} & =1=\frac{1 \mathrm{gal}}{8 \mathrm{p} \mathrm{ints}} \\
10 \text { millimeters } & =1 \text { centimeter } & \frac{10 \mathrm{~mm}}{1 \mathrm{~cm}} & =1=\frac{1 \mathrm{~cm}}{10 \mathrm{~mm}}
\end{array}
$$

Because the conversion rate is equal to 1, multiplying by one of the rates maintains equivalence. For example,

## EXAMPLE 2

a. Convert 4 miles to yards

4 miles $\left(\frac{1760 \mathrm{yd}}{1 \mathrm{mile}}\right)=7040$ yards
b. Convert $3 \frac{1}{2}$ hours to minutes
$3 \frac{1}{2}$ hours $=\left(3 \frac{1}{2} \mathrm{hr}\right)\left(\frac{60 \mathrm{~min}}{1 \mathrm{hr}}\right)=\frac{7}{2} \cdot 60 \mathrm{~min}=7 \cdot \frac{1}{2} \cdot 60 \mathrm{~min}=$
$7 \cdot 30 \mathrm{~min}$
$=210$ minutes .

## EXAMPLE 3

Convert $5 \frac{1}{3}$ yards to $\frac{16}{3}$ yards. Then multiply $\frac{16}{3}$ yards by the conversion factor ( $3 \mathrm{ft} / /(1 \mathrm{yd})$ to convert to feet: $\left(\frac{16}{3} \mathrm{yds}\right) \cdot(3 \mathrm{ft}) /(1 \mathrm{yd})=16 \mathrm{ft}$.

Now to convert feet into inches, use the conversion unit (12 in)/(1 1 ft ).
$(16 \mathrm{ft}) \cdot(12 \mathrm{in}) /(1 \mathrm{ft})=192 \mathrm{in}$.

## EXERCISES

1. a. $\frac{3}{5}$ or $3: 5$ shirts to pants c. $\frac{5}{2}$ or $5: 2$ pants to hats
b. $\frac{5}{3}$ or $5: 3$ pants to shirts
d. $\frac{\frac{2}{5}}{5}$ or $2: 5$ hats to pants $\quad$ f. $\frac{2}{3}$ or $2: 3$ hats to shirts
2. a. The ratio of those who get candy to those who do not is $2: 3$ or $\frac{2}{3}$
b. $\frac{2}{5}(25)=10$ students get candy.

## EXAMPLE 3

Susan has $5 \frac{1}{3}$ yards of cloth. How many feet of cloth is this? How many inches of cloth does she have?

## SOLUTION

Convert $5 \frac{1}{3}$ yards to $\frac{16}{3}$ yards. Then multiply $\frac{16}{3}$ yards by the conversion factor $\frac{3 \mathrm{ft}}{1 \mathrm{yd}}$ to convert to feet:

$$
\frac{16}{3} \mathrm{yds} \cdot \frac{3 \mathrm{ft}}{1 \mathrm{yd}}=16 \mathrm{ft} .
$$

Now convert feet into inches, using the conversion unit $\frac{12 \mathrm{in}}{1 \mathrm{ft}}$ :

$$
16 \mathrm{ft} \cdot \frac{12 \mathrm{in}}{1 \mathrm{ft}}=192 \mathrm{in} .
$$

## EXERCISES

1. Gladys has 3 shirts, 5 pairs of pants and 2 hats in a suitcase.
a. What is the ratio of shirts to pants?
b. What is the ratio of pants to shirts?
c. What is the ratio of pants to hats?
d. What is the ratio of hats to pants?
e. What is the ratio of shirts to hats?
f. What is the ratio of hats to shirts?
2. Mr. Morton decides to give candy to only 2 out of every 5 students in his class.
a. What is the ratio of students who receive candy to those who don't?
b. If the class has 30 students, how many will receive candy?
3. Generally, fractions and ratios should be simplified.

One possible way is to suppose there are 100 marbles with 20 red, 30 blue and 50 green.
a. $\frac{30}{1500}=\frac{3}{10}$ or $3: 10$.
b. $\frac{50}{300}=\frac{10}{2}$ or $1: 2$.
c. $\frac{3 \theta 0}{20}=\frac{3^{2}}{2}$ or $3: 2$ d. $\frac{20}{50}=\frac{2}{5}$ or $2: 5 \quad$ e. $\frac{50}{30}=\frac{5}{3}$ or $5: 3$
f. \# of red marbles $=.2 \mathrm{~T} ; \#$ of green marbles $=.5 \mathrm{~T} ; \#$ of blue marbles $=.3 \mathrm{~T}$
5. $14 \mathrm{mi} / \mathrm{hr}$. Use $\mathrm{d}=\mathrm{r} \bullet \mathrm{t} \longrightarrow \mathrm{d}=(14 \mathrm{mi} / \mathrm{m})\left(2 \frac{1}{4}\right)=(14)\left(\frac{9}{4}\right)$ miles $=31.5$ miles
6. $\quad \$ 4.05=\$ 0.27$ per oz. $\quad \frac{\$ 1.89}{8}=\$ 0.24$ per oz. $\quad \frac{\$ 5.60}{20}=\$ 0.28$ per oz.
3. The table below hangs on the wall of a bakery. The baker claims that each offering yields the same price per donut. Is she correct? Explain how you decided.

| Donuts | Price |
| :---: | :---: |
| 5 | $\$ 2.00$ |
| 8 | $\$ 3.20$ |
| 12 | $\$ 5.00$ |
| 20 | $\$ 8.00$ |

4. A bag contains red, blue and green marbles. Twenty percent of the marbles are red, thirty percent are blue and fifty percent are green. Draw a picture using area to represent the portion of each color of marble. Does the picture help to determine the ratios?
a. What is the ratio of blue marbles to the total number of marbles?
b. What is the ratio of green marbles to the total number of marbles?
c. What is the ratio of blue marbles to red marbles? $3: 2$
d. What is the ratio of red marbles to green marbles? 2:5
e. What is the ratio of green marbles to blue marbles? 5:3
f. Let $T$ be the total number of marbles. Write an expression using $T$ that describes the number of red marbles. Then, write an expression using $T$ that describes the number of green marbles, and then of blue marbles.
5. Steve rides his bike 42 miles in 3 hours. How far will he ride at a constant rate for 1 hour? How far does he travel in $2 \frac{1}{4}$ hours? $\quad 14$ miles; 31.5 miles
6. Ron went to the grocery store to purchase cereal. He found 15 ounces for $\$ 4.05,8$ ounces for $\$ 1.89$ and 20 ounces for $\$ 5.60$. What is the best price for Ron's cereal? What is the unit cost for each container of cereal?
7. Teachers, you should encourage your students to use a calculator on this exercise. This exercise requires a lot of computation which could be tedious to students. The purpose of this problem is to have students compare ratios, not to give up because of a lack of a computational tool.
Assume that none of the freshmen at Texas Technical College are double-majoring.
a. True. Theratio ofScience and LiberalArts majorstototal students is $\frac{938}{1200}=\frac{469}{600}=0.77816 \overline{6}>0.75=\frac{3}{4}$.
b. False. The ratio of international students to total students is $\frac{102}{1200}=\frac{17}{200}=0.85<0.0909 \overline{09}=\frac{1}{11}$.
c. True. The ratio of music to non-music majors is $\frac{150}{1050}=\frac{1}{7}$. Students may add the other two majors and compare. This idea is okay but note the total does not add up to 1200 .
d. False. The ratio of science to non-science majors is $\frac{525}{675}=\frac{7}{9}$ which is not $5: 6$. Students may add the other two majors and compare. This idea is okay but note the total does not add up to 1200 .
e. False. The ratio of males to females is $\frac{570}{630}=\frac{19}{21}$ or 19:21 which is not 19:20.
f. False. The ratio of Texans to non-Texas is $\frac{782}{413} \approx 1.87081>1.87$ which is a slightly higher ratio than 1.87:1.
g. You can use the ratio $\frac{4}{5}$ to compute the number of students who normally graduate in 4 years:
$\frac{4}{5} \cdot 1200=960$ students
h. $\frac{3}{5} \cdot 1200=720$ students
8. 

a. 20
b. 1300
c. 250,000
d. 485
9. C
10. Recall that there are 5280 feet to 1 mile.
a. $60 \frac{\text { miles }}{\text { hour }} \cdot 5280 \frac{\text { feet }}{\text { mile }}=316800 \frac{\text { feet }}{\text { hour }}$
b. $60 \frac{\text { miles }}{\text { hour }} \cdot 5280 \frac{\mathrm{feet}}{\text { mile }}=316800 \frac{\mathrm{feet}}{\text { hour }} \cdot \frac{1 \text { hour }}{3600 \text { seconds }}=88 \frac{\mathrm{feet}}{\text { sec ond }}$
7. The entering freshman class of Texas Technical College has 1,200 students. Of those, 782 are from Texas and 102 are international students. There are 525 science majors, 413 liberal arts majors, and 150 music majors. There are 630 females. Determine whether the statements in parts a-f are true or false, and explain your reasoning.
a. More than 3 out of 4 students are science or liberal arts majors.
b. More than 1 out of every 11 students is an international student.
c. The ratio of music majors to non-music majors is 1:7.
d. The ratio of science majors to non-science majors is $5: 6$.
e. The ratio of male to female freshman is 19:20.
f. The ratio of Texan to non-Texan is less than 1.87 to 1 .
g. Four out of five students usually graduate in 4 years. How many will that be from this class?
h. Three out of five students who enter as freshman usually go to graduate school. At that rate, how many of the entering class of 1,200 students will go to graduate school?
8. Use dimensional analysis to convert the following:
a. $2 \frac{1}{2}$ gallons $=$ $\qquad$ pints
b. $\quad 1.3 \mathrm{~km}=$ $\qquad$ meters
c. $\quad 2.5 \mathrm{~km}=$ $\qquad$ cm
d. $48.5 \mathrm{~cm}=$ $\qquad$ m
9. Which of the following can be used to find $y$, the number of yards in 7 miles?

Explain your answer.
a. $5280 \div 3$
b. 7(5280)
c. 1760(7)
d. 7(3)(5280)
10. Manuel is driving his car at 60 mph .
a. How many feet does he travel in one hour?
b. How many feet does he travel in one second?
11. Using 500 at bats, $\frac{326 \text { hits }}{1000 \text { atbats }} \cdot 500$ at bats $=\frac{326}{2}=163$ hits.
12. $B$
13. Because the ratio of Bobbie's goals to the rest of the team is $3: 7$, he makes $3 / 10$ of the goals. Because the ratio of Marie's goals to the rest of the team is $4: 5$, she scores $4 / 9$ of the goals. Together, they score $3 / 10+4 / 9$ or $67 / 90$ of the total number of goals. This results in 67 goals for Bobbie and Marie and 23 goals for the rest of the team, or a 67:23 ratio.
14. See answer on Page 972
15. a. We write all information given in the problem:
$1.5 \mathrm{mph}=$ rate Shawn swims $\quad 8 \mathrm{mph}=$ rate Shawn runs
$2 \mathrm{mph}=$ rate Jeff swims $\quad 10 \mathrm{mph}=$ rate Jeff runs
We let $\mathrm{t} 1=$ time Shawn swims and $\mathrm{t} 2=$ time Shawn runs.
We let $\mathrm{t} 3=$ time Jeff swims and $\mathrm{t} 4=$ time Jeff runs.
With Shawn's 10 minute $=\frac{1}{6}$ hours head start and using the distance formula $d=r t$, we get the following: $\frac{1}{2}$ mile $=(\mathrm{t} 1) \mathrm{hr} \cdot \frac{9}{2} \frac{\mathrm{mi}}{\mathrm{hr}}$ which implies $\frac{1}{3}=\mathrm{t} 1$.
During Jeff's swim, $\frac{1}{2}$ mile $=(\mathrm{t} 3) \mathrm{hr} \cdot 2 \frac{\mathrm{mi}}{\mathrm{hr}}=2(\mathrm{t} 3) \mathrm{mi}$ and $\frac{1}{4}=\mathrm{t} 3$.
Since Shawn has a $\frac{1}{6}$ hour head start, he finishes the swim $\frac{1}{3}-\frac{1}{6}=\frac{1}{6}$ hours after Jeff starts. Since Jeff swims for $\frac{1}{4}$ hours, Jeff swims $\frac{1}{4}-\frac{1}{6}=\frac{1}{12}$ hour $=5$ minutes longer. During this 5 minutes, Shawn has been running. So when Jeff finished swimming, Shawn was $8 \frac{\mathrm{mi}}{\mathrm{hr}} \cdot \frac{1}{12} \mathrm{hr}=\frac{2}{3}$ miles ahead.
b. Shawn ran $5 \mathrm{mi}=(\mathrm{t} 2) \mathrm{hr} \cdot 8 \frac{\mathrm{mi}}{\mathrm{hr}}$, so that $\frac{5}{8} \mathrm{hr}=\mathrm{t} 2$. Jeff ran $5 \mathrm{mi}=(\mathrm{t} 4) \mathrm{hr} \cdot 10 \frac{\mathrm{mi}}{\mathrm{hr}}$ or $\frac{1}{2}=\mathrm{t} 4$.

The total time Jeff raced was $t 3+t 4=\frac{1}{4}+\frac{1}{2}=\frac{3}{4}$. The total time Shawn raced minus his head start was $\frac{1}{\beta 9}-\frac{1}{98}+\frac{5}{8}=\frac{8-4+15}{24}=\frac{19}{24}$. Subtracting Shawn's time from Jeff's time, you get
19 $\frac{19}{24}-\frac{3}{4}=\frac{\beta 9-98}{24}=\frac{1}{24}$ hours that Shawn still has yet to run after Jeff has completed the race. The distance that Shawn had left to run when Jeff finished is $8 \frac{\mathrm{mi}}{\mathrm{hr}} \cdot \frac{1}{24}=\frac{1}{3}$ mile.
11. In baseball, a batting average is computed by dividing the number of hits by the number of times at bat, and then multiply by 1,000 . Tony has a 326 batting average. How many hits would he have in a typical baseball season of 500 times at bat?
12. Jasmine is building birdhouses. It takes her 4 hours to build 7 houses. Which of the following is an equivalent rate? Explain your answer.
a. 10 hours to build 16 houses
b. 28 hours to build 49 houses
c. 18 hours to build 35 houses
d. 5 hours to build 8 houses
13. Bobbie and Marie play soccer for the same team. The ratio of Bobbie's goals scored to the rest of the team, including Marie, is $3: 7$. The ratio of Marie's goals scored to the rest of the team, including Bobbie, is $4: 5$. What is the ratio of goals scored by Bobbie and Marie to the rest of the team?

## 14. Ingenuity: Geometric Series

It is a curious fact that in arithmetic sometimes sums of fractions go on forever and are still relatively small. Find the value of the following infinite sum. It may help to visualize the sum geometrically.

$$
\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots
$$

## 15. Investigation:

Jeff and Shawn are going for a half-mile swim, then a 5 -mile run. Shawn swims at a rate of 1.5 mph and runs at a rate of 8 mph . Jeff swims at rate of 2 mph and runs at a rate of 10 mph . Jeff gives Shawn a 10 -minute head start.
a. Who finished the swim first? How much longer did it take the second swimmer to finish? How far ahead was the first person?
b. Who finished the run first? How long did the winner wait for the other to finish? How far ahead did the winner finish?
16. The ratio of math books to physics books is $7: 3$. Why?

Let $x=$ the number of math textbooks. So $x=$ the number of physics books also. Shrenik chose $\frac{x}{3}$ books and Monica $\frac{X}{7}$ books. The ratio of the number of books Shrenik chose to number of books Monica chose is
$\frac{\frac{x}{3}}{\frac{x}{7}}=\frac{x}{3} \cdot \frac{7}{x}=\frac{7}{3}$ or $7: 3$.

## Ingenuity: Geometric Series

14. This is a challenging problem. You must use a little imagination here. We start by letting the sum be equal to $x=$ $\left(\frac{1}{3}+\frac{1}{3^{2}}+\frac{1}{3^{3}}+\frac{1}{3^{4}}+\frac{1}{3^{5}} \ldots\right)$. Then multiplying the sum by 3 , we get $3 x$ to be $\left(\frac{1}{3}+\frac{1}{3^{2}}+\frac{1}{3^{3}}+\frac{1}{3^{4}}+\frac{1}{3^{5} \ldots}\right)$ $\cdot 3$, which is equal to $\left(\frac{3}{3}+\frac{3}{3^{2}}+\frac{3}{3^{3}}+\frac{3}{3^{4}}+\frac{3}{3^{5}} \ldots\right)$. This simplifies to $3 x=\left(1+\frac{1}{\beta^{3}}+\frac{1}{\frac{\beta}{}^{2}}+\frac{1}{\frac{\beta}{}^{3}}+\frac{1}{\beta^{4}}+\frac{1}{3^{5}} \ldots\right)$ . Notice that the sum starting with the second term is $x$. Thus, $3 x=1+{ }_{( }\left(\frac{p}{3}+\frac{p}{3^{2}}+\frac{p}{3^{3}}+\frac{p}{3^{4}}+\frac{p}{3^{5}} \ldots\right)=1+$ $x$. Therefore we get $3 x=x+1$. Solving for $x$, you get $2 x=1$ and $x=\frac{1}{2}$.
See visual model on the next page.
Visual Model: In each step, one-third of a third from the previous step is shaded. In Step 1, one-third of the whole is shaded. The x marks a third that will not be touched, and the center third is the basis for Step 2. In Step 2, the center third is divided into thirds; one-third is shaded, one other third is marked with an $x$ that it will not be touched, and the center third will be the basis for Step 3. As this process continues, the shaded area of the whole approaches $\frac{1}{2}$.

15. Fineas and Monica go to the library. Fineas goes to the math books section and chooses every third book. Monica goes to the physics section and checks out every seventh book. If the library has the same number of math and physics textbooks, what is the estimated ratio of the number of books Fineas checks out to the number of books Monica checks out?

# Ratios in the World 



Objective: The students will research ratios in the real world.

## Materials:

Magazines, newspaper or internet sites for news stations
Scissors, if using newspapers or magazines
Printer, if using computers

## Activity Instructions:

1) The students will use the materials above to search for at least 10 examples the use of ratios in the real world.
2) After collecting this information, group the students and have them share what they found.
3) After a sufficient amount of time spent sharing, ask if there are any volunteers who would like to share a specific example of a ratio in the real world that they found particularly interesting. This could possibly start a great discussion about why ratios are important and when the students might be using this math skill when they grow up. The goal is to spark interest so students become excited about the information they will learn in Chapter 10.

## Section 10.2 - Rates of Change and Linear Functions

## Big Idea:

Understanding how rates of change connect to linear functions

## Key Objectives:

- Use ratios to describe behaviors of linear graphs.
- Use graphs to relate ratios and rates of change.
- Relate real-life situations to linear functions.
- Foreshadow slope by informally introducing the concept as a "rate of change."


## Materials:

Graph paper, Stopwatch or clock with a second hand, Measuring tape, Calculator, an Area like a hall, cafeteria or the outdoors

## Pedagogical/Orchestration:

- This section relates rates of change to linear functions, both algebraically and graphically. It then uses the function to answer questions from diverse real-life situations.
- Since function notation has not been used since Chapter 4, a little refresher on the use of function notation would be helpful before Exploration 1 Step 5 . Something like this can be said: Remember a function is a rule which assigns to each input value a unique output value. For example, the function $\mathrm{F}(x)=3 x$ means each input $x$ is assigned a "partner" output value which is equal to $3 x$. We read this as F of $x$ equals $3 x$.
- In the Example under Exploration 4, ask students to find Jane's unit rate and explain what it means.


## Activities:

"Fundraiser Activity" at the end of the section and on CD (a long-term project that will require a few days.)
"My House" at the end of the section and on CD
"Rectangle Activity" at the end of the section and on CD

## Exercises:

Warning on Exercise 8 that teacher may need to clarify to students that the bread is being divided equally, yet before the 3rd student showed up the other two students were not going to be sharing. The "obvious" answer is not correct.

## Vocabulary:

functions, unit rate, equivalent ratios

## TEKS:

$6.2(C) ; \quad 6.5 ; \quad 6.11(A)(C) ; \quad 6.12(A) ; \quad 7.2(D) ; \quad 7.3(B) ; \quad 7.5(A)(B) ; \quad 7.13(A)(C)(D) ; \quad 7.14(A) ; \quad 8.2(D) ;$
$8.3(\mathrm{~A}) ; 8.5(\mathrm{~A}) ; 8.14(\mathrm{~A}) ; 8.15(\mathrm{~A})$

## WARM-UPS for Section 10.2

1. The triangle $A B C$ was translated to triangle $A^{\prime} B^{\prime} C^{\prime}$. Which of the following best describes the translation
a. 3 unites left and 4 units down
b. 4 units left and 5 units up
c. 4 units right and 5 units up
d. 3 units right and 4 units down

## Ans: b


2. Marie has 60 math problems for homework. She finishes $\frac{1}{3}$ before supper. After her shower, Marie works $\frac{1}{2}$ of the problems she has left. The next morning she manages to finish $\frac{4}{5}$ of the remaining problems before class. How many remain unfinished? Therefore $20+20+16=56 \longrightarrow 60-56=\underline{4}$ )

## Launch for Section 10.2:

Working in groups of three, mark off a distance of 10 feet.
Step 1: Have one team member walk slowly for 10 feet. Have another team member record the time.
Step 2: Record the time for another walker to walk at a regular pace for 10 feet.
Step 3: Record the time for a third walker to walk quickly for 10 feet.
Step 4: Use the times to estimate the time for each walker to walk 20 feet, 30 feet and 40 feet, assuming the walker neither speeds up nor slows down when walking farther.
Step 5: Make three separate graphs, one for each walker.
What quantities are changing in each graph? What remains the same? Determine the rate of change using the quantities that you identified. Locate the rate of change in your graph. Determine the unit rate of change on your graph, if possible. Use your calculator, if it helps.

Use the rates of change to make three equations of the three walking rates.
The two quantities are time and distance. We assume the walker does not change his/her rate of walking, so the rate remains the same. The ratio should correspond to the sides of the "triangles" formed as students travel from the origin to each point. The unit rate might be harder for the student to see because the triangle is small, created by 1 on the $x$-axis and 10 feet divided by the student's walking time on the $y$-axis.

For problems done in class, use graphing calculators. Any problems done for homework need to be done on graph paper.

Overall ratio of black to white marbles is 6:8 or 3:4

They are equivalent.

EXPLORATION 1
Step 1: Expect 2 or 1. If your students say 1, then the corresponding amount of rice (second column entry) will be $\frac{1}{2}$

## SECTION 10.2 RATES OF CHANGE AND LINEAR FUNCTIONS

Let us continue to look at comparing quantities while also looking at changes that occur in those quantities. For example, if a bag contains 3 black marbles and 4 white ones, the ratio of black to white marbles is $3: 4$. Pictorially, a $3: 4$ ratio could look like this:


Notice that for every three black marbles, there are four white ones. Another bag has double the number of black and white marbles. Draw a picture like the one above to represent the ratio of marbles in this bag. Are there still four white marbles for every three black ones? What is the overall ratio of black to white marbles? What is the relationship between the two ratios? Such ratios are equivalent because they form equivalent fractions.

## EXPLORATION 1

A restaurant makes and sells a famous dish that contains rice and beans. The ratio of rice to beans in its secret recipe is $1: 2$, and the ratio of beans to rice is $2: 1$.

| Cups of beans | Cups of rice | $\frac{\text { rice }}{\text { beans }}$ |
| :---: | :---: | :---: |
| 1 |  |  |
| 2 | 1 |  |
| 3 | 2 |  |
| 4 |  |  |
| 5 | $y=$ |  |
| 6 |  |  |
| $x$ |  |  |

Step 1: Complete the table of possible amounts of rice and beans that the chef uses to make the dish.

How did you organize the first column of numbers? What is the first number in the beans column?

Step 3: The students should notice that ratio of rice to beans, 1:2, is the ratio of the second column numbers to the first column numbers. Recall the idea of unit rate from Section 10.1 and have students articulate that the rate of change or increase is the ratio of $\frac{1}{2}$ cup of rice per 1 cup of beans. If you increase the amount of beans by 1 cup, the amount of rice will increase by $\frac{1}{2}$ cup. This is called the unit rate of increase.

Step 4: Discuss whether values could be negative and what quadrants will be used during graphing. This would also be a good place to discuss what would happen if you had 1 cup of beans, or $\frac{1}{2}$ cup of beans so that the students could see that you don't have to have only 2 cups of beans and 1 cup of rice to have a 2 to 1 ratio of beans to rice.

Step 5: Ask your students if they see the unit rate of change in this formula. $R=\frac{1}{2} B$ would be an excellent rule for a student answer to the question posed.

Step 6: $1 \frac{3}{4}$

The walking unit rate is two feet per second.

If necessary, let students know we say "miles per minute" instead of "miles per 1 minute." The " 1 " is assumed. The biking unit rate is 20 miles per hour.

Step 2: Use the numbers in the table as coordinates of points. Make a graph using the data points. For each point, the number of cups of beans is the $x$-coordinate and the number of cups of rice is the $y$-coordinate. Describe the graph.
Step 3: For each of the rows in the table, what is the ratio of rice to beans? How is the second column of amounts of rice changing?
Step 4: Could you start with a smaller amount of beans? Pick two possible smaller amounts of beans and find the corresponding amounts of rice.

Step 5: The graph is a set of points on a straight line. Define $R(x)$ as the number of cups of rice needed for $x$ cups of beans in the recipe. Write a rule for this linear function $R$. What is the ratio of rice to beans?
Step 6: Use the rule for $R$ to compute the amount of rice needed for $3 \frac{1}{2}$ cups of beans. Use the graph to confirm that your answer makes sense.

This exploration is an example where two quantities change but the relationships or rates of change between them remain the same. The quantities have a constant rate of change and result in graphs that are lines. There are other non-linear ways in which two quantities can vary. You will study these interesting relationships in algebra.

Another setting for rates of change is in movement, like walking, biking or riding in a car. In each of these instances some distance is traveled over a time interval. For example, you might walk 20 feet in 10 seconds, bike 10 miles in 30 minutes, or travel 60 miles in a car in 1 hour. Notice in each example that there are two quantities: one is a length unit and the other a time unit. Terms like feet per second, miles per minute or miles per hour all refer to rates, which were discussed in a previous section. Walking 20 feet in 10 seconds can be restated as a certain number of feet walked in 1 second, which is a unit rate as we discussed in 10.1. What is the walking unit rate? the biking unit rate?

Information about distance and time is useful in finding rates of change.

## EXAMPLE 1

Priscilla takes a long walk on the weekend, walking at a steady rate of 4 miles per hour. Using the formula $d=r t$, if $r=4$ notice that the distance $d$ depends on the amount of time $t$ Priscilla walks. This process is like the frog-jumping model on the number line from Section 4.1.


From the picture, make the following table:

| Time t | Distance d | Rate $\frac{\mathrm{d}}{\mathrm{t}}$ |
| :---: | :---: | :---: |
| 0 | 0 |  |
| 1 | 4 | $\frac{4}{1}=4$ |
| 2 | 8 | $\frac{8}{2}=4$ |
| 3 | 12 | $\frac{12}{3}=4$ |
| 4 | 16 | $\frac{16}{4}=4$ |
| 5 | 20 | $\frac{20}{5}=4$ |
| 6 | 24 | $\frac{24}{6}=4$ |

Now represent the situation as a graph


Priscilla walks four miles each hour. Looking at the table, it is easy to see that after each hour, Juan's distance has changed by 4 miles. As in Section 5.5 , the constant change in the second coordinates $d$ is called a constant rate of change for the distance function $d$. The third column $r$ gives us the same constant. From Section 4.5, this number is the constant of proportionality. For linear functions in the form $y=m x$, the two constant properties are the same and are both equal to $m$. However, a general linear function $y=m x+b$ when $b$ is not 0 has a constant rate of change but does not have a constant of proportionality. Explain why.

## PROBLEM 1

In the graph below, $d$ represents the distance traveled by a bicycle rider and $t$ represents the number of hours ridden. What is the rate in miles per hour? Use this rate to write a formula for $d$ in terms of $t$.


## EXAMPLE

Students may use a table for this function to compute C(7). First, they should fill in the information they know, 4 drinks cost $\$ 9$, and use that information to find other values on the table. What do you notice about the third column?

| $x$ | $C(x)$ | Ratio <br> $C(x) / x$ |
| :--- | :--- | :--- |
| 1 | 2.25 | 2.25 |
| 2 | 4.50 |  |
| 3 | 6.75 |  |
| 4 | 9.00 |  |
| 5 | 11.25 |  |
| 6 | 13.50 |  |
| 7 | 15.75 |  |

## EXAMPLE 2

Jane bought 4 drinks for \$9.
a) What is the cost of 1 drink? What is the rate of cost for drink?
b) Fill in the following table.

| $x$ | $y=C(x)$ | $y \div x$ |
| :---: | :---: | :---: |
| 0 | 0 |  |
| 1 |  |  |
| 2 |  |  |
| 3 | 9 |  |
| 4 |  |  |
| 7 |  |  |

c) Does the cost function have a constant rate of change? If so, what is it? Where is it in the equation?

What do you notice about the third column?

## SOLUTION

We form the rate of $\frac{\$ 9}{4 \text { drinks }}$, so the unit rate is $\$ 2.25$ per drink. Use the unit rate of $\$ 2.25$ per drink to find a formula for the function $C$ where the $\operatorname{cost} C(x)$ is the cost for $x$ drinks. Use the function $C(x)=\frac{9}{4} x=2.25 x$ to find that Jane spent $\$ 2.25(7)=\$ 15.75$.

## PROBLEM 2

Make up table (with at least 5 points) and write the formula for a linear function that has a constant rate of change of 2.5.

## PROBLEM 3

Find the constant rate of change for the linear functions represented by their graphs on the coordinate system below:


## PROBLEM 4

In the graph below, $y$ represents the number of teaspoons of honey used to make $x$ number of cupcakes. What is the ratio of teaspoons of honey to number of cupcakes? Use this ratio to write a formula for $y$ in terms of $x$.


## EXERCISES

## EXERCISES

1. Write an equation and make a table with five points for a linear function with both a constant rate of change and proportionality of 6 .
2. Write an equation and make a table with five points for a linear function with both a constant rate of change and proportionality of -3 .
3. For each of the following linear functions, determine both the constant rate of change and the constant of proportionality, if it exists. If there is no constant of proportionality, explain why it doesn't exist.
a. The function given by $y=7 x$
b. The function given by $y=-5 x$
c. The function given by $y=3 x+2$
d. The function given by $y=\frac{x}{4}=\frac{1}{4} \cdot x$
e. The function given by $y=\frac{3 x}{2}$
4. The table below is from a linear function. What is the constant rate of change for this function? What is the constant of proportionality? What is the ratio of $y$ to $x$ ? What is the linear function?

| $x$ | $y$ | $\frac{y}{x}$ |
| :---: | :---: | :---: |
| -2 | 8 |  |
| -1 | 4 |  |
| 0 | 0 |  |
| 1 | -4 |  |
| 2 | -8 |  |

5. The table below is a record of the number miles a school bus has traveled during a band trip. What is the constant rate of change for the function? What is the formula for the distance d in miles as a function of time $t$ in hours? What is the ratio of distance to time?

| Hours | Distance in miles | $\frac{d}{t}$ |
| :---: | :---: | :---: |
| 0 | 0 |  |
| 1 | 64 |  |
| 2 | 128 |  |
| 3 | 192 |  |

1. a. $\$ 35.75$
b. $\frac{\$ 28}{8} \cdot 5=\$ 17.50$

| Hours | Distance in miles | $\frac{d}{t}$ |
| :---: | :---: | :---: |
| 4 | $?$ |  |

6. The picture below represents the distance a jackrabbit travels. Each hop covers the same distance. If the scale is in feet, what is the constant rate of feet-per-hop? Write a formula for the distance as a function of the number of hops.

7. For each of the linear functions given by graphs below, determine the constant rate of change and the constant of proportionality.

8. A coffee shop sells hot chocolate for $\$ 2.75$ per cup and 8 cups of regular coffee for \$28.
a. How much does it cost to buy 13 cups of hot chocolate?
b. How much do 5 cups of regular coffee cost?
c. How much does each cup of coffee cost?
d. Make a table for each of the cost functions.
e. Calculate formulas for the costs of hot chocolate and for coffee.
9. $\$ 48$; Discuss how linear function can be read: unit rate $=\frac{\$ 18}{3 y a r d s}=6 \$ / y$ ard: $C(x)=6 x$. Therefore, $C(8)=6(8)=48$
10. a. $\$ 1, \$ 1.45$
b. $C(x)=\$ 0.70+\$ 0.05 x$
11. C
f. Graph each of the cost functions on the same coordinate system.
g. Determine the constant of proportionality for each of these linear functions.
12. In the graph below, $y$ represents the cups of sugar and $x$ represents the number of kilograms of cake batter made at a large baker. What is the ratio of sugar to cake batter? Use this ratio to write a formula for $y$ in terms of $x$.

13. Sally buys 3 yards of fabric for $\$ 18$. How much will 8 yards of fabric cost?
14. Molly's Binding binds writing assignments for students. The company charges $\$ 0.70$ for binding one writing assignment and, in addition, $\$ 0.05$ per page.
a. How much does it cost to bind an assignment with 6 pages? 15 pages?
b. Make a table for this cost function with at least 6 inputs/outputs.
c. Write a rule for the cost function $C$ where $C(x)$ is the cost of binding a writing assignment with $x$ pages.
d. Draw a graph of this cost function.
e. What is the constant rate of change? Does the cost function have a constant of proportionality? Explain.
15. Zack's bicycle wheel can travel about 8.5 feet per revolution. Which statement is best supported by this information? Explain your answer.
a. The wheel can travel about 415 feet in 50 revolutions.
b. The wheel can travel about 72 feet in 8 revolutions.
16. The best price is $\$ 4.05$ for for 15 ounces. The unit costs are $\$ 0.27$ and $\$ 0.28$ respectively.

| oz. of cereal | price (\$4.05/150z) | price $(\$ 5.60 / 200 z)$ |
| :--- | :--- | :--- |
| 1 | $\$ 0.27$ | $\$ 0.28$ |
| 2 | 0.54 | 0.56 |
| 3 | 0.81 | 0.84 |
| 4 | 1.08 | 1.12 |
| 5 | 1.35 | 1.40 |
| 10 | 2.70 | 2.80 |
| 15 | 4.05 | 4.20 |
| 20 | 5.40 | 5.60 |

6. C
7. 

a. $\quad \frac{39.00 \text { dollars }}{6 \text { shirts }}=\$ 6.50 /$ shirt
b. $\quad(\$ 6.50 /$ shirt) $(4$ shirts $)=\$ 26.00$
c. $\quad C(x)=6.50 x$
8. Unit rate $=\frac{11.20 \text { dollars }}{8 \text { drinks }}=\$ 1.40 /$ drink so $(\$ 1.40 /$ drink $)(20$ drinks $)=\$ 28.00$.
9. $P(5)=2 \mathrm{~kg}, P(8)=3.2 \mathrm{~kg}, P(x)=\frac{2}{5} \mathrm{x}=0.4 \mathrm{xkg}$
c. The wheel can travel about 297.5 feet in 35 revolutions.
d. The wheel can travel about 300 feet in 36 revolutions.
13. Ron went to the grocery store to purchase cereal. He found 15 ounces of brand $A$ for $\$ 4.05$ and 20 ounces of brand $B$ for $\$ 5.60$. Which brand of cereal has the best price for? What is the unit cost of each container of cereal? Make a table and determine the unit rate.
14. In a factory, one machine makes flashlights at a rate of 140 flashlights per hour, and another machine makes the same flashlights at a rate of 175 flashlights per hour. Which of the following equations can be used to find $t$, the total number of flashlights both machines will make in 6 hours? Explain your answer.
a. $(140+175) \div 6$
b. (175)(6)-(140)(6)
c. $(140+175)(8)$
d. $140(8)+175$
15. Sandra bought 6 T-Shirts for $\$ 39.00$.
a. How much does one T-Shirt cost?
b. How much would 4 T-Shirts cost?
c. What is the formula for $C(x)$, the cost of $x$ T-Shirts?
d. What is the constant of proportionality for this cost function?
16. Mike bought 8 fruit drinks for $\$ 11.20$. How much would 20 drinks cost?
17. Scientists developed a new variety of flower. They found that $40 \%$ or $\frac{2}{5}$ of the seeds produced plants with purple flowers and the rest produced plants with white flowers. Let $P(x)$ be the number of kilograms of seeds that will produce purple flowers from a bag with $x$ kilograms of the new variety's seeds. How many kilograms of purple seeds will there be in a 5 kg bag, 8 kg bag, and 1 kg bag of seed?
a. Write a rule for $P(x)$, where $P$ is the purple flower seeds.
b. Fill in the table below.
10. A
$\begin{array}{lll}\text { 11. a. } & C(x)=\frac{16}{8} x=2 x ; & C(7)=2 \cdot 7=14 \\ \text { b. } & P(x)=\frac{2}{8} x=\frac{1}{4} x ; & P(12)=\frac{1}{4} \cdot 12=3\end{array}$

| $x$ | $y=P(x)$ | $y \div x$ |
| :---: | :---: | :---: |
| 0 |  |  |
| 1 |  |  |
| 2 |  |  |
| 5 |  |  |
| 8 |  |  |

c. Does P have a constant of proportionality?
18. A truck driver travels at an average speed of 62 miles per hour. Which expression can be used to find $d$, the distance the truck driver will travel in 217 hours? Explain your answer.
a. 62(217)
b. $62 \div 217$
c. $217 \div 62$
d. $217+62$
19. Tommy's Taters brand of potato chips has 8 grams of fat, 16 grams of carbohydrates, and 2 grams of protein per serving.
a. Write a function $C(x)$ for the number of grams of carbohydrates if $x$ grams of fat are eaten. What is $C(7)$ ?
b. Write a function $P(x)$ for the number of grams of protein if $x$ grams of fat are eaten. What is $P(12)$ ?
12. a. 120
b. 280
c. $\frac{3}{7}$

Make students check for reasonableness of their answer here. A common mistake will result in an answer of $\frac{7}{3}$.
13. Let $L=$ amount of low-grade cement, $M=$ amount of medium-grade cement, and $H=$ amount of high grade cement. Then $L_{1}=3 H, M=2 H, L+M+H=x, 3 H+2 H+H=x, 6 H=x$. So $H=\frac{1}{6} x, M=\frac{2}{6} x=\frac{1}{3} x$, and $L=\frac{3}{6} x=\frac{1}{2} x$.
Ingenuity
14. After the third student arrives, each student gets $\frac{8}{3}=2 \frac{2}{3}$ loaves of bread. The 5 -loaf student keeps $2 \frac{2}{3}$ loaves and gives $5-2 \frac{2}{3}=2 \frac{1}{3}$ loaves, and the 3-loaf student keeps $2 \frac{2}{3}$ and gives $3-2 \frac{2}{3}=\frac{1}{3}$ loaves. So the 5 -loaf student gives 7 times as much as the 3 -loaf student. Thus the 5 -loaf student gets 7 cents and the 3 -loaf student gets 1 cent.


## Investigation

15. a. 17 DVDs per 1 dollar change.
b. $\quad D(x)=1360-17 x$

The function $D(x)$ is called a demand function in economics. This function gives the relationship between the price of a good $(x)$ and the quantity of the good consumers demand (i.e. are willing and able to pay) at that price. The negative slope of the demand function reflects the fact that all things being equal, consumers will demand less of a good when its price increases.
20. The Slugger Company has enough employees to devote 1680 work-hours a day making baseball bats. It takes 6 work-hours to make a wooden baseball bat, and it takes 14 work-hours to make an aluminum bat.
a. How many aluminum bats can the company make a day if it only makes aluminum bats?
b. How many wooden bats can it make if it makes no aluminum bats?
c. How many aluminum bats can it make per wooden bat?
21. A cement company makes three grades of cement. The company makes three times as much low-grade as high-grade cement, and it makes twice as much medium-grade as high. The company produces on average $x$ tons of cement per day. How much of each grade of cement does it make each day? Express each amount as a fraction of $x$.

## 22. Ingenuity:

Here is a problem students in Baghdad solved over 2000 years ago.
Two students sat down to eat. One had five loaves of bread and the other, three. Just as they were about to begin, a third student came in and asked to eat with them, promising to pay eight cents for her part of the meal. If they ate the same amount and consumed all the bread, how should the two original students divide the eight cents?

## 23. Investigation:

People in San Marcos want 1360 DVDs if they are given away for free. However, if the cost of DVDs increases, the number of people who want them decreases at a constant rate as the cost increases. If the cost is increased to $\$ 80$, no one in San Marcos wants any DVDs.
a. For every dollar the price of DVDs increases, by how much does the amount of DVDs wanted change?
b. Write a function $D(x)$ expressing the number of DVDs people in San Marcos want if a DVD costs $x$ dollars.

## EXPLORATION 2

The unit rate for salt per oregano is. The graph is $y=\frac{2}{3} \times \frac{\frac{2}{3} \text { tsp. of salt }}{1 \text { tsp.oforegano }}$.

For every 2 cups of beans, use 1 cup of rice, or for each cup of beans, use $\frac{1}{2}$ cup of rice. The unit rate for rice is $\frac{1}{2}$ cup of rice per cup of beans. The change is multiplicative and not additive. Two cups of beans doubles the amount of rice. Adding a cup of beans does not add a cup of rice. The unit rate for salt per tablespoon of oregano
is. $\frac{2}{3}$ tsp.ofsalt
1 tsp. oforegano

## EXPLORATION 3

Students might notice that the 2:1 ratio graphs a "flatter" line than the 1:2 ratio, which graphs a "steeper" line. The graphs cross the vertical axis at the origin and the changes in $x$ and $y$ are multiplicative of each ratio.

Students should realize:
(1) that these lines all pass through the origin, or have a y-intercept of 0 ,
(2) that the ratio is in the equation, either $y=\frac{1}{2} x$ or $y=\frac{2}{3} x$,
(3) the ratios $\frac{1}{2}$ and $\frac{2}{3}$ are the rates of change for each of the lines. Later the rate of change of a line will be called the slope.

## 24. Investigation:

In the same dish, the recipe calls for 2 teaspoons of oregano for each 3 teaspoons of salt. Repeat the first four steps from Exploration 1 for these two ingredients. Include a graph where the $x$-coordinate is the amount of salt and the $y$-coordinate is the amount of oregano.
Compare your graph and table to those from Exploration 1. What changes and how? What remains the same? Describe in words what the rate of change for each graph means.
25. Investigation:

Make a table and draw the graph using points with a $2: 1$ ratio between the first and second coordinates. Do the same for a graph with the ratio $1: 2$. What do you notice about these graphs? Write an equation for each of these functions. What do the graphs tell you about the relationship between the ratios $2: 1$ and $1: 2$ ?

# FUNDRAISER 



Objective: The students will simulate a mock fundraiser and analyze the experience using their knowledge of linear functions.

## Materials:

Graph Paper
Paper and Pencil

## Activity Instructions:

1) Have the students divide into groups of three or four. Each group will decide on a particular item they would like to sell for their simulated fundraiser. Explain to the students that they must decide upon a single item to sell so that they can follow the instructions of the activity. Examples might include a carnation, a candy bar, a balloon or a candy cane.
2) Each group will then research how much it costs to buy at least 200 of whatever they have chosen to sell. Encourage them to get a competitive bid on a large number of items.
3) After the research, each group will find the function that describes their particular fundraiser. Each function will reflect a start up cost of $\$ 3$ to pay for the posters that will be posted around the school to advertise their fundraiser. The rest of their function will depend upon the unit cost of the item they are selling.
4) Next, each group will decide on a good selling price for their item. The price needs to be appropriate for sales to middle school students, but should also take into consideration making a maximum profit. Warn them that lack of realistic pricing or cost could affect the quality of their project.
5) Each group will now create two function tables for each of their cost and price functions. Label the function from step \#3 as the cost function, and label the function from step \#4 as the price function. Complete the functions for input values of $1,2,3,4,5,6,7,8,9,10,20,50,100,200$, and $n$.
6) Last, each group will graph both functions on one coordinate plane. Students should color code each function line so that they are each easily recognizable. Once the graphs are completed, ask each group how they would figure the profit from the cost and price graphs. Then ask the class how it might decide which item to choose for a fundraiser.

Teacher Edition Section 10.2 Rates of Change and Linear Functions

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## RECTANGLE ACTIVITY

Objective: Students will measure a rectangle using standard and nonstandard units. They will discover that the ratio of length to width of a rectangle is constant and related to a linear function.

## Materials:

Copy of rectangle \& table
Grid paper
Rectangle activity worksheet
Rectangle (1 per student or 1 per group)
Rulers
At least 4 non-standard measuring units (crayons, pencils, paperclips, sticky notes, etc.)

## Activity Instructions:

Students measure a rectangle using six different measuring devices (four non-standard units, inches, and cm ). They will record their measurements on their worksheet.

Once they complete their chart, class will discuss how they will use the ordered pairs of (length, width) to create a graph. Have students graph their six ordered pairs on a coordinate grid. Teacher will complete a scatter plot on 1 -inch chart paper based on student measurements. If pairs of students have used different units of measure, you may be able to display a scatter plot with more than six points by aggregating the measurements from the entire class.

Teacher will ask: "Do the points appear to be random, or do they seem to follow a pattern?" Students should recognize that the points follow a linear path. Then ask, "What might explain the pattern formed by the points?"

One pattern is the ratio of the change in width to the change in length, or slope, which is constant. Although the measurements may have changed because of the units, the ratio of length to width does not change. Students will connect their results to a function with a constant rate of change, which makes it linear.

Ask students to predict the results using other units of measure. For instance, ask, "The length of the rectangle measured approximately 9 nickels. What is the width of the rectangle in nickels?" (Approximately 6 nickels.)

To help with these predictions, students should draw a "line of best fit." Students can estimate this line and draw it with a ruler.

Since the points form a pattern, students should realize that a rule relates the length and width for this rectangle. Ask students if they can determine the rule.
Students should discover that the length is always 1.5 times the width, regardless of the unit of measure. Written algebraically, $L=1.5 \mathrm{~W}$.

Note: Teacher may recycle this activity by changing the size of the rectangle. Students repeat the steps and relate it to linear functions (Section 10.3)

EXTENSION: This activity can be used again with discussion on scatter plots and line of best fit.

## Rectangle Activity Worksheet

First measure your rectangle with a ruler. Record your measurements below in both inches and centimeters.

|  | Inches: | Centimeters: |
| :---: | :---: | :---: |
| Length $=$ |  |  |
| Width $=$ |  |  |

Now measure your rectangle with 4 other units (be sure to record your units on the table!) Record your measurements in the tables below.

Unit \#1:
Unit \#2:

|  |  |  |
| :---: | :--- | :--- |
| Length $=$ |  |  |
| Width $=$ |  |  |

Unit \#3:

|  |  |  |
| :---: | :--- | :--- |
| Length $=$ |  |  |
| Width $=$ |  |  |

## Shape for Rectangle Activity



## Section 10.3 - Proportions

## Big Idea:

Understanding and using proportional thinking to solve problems

## Key Objectives:

- Understand the definition and use of a proportion.
- Represent proportional relationships using tables.
- Represent proportional relationships using unit rates.
- Solve proportions in the context of real-life situations.


## Materials:

Two-column tables from CD, Maps

## Pedagogical/Orchestration:

- This section involves developing proportions through patterns in tables and unit rates. Proportions are a good mathematical tool that middle school students should learn to use. This section is rich with situations that call for proportional thinking and mathematics.
- Be aware of the prerequisites needed for students to be successful in understanding ratios, such as knowing the proper way to set up a proportion, and solve for the unknown. Guide the students through the minilesson included before the examples to help students understand the Big Idea.


## Internet Resources:

Jeopardy Game: Rates, Ratios, and Proportions- http://www.quia.com/cb/158527.html Rags to Riches: Solving Proportions- http://www.quia.com/rr/35025.html

## Activity:

"Just How many Boys and Girls Are There?" on CD and at the end of the section
"Marbles in a Bag" on CD and at the end of the section
"Proportions with Measurement Conversion" on CD and at the end of the section

## Exercises:

Exercise 13 uses the proportional reasoning that the number of fish tagged out of the population will equal the number of tagged fish caught out of the sample. In other words, $\frac{30}{\mathrm{p}}=\frac{3}{50}$. The following website/activity can help students understand this type of population problem: http://illuminations.nctm.org/LessonDetail.aspx?id=L721

## Vocabulary:

proportions

## TEKS:

$6.2(C) ; \quad 6.3(A)(C) ; \quad 6.4(A) ; \quad 6.5 ; \quad 6.11(A)(B)(C) ; \quad 7.2(D)(F) ; \quad 7.3(B) ; \quad 7.13(A)(B)(C)(D) ; \quad 7.14(B) ;$
8.2(C,D); 8.3(B); 8.14(A,B)

## WARM-UPS for Section 10.3 (Proportion)

1. Brian rode 18 miles on his 90 minute ride. If he rode at a steady pace, what was his speed in miles per hour? Ans: $(18$ mile $) \div\left(\frac{3}{2}\right.$ hours $)=(18)\left(\frac{2}{3}\right)=12 \mathrm{mi} / \mathrm{hr}$
2. What is the maximum number of $\frac{1}{5}$ meter length pieces of rope that can be cut from a $6 \frac{2}{3}$ meter length of rope?
a. 31 pieces
b. 32 pieces
c. 33 pieces
d. 34 pieces

Ans: (c) because $\left(6 \frac{2}{3}\right) \div\left(\frac{1}{5}\right)=100 \div 3=33 \frac{1}{3}$
3. Marching in a parade, the tuba players to 8 steps every 12 seconds. How many steps did they take in 45 seconds?
a. 30 steps
b. 32 steps
c. 33 steps
d. 35 steps

Ans: (a) because $\frac{8}{12}=\frac{x}{45}$ and so $x=30$.

## Launch for Section 10.3:

"Mrs. Saenz has a class whose ratio is 1:2 boys to girls. List 5 possibilities for Mrs. Saenz's class." Teachers should write possibilities that students state. Have a discussion on what they all mean. Do they agree with all of them?

Lead your students in a discussion of how they might solve this equation. The discussion might include the following: Multiply both sides of the equation by 2 . Or multiply the right side by 1 in the form of $\frac{2}{2}$ to produce the equation $\frac{x}{12}=\frac{100}{2}$ and then note that the numerators must be equal, so $x=100$. Some students might know that the actual distance x is equal to 100 miles using arithmetic, not algebra.

## SECTION 10.3 PROPORTIONS

When you look at a map of Texas, you know that the actual state is much larger than the map. For example, one inch can represent 50 miles, according to a scale designation on the map legend. That means that the ratio of the map distance to the actual distance is 1 inch to 50 miles. This ratio is written $1: 50$ or $\frac{1}{50}$.

Using this information, what actual distance does 2 inches represent? This time, writing the information as a ratio of actual distance to the map distance, the fraction is $\frac{x}{2}$, where $x$ is the actual distance in miles, represented by 2 inches on the map. Using the scale of 50 miles to 1 inch from the map, combine the two ratios in the equation $\frac{x \text { miles }}{2 \text { inches }}=\frac{50 \text { miles }}{1 \text { inch }}$. Solve the equation for $x$. We will explore a way to solve equations like this later in this section.

In a proportion, each side of the equation is a ratio. Sometimes, a proportion can compare two different types of the same units, like inches to inches and
 $\frac{2 \text { in }}{1 \text { in }}=\frac{x \mathrm{mi}}{50 \mathrm{mi}}$

Sometimes, the first ratio in the proportion compares different units. If the first ratio compares inches to miles, then the second ratio also compares inches to miles: $\frac{1 \text { in }}{50 \mathrm{mi}}=\frac{2 \mathrm{in}}{x \mathrm{mi}}$. If the first ratio compares miles to inches, then the second ratio also compares miles to inches: $\frac{50 \mathrm{mi}}{1 \mathrm{in}}=\frac{x \mathrm{mi}}{2 \mathrm{in}}$. It does not matter which quantity, miles or inches, is in the numerator. The important thing is that the fractions are equivalent.

## DEFINITION 10.1: PROPORTION

A proportion is an equation of ratios in the form $\frac{a}{b}=\frac{c}{d}$, where $b$ and $d$ are not equal to zero.

## EXAMPLE 1

Four water bottles costs $\$ 6$. How much will it cost to buy 10 of the same water bottles?

Placing the unknown or variable in the numerator makes solving the problem simpler for students to see and work through.

## SOLUTION

One way to set up the ratio is to use number of water bottles to cost.
We use the ratio $\frac{4 \text { bottles }}{6 \text { dollars }}$ and then equate this to $\frac{10 \text { bottles }}{x \text { dollars }}$, where $x$ represents the cost of 10 water bottles.

The proportion we have is:
$\frac{4 \text { bottles }}{6 \text { dollars }}=\frac{10 \text { bottles }}{x \text { dollars }}$ Note that both sides of the equation have the same units: bottles per dollar. Thus, we can drop the units.
$\frac{x}{1} \cdot\left(\frac{4}{6}\right)=\frac{x}{1} \cdot\left(\frac{10}{x}\right) \quad$ Multiply both sides by $\frac{x}{1}$.
$\frac{4 x}{6}=\frac{10 x}{x} \quad \frac{x}{x}$ equals 1 , so we are left with 10 on the right side of the equation.
$\frac{4 \mathrm{x}}{6}=10 \quad$ Multiply both sides by 6
$6\left(\frac{4 x}{6}\right)=6 \cdot 10 \frac{6}{6}$ equals 1 , so we are left with $4 x$ on the left side of the equation.
$4 x=60 \quad$ Divide both sides of the equation by 4.
$\frac{4 \mathrm{x}}{4}=\frac{60}{4} \quad$ Again, since $\frac{4}{4}$ equals 1 , the left side of the equation is $x$.
$x=15 \quad$ The situation is $x=15$.
Alternate method: Or we can use an alternate ratio of cost to water bottles $\frac{6 \text { dollars }}{4 \text { bottles }}$ The proportion then looks like:
$\frac{6 \text { dollars }}{4 \text { bottles }}=\frac{x \text { dollars }}{10 \text { bottles }}$ Note that both sides of the equation have the same units: dollars per bottle. Thus, we can drop the units.
$\frac{6}{4}=\frac{x}{10}$

This is a very important point in understanding proportions. Make sure your students see how varied a correct proportion can be and what the big mistake is - not to match the units in the fractions' numerators and denominators. The ratio of cost per bag of chips, $\frac{2.79 \text { dollars }}{3 \text { bags }}$ (or $\$ 0.93$ per bag), is a unit rate, as is, $\frac{3 \text { bags }}{2.79 \text { dollars }}$.

$$
\begin{aligned}
& \frac{10}{1}\left(\frac{6}{4}\right)=\frac{10}{1} \frac{x}{10} \quad \frac{10}{10} \text { equals } 1 \text {, so the right side of the equation is } \\
& \frac{60}{4}=x \\
& 15=x
\end{aligned} \begin{aligned}
& \text { simplifying the left side of the equation } \\
& 15=
\end{aligned}
$$

Notice the result is the same regardless of the ratio set up. In fact, there are other proportions that can be used as you can see in Example 2. Which of the two choices required fewer steps to solve? Why do you think this happened?

## EXAMPLE 2

Set up the following problem using a proportion.
Three bags of chips cost $\$ 2.79$. How much do 7 bags of chips cost?

## SOLUTION

In the chips problem, any one of these proportions will correctly determine $x$, the cost of 7 bags of chips. The possible proportions are indicated:
a. $\frac{\text { bags of chips }}{\text { bags of chips }}=\frac{\text { cost }}{\operatorname{cost}} \rightarrow \frac{3}{7}=\frac{2.79}{x}$
b. $\frac{\text { bags of chips }}{\operatorname{cost}}=\frac{\text { bags of chips }}{\operatorname{cost}} \rightarrow \frac{3}{2.79}=\frac{7}{x}$
c. $\frac{\text { cost }}{\text { bags of chips }}=\frac{\text { cost }}{\text { bags of chips }} \rightarrow \frac{x}{7}=\frac{2.79}{3}$
d. $\frac{\text { cost }}{\operatorname{cost}}=\frac{\text { bags of chips }}{\text { bags of chips }} \rightarrow \frac{x}{2.79}=\frac{7}{3}$

Use one of the proportions to solve for $x$. Verify that the cost for 7 bags of chips is the same when each proportion is solved. Which of the possible proportions above involved a unit rate?

Setting up the correct proportion is often the hardest part of solving a proportion problem. Some proportions are easier to solve than others because of the way they are set up. Which of the proportions above was easiest to solve and why?

## EXAMPLE 3

Marla estimates her party guests will consume an average of a pint of punch each, so she will need 28 pints of punch. She has a family recipe that makes one gallon of punch. How many gallons of punch does she need for the party?

## SOLUTION

Recall there are four quarts in a gallon and two pints in a quart. So there are eight pints in a quart. To calculate the number of gallons of punch needed, set up a proportion

$$
\begin{aligned}
& \text { Let } \mathrm{g}=\text { number of gallons of punch needed } \\
& \qquad \frac{\mathrm{g} \text { gallons }}{28 \text { pints }}=\frac{1 \text { gallon }}{8 \text { pints }}
\end{aligned}
$$

So, g gallons $=\left(\frac{1 \text { gallon }}{8 \text { pints }}\right)(28$ pints $)=\left(\frac{28}{8}\right)$ gallons $=3.5$ gallons.
Marla needs to make 3.5 gallons of punch.

## ALTERNATE SOLUTION

Another way to approach this problem is to use the unit rate of conversion, which in this problem is $\frac{1 \text { gallon }}{8 \text { pints }}$. To convert 28 pints into gallons, multiply the quantity of punch in pints times the unit rate in gallons per pint:

$$
(28 \text { pints })\left(\frac{1 \text { gallon }}{8 \text { pints }}\right)=\left(\frac{28}{8}\right) \text { gallons }=3.5 \text { gallons }
$$

Notice that in each solution the pint units canceled through multiplication leaving the gallon units

## EXAMPLE 4

A colony of leafcutter ants cuts up 4 leaves in 7 minutes. How many leaves does the colony cut in an hour?

## SOLUTION

## Tabular Method:

Construct a table to record the time and the number of leaves cut.

| Time in minutes | Number of leaves cut |
| :---: | :---: |
| 0 | 0 |
| 7 | 4 |
| 14 | 8 |
| 21 | 12 |
| 28 | 16 |
| 35 | 20 |
| 42 | 24 |
| 49 | 28 |
| 56 | 32 |
| 63 | 36 |

Notice that the problem asked how many leaves the colony cut in 1 hour or 60 minutes. The table shows that the number of leaves must be between 32 and 36 leaves. This gives a good estimate for the solution, but there is a way to get an exact answer.

## Unit Rate Method:

Set up a proportion that compares the ratio of leaves to minutes. Note that 1 hour must be converted into 60 minutes.

Discuss how the table is helpful to see the pattern, but does not easily show the number of leaves cut in 60 minutes

## EXAMPLE 5

Unit rate method:
Unit rate $=\frac{1}{4}=\frac{1}{4} \bullet \frac{1}{3}=\frac{1}{12} \frac{\mathrm{mi}}{\mathrm{min}}$
$d=r t \longrightarrow=\left(\frac{1^{4}}{12}\right)(45$ mins $)=\frac{\mathrm{m}^{\mathrm{tn}}}{12}$ miles $=3 \frac{3}{4}$ miles.

Because the ants cut 4 leaves in 7 minutes, using division, the ants must cut $\frac{4}{7}$ of a leaf in 1 minute. This is the unit rate or the number of leaves cut per minute. If the ants keep cutting at this rate, they will cut 60 times this number of leaves in 60 minutes. Call the number of leaves cut in 60 minutes $x$. Then

$$
x=\frac{4 \text { leaves }}{7 \mathrm{~min}} \cdot 60 \mathrm{~min}=\frac{240}{7} \text { leaves }=34 \frac{2}{7} \text { leaves }
$$

## Proportion Method:

Set up a proportion by comparing amounts for the two different times.
The ants cut 4 leaves in 7 minutes. How many leaves $x$ will the ants cut in 60 minutes?

$$
\frac{x \text { leaves }}{60 \mathrm{~min}}=\frac{4 \text { leaves }}{7 \mathrm{~min}}
$$

To solve, multiply both sides of the equation by the denominator 60 .

$$
x=\frac{4}{7} 60 \cdot=\frac{240}{7}=34 \frac{2}{7} \text { leaves. }
$$

This proportion method involves the rate of change in the form of leaves cut per unit time or minute. This is a rate of change like miles per hour or mph.

## EXAMPLE 5

Leo runs $\frac{1}{4}$ of a mile in 3 minutes. How many miles will he run in 45 minutes, assuming he continues to run at the same rate?

Use table, proportion and unit rate for the problem.

## SOLUTION

To see a visual way of organizing the problem, set up a table. Make a table with time and distance as headers.

| Time <br> (in minutes) | Distance (in <br> miles) |
| :---: | :---: |
| 0 | 0 |
| 3 | $\frac{1}{4}$ |
| 6 | $\frac{2}{4}=\frac{1}{2}$ |
| 9 | $\frac{3}{4}$ |
| 12 | $1 \frac{1}{4}$ |
| 18 | $1 \frac{1}{2}$ |
| 21 | $\frac{3}{4}$ |


| Time <br> (in minutes) | Distance (in <br> miles) |
| :---: | :---: |
| 24 | 2 |
| 27 | $2 \frac{1}{4}$ |
| 30 | $2 \frac{1}{2}$ |
| 33 | $2 \frac{3}{4}$ |
| 36 | 3 |
| 39 | $3 \frac{1}{4}$ |
| 42 | $3 \frac{1}{2}$ |
| 45 | $3 \frac{3}{4}$ |

You can also use proportions to determine the distance Leo runs in 45 minutes. Let x represent the distance Leo runs in 45 minutes. You can use the proportion $\frac{x \text { miles }}{45 \text { minutes }}=\frac{1}{4}$ miles. Multiply both sides of the equation by 45 .
The resulting equation is: $x=\frac{\frac{1}{4}}{3}(45)=\left(\frac{1}{4}\right)\left(\frac{1}{3}\right)(45)=\frac{45}{12}=3 \frac{9}{12}=3 \frac{3}{4}$ miles.
Can you use the Unit Rate Method on this problem?

## EXAMPLE 6

Lucinda is studying prairie dog populations in Colorado. She captures and tags 15 prairie dogs and then releases them back into the wild. Two weeks later she captures 35 prairie dogs and discovers 3 are tagged. What is the approximate population of prairie dogs in the region?

Unit Rate Method:
$\frac{1}{4} \div 3=\frac{1}{4} \frac{1}{3}=\frac{1}{12}$ miles $/ \mathrm{min}$.
$\frac{1}{12} \frac{45}{1}=\frac{45}{12}=\frac{15}{4}=3 \frac{3}{4}$ miles in 45 minutes

## SOLUTION

Setting $x$ equal to the approximate total population of prairie dogs in the area,

$$
\frac{x \text { total prairie dogs }}{15 \text { tagged prairie dogs }}=\frac{35 \text { total prairie dogs }}{3 \text { tagged prairie dogs }}
$$

To solve, multiply each side of the equation by the denominator 15 .

$$
x=\frac{35}{3} \cdot 15=\frac{525}{3}=175 \text { prairie dogs }
$$

## EXPLORATION 1

Materials: You will need a map of any region that contains a legend with the distance scale and a ruler or tape measure in the same unit system as the map.
Step 1: Find the legend in the map and write a ratio that relates the map measure to the actual measure.

Step 2: Use a measuring instrument to measure the straight-line distance between two major cities on the map.

Step 3: Determine the actual straight-line distance between the cities using proportions.
Step 4: Repeat Steps 2 and 3 with two other cities.
What are the actual straight-line distances between the cities that you chose?

## EXPLORATION 2

| Object | Actual size (miles) | Scaled size (cm) |
| :--- | :--- | :--- |
| Earth Diameter | 8,000 | 30 |
| Top of the atmosphere | 100 | 0.375 |
| Space shuttle orbit height | 200 | 0.75 |
| Height of satellite | 18,000 | 67.5 |
| Moon diameter | 2,100 | 7.875 |
| Distance from Earth to Moon | 240,000 | 900 |

EXERCISES

1. 6
2. $\$ 6=x$
$\frac{x \text { dollars }}{8 \text { peaches }}=\frac{9 \text { dollars }}{12 \text { peaches }}$
3. a. $\frac{3}{5}=\frac{42}{9} ; g=70$
b. $\frac{b}{70}=\frac{8}{5} ; b=112$

## EXPLORATION 2

From Math Explorer, December 1999, vol. 2.3
A globe is approximately 30 cm in diameter. Using that measurement, calculate the scaled size for the measurements indicated in the table.

| Object | Actual size <br> (miles) | Scaled size (cm) |
| :--- | ---: | :---: |
| Earth diameter | 8,000 | 30 |
| Top of the atmosphere | 100 |  |
| Space shuttle orbit height | 200 |  |
| Height of satellite | 18,000 |  |
| Moon diameter | 2,100 |  |
| Distance from Earth to Moon | 240,000 |  |

## EXERCISES

1. Alvin the squirrel picks as many acorns as he can hold, then brings them home. For every seven acorns he picks, he loses two but manages to get the remaining five home. After one hour Alvin has fifteen acorns stored in his home. How many acorns has he dropped?
2. Farmer Al sells peaches at $\$ 9$ a dozen at the farmer's market. At that price, what is the cost of 8 peaches? At that rate how much does each peach cost?
3. A large restaurant makes a soup that uses twelve pounds of carrots for each fifteen pounds of potatoes. What is the proportion of potatoes to carrots in the soup? What is the rate of pounds of carrots per pound of potatoes?
4. A toy manufacturer makes colored marbles and packages them. Each pac kage contains red, blue, and green marbles, the ratio of red to green is 3:5, and the ratio of blue to green is $8: 5$. If the package contains 42 red marbles:
a. How many green marbles does each package contain?
b. How many blue marbles does each package contain?
5. B. 24 minutes
6. C. 225 books
7. It takes 120 minutes to wash 30 vehicles at a car wash. At this rate, how many minutes does it take to wash 6 vehicles? Select the best choice and explain your answer.
a. 30 minutes
b. 24 minutes
c. 28 minutes
d. 6 minutes
8. Gabby can assemble 10 music books in 8 minutes. At this rate, how many music books can she assemble in 3 hours? Select the best choice and explain your answer.
a. 24
b. 4
c. 225
d. 300
9. Use proportions to convert the original units to the new units of measure:
a. Convert 11 quarts to gallons.
b. Convert 8 feet to yards.
c. Convert 8 yards to feet.
d. Convert 150 seconds to minutes.
e. Using the fact that 440 yards is equivalent to $1 / 4$ mile, find the number of yards are in a mile. How can you compute the same answer mentally?
f. Using the answer in e, compute the number of feet in a mile. Explain how to compute the same answer mentally.
g. Convert 84 hours to days.
10. Use each given unit rate to convert the following quantities to the new unit of measure:
a. $\left(\frac{12 \text { inches }}{\text { foot }}\right)$; Convert 4 feet into inches.
b. $\left(\frac{12 \text { inches }}{\text { foot }}\right)$; Convert $2 \frac{1}{4}$ feet into inches.
c. $\left(\frac{1 \text { foot }}{12 \text { inches }}\right)$; Convert 30 inches into feet.
11. a. $x=9$
b. $x=20$
c. $x=10$
d. $x=22.5$

This is an opportunity to review the idea that $\frac{\mathrm{a}}{\mathrm{b}} \cdot b=a$ and if $a=b$, then $a \cdot c=b \cdot c$.
7. $14 \frac{80}{6}=13 \frac{1}{3}$. One chaperone for the last 2 students.
8. 120
9. Monitor how students think about this problem. Does anyone use a picture? Share approaches that are different. $15\left(\frac{2}{3}\right)=10$ box kites. As a proportion: $\frac{2}{3}=\frac{x}{15} ; \frac{2(15)}{3}=x ; 10=x$

$$
\frac{1_{1} \text { green }^{12} \text { green marbles }}{4 \text { marbles }}=\frac{3 \text { green }}{x \text { marbles }} ; \frac{1}{4}=\frac{3}{x} ; x=3(4)=12
$$

$\frac{\text { 11. } \mathrm{x} \text { boys boys }}{18 \text { students }}=\frac{1 \text { boy }}{3 \text { students }} ; x=\frac{18}{3}=6$ boys
12. a. $\frac{7}{34}$ or a little over a fifth of a gallon
b. $\$ 0.83$
13. 500
$\frac{30}{x}=\frac{3}{50} ; 500=x$
14. about 453

Let $b=$ number of times at bat. $\frac{3}{10}=\frac{136}{b}$
d. $\left(\frac{1 \text { foot }}{12 \text { inches }}\right)$; Convert 15 inches into feet.
e. $\left(\frac{24 \text { hours }}{\text { day }}\right) ;$ Convert 4 days into hours.
f. $\left(\frac{24 \text { hours }}{\text { day }}\right)$; Convert $1 \frac{1}{2}$ days into hours.
g. $\left(\frac{1 \text { day }}{24 \text { hours }}\right)$; Convert 8 hours into days.
9. Solve the following equations.
a. $\frac{3}{4}=\frac{x}{12}$
b. $\frac{5}{9}=\frac{x}{36}$
c. $\frac{25}{x}=\frac{5}{2}$
d. $\frac{8}{5}=\frac{36}{x}$
10. If a museum requires at least one chaperone for every six students, what is the minimum number of chaperones required for a field trip with 80 students?
11. If six shepherds can watch 180 sheep, how many sheep can four shepherds probably watch?
12. Jane has 15 kites. She says $\frac{2}{3}$ of them are box kites. How many box kites does she have?
13. Bobby has 3 green marbles. He says that $\frac{1}{4}$ of his marbles are green. How many marbles does he have?
14. A class has 18 students and $\frac{1}{3}$ of them are boys. How many boys are there in the class? Explain your thinking on this problem.
15. The mileage for Dr. Thompson's new car is $34 \frac{\text { miles }}{\text { gallon }}$. That is the average mileage per gallon in France, Denmark and Italy, where the cost of gas is over $\$ 4.00$ a gallon. When returning from a research expedition in the desert, her car runs out of gas seven miles from the nearest gas station. She plans to walk to the gas station, bring back just enough fuel to drive her car to the gas station, and refill there.
a. How much gas does she need to buy?
b. What is the least amount of money she needs to buy the necessary gas?
16. Heather is studying the fish population for Canyon Lake. She catches 30 fish and tags them. After a month, she catches another 50 and discovers that 3 of them are tagged. What is the approximate population of fish in the lake?
17. Ernie averages 3 hits for every 10 times he goes to bat during the season. If he gets 136 hits over the whole season, about how many times did he bat?
15. a. 3.00 ERA $=\frac{\text { runs }}{\text { game }}=\frac{\text { runs }}{\text { inning }} \times \frac{\text { innings }}{\text { game }} \frac{28}{84} \times \frac{9}{1}=$ ERA
b. 40 ERA $\times$ number of games $=3 \times \frac{120}{9}$
c. Set up $\frac{2.4}{9}=\frac{x}{210}$ then $x=56$.
16. $81 \mathrm{r}: \mathrm{b}=4: 5 \quad \frac{\mathrm{~b}}{\mathrm{a}}=\frac{5}{9}=\frac{45}{\mathrm{a}}$
17. $6 \frac{1}{8} \quad \frac{48}{1.75}=\frac{168}{s} \quad 6.125=6$
18. C
19. between $\frac{\frac{2}{3} b+b+\frac{2}{7} b}{\left(\frac{8}{3}\right) \frac{2}{7} b} \cdot \frac{21}{21}=\frac{14 b+21 b+6 b}{16 b}=\frac{41}{16}$ and $1 \frac{1}{4}$.

Let $C=$ number of candy bars. $C=\frac{160}{10}=16$ candy bars. $\frac{16}{26.2}$ is approximately 0.6 candy bars per mile.

Ingenuity: Geometric Series
20. Possibly: First, find the changes in the different random speeds. Is there a multiplicative relationship between the differences? If so, is there a "unit" change? Use it, going backward and forward, to decide the number of seconds for each measurement. All three differences in speed $(48,32,16,48)$ have a common factor of 16 (but also 8, 4 and 2). Most likely, you will need to know the times.
18. In baseball, the Earned Run Average, or ERA, is determined by the average number of earned runs a pitcher allows during a 9-inning game over a period of time. Roy has allowed 28 earned runs in 84 innings.
a. What is Roy's ERA?
b. Generally, how many earned runs does Roy allow in 120 innings?
c. Roy's teammate, Roger, has an ERA of 2.40. On average, how many earned runs does Roger allow in 210 innings?
19. Marcy has a bag of red and blue jacks. The ratio of red jacks to the total number of jacks is $4: 9$. If she has 45 blue jacks, how many jacks does she have in all?
20. A recipe for 48 cookies requires $1 \frac{3}{4}$ cups of sugar. How much sugar is needed to make 168 cookies?
21. Mrs. Jackson bought 12 pounds of potatoes for $\$ 5$. 16 . Which of the following represents the same price per pound? Select the best choice and explain your answer.
a. 8 pounds of potatoes for $\$ 3.28$
b. 10 pounds of potatoes for $\$ 4.00$
c. 15 pounds of potatoes for $\$ 6.45$
d. 9 pounds of potatoes $\$ 4.05$
22. George is running a marathon, a 26.2-mile. Researchers have determined that to run in a marathon, it is generally safe to eat up to one power bar for every ten pounds a person weighs. George weighs 160 pounds. According to these guidelines, over the whole race, approximately what part of a power bar would he eat per mile?

## 23. Ingenuity:

Reagan dropped a rock from the top of a high cliff. The speed of the rock was recorded at random times while the rock was falling. The first 5 speeds recorded, in feet per second, were $0,48,80,96$ and 144 .

If the rate of change in speed is constant, is there enough information to find how fast the speed is changing? If there is, what is the rate of change? If not, what information do you need?

## Investigation

21. $\frac{41}{16}$ Substituting $\frac{2}{3} b$ for $a, \frac{2}{7} b$ for $c$, and $\frac{8}{3}\left(\frac{2}{7} b\right)$ for $d$ in the fraction,

$$
\frac{\frac{2}{3} b+b+\frac{2}{7} b}{\left(\frac{8}{3}\right) \frac{2}{7} b} \cdot \frac{21}{21}=\frac{14 b+21 b+6 b}{16 b}=\frac{41}{16}
$$

An alternate solution is given by taking the $\frac{a+b+c}{d}$ apart as the sum of three fractions:
$\frac{a}{d}+\frac{b}{d}+\frac{c}{d}$.
$\frac{c}{d}=\frac{3}{8}$ by the given.
$\frac{b}{c} \times \frac{c}{d}=\frac{7}{2} \times \frac{3}{8}=\frac{21}{16}$ but
$\frac{b}{c} \times \frac{c}{d}=\frac{b c}{c d}=\frac{b}{d}$.
So $\frac{b}{d}=\frac{21}{16}$.
$\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d}=\frac{2}{3} \times \frac{7}{2} \times \frac{3}{8}=\frac{7}{8}$ but
$\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d}=\frac{a}{d}$.
So $\frac{a}{d}=\frac{7}{8}$.
Using the computed values, we have $\frac{a+b+c}{d}=\frac{a}{d}+\frac{b}{d}+\frac{c}{d}=\frac{7}{8}+\frac{21}{16}+\frac{3}{8}=\frac{14+21+6}{16}=\frac{41}{16}$
24. Investigation:

Four numbers $a, b, c$ and $d$ satisfy the relationships $\frac{a}{b}=\frac{2}{3}, \frac{b}{c}=\frac{7}{2}$ and $\frac{c}{d}=\frac{3}{8}$. Find $\frac{a+b+c}{d}$.

## JUST HOW MANY BOYS AND GIRLS ARE THERE?

Objective: The students will collect data to form a ratio of how many boys compared to girls there are in their world. They will then take this ratio, and use it in a proportion to figure out how many boys compared to girls there could possibly be in the world.

## Materials:

Paper and Pencil
Computer, or some other resource, to figure out how many people are in their state, country and then the whole world.

## Activity Instructions:

1) Ask the students to count how many boys and how many girls are in their class. The students will write this comparison as a ratio, and make sure that their ratio is simplified. Write the ratio three different ways: boys : girls, boys : total, and girls : total.
2) Next, have the students make the same three ratios using data from the school's front office regard ing how many boys and girls there are in their whole school. Compare the ratios. How are they similar, how are they different, what are the possible reasons that could have caused the similarities or differences?
3) For an extension, have the students research how many people were in their state during the last census. Using the original ratios either from the class or from the school, the students will use a propor tion to estimate how many males are in their state compared to how many females.
4) Continue the activity with research about the country's population and then the whole world.
5) Give bonus points for any student who brings in documented evidence of the real state, national or world population data. As an extension, you might ask them to find out the consequences of the "one family, one child" policy in China.

## MARBLES IN A BAG

Objective: Students will find ratios of marbles to figure out an unknown amount of one color of marbles in a bag. They will compare two ratios together and solve the proportion.

## Materials:

Marbles (yellow, red, blue, green) (at least 10 of each)
Colors or markers

## Activity Instructions:

Teacher arranges students in groups of 3 or 4 per group. Each group receives a bag of about 20 marbles. Teacher sets up this activity by choosing how many marbles of each color to use. One suggestion is to place the following amount of marbles in the bag: 4 yellow, 7 blue, 3 red, leave green as unknown.

Students will answer the following questions about the marbles in the bag:
Set up each ratio and simplify as needed.

1) What is the ratio of yellow marbles in the bag? $\frac{4}{20}=\frac{1}{5}$
2) What is the ratio of blue marbles in the bag? $\frac{7}{20}$
3) What is the ratio of red marbles in the bag? $\frac{3}{20}$
4) What is the ratio of green marbles in the bag? $\frac{6}{20}$

If one marble of each color is removed from the bag, what will be the new ratios of each color of marbles in the bag?
a. yellow:
b. blue:
$\qquad$
c. red: $\qquad$
d. green:

Note: Students may draw \& color marbles as a solution strategy.

## PROPORTIONS WITH MEASUREMENT CONVERSIONS

Here are some sample problems to supplement Chapter 10 proportions along with measurement conversions.

1. Sarah has a recipe for Italian dressing. It calls for mixing 5 tablespoons of oil to 2 tablespoons of vinegar. How many tablespoons of oil should Sarah mix with 5 tablespoons of vinegar?
2. A car travels 240 miles in 4 hours. At the same rate of speed, how far can it travel in 6 hours?
3. Sandra walks 3 miles each week. How many feet does she walk?
4. Marks cats eat 12 pounds of cat food in 30 days. How many pounds of food would you expect his cats to eat in 90 days?
5. How many pounds are in 128 ounces? ( 1 pound $=16$ ounces)
6. How many centimeters are in 5 meters?
7. Jared ran a distance of 3,500 meters. How many kilometers did he run?
8. A recipe calls for 20 ounces of sugar. How many cup of sugar is that?
9. Jade ran 3 miles in a Cancer Awareness Charity marathon. How many yards did she run?
10. Ralph is 1.5 meters tall. What is his height in centimeters?
11. At Tom's birthday party, the children drank 6 gallons of punch. How many cups of punch is that?
12. A brownie recipe calls for 250 grams of chocolate. How many kilograms is that?
13. How many milligrams are in 0.5 grams?
14. Sam's mom bought 3 yards of trim for a quilt. How many inches are in 3 yards?
15. An aquarium holds 20 liters of water. How many milliliters does the aquarium hold?
16. Jerome lives 6 kilometers from his cousin. How many meters does he live from his cousin?
17. Danette needs 0.2 kilograms of chocolate chips for a recipe. How many grams of chocolate chips does she need?
18. How many ounces are in 3 pints?
19. How many millimeters are in 0.1 meters?
20. A cup holds 500 milliliters. How many liters does the cup hold?

## Section 10.4 - Percents and Proportions

## Big Idea:

Connecting percents to ratios, exploring percent increase and decrease, and solving percent problems with proportions and linear equations

## Key Objectives:

- Find percent of a number.
- Find percentage increase and percentage decrease.
- Use percentages in the context of real-life situations.


## Pedagogical/Orchestration:

- This section explores computing percents and percentage increase and decrease, one of the most important applications of percentages. A $20 \%$ increase in price added to the original price, equals $120 \%$ of the original price. A $20 \%$ decrease in price yields a new amount that is $80 \%$ of the original price. For students who struggle with the next step, the altered price, ask them what they would do if the original price were $\$ 100$. They can almost always tell you. Then ask them how they came up with their answer, and they will tell you how to work the problem.
- We will develop 2 methods to solve percent problems, using proportions and linear equations.
- Encourage all your students who do not understand percents to make the following analogous relationship: decimals are to percents as dollars are to cents. 1.25 is to $125 \%$ just as $\$ 1.25$ is to $125 \phi$.


## Vocabulary:

percent, percentage increase, percentage decrease, mixed fraction, proper fraction, sales tax

## TEKS:

$6.1(A)(B) ; \quad 6.2(D) ; \quad 6.3(A)(B) ; \quad 7.1(B) ; \quad 7.2(A)(F)(G) ; \quad 7.3(A) ; \quad 7.5(B) ; \quad 7.13(A)(B)(C) ; \quad 7.15(B) ; \quad 8.1(B) ;$
8.2(A,C,D); 8.14(A); 8.15(A)

## WARM-UPS for Section 10.4

1. Raphael rides his bike at a constant speed on a flat road. He burns 810 calories in a 90 minute bike ride. How many calories does he burn in 35 minutes?
a. 305 calories
b. 310 calories
c. 315 calories
d. 320 calories

Ans: (c) because $\frac{810}{90}=\frac{x}{35}$ and so $x=315$.
2. Jerry left a $\$ 3.00$ tip at a cafe. If he calculated that this was a $12 \%$ tip, what was the original bill?
a. $\$ 30$
b. $\$ 28$
c. $\$ 35$
d. $\$ 36$

Ans: (a) because $0.12 \mathrm{x}=3.00$ and so $\mathrm{x}=30$.
3. Forty percent of a class is in the band. If there are 14 band members in the class, how many students are in the class? Ans: $0.40 \mathrm{x}=14$ so $\mathrm{x}=14 \div 0.4=35$ students in the class

## Launch for Section 10.4:

Today we are going to draw a model to represent $25 \%$ of 28 . Draw a rectangle on your paper and divide it into 4 equal sections by drawing vertical lines.. Label the marks like a ruler with percents on the top and fractions on the bottom.


How can we represent $25 \%$ of 28 on this model? (Responses may include: $28 / 4=7$. Write the numbers 7, 14, 21, 28 in the appropriate box.)

Thus, $25 \%$ of 28 is 7 .
What does 14 represent? ( $50 \%$ of 28)
What does 21 represent? ( $75 \%$ of 28)

## Hands on Banking Activities

The following activities are available on http://www.handsonbanking.org

## Budgeting

## Unit Overview

In these lessons, middle-school students (grades 6-8) are introduced to a personal budget. At the end of these lessons, students will be able to explain the purposes of budgeting and basic budgeting strategies. Students will be able to create their own personal budgets.

In the online/CD-ROM version of the Hands on Banking program, there are eight lessons that are condensed into two sections, below.

## Section 1: Understanding and Creating Budgets

Individuals use budgets to itemize and manage their income, expenses, and savings. To be financially sound, it's important to spend less than you earn. Students identify fixed, flexible, and discretionary expenses. Students create a personal budget showing income, expenses, and savings.

## Section 2: Using a Budget

Students apply what they know about budgets to make sound financial decisions.

## Learning Objectives

The financial-literacy objective of these lessons is for students to recognize that a major factor in being financially solvent is to spend less than one earns and to save the difference. A personal budget is a tool that can assist an individual stay within his or her income.

The mathematical objective of these lessons is for students to compute the sum or difference of whole numbers and positive decimals to two places.

## Alignment with Educational Standards

National Council of Economic Education and the National Association of Economics Educators and the Foundations for Teaching Economics, Voluntary National Content Standards in Economic, (1997), Grade 8:

- Content Standard 2, "To determine the best level of consumption of a product, people must compare the additional benefits with the additional costs of consuming a little more or a little less."

JumpStart Coalition for Personal Financial Literacy, National Standards in K-12 Personal Finance Education (2007), Grade 8 Standards:

- Planning and Money Management

National Council of Teachers of Mathematics Principles and Standards for School Mathematics, (2000), Grades 6-8:

- Number and Operation Expectations, "[Students will] work flexibly with fractions, decimals, and percents to solve problems .... [Students will] select appropriate methods and tools for computing with fractions and decimals from among mental computation, estimation, calculators or computers, and paper and pencil, depending on the situation, and apply the selected methods."
- Problem-Solving Expectations: "Solve problems that arise in mathematics and in other context; apply and adapt a variety of appropriate strategies to solve problems."
- Connections Expectations: "Recognize and apply mathematics in contexts outside of mathematics."


## Section 1: Understanding and Creating Budgets

Individuals use budgets to itemize and manage their income, expenses, and savings. To be financially sound, it's important to spend less than you earn. Students identify fixed, flexible, and discretionary expenses. Students create a personal budget showing income, expenses, and savings.

## Opening Questions

Use these or similar questions to start students thinking about this concept and how it relates to them:

## Key Points

- A personal budget:
- helps you identify how you spend your money and how much you spend in a given period of time;
- helps you plan the savings you'll need for unexpected expenses or changes in income; and,
- helps you make decisions about your money both today and as your situation changes over time.
- Remember, your budget is a general plan. If your expenses change, or if you have an emergency expense, your budget will have to change, too. So try to allow yourself a few dollars left over every month for pocket change - or for the unexpected.
- What information do you need to make a personal budget? You need to know:
- how much money you have coming in during a given period of time, that is, your income;
- how much money you have going out in a given period of time, that is, your expenses; and,
- how you can adjust your spending habits to save for unexpected events and get the most value for your money.


## - Types of Expenses

- Fixed Expenses: These expenses occur regularly and don't change from month to month. Examples of fixed expenses are rent and car payments.
- Flexible Expenses: Liked fixed, flexible expenses occur on a regular basis. The difference is that with flexible expenses, you have some control over how much you spend. Examples of flexible expenses include food and gasoline.
- Discretionary Expenses: This is money that you choose to spend - like money for movies or having pizza with friends. It also includes the money that you save.


## Activity

Students use the following worksheet to analyze a personal 4 -week budget. The teacher's copy of this activity folllows the students' worksheet.
(Please download handouts from http://www.handsonbanking.org/en/instructional-resources.html under "Teen Instructor Guide")

## Saving and Checking Guide

## Unit Overview

In these lessons, middle-school students (grades 6-8) explore and compare savings and checking accounts. At the completion of this unit, student will be able to explain how savings and checking accounts works, describe the benefits of using these basic accounts to manage their money, and complete the forms necessary for making deposits and withdrawals from these accounts. Student will do the necessary calculations to balance a checking account.

In the online/CD-ROM version of the Hands on Banking program, there are 16 lessons that are condensed into 3 sections, below.

## Section 1: Savings Accounts

Saving money is an important step toward financial well-being. In this section, students recognize the purpose of savings accounts, how to make deposits and withdrawals, and how to manage a savings accounts. Students will do calculations to compare simple and compound interest.

## Section 2: Checking Accounts

Checking accounts are another tool banks provide to help individuals manage their finances. Students will investigate the basics of checking accounts and practice writing checks.

## Section 3: Balancing a Bank Account

Balancing a bank account is an important and basic financial skill. Students will perform the necessary computations to balance a checking account.

## Learning Objectives

The financial-literacy objectives of the lessons are for student to recognize the services banks provide and how to use these services effectively, and that savers earn compound interest on principal and previously earned interest.

The mathematical objective of these lessons is for students to compute the sum or difference of whole numbers and positive decimals to two places.

## Alignment With Educational Standards

National Council of Economic Education and the National Association of Economics Educators and the Foundation for Teaching Economics, Voluntary National Content Standards in Economics (1997), Content Standard 10:

- Grade 4: "Banks are institutions where people save money and earn interest, and where other people borrow money and pay interest."
- Grade 8: "Banks and other financial institutions channel funds from savers to borrowers and investors."

JumpStart Coalition for Personal Financial Literacy, National Standards in K-12 Personal Finance Education (207), Grade 8 Standards:

- Planning and Money Management
- Saving and Investing

National Council of Teachers of Mathematics Principles and Standards for School Mathematics, (2000), Grades 6-8:

- Number and Operations Expectations, "(Students will" work flexibly with fractions, decimals, and percents to solve problems .... (Students will) select appropriate methods and tools for computing with fractions and decimals from among mental computation, estimation, calculators or computers, and paper and pencil, depending on the situation, and apply the selected methods."
- Problem-Solving Expectations: "Solve problems that arise in mathematics and in other contexts; apply and adapt a variety of appropriate strategies to solve problems."
- Connections Expectations: "Recognize and apply mathematics in contexts outside of mathematics."


## Section 1: Saving Accounts

Saving money is an important step toward financial well-being. In this section, students recognize the purpose of savings accounts, how to make deposits and withdrawals, and how to manage a savings account.

## Opening Questions

Use these or similar questions to start students thinking about this concept and how it relate to them:

- Are you saving money for something you want or need? Describe how you are managing to save money.
- Why would you recommend opening a savings account to someone who doesn't have one yet?
- Let's say you have some money in a savings account, and you want to take some of the money out. Where would you go to do that, and what would you have to do?
- Even though the purpose of the account, and start putting money in and taking money out, who's going to keep track of how much money you have in the account?
- What do the initials "ATM" " stand for, and what's the purpose of an ATM? What banking transactions can people do at an ATM?


## Key Points

## Savings and Interest

- Saving means putting money aside for future use. Banks and other financial institutions offer incentives for people to keep their money in a savings account. These incentives are referred to as earning interest.
- The amount of interest people will earn depends on a number of factors, including the type of savings account they have, which financial institution has the account, and how long they keep their money in the account.
- Banks pay interest on money put into savings accounts because the bank is able to use the money to make loans to other customers.
In fact, banks pay their customers for the privilege of using their money.
(Please download handouts from http://www.handsonbanking.org/en/instructional-resources.html under "Teen Instructor Guide")

PROBLEM 1
a. $\frac{30}{75}=.4=40 \%$
b. $\frac{45}{75}=.6=60 \%$

## SECTION 10.4 - PERCENTS AND PROPORTIONS

In this chapter, we have been comparing quantities using ratios and proportions. It is also possible to make comparisons using percents. Ratios can be used to compare a part to the whole or to compare one part to another part. A percent is a ratio of a part to the whole. For example, in a class of 18 girls and 12 boys, the ratio of girls to total students is $18: 30$, or equivalently, $\frac{18 \text { girls }}{30 \text { students }}$. This ratio or rate is equivalent to the fraction $\frac{3}{5}$, the decimal 0.60 , and represents $60 \%$. The whole is the original amount, called its base. In this case, the base is the class size 30 students. Likewise, the rate $\frac{12 \text { boys }}{30 \text { students }}$ is equivalent to the fraction $\frac{2}{5}$, and the decimal .4 , and represents $40 \%$. In each case, we are comparing a part of the class to the whole class. We say that $60 \%$ of the class is female, which is equivalent to the statement that $\frac{3}{5}$ of the class is female. From Chapter 9 , we know that to compute $\frac{3}{5}$ of 30 is a multiplication problem:

$$
\left(\frac{3}{5}\right)(30)=18 .
$$

We can compute the number of girls by using the decimal form of $\frac{3}{5}$ :

$$
(0.60)(30)=18 .
$$

We say that " $60 \%$ of the class is female," and that $60 \%$ of 30 is 18 .

## PROBLEM 1

Joe and Moe went to the pizza arcade. Their parents bought 75 tickets for them to spend. Joe used 30 tickets while his brother used the rest.
a. What percent of the tickets did Joe use?
b. What percent of the tickets did Moe use?
c. How could you solve for these problems using proportions?

## PERCENT INCREASE AND PERCENT DECREASE

Percents can also be used to express the change between two quantities. Percent change is the ratio of the amount of change compared to the original quantity.

$$
\text { Percent change }=\frac{\text { amount of change }}{\text { original amount }}
$$

## EXAMPLE 1

A store receives a shipment of summer shirts. Each shirt costs the store $\$ 80$ wholesale. They sell the shirts for \$104. What was the percent markup, or percent increase?

## SOLUTION

Amount of change $=$ new amount - original amount $=\$ 104-\$ 80=\$ 24$
Original amount $=\$ 80$
Percent change (increase) $=\frac{24}{80}=0.3=30 \%$

## EXAMPLE 2

At the end of summer, a store manager decides to mark down the price of a dress from $\$ 120$ to $\$ 84$. What was the markdown, or percent decrease?

## SOLUTION

Amount of change $=\$ 120-\$ 84=\$ 36$
Original amount $=\$ 120$
Percent change (decrease) $=\frac{36}{120}=0.3=30 \%$

PROBLEM 2
$\frac{32}{640}=0.05=5 \%$; Percent Increase

## PROBLEM 3

$\frac{5}{45}=0.1111 \ldots$ So, approximately $11.1 \%$; Percent Decrease

## PROBLEM 2

Molly Drilling Company employed 640 people. When the oil boom began, their workforce grew to 672 people. What was the percent change in the number of employees? Was this a percent increase or decrease?

## PROBLEM 3

Briley Machine Shop builds oil field drill bits. They had 45 bits in stock. Three weeks later, their stock contained 40 drill bits. What was the percent change? Was this a percent increase or decrease?

Sometimes in a problem, we are given the percent that the part is of the whole and the amount of the whole. We can use the percent to compute the amount of the part.

## EXAMPLE 3

Suppose you know that a class of 30 students conducted a survey about pets and $80 \%$ of the students in the class had a dog. How do you calculate the number of students in this class that have a dog?

## SOLUTION

We shall solve this problem two ways.
Method 1: The first method uses proportions. First, we define a variable $x$.

$$
\text { Let } x=\text { the number of students in class that have a dog. }
$$

The ratio of students with dogs to total students can be written two ways:
$\frac{80}{100}=\frac{4}{5}$ and $\frac{x}{30}$. Thus, we can set up the proportion:

$$
\frac{4}{5}=\frac{x}{30}
$$

Discuss the similarity of these two approaches.

## PROBLEM 4

Method 1: Let $\mathrm{x}=$ \# of students participating in athletics. So, $\frac{4}{10}=\frac{x}{35}$ and $\left(\frac{4}{10}\right) 35=x=14$. Method 2 : $40 \%=0.40=$ percent of 35 students participating in athletics. So, ( 0.40 )(35) $=14=$ number of students participating in athletics.
Reflect on the advantage of each method.

Multiplying both sides by 30, we get

$$
\frac{4}{5}(30)=x . \text { Thus, } x=24
$$

Method 2: Another way to think of this question is to ask: "What is $80 \%$ of 30 ?" If we used the fractional equivalent of $80 \%, \frac{80}{100}=\frac{4}{5}$, then the question becomes: "What is $\frac{4}{5}$ of 30 ?" From Chapter 9, we know this means that we use multiplication to compute the number of students who have a dog.

$$
\frac{4}{5}(30)=24=\text { number of students with a dog }
$$

Thus, computing the percent of a number involves multiplication of the percent (in its decimal form) by the quantity that answers the question: " $80 \%$ of what?" The answer is $80 \%$ of 30 , which is equivalent to the product $(.80)(30)=24$. Note that 30 students is the base of $80 \%$. We always follow the computation of a percent times its base with a description that explains what the product represents. This written description is very helpful to remember what the product means.

## PROBLEM 4

If $40 \%$ of a group of 35 students participate in athletics, how many of these 35 students participate in athletics? Solve this using both methods.

## EXAMPLE 4

A region averages 28 inches of rain each year. Last year, there was a 30\% increase above the average rain. How much rain did the region receive?

## SOLUTION

Method 1: We let $x=$ rainfall last year. The normal rainfall is $100 \%$ and the increase in rainfall last year was 30\%, the total rainfall last year was 130\%. Thus, the ratio of $\frac{130 \%}{100 \%}$ is the ratio of the amount of rainfall last year compared to the average rainfall. Thus, we can write the proportion:

$$
\frac{130}{100}=\frac{x}{28}
$$

Rewriting this equation, we get $(1.3)=\frac{x}{28}$. Solving this equation, we get

$$
(1.3)(28)=x
$$

Thus, $x=36.4$ inches.

## PROBLEM 5

Method 1: Let $\mathrm{x}=$ total bill. So, $\frac{115}{100}=\frac{\mathrm{x}}{22}$, and $(1.15)(22)=25.30$.
Method $2:(0.15)(22)=3.30=$ amount of the tip. So, the total bill is the sum $\$ 22.00+\$ 3.30=\$ 25.30$.

## EXAMPLE 5

Method 1: Let $\mathrm{x}=$ total bill. So, $\frac{108}{100}=\frac{\mathrm{x}}{12}$, and $\mathrm{x}=(1.08)(12)=12.96$.
Method 2: $(0.08)(12)=0.96=$ amount of the tax. So, the total bill is the sum $\$ 12.00+\$ 0.96=\$ 12.96$.

## EXPLORATION 1

a. $\quad(0.30)(100)=\$ 30=$ dollar of mark up
$\$ 130=100+30=$ wholesale price + markup $=$ selling price
b. $\quad(130)(0.30)=\$ 39=$ amount of discount
$130-39$ = selling price - discount $=$ sale price
$\$ 91$ = sale price
100-91 = \$9 = amount of loss

Method 2: We compute $30 \%$ of 28 to get $(0.30)(28)=8.4=$ amount of increase of rain.

So, the total rainfall in this region last year was $28+8.4=36.4$ inches of rain. Notice the importance of writing a description of what $(0.30)(28)=8.4$ represents.

## PROBLEM 5

Jack and Jill ate at the Hill Restaurant and their bill was $\$ 22.00$. If they include a $15 \%$ tip for the waiter, how much is their payment?

Another example of percent increase is computing sales tax to include in the overall price of an item.

## EXAMPLE 5

Vera buys a CD for $\$ 12.00$. The sales tax rate is $8 \%$. How much tax does she pay? What is her final bill?

## SOLUTION

The tax paid is $8 \%$ of $\$ 12.00$, or $(12.00)(0.08)=\$ 0.96$. So her final bill is

$$
\$ 12.00+\$ 0.96=\$ 12.96 .
$$

You could also compute the final bill directly as $(12.00)(1.08)=\$ 12.96$. Explain why.

## EXPLORATION 1

A store receives a shipment of summer shirts. Each shirt costs the store $\$ 100$ wholesale. The shirts are marked up $30 \%$.
a. What is the selling price for each shirt?
b. At the end of the summer, the store puts the shirts on sale at a $30 \%$ discount from the selling price. What is the discounted price for each shirt? Does the store make a profit, suffer a loss, or break even on the discounted shirts?

To compute the selling price in the Exploration, the store must mark up the price to the selling price. This requires that you take $30 \%$ of $\$ 100$ to compute the amount of markup and then add this to the wholesale price to get the selling price. Other examples of this kind of percentage increase can be seen in adding a tip to a bill at a restaurant or in computing the cost of an item at a store with the tax included. The discounted price in part b of the Exploration is an example of a percentage decrease. You multiply the percent of discount $30 \%$ by the selling price and then subtract this amount from the selling price to find the discounted price.

## EXAMPLE 6

Suppose $80 \%$ of the students in a class have backpacks and we know that 20 students have backpacks. How many students are in the class?

## SOLUTION

In each problem involving a percent, we have identified the "base" of the percent and then multiplied the percent times this base to get a useful quantity. What is the base of $80 \%$ in this problem? We need to compute $80 \%$ of the number of students in the whole class. But this is the number that we want to compute. We let $x$ represent the number of students in the whole class:

$$
\text { Let } x=\text { the number of students in the whole class. }
$$

Method 1: The number $x$ represents $100 \%$ of the class. The ratio of students with backpacks to the whole class is $\frac{20}{x}$. This is equivalent to the ratio $\frac{80 \%}{100 \%}$ . Thus, we set up the equation:

$$
\frac{80}{100}=\frac{20}{x} \quad \text { or equivalently } \frac{4}{5}=\frac{20}{x} .
$$

Solving this equation, we multiply each side by $5 x$ to get

$$
4 x=100 . \text { Thus, } x=25 \text {. }
$$

Method 2: The base of $80 \%$ is the number of students in the whole class, $x$. Therefore, we compute $80 \%$ of $x$ and write a description of what this product means:

$$
(0.80)(x)=0.8 x=\text { the number of students with backpacks. }
$$

a. $(100)(0.2)=20$
(80) $(0.4)=32$
Final price $=80-32=48$
b. $(100)(0.4)=40$
$(60)(0.2)=12$
so, $100-40=60$ final sale price

## PROBLEM 7

Let $x=$ wholesale price.
Method 1: Set up the proportion of $\frac{140 \%}{100 \%}$ is equivalent to $\frac{56}{x}$. So, $1.4=\frac{56}{x}$ and $x=40$.
Method 2: Multiplying $40 \%$ of the whole, we get $0.4 x=$ amount of markup. Thus, $x+0.4 x=$ wholesale price + mark-up $=$ selling price. So, $1.4 x=56$ and $x=40$.

We know that 20 students have backpacks so we can make an equation: $0.8 x=$ 20.

In solving this equation, we divide both sides by 0.8 to get:

$$
\frac{0.8 x}{0.8}=\frac{20}{0.8} .
$$

Thus, $x=25$. We have calculated that the whole class must have 25 students and that $80 \%$ of these, 20 students, have backpacks.

## PROBLEM 6

a. Store A has a $\$ 100$ skirt on sale for $20 \%$ off. At the end of the summer, it offers an additional $40 \%$ off then reduced price. What is the final sale price?
b. Suppose Store B has a $\$ 100$ skirt on sale for $40 \%$ off. At the end of the summer, it offers an additional $20 \%$ off the first sale price. What is the final sale price for this skirt?
c. Compare the answers in parts $a$ and $b$.

## PROBLEM 7

A store manager marks a cell phone $40 \%$ above the wholesale cost to the selling price of $\$ 56$. What was the original wholesale price of the cell phone?

## exploration 2: Sales, Rebates, and Coupons

Stores are eager to promote new business and offer various ways to encourage new customers to shop with them. Some of these ways include special sales, coupons, and rebates. In a sale, the store reduces the selling price, usually by some percentage discount. A coupon could either provide a percentage discount or a fixed amount off. A rebate is money that the customer receives, typically from the manufacturer, after making the purchase. Generally, rebates require the customer's further action.

Stores are eager to get your business and offer ways to encourage you to shop with them. An item may have a certain price tag but if it is on sale, a newspaper has a special coupon, or there is a rebate offer, what is the actual cost to you the customer?

For example, here are three stores with their various promotions:

- Pascal Shop offers $15 \%$ off on all the items in its store.
- Euclidmart has a $5 \%$ store coupon in the newspaper that can be used on all clearance where you can take $10 \%$ off the tag price.
- Fibonacci 's Department Store is giving a rebate of $\$ 15$ on certain items.


## PROBLEM 8

Suppose Ruth is considering buying four items available at all three stores for $\$ 15$, $\$ 45, \$ 100$, and $\$ 200$.

Use the table below to compare the actual dollar amount that Ruth will pay at each of the stores with their promotions. Assume that the coupon can be used with any purchase from Euclid Mart, and that each of the items at Fibonacci's has a rebate

| Cost of Item | $\$ 15$ | $\$ 45$ | $\$ 100$ | $\$ 200$ |
| :--- | :--- | :--- | :--- | :--- |
| Pascal's Shop <br> $15 \%$ off Sale |  |  |  |  |
| Euclidmart 5\% off <br> in addition 10\% <br> off |  |  |  |  |
| Fibonacci's <br> Department Store <br> $\$ 15$ rebate |  |  |  |  |

Analyze each of the monetary incentives-sales, rebates, and coupons. Discuss which offer is the best for each item and why. Should Ruth shop at just one store for all of her items? Generally, how much does the cost of the item affect your choice of store?

What is the advantage of a sale or coupon to a rebate? Why might some stores offer rebates and not sales?

## EXAMPLE 7

Henry's Clothing store is running a sale of $60 \%$ off of the list price. Marvin's Clothing store has already reduced the price by $25 \%$, and has a new sale offering $40 \%$ off of the marked price. If the original price of a shirt is $\$ 75$ at both stores,
a. How much is the final discount at each store?
b. Which store has the lower price, or are the prices be the same? Explain

## SOLUTION

a. At Henry's, the discount is $60 \%$ of $\$ 75,(0.60) 75=\$ 45$.

At Marvin's, the original discount is $25 \%$ of $75,(0.25) 75=\$ 18.75$. So the original sale price is $75-18.75=\$ 56.25$. Marvin's second discount is $40 \%$ from the sale's price, so the additional discount is ( 0.40 ) $56.25=$ $\$ 22.50$. The total discount taken is $18.75+22.50=\$ 41.25$.
b. Taking $60 \%$ off the original price leads to a final sales price of $\$ 30$ at Henry's.
Taking a discount of $25 \%$, followed by an additional discount of $40 \%$ leads to a final sales price of $\$ 33.75$. It is better to take $60 \%$ off of the original price-the shirt will be cheaper at Henry's.
Now explore and compare different types of promotions that a store might run.

## PROBLEM 9

Fermat Market offers a rebate of $\$ 50$ for each of your purchases while Euclidmart has $10 \%$ off sale for each item purchased. Use the table below to determine what is the actual price you will pay for each item with the indicated price tag.

| Item's price tag | Fermat Market | Euclidmart |
| :--- | :--- | :--- |
| $\$ 100$ |  |  |
| $\$ 200$ |  |  |
| $\$ 400$ |  |  |
| $\$ 500$ |  |  |
| $\$ 600$ |  |  |

1. Discuss which store provides a better savings. Explain why. What is your strategy for where you would shop and why?

## EXERCISES

Encourage your students either to draw a picture of the situation representing the situation as a way to working the exercises. Have them use both methods in solving the problems.
5. a. The discount is $20 \%$ off of the original price, $20 \%$ of $\$ 200$ is $\$ 40$.
b. The sale price is the original price minus the discount, $\$ 200-\$ 40=\$ 160$.
c. The $\frac{\text { Sale Price }}{\text { Original Price }}$ is $\frac{\$ 160}{\$ 200}$ or $80 \%$.
2. Both stores have an item that is listed at the price of $\$ 480$. If the stores provide the incentives mentioned above, which store would you go in order to get the most in savings? Explain why.
3. Both stores have an item that is listed at the price of $\$ 563$ ? If the stores provide the incentives mentioned above, which store would you go in order to get the most in savings? Explain why.

## EXERCISES

1. A pet store owner had 45 parrots in her store. She received 9 more parrots in a shipment. What was the percent of increase in the number of parrots? 20\%
2. A shirt was selling regularly for $\$ 40$. The store manager put up a sign offering a discount of $\$ 5$. What was the percent of discount? 12.5\%
3. The wholesale price of a refrigerator is $\$ 480$. If the store marked the selling price as $\$ 590$, what was the percent of markup? 22.9\%
4. Kendall has a collection of pets. She has 4 dogs and 9 fish. What percent of animals are dogs? What percent are fish? 30.8\%; 69.2\%
5. A department store has a dress that is originally priced at $\$ 200$. A sale offers a discount of $20 \%$ off the original price.
a. How much is the discount in dollars?
b. What is the sale price?
c. What percentage of the original price does the sale price represent?
6. A food store offers a $\$ 5$ rebate on any purchase over $\$ 10$. It also gives a $25 \%$ rebate off the sales price for any purchase. Unfortunately, you may use only one of the offers. How much is each rebate for
7. Teachers, make sure all of your students understand the relationship between wholesale and retail and the vocabulary involved.
The retail price for the printer is $\$ 180$ plus $28 \%$ of $\$ 180$, a total of $\$ 230.40$.
8. Ask if they can compute them mentally. Do they see connections to fractions?
a. 7
g. $\quad 11$
m. 79
s. $x=84$
b. 6.4
h. 44
ก. 8
c. 26.3
i.
12
9. 42
t. $A=80$
d. 2.54
j.
6
p. 12
e. 12
k. $\quad 3$
q. 8
f. 18
I. 6
r. 15
10. a. 80 of the 200 marbles are red.
b. 104 of the 260 marbles are red.
c. There are a total of 160 marbles.
11. Amy made 63 of the 70 free throws attempted. Students can draw an area or linear model to represent this problem.
12. Of the 100 gallons, only 30 gallons are pure alcohol and 70 gallons are pure water.
a. A $\$ 10$ purchase?
b. A $\$ 25$ purchase?
c. Is there a cost that would give the customer the same rebate with either offer? What is that cost?
13. Connie's grocery bill totals $\$ 125.24$. However, she has three coupons, one for $\$ 1.50$, one for $\$ 2.00$, and one for a free $\$ 12$ turkey for a $\$ 19.50$ purchase she made.
a. What is the total of all her coupons?
b. How much will her grocery bill be when she subtracts the coupons?
c. What percent discount equals the coupons' savings?
14. A printer costs a store $\$ 180$ wholesale. The store's retail markup is $28 \%$. What is the retail price for this printer?
15. Compute.
a. $10 \%$ of 70
b. $10 \%$ of 64
c. $10 \%$ of 263
d. $10 \%$ of 25.4
e. $50 \%$ of 24
f. $75 \%$ of 24
g. $20 \%$ of 55
h. $80 \%$ of 55
i. $12.5 \%$ of 16
j. $37.5 \%$ of 16
k. $33 \frac{1}{3} \%$ of 9
l. $66 \frac{2}{3} \%$ of 9
m. $79 \%$ of 100
n. $16 \%$ of 50
o. $21 \%$ of 200
p. $8 \%$ of 150
q. $40 \%$ of what is 20
r. $20 \%$ of what is 3
s. $50 \%$ of $x$ is 42 . What is $x$ ?
t. $35 \%$ of A is 28 . What is A ?
16. A bag of red and blue marbles contains $40 \%$ red marbles.
a. If there are 200 marbles total, how many of them are red?
b. If there are 260 marbles in all, how many of them are red?
c. If there are 64 red marbles, how many marbles are in the bag?
17. Amy plays on the school basketball team. She shot a total of 70 free throws this season and made $90 \%$ of them. How many free throws did she make? Draw a picture that represents the quantities in this problem.
18. A lab has an alcohol solution (alcohol mixed with water) of 100 gallons. This solution contains $30 \%$ alcohol. How much of the solution, in gallons, is pure alcohol? How much of the solution, in gallons, is pure water? Draw a picture of the solution, showing the amount of alcohol and water present.
19. The milk contains $\frac{3}{5}$ or 0.6 gallons of milk fat or $(0.20)(30)=0.6$ gallons of milk fat.
20. $(\$ 45.60)(0.15)=\$ 6.84$. So, $\$ 36.50+\$ 6.84=\$ 52.44$
21. $(\$ 36.50)(0.08)=\$ 2.92 . \mathrm{So}, \$ 36.50+\$ 2.92=\$ 39.4$
22. a. $\$ 2.18$
b. $\$ 83 \quad \frac{t}{p}=\frac{0.08}{1.00}=\frac{6.64}{p}$
c. $\$ 11.40 \quad \mathrm{p} \quad 1.00 \quad \mathrm{p}+0.08 p=153.90 \quad 1.08 p=153.90 \quad 0.08 p$
d. $\$ 78.75$
23. Let $\mathrm{x}=$ total marbles. Method $1: \frac{40}{100}=\frac{30}{\mathrm{x}}$. Solving proportional equation, we get $\mathrm{x}=75$.

Method 2: .40x = \# marbles lost = 30. So, $x=75$ marbles.
16. Let $\mathrm{x}=$ original weight. So, $.08 \mathrm{x}=12$, or $\mathrm{x}=150$ pounds.
17. You can visually solve this problem using area model. A rectangle divided into 5 equal parts, each part representing $\frac{1}{5}$ or $20 \%$ of the original price. The 4 parts left represent the sale price, $\$ 40$. You can answer both questions from these pictures. Alternate: Let x be the original price, $0.20 \mathrm{x}=$ amount of discount. So, $\mathrm{x}-0.20 \mathrm{x}=$ $0.8 x=$ original price - amt. of discount $=$ sale price $=\$ 40$. Solve the equation $0.8 x=40$ by dividing $0.8 x$ and 40 by 0.8 . This shows that the mp3 player has an original price of $\$ 50$. The amount of discount was $\$ 10$.
18. Let $x=$ original price. $(0.20) x=$ amount of discount. So $x-0.20 x=$ sale price.
$0.8 x=50 . S 0,50 \div 0.8=\$ 62.50$ and the original price is $\$ 62.50$.
19. The percent markup is $35 \%$. The difference between the retail price and the wholesale price is $\$ 126$ which is $35 \%$ of the wholesale price of $\$ 360$.
13. The cafeteria has 30 gallons of milk that contains $2 \%$ fat. How much fat does the 30 gallons of milk contain?
14. A family ate out after a soccer game. Their bill was $\$ 45.60$. If they included a $15 \%$ tip, what was their final bill? How much tip did they give the waitress?
15. Amanda bought a new blouse for $\$ 36.50$. If the sales tax is $8 \%$, how much was her final bill? How much tax did she pay?
16. The tax rate in Gainesville is $8 \%$.
a. What is the tax on a $\$ 27.25$ purchase?
b. If the tax on an item was $\$ 6.64$, how much did the item cost?
c. What is the amount of tax paid if the total cost is $\$ 153.90$ ?
d. How much does the item cost if the total cost is $\$ 85.05$ ?
17. Juan lost $40 \%$ of his marbles. If he lost 30 marbles, how many marbles did he have originally? Use both methods.
18. Bill lost 12 pounds on his diet. If he lost $8 \%$ of his weight during this diet, what was his original weight?
19. An mp3 player is on sale for $\$ 40$ after a $20 \%$ discount. What was the original price? What was the amount of the discount? Show how a picture can help solve this problem.
20. The discount during a summer clearance sale is $20 \%$ and the sale price is $\$ 50$. What was the original price?
21. A computer costs a store $\$ 360$ wholesale and is sold for $\$ 486$ retail. What is the percent markup?
22. Sunshine Department Store offers a rebate of $\$ 20$ for one purchase over $\$ 25$. Joe's Department Store has a $25 \%$ off sale for any one item. Sam's Department Store has a $\$ 10$ coupon for each item purchased over $\$ 50$. Use the table below to determine what is the actual price paid for each item with the indicated price tag. Assume the cost of sending in the paperwork for a rebate is $\$ 1$.
20. $(\$ 14.95)(0.20)=\$ 2.99$ and so $\$ 14.95+\$ 2.99=\$ 17.94$
21. $(\$ 500)(0.09)=\$ 45$ and so $\$ 500+\$ 45=\$ 545$
22. Let $\mathrm{x}=$ wholesale price. So, $.45 \mathrm{x}=$ mark-up and so $\mathrm{x}+0.45 \mathrm{x}=1.45 \mathrm{x}=$ wholesale price + mark-up $=$ selling price. So, $1.45 \mathrm{x}=92.80$ and $\mathrm{x}=64$.
23. Have students work with the original price of $\$ 100$ and experiment with doing the markup and discount.

If we let OP be the original price, then we can ask students - what does a markup do to the original price? What will the discount do to the new markup price?
The markup will add $10 \%$ of the original price (or $\frac{1}{10}$ of the original price) and thus the marked up price is $\mathrm{OP}+\frac{1}{10} \mathrm{OP}$ or $110 \%$ of OP .
The discount will take away $10 \%$ of the new markup price of $\mathrm{OP}+\frac{1}{10} \mathrm{OP}$ or $110 \% \mathrm{OP}$
$\left(O P+\frac{1}{10} O P\right)-\frac{1}{10}\left(O P+\frac{1}{10} O P\right)=O P+\frac{1}{10} O P-\frac{1}{10} O P-\frac{1}{100} O P=O P-\frac{1}{100} O P$. which is not equal to the original price.
24.
(54) $(0.10)=5.40$
$60-6=54$
First sale price
Final sale price

Note: The answer is not $20 \%$ off the original price
25. Encourage your students to draw a visual model as a possible means to solve this problem. Also encourage the use of the model to help write an equation to solve the problem.

Allison had 60 dolls before she bought the new dolls. An equation students could use to model the problem is $96=160 \%$ of x where x is the number of dolls Allison had in her original doll collection.
26. Jameer's average was 25 points.
27. He had 120 cards.

Let $\mathrm{x}=$ total $\#$ of cards. $(0.70) \mathrm{x}=$ cards he sold. $\mathrm{x}-0.7 \mathrm{x}=\#$ of cards left $=0.3 \mathrm{x}$.
So, $0.3 x=36$. Therefore, $x=36 \div 0.3=120$ cards.
28. Janice previously had 80 marbles.

|  | Sunshine | Joe's | Sam's |
| :--- | :--- | :--- | :--- |
| $\$ 25$ |  |  |  |
| $\$ 50$ |  |  |  |
| $\$ 100$ |  |  |  |
| $\$ 200$ |  |  |  |
| $\$ 400$ |  |  |  |

How do these different promotions compare? Explain how to spend the least amount of money.
23. Mr. Robinson ate supper at a locally owned restaurant. His meal cost $\$ 14.95$ and he included a $20 \%$ tip. How much was his final bill?
24. Ms. Puente bought a new TV for $\$ 500$. How much sales tax did she pay if the sales tax in her town is $9 \%$ ? What was her final bill? What would her final bill have been if the sales tax was $8 \frac{1}{4} \%$ ?
25. A pair of jeans was marked-up $45 \%$ to a selling price of $\$ 92.80$. What was the original wholesale cost of this pair of jeans?
26. A clothing store marks a dress up $10 \%$ right before a $10 \%$ discount sale. Does the dress cost the same on sale as it did before the markup? Explain.
27. Jan finds a $\$ 60$ dress on sale for $10 \%$ off. She offers to buy the dress it the store manager gives her another $10 \%$ off the sale price. What would be her new sale price?
28. Allison bought some new dolls to add to her collection. She increased her collection by $60 \%$ and has 96 dolls now. How many dolls did she have before she bought the new dolls?
29. Jameer scored $36 \%$ more points than his average basketball score this past game. He finished the game with 34 points. What was Jameer's average before the game?
30. Toby sold $70 \%$ of his baseball cards and was left with 36 cards. How many cards did he have in his collection before the sale?
31. Janice lost $35 \%$ of her marbles. She has 52 marbles now. How many marbles did she originally?
29. \$7,200
30. \$306
31. a. The 48 -gallon rain barrel contains more water. We say 2 containers have the same fullness if the percent the water is of the capacity of each is the same. The barrel is $50 \%$ full and the jug is $60 \%$ full, so the jug is more full than the barrel.
b. The barrel still contains more water, even though we are now comparing 2 gallons in the jug versus 23 gallons in the barrel. But, in terms of fullness we are now comparing $\frac{2}{5}$ to $\frac{23}{48}$, which of these is greater? Draining 1 gallon from the jug has a bigger effect because 1 gallon is a larger portion of the jug's capacity than of the barrel's capacity.
c. Since the 48 -gallon barrel is only $50 \%$ full, how many gallons does the 5 -gallon jug need to be only $50 \%$ full? $50 \%$ of 5 gallons is 2.5 gallons.
d. If we let $x=$ the amount of rain needed to make the barrel as full as the jug
$\frac{3}{5}=0.60=60 \%=$ the percent of the jug that is water.
$\frac{24+x}{48}=$ the fraction of the barrel that is water if we add $x$ gallons of rain.
These must be the same if the two containers have the same percent of water.
Thus, $0.60=\frac{24+x}{48}=(24+x) \div 48$. Using the missing factor model for multiplication, we get:
$(0.60)(48)=(24+x)$ and $28.8=24+x$. Thus, $4.8=x$.
Another approach students could take is to say that the jug is only $60 \%$ full and after calculating what $60 \%$ of 48 gallons is they can see that $60 \%$ of 48 gallons is 28.8 gallons, there are only 24 gallons in the barrel and we need 4.8 more gallons. Students can discuss how these approaches are different and how they are the same.
32. Teachers, make sure your students understand that a 20\% alcohol solution means it's 20\% alcohol and 80\% water.
a. Using graph paper and the area model, let 10 squares equal a gallon, so 3 gallons $=30$ squares and 6 gallons $=60$ squares. Shade the part that is pure alcohol in each.
b.

| Tub | \# of gallons | \% alcohol |
| :--- | :--- | :--- |
| $20 \%$ Solution | 3 | $(0.2)(3)=0.6$ |
| $40 \%$ Solution | 6 | $(0.4)(6)=2.4$ |
| Resulting Solution | 9 | 3 |

c. 3 gallons
d. 9 gallons
$\begin{array}{ll}\text { e. } & \frac{1}{3} \\ \text { f. } & 33 \frac{1}{3} \%\end{array}$
32. Kirsten is a real estate agent and earns $4 \%$ commission for each property she sells. If she sells a house for a client for $\$ 180,000$, how much money will she earn?
33. Ian put $\$ 300$ into a savings account at his bank. If he leaves the money in the bank for an entire year he will earn $2 \%$ interest. How much money will he have in the account in total at the end of one year?

## 34. Ingenuity:

The Guerrero family has a 48 -gallon rain barrel that contains 24 gallons of water and a 5 -gallon water jug that contains 3 gallons of water.
a. Which container has more water? Which container is fuller?
b. If we drain a gallon of water from each, does this change the second answer to the previous problem? Explain your reasoning.
c. How many gallons of water should the Guerreros have in the 5 -gallon jug to make it as full as the 24 gallons in the 48 gallon barrel?
d. How many more gallons of rain do the Guerreros need to catch in the barrel in order to for it to be as full as the jug is?

## 35. Investigation:

Ms. Campos, the science teacher, mixed 3 gallons of a 20\% percent alcohol solution with 6 gallons of a $40 \%$ alcohol solution.
a. Draw a picture of the mixtures showing the alcohol amount and the water amount.
b. Make a table or a diagram that indicates the number of gallons of the different solutions, the number of gallons of the alcohol, and the corresponding percentage concentrations. Notice that there are three solutions: the $20 \%$ solution, the $40 \%$ solution and the resulting solution.
c. How many gallons of alcohol are in the resulting solution?
d. How many gallons of solution are there total?
e. What fraction of the resulting solution consists of alcohol?
f. What percent alcohol is the resulting solution?

## Section 10.5 - Scaling

## Big Idea:

Using rates and ratios to scale dimensions and geometric figures

## Key Objectives:

- Understand the relationship between scaling dimensions and areas of geometric figures.
- Recognize, compute and work with scale factors.


## Materials:

Grid paper, "Sorting Rectangles" sheet from the CD, Table for Exploration 3 from the CD, 2-Column Tables from the CD, Measuring tape or yardsticks for the Investigation

## Pedagogical/Orchestration:

This section introduces the concept of scaling geometrically. It lays the groundwork for the relationship between a change in dimension, a change in linear measure and a change in area. It also introduces the concept of scale factor and similarity based on scale.

## Activities:

"Scaling Activity" at the end of the section on CD
"Sorting Rectangles" activity in Exploration 1 has handout of rectangles on CD.

## Vocabulary:

scale factor

## TEKS:

$6.3(B)(C) ; \quad 6.4(A) ; \quad 7.3(B) ; \quad 7.4(A)(B)(C) ; \quad 7.9(A) ; \quad 7.13(A)(C)(D) ; \quad 7.14(A)(B) ; \quad 7.15(A)(B) ; \quad 8.3(A) ;$
8.4(A); 8.5(B); 8.10(A); 8.14(A,B); 8.15(A); 8.16(A)

## WARM-UPS for Section 10.5

1. A group of students are surveyed about two sports. A total of 25 of them liked football, 40 of them liked soccer, 15 of them liked both and 20 students did not like either. How many students were surveyed?
a. 70 students
b. 80 students
c. 90 students
d. 100 students

Ans: (a) They can draw a Venn diagram: 10 football only +25 soccer only +15 both +20 neither $=70$
2. The price of a stock on Wall Street started at $\$ 100$ per share and dropped in value by $10 \%$ on Tuesday. The next day the stock increased in value by $10 \%$. What was the value of the stock at the end of Wednesday? Explain.
a. $\$ 99$
b. \$100
c. \$101
d. \$81

Ans: (a) because $10 \%$ of $\$ 100$ is $\$ 10$ and so the beginning price on Wednesday is $\$ 90.10 \%$ of $\$ 90$ is $\$ 9$ and so the value at the end of Wednesday is $\$ 90+\$ 9=\$ 99$.

## Launch for Section 10.5:

Exploration 1 in Section 10.5 is a great way to launch this lesson. Give students copies of the rectangles on that page asking them to cut and sort the rectangles in any way that seems natural. Discuss with the students how they decided to sort them. Make sure to ask Question 2 in the exploration: "What are the attributes of a rectangle? Which did you use to sort?" Students may have sorted by size or by shape. Even though every figure is a rectangle, lead them to discover and discuss what it means when we say that two rectangles have the same shape. Do not talk about scale factor at this time. If you have an overhead projector, it is helpful to have one rectangle drawn on the board and project on the board another rectangle that is smaller than the one drawn on the board. Ask the students if the rectangles are the same shape and then back up the overhead projector making the projected image larger until it is the same size and shape as the drawn rectangle. You can do the same thing with another rectangle that when enlarged will not be able to exactly cover the drawn rectangle (non-similar). Do not bring up vocabulary at this time as it will be developed in Chapter 11. Tell your students that today they will be comparing rectangles in order to answer the question, "What makes one rectangle look like it has the same shape as another rectangle?" Then proceed to Exploration 2.

Pre-make sets of rectangles from Rectangles for Exploration 1 Activity at end of section. If possible, laminate for classroom set use. Alternatively, assign as homework at the end of Section 10.3 so classtime is not used to cut these out.

## EXPLORATION 1: SORTING RECTANGLES

Hand out grid paper with rectangles from CD.
Your students might sort by size or by shape. Even though every figure is a rectangle, lead them to discover and discuss what it means when we say that two rectangles have the same shape. It might be necessary to make even more rectangles as examples. Do not talk about scaling factor yet.

## SECTION 10.5 SCALING

So far in this chapter you examined rates and ratios, two of the most important ways to relate two quantities. For example, relating the number of miles to number of hours is called a rate of miles per hour. Other examples of rates and ratios included feet per yard, miles per hour, and even the number of Texas students to the number of international students. In this section, you will explore relationships that occur in geometry by observing relationships involving rectangles.

## EXPLORATION 1: SORTING RECTANGLES

1. On a printed copy of the shaded rectangles below, cut and sort these rectangles in any way that seems natural. Discuss how you decided to sort them.


Did your students look at length, width, perimeter, or area? If not, it would be useful to bring these attributes into their discussion. Recall in Section 6.4, students explored what "biggest" meant among a set of rectangles.

## EXPLORATION 2: ATTENDING TO SHAPES AND SIZES

Hand out grid paper with rectangles from CD.

1. B, C, E, and G. Have students discuss their answers in small groups and come to a consensus. Then have groups report their findings. These answers are for teachers' convenience, teachers are to facilitate discussion to these ideas throughout this exploration.
2. Your students should notice that the dimensions of the starred rectangles are multiples of the dimensions of $Z$. This is the time for the groups to self correct if they selected some incorrect rectangles or left some out. The rectangles that are the same shape are B, C, E, G.
3. What are the attributes of a rectangle? Which did you use to sort? Explain.

## EXPLORATION 2: ATTENDING TO SHAPES AND SIZES

Now consider the following rectangles from a printed copy:


1. Which rectangles have exactly the same shape as rectangle $Z$ ? Explain your answer.
2. Copy and complete the table. What do you notice? What do rectangles B, C, E, and G have in common?

| Rectangle | Length | Width |
| :---: | :---: | :---: |
| $Z$ | 3 | 2 |
| $A$ |  |  |
| $B$ |  |  |
| $C$ |  |  |
| $D$ |  |  |
| $E$ |  |  |
| $F$ |  |  |
| $G$ |  |  |

TEKS refer to scale factor $k$ as the "constant rate of proportionality".
4. Teachers, expect students to divide length of $B$ by length of $Z$, or answer the question: 3 times what $=6$ ? or solve the equation: $3 x=6$.
5. No, each of these rectangles do not have length and width that are multiples of 3 and 2 .

If $s$ is greater than 1 then the scale factor enlarges or stretches the figure. If the scale factor is less than 1 the scale factor will shrink the figure.

Problem 1:
Scale factor from $P$ to $R$ is 2 .
$R$ to $T$ is 2 .
T to T is 1 .
$R$ to $P$ is $\frac{1}{2}$
$P$ to $T$ is 4.
$T$ to $P$ is $\frac{1}{4}$

You probably discovered that the dimensions of rectangles B, C, E, and G are multiples of the dimensions of rectangle $Z$.

| Rectangle | Length | Width |
| :---: | :---: | :---: |
| $Z$ | 3 | 2 |
| $B$ | 3.2 | 2.2 |
| $C$ | 3.3 | 2.3 |
| $E$ | 3.4 | 2.4 |
| $G$ | 3.5 | 2.5 |

The bold numbers for each rectangle are called the scale factors, or the constant rate of proportionality, of the dimensions of the original rectangle $Z$.
4. For each rectangle, how can you find the scale factor that relates it to $Z$ ?
5. Do rectangles $A, D$, and $F$ have scale factors that relate to $Z$ ? Explain.

## DEFINITION 10.2: SCALE FACTOR

If rectangle $A$ has dimensions $b$ and $h$, two positive numbers, and rectangle $B$ has dimensions $k b$ and $k h$, where $k$ is also a positive number, then $k$ is the scale factor from $A$ to $B$. It is also called the constant rate of proportionality.

## PROBLEM 1

Suppose we start with the rectangle $P$ below. What is the scale factor from $P$ to $R$ ? From $R$ to $T$ ? From $T$ to $T$ ? From $R$ to $P$ ? From $P$ to $T$ ? From $T$ to $P$ ?

## EXPLORATION 3

3. Students should discover a rectangle with length 2 and width 4 or length 1 and width 2.


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## EXPLORATION 3

1. Make a 4 by 8 rectangle on blank grid paper and label it $R$. Create 5 more rectangles that have the same shape but different size from $R$ and label them $A, B, C, D$ and $E$. What is the scale factor of each new rectangle in relation to the rectangle $R$ ?
2. Copy the table below and complete the data.

| Rectangle | Length | Width | Scale Factor | Perimeter | Area |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $R$ | 4 | 8 | 1 | 24 | 32 |
| $A$ |  |  |  |  |  |
| $B$ |  |  |  |  |  |
| $C$ |  |  |  |  |  |
| $D$ |  |  |  |  |  |
| $E$ |  |  |  |  |  |
| $F$ |  |  |  |  |  |

Did you create a rectangle that is smaller than $R$ ? If not, make one now and label it $F$.
3. What is the scale factor from $R$ to $F$ ? Make a conjecture about scale factors less than 1.
4. $G$ is a $6 \times 12$ and scale factor from $R$ to $G$ is $\frac{6}{4}=\frac{3}{2}=1.5$
5. See if your students compute the dimensions first and then draw the new rectangle or if they try to draw the rectangle first by visually interpreting what a scale factor of $\frac{5}{2}$ does.
6. New rectangle is 3 by 6 and scale factor from $R$ to $J$ is $\frac{3}{4}=0.25$

A scaling activity is available in the Supplement Notebook that connects scaling to cartoon characters and nature.
4. Make a rectangle the same shape as $R$ that is 6 units long. Label it G . What is the scale factor? from R to G ?
5. Make a new rectangle using a scale factor of $\frac{5}{2}$. Label it $H$. What are the dimensions of this new rectangle? How can you check to see if the scale factor is $\frac{5}{2}$ ?
6. Make a rectangle with the same shape as $R$ that is 3 units long. Label it J. What is the scale factor for this rectangle?

## EXAMPLE 1

Sue is making a dessert from a recipe that will serve 3 people. The recipe uses 5 tablespoons of chocolate, and 2 cups of milk.
a. Sue wants to serve six people. How much milk will she need?
b. Sue decides to invite four more people. How much milk will she need now? What is the scale factor between her final recipe and the original?
c. On another occasion, Sue uses 7 tablespoons of chocolate. How much milk should she use this time?

## SOLUTION

a. When Sue doubles the people, the scale factor is 2 . So the amount of milk will be $2 \bullet 2$ cups $=4$ cups of milk.
b. What is the scale factor between 4 people and 10 people? Call the scale factor $x$.

Set $4 \bullet x=10$. Dividing both sides of the equation by $4, x=\frac{10}{4}=2.5$.
Using the scale factor 2.5 , the recipe will use $2.5 \bullet 2=5$ cups of milk.
c. The recipe uses a ratio of 5 tablespoons chocolate for every 2 cups of milk. What is the scale factor when Sue increases the chocolate to 7 tablespoons? Call this scale factor $x$. Set $5 \bullet x=7$, so $x=\frac{7}{5}$. The amount of milk will be $2 \bullet \frac{7}{5}=\frac{14}{5}=2 \frac{4}{5}$ cups of milk.

Alternatively, let $C=$ the number of cups of milk needed with 7 tablespoons of chocolate. Then

## PROBLEM 1

A. Rectangle $N$ new dimensions: 3 units $\times 5.4$ units
B. Rectangle P new dimensions: 7.5 units $\times 13.5$ units

PROBLEM 2
A. 3
C. $\frac{2}{3}$
D. 1.5
E. They are reciprocals of one another.

$$
\begin{aligned}
5: 2 & =7: C \\
\frac{5}{2} & =\frac{7}{C}
\end{aligned}
$$

Multiplying both sides of the equation by 2 C

$$
\begin{aligned}
\left(\frac{5}{2}\right) \cdot 2 C & =\left(\frac{7}{C}\right) \cdot 2 C \\
5 C & =14 \\
C & =\frac{14}{5}
\end{aligned}
$$

## PROBLEM 2

Make a 5 units $\times 9$ units rectangle on grid paper and label it Rectangle M .
A. Construct a new rectangle using a scale factor of 0.6 . Label the new dimensions and name it Rectangle N .
B. Construct another rectangle using a scale factor of 1.5 . Label the new dimensions and name it Rectangle $P$.

## PROBLEM 3

Suppose Rectangle A is a 6 by 9 rectangle. Draw a Rectangle B so that the scale factor from $A$ to $B$ is $\frac{1}{3}$.
A. What is the scale factor from $B$ to $A$ ?
B. Draw a Rectangle $C$ so that the scale factor from $B$ to $C$ is 2 .
C. What is the scale factor from A to C ?
D. What is the scale factor from C to A ?
E. What do you notice about the scale factors from A to B and B to A? From $A$ to $C$ and $C$ to $A$ ?

## EXERCISES

1. a. Since $\frac{9}{6}=\frac{3}{2}=1.5$ and $\frac{12}{8}=\frac{3}{2}=1.5$, the scale factor is $\frac{3}{2}=1.5$.
b. Since $\frac{18}{9}=\frac{2}{1}=2$ and $\frac{24}{12}=\frac{2}{1}=2$, the scale factor is 2 .
c. Since $\frac{18}{6}=\frac{3}{1}=3$ and $\frac{24}{8}=\frac{3}{1}=3$, the scale factor is 3 .
2. a. Suppose a rectangle has length $L$ and the scale factor 1.8 . The difference between the new length $1.8 L$ and the old length $L$ is $1.8 L-L=0.8 L$.
The percent increase is $\frac{0.8 \mathrm{~L}}{\mathrm{~L}}=0.8$ which is equivalent to an $80 \%$ increase.
b. The new length is 0.8 L . So the amount of decrease is 0.2 L .

The percent increase is $\frac{0.2 \mathrm{~L}}{\mathrm{~L}}=0.2$ which is equivalent to a $20 \%$ decrease.
c. The difference between the new length and old length is $2.5 L-L=1.5 L$.

The percent increase is $\frac{1.5 L}{L}=1.5$ which is equivalent to a $150 \%$ increase.
d. The difference between the new length and old length is $3 L-L=2 L$.

The percent increase is $\frac{2 L}{L}=2=2.00$ which is equivalent to a $200 \%$ increase.
3. a. $\frac{2}{3}$
b. $\frac{3}{2}$
c. $\frac{4}{3}$
d. $\frac{3}{4}$
e. $\frac{1}{2}$
f. 2

## PROBLEM 4

A. Draw a rectangle that is 3 inches by 4 inches. In a map with a scale factor of 1 inch $=2$ miles, how big an area does the rectangle represent?
B. What are the dimensions of a rectangle that represents an area 5 miles wide and 8 miles long on the map? What is the area of the scaled rectangle? What is the area represented by the map?

## EXERCISES

1. Three rectangles have the following dimensions:

Rectangle A: 6 cm by 8 cm
Rectangle B: 9 cm by 12 cm
Rectangle C: 18 cm by 24 cm
Determine the scale factor, if there is one, from
a. Rectangle $A$ to Rectangle $B$.
b. Rectangle $B$ to Rectangle $C$.
c. Rectangle $A$ to Rectangle $C$.
2. In each problem below, apply the specified scale factor to a rectangle. What is the percent increase or decrease in the length of the new rectangle?
a. scale factor of 1.8
c. scale factor of 2.5
b. scale factor of 0.8
d. scale factor of 3
3. Draw a 6 -by- 15 rectangle and label it A , a 4 -by- 10 rectangle and label it B , and a 8 -by-20 rectangle and label it C.
a. What is the scale factor from $A$ to $B$ ?
b. What is the scale factor from $B$ to $A$ ?
c. What is the scale factor from A to C ?
d. What is the scale factor from C to A ?
e. What is the scale factor from C to B ?
f. What is the scale factor from B to C ?
4. To help work the problem, call the dimensions of $A$ length $L$ and width $W$. The rectangle $A$ has $L=4$ and $W=6$. When we increase the length by $50 \%$, the increase in length is $4 \cdot(0.50)=2$ so that the new length is $4+2=6$. When we increase the width by $50 \%$, the increase in width is $6 \cdot(0.50)=3$ so that the new width is $6+3=9$. The ratio of the new length to the old length is $\frac{6}{4}=\frac{3}{2}$, that is $4 \cdot \frac{2}{3}=6$. The ratio of the new width to the old width is $\frac{9}{6}=\frac{3}{2}$, that is $6 \cdot \frac{3}{2}=9$. So the scale factor is $\frac{3}{2}$.
5. The rectangle $A$ has length $L$ and width $W$. When we increase the length by $40 \%$, the increase in length is $L \cdot(0.40)=0.40 L$ so that the new length is $L+0.4 L=1.4 L=\frac{7}{5} L$. When we increase the width by $40 \%$, the increase in width is $W \cdot(0.40)=0.40 \mathrm{~W}$ so that the new width is $\mathrm{W}+0.4 \mathrm{~W}=1.4 \mathrm{~W}=\frac{7}{5} \mathrm{~W}$. The ratio of the new length to the old length is $\frac{1.4}{1}=\frac{7}{5}$. So the scale factor is $\frac{7}{5}$.
6. The rectangle $A$ has $L=15$ and $W=10$. When we decrease the length by $40 \%$, the decrease in length is 15 . $(0.40)=6$ so that the new length is $15-6=9$. When we decrease the width by $40 \%$, the decrease in width is $10 \cdot(0.40)=4$ so that the new width is $10-4=6$. The ratio of the new length to the old length is $\frac{9}{15}=$ $\frac{3}{5}$, that is $15 \cdot \frac{3}{5}=9$. The ratio of the new width to the old width is $\frac{6}{10}=\frac{3}{5}$, that is $10 \cdot \frac{3}{5}=6$. So the scale factor is $\frac{3^{5}}{5}$.
8. a. The perimeter of rectangle $T$ is $P(T)=2 L+2 W+2 L+2 W=4 L+4 W$. The perimeter of rectangle $S$ is $P(S)=L+W+L+W=2 L+2 W$. Notice that $P(T)=2 P(S)$, that is the perimeter of rectangle $T$ is twice the perimeter of rectangle $S$.
b. The area of rectangle $T$ is $\mathrm{A}(T)=2 \mathrm{~L} \cdot 2 \mathrm{~W}=4 \mathrm{~L} \cdot \mathrm{~W}$. The area of rectangle S is $\mathrm{A}(S)=\mathrm{L} \cdot \mathrm{W}$. Notice that $A(T)=4 \cdot A(S)$, that is the area of rectangle $T$ is four times the area of rectangle $S$.
c. $100 \%$
d. $300 \%$

9d. They all have the same shape. The lengths of the sides in figure $b$ are twice the lengths of the corresponding sides in figure a . The lengths of the sides in figure c are half the lengths of the corresponding sides in figure a .
4. Draw a 4 -by-6 rectangle and call it $A$. Increase the length and the width of $A$ by $\frac{1}{2}$ to form rectangle $B$. What is the scale factor from $A$ to $B$ ? What is the scale factor from $B$ to $A$ ?
5. Increase the length and the width of rectangle $A$ by $\frac{2}{5}$ to form rectangle $B$. What is the scale factor from $A$ to $B$ ? (Call the dimensions of $A$ length $L$ and width $W$.)
6. Decrease the length and the width of $15 \times 10$ rectangle $A$ by $\frac{2}{5}$ to form rectangle $B$. What is the scale factor from $A$ to $B$ ? What is the scale factor from $B$ to $A$ ?
7. Make a 2 -by-3 rectangle and call it R. Make a new rectangle using a scale factor of 3 , and label it $S$. What is the perimeter of S ? the area of S ? How do the perimeter and area of $S$ compare to the perimeter and area of $R$ ?
8. For rectangle $S$ with length $L$ and width $W$, use a scale factor of 2 to make a larger rectangle $T$ (sketch a picture for this problem.)
a. Compute and compare the perimeters of both rectangles. What do you notice?
b. Compute and compare the areas of both rectangles. What do you notice?
c. Compare the perimeter of rectangle $S$ to $T$. Determine the percent increase in the perimeter.
d. Compare the area of rectangle $S$ to $T$. Determine the percent increase in the area.
9. On a coordinate plane, plot the points $A(1,0), B(4,0), C(5,5), D(0,3)$ and $E(0,1)$.
a. Draw line segments connecting these points in alphabetical order.
b. Double each coordinate of the given points $A, B, C, D$, and $E$. Plot these points and draw line segments connecting the points as you did in part a.
c. Find half of each coordinate of the given points $A, B, C, D$, and $E$. Plot these points and draw line segments connecting the points as you did in part a.
d. What do you notice about the figures in parts $a, b, a n d c$ ?
10. $18^{\prime \prime} \times 36^{\prime \prime}$

## 11. Ingenuity

When approaching such problems, it is almost always helpful to first look at smaller examples and try to extrapolate a pattern from them. Follow the suggestion by finding the first 6 partial sums. They should look as if they are approaching $1 / 2$ but have no easily-discernable pattern. Find out what the partial differences are between the partial sums and $1 / 2$. All of this could be a source of class discussion with student suggestions.

Writing the product of the first 9 factors, you get a10 $=\frac{3}{4} \frac{8}{9} \frac{15}{16} \frac{24}{25} \frac{35}{36} \frac{48}{49} \frac{63}{64} \frac{80}{81} \frac{99}{100}$. Shifting numerators over by one fraction, you get $\frac{1}{4} \frac{3}{9} \frac{8}{16} \frac{15}{25} \frac{24}{36} \frac{35}{49} \frac{48}{64} \frac{63}{81} \frac{80}{100} \frac{99}{1}$. Simplifying each of
the interior fractions, you get $\frac{1}{4} \frac{1}{3} \frac{2}{4} \frac{3}{5} \frac{4}{6} \frac{5}{7} \frac{6}{8} \frac{7}{9} \frac{8}{10} \frac{9}{1} \frac{11}{1}$. In general the nth member
of the sequence is $\frac{1}{4} \frac{1}{3} \frac{2}{4} \frac{3}{5} \frac{4}{6} \frac{5}{7} \frac{6}{8} \frac{7}{9} \frac{8}{10} \cdots\left(\frac{n-2}{n}\right) \frac{n-1}{1} \frac{n+1}{1}$ which is equal to
$\frac{1}{2} \frac{1}{2} \frac{2}{4} \frac{3}{5} \frac{4}{6} \frac{5}{7} \frac{6}{8} \frac{7}{9} \frac{8}{10} \ldots \frac{n-2}{n} \frac{n-1}{1} \frac{n+1}{1}=\frac{(n-1)!(n+1)}{2 \cdot n!}=\frac{n+1}{2 n}$. Using this formula, you see that
$\mathrm{a} 2=\frac{3}{4}, \mathrm{a} 3=\frac{4}{6}, \mathrm{a} 4=\frac{5}{8}$ and so on.

## 12. Investigation

Be sure to provide measuring tape or yardsticks for students to use. You can also ask students to compare data on a few rooms. This may be a fun activity for your students to talk about math affecting their everyday lives. What scale factor would they need to use to have 4 times as much space in their bedroom?
a. A scale factor that is twice the dimensions of the original yields an area that is 4 times the original area. So the new dimensions of the room should be $2 \mathrm{~L} \cdot 2 \mathrm{~W}$, where L and W are the dimensions of the original room.
b. A scale factor of 4 makes the perimeter 4 times the original perimeter. So the dimensions of the bigger room is $4 \mathrm{~L} \cdot 4 \mathrm{~W}$, where the original room has dimensions L and W .
10. Nama is making a dog house for her dog Cloud using scale drawings. If the dimensions on the pattern are $1 \frac{1}{2}$ inches by 3 inches, what will the final dimensions of the dog house be using a scale factor of 12 ?
11. Ingenuity:

What is the value of the product

$$
\left(1-\frac{1}{2^{2}}\right)\left(1-\frac{1}{3^{2}}\right)\left(1-\frac{1}{4^{2}}\right)\left(1-\frac{1}{5^{2}}\right) \ldots\left(1-\frac{1}{101^{2}}\right) ?
$$

Look at other products of this form. Do you see a pattern?

## 12. Investigation:

Measure the dimensions of an interesting room in your house. Using a scale of 1 inch: 2 feet, draw the floor plan. Include doorways, closets, windows, and furniture.
a. Find the scale factor that makes the area of the room 4 times the original area. What are the dimensions of the bigger room?
b. Find the scale factor that makes the perimeter 4 times the original perimeter. What are the dimensions of the bigger room?

## Rectangles for EXPLORATION 1

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Objective: Students will continue exploring ratios and rates with respect to geometry by scaling images.

## Materials:

Grid Paper
Poster Board
Small Cartoon Images

## Activity Instructions:

Have student bring to class simple images of cartoon characters or other animated figures (the teacher may want to have some extra images for students who forget). Have each student place points around his/her picture and place the picture on grid paper. Students should determine the coordinates for each point on the picture. Have each student write a list of coordinates for their picture and then apply a scale factor of 2 to their list, creating a new list of coordinates. The students should plot the new coordinates on a second sheet of grid paper. **The teacher may also ask the students to only apply the scale factor to the $x$-coordinate leaving the $y$-coordinate as it is or to perform the activity with the coordinate $(0,0)$ in the center of the image. ** Discuss with the students what happened to their images as they performed this activity.

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Teacher Edition
Section 10.5 Scaling

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## MY HOUSE



Objective: The students will use their knowledge of measurement and scaling to design an accurate floor plan of their house.

## Materials:

Measuring tools, either ruler, yard stick, measuring tape or meter stick
Paper and Pencil
White Butcher Paper

## Activity Instructions:

1) First, the students will go home and take measurements of all of the rooms in their house. They may need their parent's assistance with this. Also, if anyone comes back to school the next day without their measurements, you can provide them with the measurements of your house so that they can participate in the activity. They should record all measurements on a sketch drawing of their house's floor plan with the measurements written in each room.
2) After collecting this data, the students will then choose an appropriate scale to use to recreate the floor plan, but this time with an accurate drawing rather than just a sketch. Once a scale is determined, hand all students a piece of white butcher paper so they can start working on their final draft of their floor plans.
3) When finished, students can use markers and/or colored pencils to decorate their scale drawings, if time permits.

## Section 10.6 - Scaling and Similarity

## Big Idea:

Discover connections between scaling and similarity

## Key Objectives:

- Understand the characteristics of similar figures.
- Use scaling to make similar figures.
- Identify and use corresponding angles and sides with similar figures.
- Use scaling and similarity to solve everyday problems.


## Materials:

Grid paper, Ruler

## Pedagogical/Orchestration:

- Similarity, one of the most useful geometric concepts, is developed in this section. As usual, the concept is discovered by students, not handed to them.
- Scaling of lines is quickly extended to two-dimensional figures by the use of pattern development. The subsequent change in scaling one-dimensionally to two-dimensionally is carefully developed.
- Scaling in two and three dimensions is a major problem traditionally for exit-level math students. Any firm foundation you can give your students will be a gift for their mathematical future.


## Activities:

"Stretch it Out" on CD and the end of the section
"Triangles from Exploration 1" on CD and at the end of the section

## Vocabulary:

scaling, corresponding angles, corresponding sides, similar

## TEKS:

$\begin{array}{llllllll}7.1(A) ; & 7.5(A)(B) ; & 7.6(A)(D) ; \quad 7.9(A) ; \quad 7.13(A)(B)(C) ; \quad 7.14(A) ; \quad 7.15(A)(B) ; \quad 8.5(A) ; \quad 8.6(A) ; \quad 8.10(A) ; \\ 8.14(A) ; \quad 8.15(A) ; \quad 8.16(A) & & \end{array}$ 8.14(A); 8.15(A); 8.16(A)

## WARM-UPS for Section 10.6

1. Mr. Jones bought 14 tickets for the football game for $\$ 33.60$. How much would it cost him to buy 24 tickets?
a. $\$ 58$
b. $\$ 57.60$
c. $\$ 57.40$
d. $\$ 57.20$

Ans: (b)
2. A frog ate 38 flies in 7 hours. At this rate, how many flies will he eat in 35 hours?

Ans: Set proportion $\frac{38}{7}=\frac{x}{35}$. So, (38)(35) $\div 7=x$ and $38(5)=x$. Thus, $x=190$ flies.

## Launch for Section 10.6:

Ask your students, "Have you ever seen a picture or diagram that is captioned Picture not drawn to scale?" If you have a picture to show as an example of this, do so, otherwise just draw a simple diagram and label it with dimensions that cannot be correct such as:

## 9 ft.

3 ft.

Ask your students, "What does it mean for a picture to be drawn to scale? How would you describe this?" Allow students to discuss this question, listening for any hint of scale factor references, or that the shape of the image should be the same as the shape of the real object. Tell your students that floor plans and maps should be drawn to scale and show them your school's evacuation plan that has a floor plan of the school. If in the school, hallway A is twice as long as hallway B, then the same relationship should hold on the floor plan. Also, when the evacuation plan is scaled correctly, if you multiply the width of the gym on the floor plan by 50 to get the width of the real gym, then you should be able to multiply any length on the map by 50 to get the actual length of the object in the school. Tell your students, "Today we will be exploring the effect of scaling to make different sized shapes that look like the original, and also see the effect scaling has on the perimeter and area of the new shapes."

What does "not drawn to scale" mean in the picture below?
100
$\square$

## EXPLORATION

a.

| Figure | Dimensions | Scale <br> Factor | Perimeter | Ratio of new <br> perimeter to <br> original perimeter |
| :---: | :---: | :---: | :---: | :---: |
| A | $2 \times 3$ | 1 | 10 units |  |
| B | $4 \times 6$ | 2 | 20 | $20 \div 10=2$ |
| C | $6 \times 9$ | 3 | 30 | $30 \div 10=3$ |
| D | $8 \times 12$ | 4 | 40 | $40 \div 10=4$ |
| E | $10 \times 15$ | 5 | 50 | $50 \div 10=5$ |

b. Perimeter: 60 , Area: 216
c. Scale factors of linear measurements, like the perimeter, are the same. Changes in area are the squares of the scale factors.

## SECTION 10.6 SCALING AND SIMILARITY

Amy's teacher asked her to draw a floor plan of her house. The boundary is a rectangle 100 feet long and 32 feet wide. In her drawing, she cannot use the actual dimensions of the house, because her paper is not 100 feet long. She decides to use a scale of one inch in her drawing to represent 8 feet in the house. After Amy has completed the drawing, filling in the walls of all the rooms in her house, how can her teacher find the dimensions of each room of Amy's house?

Scaling is the process of taking a figure, like the floor plan of a house, and using it to make a new figure, for example the real floor plan. In Section 10.4, you learned how to use a scale factor to make a smaller and a larger rectangular figure with the same shape.

## EXPLORATION 1

a. Draw rectangle A. Apply the scale factor to each new rectangle. Draw and label appropriately. Use the new rectangle to complete the tables

| Figure | Dimensions | Scale <br> Factor | Perimeter | Ratio of new <br> perimeter to <br> original perimeter |
| :---: | :---: | :---: | :---: | :---: |
| A | $2 \times 3$ | 1 | 10 units |  |
| B |  | 2 |  |  |
| C |  | 3 |  |  |
| D |  | 4 |  |  |
| E |  | 5 |  |  |


| Figure | Dimensions | Scale <br> Factor | Area | Ratio of new <br> area to <br> original area |
| :---: | :---: | :---: | :---: | :---: |
| A | $2 \times 3$ | 1 | 6 sq. units |  |
| B | $4 \times 6$ | 2 | 24 | $24 \div 6=4$ |
| C | $6 \times 9$ | 3 | 54 | $54 \div 6=9$ |
| D | $8 \times 12$ | 4 | 96 | $96 \div 6=16$ |
| E | $10 \times 15$ | 5 | 150 | $150 \div 6=25$ |


| Figure | Dimensions | Scale <br> Factor | Area | Ratio of new <br> area to <br> original area |
| :---: | :---: | :---: | :---: | :---: |
| A | $2 \times 3$ | 1 | 6 sq. units |  |
| B |  | 2 |  |  |
| C |  | 3 |  |  |
| D |  | 4 |  |  |
| E |  | 5 |  |  |

b. Using your information, make a prediction for the Perimeter and Area for a new rectangle $F$ with a scale factor of 6 .
c. What is the relationship between the scale factor and the perimeters of the rectangles? What is the relationship between the scale factor and areas of the scaled rectangles?

When an $n$-by- $m$ rectangle is scaled in each direction using the scale factor $k$, the old perimeter and the new perimeter are

$$
\begin{aligned}
& P_{\text {old }}=2 n+2 m \text {, and } \\
& P_{\text {new }}=2(k \cdot n)+2(k \cdot m)=k \cdot(2 n+2 m)=k \cdot P_{\text {old }} .
\end{aligned}
$$

## The new perimeter is the old perimeter multiplied by $k$.

The old area and the new area are

$$
\begin{aligned}
& A_{\text {old }}=n \cdot m, \text { and } \\
& A_{\text {new }}=(k \cdot n)(k \cdot m)=k^{2}(n \cdot m)=k^{2} \cdot A_{\text {old }} .
\end{aligned}
$$

The new area is equal to the old area multiplied by $\mathbf{k}^{2}$. The area has been scaled by the same number in each direction. It is also possible to scale by different numbers in each direction, but the resulting figure will not look like an enlarged or reduced copy of the original shape.

Explain to students they can mark equal angles using little arcs.
b. The angle pairs have the same measure

## EXPLORATION 2

What do you notice about the triangles below?

a. Measure the angles and the side lengths of triangle $A B C$ and triangle DEF.
b. Compare the following angle pairs, $\angle \mathrm{A}$ to $\angle \mathrm{D}, \angle \mathrm{B}$ to $\angle \mathrm{E}$, and $\angle \mathrm{C}$ to $\angle \mathrm{F}$. What do you notice?
c. Compare the side lengths $A B$ to $D E, B C$ to $E F$, and $A C$ to $D F$. What do you notice?
d. Compute the ratio of each pair of side lengths from part c. What pattern do you notice in these ratios? Do you see a scale factor from triangle $A B C$ to triangle $D E F$ ? Do you see a scale factor from triangle $D E F$ to triangle $A B C$ ?
e. Graph the two triangles on a coordinate plane and discuss your findings. How do the area and perimeter compare? Explain using algebraic notation.

The smallest angles in triangles $A B C$ and $D E F$ had the same measure, so they represent corresponding angles of both triangles. Similarly, the shortest sides in triangles $A B C$ and $D E F$ had the same ratio as the longest sides in triangles $A B C$ and $D E F$. That characteristic makes them corresponding sides. Look again at the picture. It's easiest to recognize corresponding sides and angles visually.

The corresponding angles have the same measure. To draw $S$ and then $T$ is not so easy. You can use a ruler and some string to try to construct $S$ and $T$ or a compass and straight edge, if you wish. The students will see that they are the same shape.

The ratio of the height will be the scale factor, the same as the lengths of the sides.


Two triangles are similar when their corresponding angles have equal measures and their corresponding sides have the same ratio. In the picture, $\angle A$ corresponds to $\angle \mathrm{D}, \angle \mathrm{B}$ corresponds to $\angle \mathrm{E}$, and $\angle \mathrm{C}$ corresponds to $\angle \mathrm{F}$. From this information, we can also determine that $\overline{\mathrm{AB}}$ corresponds to $\overline{\mathrm{DE}}, \overline{\mathrm{AC}}$ corresponds to $\overline{\mathrm{DF}}$, and $\overline{\mathrm{BC}}$ corresponds to $\overline{\mathrm{EF}}$. Choosing corresponding sides is essential in working with similar triangles.

THEOREM 10.1: TRIANGLE SIMILARITY THEOREM
If two triangles have corresponding angles of the same measure, then the ratios of their corresponding sides are the same. Conversely, if two triangles have corresponding sides with the same ratio, then the triangles' corresponding angles have equal measure.

Triangle $S$ has sides measuring 2,3 and 4 units, and triangle $T$ is triangle $S$ scaled by a factor of 3 . Triangle $T$ has corresponding sides that measure $3(2)=6$, $3(3)=9$ and $3(4)=12$ units. Although the lengths of the sides are all different, the ratios of corresponding sides are the same, because $\frac{6}{2}=\frac{9}{3}=\frac{12}{4}=3$. The ratio of the corresponding sides is the scale factor, 3 .

What is the effect on the angles of the change from triangle $S$ to $T$ ?
In summary, for similar triangles, one of the triangles can be created from the other by stretching or compressing the sides by the same scale factor, without changing the measures of the angles. The triangles have the same shape, but different sizes.

You might call your students' attention to the fact that many times in mathematics, facts like the properties of fractions are also important when they use the properties in another context, like geometric shapes.

Because the scaling relationship holds for all similar triangles, when you find the ratio of any two sides of a triangle, this will be the same ratio for the corresponding sides in any similar triangle. Generalizing the example above, call the lengths of the sides of the first triangle $a, b$ and $c$. With a scale factor of $k$, the similar triangle will have sides that measure $k a=A, k b=B$ and $k c=C$, respectively. In the picture below, the original triangle has been scaled by a factor $k=2$ : Each side is twice as long.


With two similar shapes, order is important. Corresponding sides must be labeled in the same order. For instance, $\Delta r s t$ is similar to $\triangle R S T$ and $\angle s t r$ has the same measure as $\angle S T R$. (By convention, $\triangle r s t$ is not said to be similar to $\triangle S T R$, because there is no fixed scale factor such that $\overline{r s}$ scales to $\overline{S T}, \overline{s t}$ scales to $\overline{T R}$, and $\overline{t r}$ scales to $\overline{R S}$.) The ratios between corresponding sides in similar triangles are equivalent.

$$
\frac{\mathrm{a}}{\mathrm{~b}}=\frac{\mathrm{ka}}{\mathrm{~kb}}, \quad \frac{\mathrm{a}}{\mathrm{c}}=\frac{\mathrm{ka}}{\mathrm{kc}} \text { and } \quad \frac{\mathrm{b}}{\mathrm{c}}=\frac{\mathrm{kb}}{\mathrm{kc}}
$$

In general, we can have similarity in any type of polygon.

## THEOREM 10.2: POLYGON SIMILARITY THEOREM

Two polygons are similar when their corresponding angles have equal measure and their corresponding sides have the same ratio.

## EXAMPLE 1

In the afternoon, a tree casts a shadow of 28 feet. If a 5 foot post casts a 4 foot shadow, what is the height of the tree? Draw a picture.

1. $\quad$ Scale factor $=5$.
2. Using a scale factor of 3 , the side corresponding to length 5 is 15 and the side corresponding to length 6 is 18.


Figure A
Figure B
From the picture above, notice that the shadows of the figures create 2 similar triangles, due to the position of the sun and the right angles between the figures and the ground. Thus, if we find the scale factor from figure $A$ to figure $B$, we can compute the height of the tree. To find the scale factor, we can compare the lengths of the shadows. $\mathrm{So}, 4 k=28$, or $k=7$. Since the heights of the figures are corresponding sides, then the height of the tree must be $5 \mathrm{ft} \cdot 7=35 \mathrm{ft}$.

However, we can use our knowledge of ratios between corresponding sides in similar triangles to set up the following proportion to solve this problem in a different way:

$$
\frac{5}{4}=\frac{x}{28} .
$$

Therefore, $x=\left(\frac{5}{4}\right)(28)=35$. So, the height of the tree must be 35 ft .

## EXERCISES

1. Suppose that $\triangle A B C$ is similar to $\triangle A^{\prime} B^{\prime} C^{\prime}$, and the side of length 10 in triangle $A B C$ has length 50 in $A^{\prime} B^{\prime} C^{\prime}$. What is the scale factor?
2. Suppose that the sides of $\triangle A B C$ are of length 4,5 , and 6 . Suppose that $\triangle A^{\prime} B^{\prime} C^{\prime}$ is similar and that the side corresponding to the side of length 4 has length 12. What are the lengths of the other two sides of $\Delta A^{\prime} B^{\prime} C^{\prime}$ ?
3. $C$ because $P_{\text {new }}=3 a+3 b+3 c=3(a+b+c)=3\left(P_{\text {old }}\right)$
4. a. Yes, they are similar with a scale factor of 2 .
b. No, because there is no single scale factor to relate 10 to 50,20 to 20 , and 25 to 40 . However, by reordering the angles of triangle DEF so that the sides are in the order 20,40 , and 50 , the scale factor of 2 will work, as in part a.
5. Answers will vary. Again, try to develop good geometric construction habits. Also, if you have time, ask students to estimate the scaled areas compared to the original. Then ask them to compare that ratio to the scale. They should discover that they are NOT equal. Some might even conjecture that ratio of the scaled areas to the original areas equal the squares of the ratios.
6. Which statement best describes the change in the perimeter of a triangle if all its side lengths are multiplied by 3 ? Select the best choice and explain your answer.
a. The new perimeter will be 9 times as large as the perimeter of the original triangle.
b. The new perimeter will be 12 times as large as the perimeter of the original triangle.
c. The new perimeter will be 3 times as large as the perimeter of the original triangle.
d. The new perimeter will be 6 times as large as the perimeter of the original triangle.
7. Triangle $A B C$ has sides of lengths 10,20 , and 25 .
a. Triangle DEF has sides of lengths 20,40 , and 50 . Are these two triangles similar? Explain.
b. Triangle DEF has sides of lengths 50,20 , and 40 . Are these two triangles similar? Is there a different ordering of the angles that would make the triangles similar? If so, what ordering would make them similar?
8. On a grid, draw a right triangle with sides of length 4 and 6 .
a. Draw two similar triangles with scale factors of 2 and 1.5. Label and measure all of the corresponding angles and compare them.
9. Use a grid to make copies of each of the triangles below. For each triangle, draw a similar triangle that uses a different scale factor.

10. A, B: scale factor $=2 ;$ C, D: scale factor $=3 ; D, F:$ scale factor $=2 ;$ F, C: scale factor $=1.5 ; E$, G: scale factor $=$ 2
11. a. The scale factor is $10: 4=5: 2=2.5$. So the dimensions of $N$ are 2.5 times the dimensions of $M$.
b. The area of M is $\frac{1}{2}(4)(3)=6 \mathrm{sq}$ units. The area of N is $\frac{1}{2}(4)(2.5)(3)(2.5)=37.5 \mathrm{sq}$ units. Both the height and base of Triangle $N$ are multiplied by 2.5 , so the area is multiplied by $2.5^{2}$.
c. The ratio of areas is $\frac{37.5}{6}=6.25=2.5^{2}$.
12. a. The scale factor is $3: 6=\frac{1}{2}=0.5$. So the dimensions of $T$ are 0.5 times the dimensions of $S$.
b. The area of $S$ is $\frac{1}{2}(6)(2)=6 \mathrm{sq}$ units. The area of T is $\frac{1}{2}(6)(0.5)(2)(0.5)=1.5 \mathrm{sq}$ units. Both the height and base of Triangle $T$ are multiplied by 0.5 , so the area is multiplied by $0.5^{2}$.
c. The ratio of areas is $\frac{3}{12}=\frac{1}{4}=\frac{1}{2}^{2}$.
13. Find all pairs of similar rectangles in the drawing below and give the scale factor for each pair.

14. On grid paper, draw a triangle $M$ with base 4 and height 3. Then draw a similar triangle $N$ with base 10.
a. Determine the scale factor $k$ from $M$ to $N$.
b. Compute the area $A_{M}$ of triangle $M$ and the area $A_{N}$ of triangle $N$.
c. What is the ratio $\frac{A_{N}}{A_{M}}$ ?
15. Draw a triangle $S$ with base 6 in., another side with length 4 in . and height 2 in. Then draw a similar triangle $T$ with base 3 in.
a. Determine the scale factor $k$ from $S$ to $T$.
b. Compute the area $A_{S}$ of triangle $S$ and the area $A_{T}$ of triangle $T$.
c. What is the ratio $\frac{A_{T}}{A_{S}}$ ?
16. Triangle ABD is similar to triangle LKM with scale factor $2: 1$

Triangle ABC is similar to triangle FHE with scale factor 2:1
Triangle MNK is similar to triangle FGH with scale factor 4:5
11. a. Let $x=$ length of each equal side of the coop in meters. The ratio of the model is $\frac{x}{12}=\frac{25}{40}$. Solve for $x$. $x=7.5 \mathrm{~m}$.
b. Find the height using the Pythagorean Theorem, $h=\sqrt{7.5^{2}-6^{2}}=\sqrt{20.25}=4.5$ meters.
12. Let $h=$ height of the light post. Draw a picture of the situation: a large triangle with height $h$ and base $10+$ $12=22$ feet that includes a smaller similar triangle with height 5 feet and base 12 feet. $\frac{5}{12}=\frac{h}{22} . h=9 \frac{1}{6}$ ft .

13. $180 \mathrm{~cm}^{2}$
14. $4: 1,2: 1$
10. Find all pairs of similar triangles in the drawing below and give the scale factor for each pair.

11. A farmer builds a chicken coop with a front that has the shape of an isosceles triangle. The base of the triangle is 12 meters long. In the plans that she used to build the barn, the base was 40 cm and the lengths of the other two sides were 25 cm .

a. What is the length of each equal side of the coop?
b. What is the height of the coop?
12. One night a woman 5 feet tall stood 10 feet from a light post. She measured the length of her shadow and found that her shadow was 12 feet long. What was the height of the light post?
13. A triangle originally has area $20 \mathrm{~cm}^{2}$. Each of its sides is increased by a scale factor of 3 . What is the area of the new triangle?
14. Two triangles are similar, and one triangle's sides are twice as long as the other triangle's. What is the relation between the areas of the triangles? What is the relation between their perimeters?
15. a. $\frac{10}{6}=\frac{5}{x}$. So, $x=3$ feet
b. $\frac{10}{6}=\frac{3}{x}$. So, $x=1.8$ feet
16. a. $\frac{4}{5}=\frac{x}{96}$. So, $x=76$.
b. $\frac{4}{5}=\frac{6}{x}$. So, $x=76$.
17. $\frac{80}{30}=\frac{x}{150} \cdot 50, x=400$.

Ingenuity
18. Mark the triangle as indicated. Mark the two angles at $D$ as right angles. Then the measure of angle $A C D$ is equal to the measure of angle $B$, and the measure of angle $B C D$ is equal to the measure of angle $A$. So all three angles are similar. Look back at the Ingenuity in Section 11.4 for help.

Investigation
19. Let $x=$ height of the pole. $\frac{x}{5.5}=\frac{20}{3}, x=5.5\left(\frac{20}{3}\right)=36 \frac{2}{3} \mathrm{ft}$.
15. If a 10 foot pole casts a 6 foot shadow, find the length of the shadows of each of the following:
a. 5 foot boy
b. 3 foot child
16. A flagpole casts a shadow of 96 feet and a 4 foot child casts a shadow of 5 feet.
a. How tall is the flagpole?
b. How long of a shadow will a 6 foot person cast at this time?
17. In downtown New York, Jack looks up from a park bench to see the top of a flagpole and behind it is the top of a building. The pole is 80 feet tall and Jack is 30 feet from the pole. If Jack is sitting 240 feet from the base of the building, how tall is the building?
18. Ingenuity:

In right triangle $A B C$, draw a perpendicular line from right angle $C$ to hypotenuse $\overline{A B}$. The perpendicular line intersects side $\overline{A B}$ at point $D$. Show that $\triangle A C D$ is similar to $\triangle A B C$ and that both are similar to $\triangle C B D$.


## 19. Investigation:

There is a tall pole with a straight wire from point $A$ on the ground to the top of the pole. The bottom of the pole is 20 feet from the wire. Sam, who is 5 feet, 6 inches tall, stands a yard away from point $A$ and the top of his head touches the wire. How high is the pole?


## Triangles from Exploration 2



## STRETCH IT OUT

Objective: The students will create similar objects making sure that all side lengths are proportional and all angle measures are equal. They will then find the perimeter and area of each shape to reinforce their knowledge of how the area and perimeter are affected when a shape is scaled proportionately.

## Materials:

Rulers, meter sticks or yard sticks
Protractors
Butcher paper, any color
Scissors
Copy of the "original" shape on the next page

## Activity Instructions:

1) This activity will probably work best if students are able to work in small groups of either two to three. Once all groups have been chosen, pass out a copy of the "original object" to each group. One per group is all that will be needed.
2) The students will first need to measure the length of each side and the measure of each angle in this original shape. If they are able to remember that opposite sides and opposite angles of a paral lelogram are congruent, they may finish measuring more quickly.
3) Once all lengths and angles are measured and checked by the teacher, then you can hand each group a large piece of butcher paper. At this point, the group may want to start working on the floor for more room. Their objective is to recreate a similar parallelogram on the butcher paper that in which the dimensions are 10 times as big as the dimensions of the original object. Remind the students at this point that their new shape should have sides that are 10 times bigger than the original, but that all angle measures should remain exactly the same. Encourage the students to ONLY use pencils at this point, as many may need to redraw the shape several times until they get it perfect.
4) Once they feel that they have recreated the new shape perfectly, ask them to calculate the area and perimeter of both objects, and record these on the butcher paper inside of the object. The perimeter of the new shape will be 10 times the perimeter of the original object; whereas, the area of the new shape will be 100 times the area of the original.
5) When finished, they can carefully cut out their new object along its edges. All group members should record their name somewhere on the paper before it is turned in to you.
6) The teacher can easily check to see if the groups have recreated the shape correctly simply by laying all of the new, bigger objects from each group one on top of the other. The stack should match up perfectly with no edges or angles sticking out.

## Original Object



1. In the 60 minute game, the Mountain Goats score 24 points and the Bobcats score 15 points. In the 80 minute game, the Mountain Goats score 32 points and the Bobcats score 20 points. The length of the game does not matter; the Mountain Goats will always win.
2. Members: $C(p)=10+0.005 p$.

Non-members: $\mathrm{C}(\mathrm{p})=0.1 \mathrm{p}$.
Solving $10+0.005 p=0.1 p$ for $p=105.26$. Worth buying a membership when making more than 105 copies.
3. a. 15 boys, 25 students total
b. 18 boys
4. 4 legs
5. $\frac{\$ 0.03}{\text { grape }} \cdot \frac{200 \text { grapes }}{3 \text { jars }} \cdot 5$ jars $=\$ 10$.
6. 80 customers
7. a. 14 counselors
b. 48 students

## REVIEW PROBLEMS

1. The Grinnell Mountain Goats and the Texas State Bobcats are playing a water polo match. The Mountain Goats score an average of 2 times every 5 minutes and the Bobcats scored an average of 1 point every 4 minutes. What is the score of the game if they played a 60 minute game? What if they played an 80 minute game? Does the length of time played change who won the game?
2. Quicko's makes copies for $\$ .10$ a page. They also allow people to pay $\$ 10$ and become Quicko's Preferred Customers and pay half a cent for each copy. Write a function for the cost per copy of members and non members. When is it worth it to buy a membership?
3. Mr. Boney's class has 3 boys for every 2 girls.
a. Suppose there are 10 girls in Mr. Boney's class. How many boys are there in the class? How many students total?
b. Suppose Mr. Boney has 30 students in his class. How many boys are there in the class?
4. The Quadlings are an alien race from the planet Quadloo. If 8 Quadlings have 32 legs, how many legs does each Quadling have?
5. Jeff's local grocery store normally sells grapes for $\$ .03$ per grape. He uses these grapes to make his delicious grape jam. If it takes 200 grapes to make 3 jars of jam, how much is cost to make 5 jars of jam?
6. If it takes 12 waiters to serve 192 customers, how many customers can 5 waiters serve?
7. At the Texas State Summer Math Camp, each study group is made up of 4 students and 1 counselor.
a. How many counselors are there if there are 56 campers total?
b. How many students are there if there are 12 counselors?
8. It costs $\$ 20$ to make 20 glasses of lemonade.

For $\$ 50$, he can make 80 glasses of lemonade.

$$
C(x)=0.50 x+10
$$

9. 189 trees
10. 35 books
11. about 38 hawks
12. $2 \frac{1}{4} \div 3=\frac{9}{4} \cdot \frac{1}{3}=\frac{9}{12}=\frac{3}{4}$ hour. $\frac{3}{4} \cdot 2$ miles $=\frac{6}{4}=1 \frac{2}{4}=1 \frac{1}{2}$ hours. It will take Peter $1 \frac{1}{2}$ to walk two miles.
13. RJ wants to run a neighborhood lemonade stand. It costs $\$ 10$ dollars for the startup costs of the stand itself, the pitchers, etc. It costs $\$ 0.50$ to make each large glass of lemonade. How much does it cost to make 20 glasses of lemonade? How many glasses can he make for $\$ 50$ ? Write a function that describes the amount of money RJ makes based on the number of glasses of lemonade that he sells.
14. Heather is working in Denali National Park for 63 days ( 9 weeks) in the summer. If she averages sighting 15 Pine Grosbeaks every 5 days, how many will she have seen over her entire time there?
15. Kate is working at a bookstore. She sells twice as many mystery books as science fiction books and she sells 10 mystery books a day. How many science fiction books does she sell in a week? Do this without figuring out the number of mystery books she sells.
16. A bird watcher sees 3 hawks every 8 miles she drives across Texas. How many hawks does she expect to see if she drives 100 miles?
17. Peter walks 3 miles in $2 \frac{1}{4}$ hours. If he walks all three miles at the same pace, how long will it take him to walk two miles?
18. A rancher increases his herd from 84 to 106 cattle. What is the percent of increase iof the herd?
19. A dress was discounted from $\$ 96$ to $\$ 84$. What was the rate (percent) of discount?
20. Sketch a rectangle $A$ with dimensions 4 by 10 units.
a. Sketch a rectangle $B$ with a scale factor of 3.5 from $A$ to $B$. Label the dimensions of rectangle $B$.
b. Sketch a rectangle $C$ with a scale factor of $\frac{1}{4}$ from $A$ to C. Label the dimensions of rectangle C .
21. Let X be a 12 by 20 rectangle. Determine which of the following rectangles are similar to rectangle $X$ and, if it is, what is the scale factor from $X$ to it?
a. Rectangle $W$ is a 3 by 4 rectangle
b. Rectangle $Y$ is a 3 by 5 rectangle
22. 2 a. Leopold's dinner weighs $1+1 \frac{1}{2}=2 \frac{1}{2}=\frac{5}{2}$ ounces. The percentage that is carrot is $\frac{\frac{1}{5}}{2}=1$. $\frac{2}{5}=0.4=40 \%$.
b. $\frac{\frac{5}{2} \text { ounces }}{\text { day }} \cdot 7$ days $=\frac{35}{2}$ ounces $=17 \frac{1}{2}$ ounces.
c. Leopold's weight in ounces is $5 \mathrm{lb} \cdot \frac{16 \text { ounces }}{\text { pound }}=80$ ounces. $\frac{80}{\frac{5}{2}}=80 \cdot \frac{2}{5}=\frac{160}{5}=32$ days.
23. $110^{\circ}$
24. $\frac{t}{20}=\frac{6}{4} ; t=30$ feet
25. $5 \frac{1}{4}$ by $7 \frac{1}{2}$
c. Rectangle $Z$ is an 18 by 30 rectangle
26. Jack and Jill ate at a cafe and the bill (including tax) was $\$ 13.50$. If they leave a $20 \%$ tip, how much was their final bill?
27. Juan and Sherry ate at a cafe and their total bill (with tip) was $\$ 19.25$. If they included a $10 \%$ tip, what was their original bill?
28. A store marks up the wholesale price of a shirt by $35 \%$ to the selling price of $\$ 75.60$. What was the wholesale price of the shirt?
29. A CD is marked down $30 \%$ to the sale price of $\$ 8.82$. What was the original price of the CD?
30. Every evening for dinner, Leopold the rabbit eats $1 \frac{1}{2}$ ounces of leafy greens, such as lettuce, and 1 ounce of carrot.
a. By weight, what percentage of Leopold's dinner consists of carrot?
b. How many ounces of vegetables does Leopold eat in one week, given that he only eats vegetables at dinner?
c. Leopold weighs 5 pounds. How long will it take him to eat his body weight in vegetables?
31. Triangle $A B C$ is similar to triangle $D E F$. What is the measure of $\angle E$ ?

32. At $10: 00$ a tree casts a 20 -foot shadow. At the same time, a 6 -foot man casts a 4 -foot shadow. Write a proportion to find the height of the tree.
33. Katrina was enlargng a photo of her mom for her mom's birthday. She enlarged the photo $150 \%$ of its original size on a copy machine. If the photo is $3 \frac{1}{2}$ inches by 5 inches, what are the dimensions of the enlarged photo?
34. Rectangle EFGH
35. a. $x=24 \mathrm{~cm}$
b. $x=2.8 \mathrm{ft}$
c. $x=5$ in
36. Which rectangle is similar to rectangle RSTU?

37. Find the value of $x$ in each pair of similar figures.
a.

b.

c. 15 in .



Section 10.1: $2 \frac{19}{20}$
Solution: Anne and Bob do $\frac{1}{3}$ of the demolition each day. All three of them together do $\frac{1}{2}\left(\frac{1}{6}+\frac{1}{3}+\frac{1}{5}\right)=\frac{7}{20}$ of the work on the first day, leaving $\frac{13}{20}$, which takes Anne and Bob $\frac{13}{20} \cdot 3=\frac{39}{20}=1 \frac{19}{20}$ more days, for a total of $2 \frac{19}{20}$.

Section 10.2: 10
Solution: We need $2 x+30=\frac{9}{5} x+32$, or $\frac{1}{5} x=2$, so $x=10$.

Section 10.3: $3 \frac{1}{5}$
Solution: Since 2 authors can write 3 pages in 4 days, each author writes $\frac{3}{2 \cdot 4}=\frac{3}{8}$ of a page each day. Thus 5 authors will write $5 \cdot \frac{3}{8}=\frac{15}{8}$ pages each day, so 6 pages will take $6 \div \frac{15}{8}=6 \cdot \frac{8}{15}=\frac{16}{5}=3 \frac{1}{5}$ days.

Section 10.5: 2 or $2 / 1$ or $2: 1$
Solution: Draw a perpendicular from $C$ to side $A B$ intersecting $A B$ at $E$. Note that $C D$ is an altitude for both triangle $A C D$ and triangle $B C D$ and that $B D=5$. The areas of triangle $A C D$ and triangle $B C D$ are respectively $\frac{1}{2}$ (CE)(10) and $\frac{1}{2}(C E)(5)$. So the ratio of their areas is $\frac{10}{5}=2$.

## CHALLENGE PROBLEMS

## Section 10.1:

Carl and Bob can demolish a building in 6 days, Anne and Bob can do it in 3, Anne and Carl in 5 . How many days does it take all of them working together if Carl gets injured at the end of the first day and can't come back?

## Section 10.2:

An estimate for converting temperatures from Celsius to Fahrenheit is to double and add 30 . Thus $0^{\circ} \mathrm{C}$ is approximately $30^{\circ} \mathrm{F}$ (actually 32 ), and $100^{\circ} \mathrm{C}$ is approximately $230^{\circ} \mathrm{F}$ (actually 212). For what temperature in ${ }^{\circ} \mathrm{C}$ is the estimate correct?

## Section 10.3:

If 2 authors can write 3 pages in 4 days, how many days does is take 5 authors to write 6 pages?

## Section 10.5:

In triangle $A B C$ the length of $A B=15 \mathrm{~cm}$. Point $D$ is a point on side $A B$ so that $A D$ $=10 \mathrm{~cm}$. What is the ratio of the area of triangle $A D C$ to the area of triangle $B D C$ ?

## Section 11.1 - Measuring Angles

## Big Idea:

Measuring angles

## Key Objectives:

- Measure angles using a protractor.
- Understand how angles are made, using rays and a vertex.
- Learn how to notate angles symbolically.
- Recognize special angles including $180^{\circ}, 90^{\circ}$, and $45^{\circ}$.
- Define and identify complementary and supplementary angles.


## Materials:

- Protractor, Straight edge, Rope or yarn


## Pedagogical/Orchestration:

- Chapter 10 is a short course in geometry. It contains much more information than the average academic year's requirements. You can treat it like a mathematical cafeteria, choosing what you need and like.
- Working with protractors and straight edges is a useful and, for most students, an interesting skill. Be firm with your students about accuracy and good work habits when drawing and measuring angles.


## Activity:

"What's Your Angle?"; "Protractor Practice"; "Angles all Around Us"; "Lesson Review"; "What's Your Angle"; and "Measuring Angles" on CD and at end of the section

## Vocabulary:

angle, ray, vertex, size of angle, protractor, degrees, straight angle, supplementary, right angle, perpendicular, complementary, acute angle, obtuse angle

## TEKS:

6.6(A); 6.8(C); 7.1(A); 7.2(D); 7.5(A,B); 7.6(A); 7.8(C); 7.9(A); 7.13(A,B,C); 7.14(A); 7.15(A,B) ; 8.5(A);8.10(A); 8.14(A); 8.15(A); 8.16(A)

## Note to teacher:

Have students draw a triangle of their choice at the end of the section in preparation for the 11.2 Launch.

## WARM-UPS for Section 11.1 (Measuring Angles)

1. Let triangle CAT be a right triangle. If angle C has measure of $37^{\circ}$, find the measure of angle T .


Ans: The measure of angle T is $53^{\circ}$.
2. Which of the following is a reasonable estimate of the measure of $P Q R$ ? Explain the reason for your choice.
a. $85^{\circ}$
b. $90^{\circ}$
c. $105^{\circ}$
d. $150^{\circ}$


Ans: (d)

## Launch for Section 11.1:

As we begin the geometry chapter, this Launch is meant as a review of terminology in relation to angles. As a Launch, you may simply brainstorm what the students know about angles, or if you are feeling adventurous, you may take the students outside and review the terms by creating "human angles." Have the students choose one person to be the vertex, and split the remaining students to form the two rays. Ask students which student will be the only person that is a member of both rays. (The vertex). Have the students stand in a straight line with each ray on either side of the vertex. They may hold on to a rope or yarn to represent the line. Tell them this is called a straight angle. Review other terms by having the students move to form a right angle, an obtuse angle, and an acute angle. You may also ask them to make a 45 degree angle and a 180 degree angle to further test their understanding. Once the students are settled back in class, tell them that they did a great job making human angles, and now they will be learning how to make precisely measured angles using a protractor, and they will be learning about some special relationships between certain angles.

## Alternate Launch for Section 11.1:

Have students brainstorm what they know about angles. Make a list of the different types of angles. Make a picture of each type of angle. Describe each type of angle. Make and fill in the table below:

| Type of angle | Example | Definition |
| :--- | :--- | :--- |
|  |  |  |

## Activity:

Have students create a foldable or flip chart of vocabulary, definitions, and drawings. Start with 2 different colors of paper alternating them for a total of 5 pages. Stack.

Vocabulary for Activity:
Point
Line
Line Segment
Ray
Vertex (Vertices)
Angle
Degree
Protractor


## SECTION 11.1 MEASURING ANGLES

How would you answer the question, "What is an angle?"
You have probably seen angles in many places in everyday life. Can you name a few of these places? In this section, you will learn what angles are, how to construct angles, and how to measure the size of an angle.


Let's begin by constructing an angle. To do this, first draw two rays from a common point $P$. A ray is part of a straight line that has a starting point and continues forever in only one direction. In the figure above, there are two rays, $\overrightarrow{P Q}$ and $\overrightarrow{P R}$. Both of these rays begin at point $P$ and pass through the points $Q$ and $R$ respectively.

To name the ray, list the starting point first and then any other point on the ray.
There are many different ways to name it, $\overrightarrow{P Q}$, or ray $P Q$. Picking another point on this ray, for instance $S$, a point between $P$ and $Q$, gives the ray $\overrightarrow{P S}$, which is the same ray as $\overrightarrow{P Q}$. However, the ray $S P$ is a different ray because it begins at point $S$. In fact, $\overrightarrow{S Q}$ is a third ray that is different from all of the rays mentioned so far.


Now we can answer our original question: "What is an angle?"

DEFINITION 11.1: ANGLE
An angle is formed when two rays share a common vertex.
The common endpoint $P$ on both rays is called the vertex of the angle. In the diagram, the rays $\overrightarrow{P Q}$ and $\overrightarrow{P R}$ form an angle called angle $Q P R$ or $\angle Q P R$. The symbol " $\angle$ " is the math symbol for the word angle. To name an angle, you can do the following three steps:

1. Write the name of one of the non-vertex points on one of the rays.
2. Write the name of the vertex.
3. Write the name of the other non-vertex point.

There can be many ways to name the same angle because there are many choices of points on the two rays in steps 1 and 3. You could also label this angle $\angle R P Q$. The order in which the points are written does not matter as long as the middle point identifies the vertex of the angle.

## EXAMPLE 1

In angle $\angle X Y Z$, identify the rays and identify the vertex.

## SOLUTION

$\overline{X Y}$ and $\overrightarrow{Y Z}$ are the rays, and $Y$ is the vertex of the angle. We can use different letters to label different angles, like $\angle Q P R$ or $\angle X Y Z$.

Sometimes a single letter is used to name an angle. Sometimes $\angle Q P R$ is called angle $P$, or $\angle P$, when there is no confusion. Rather, the letter $P$ is the name for the angle, so $\angle Q P R=\angle P$. For instance, in the triangle on the left, $\angle Q P R$ is the same as $\angle P$. In the triangle on the right, however, $\angle Q P R$ is the largest angle, but $\angle P$ could be one of three angles.


Once you understand the definition of an angle, the next step is to measure the size of the angle. Angles are measured as part of a circle. A circle is divided into 360 equal parts. Each part is one degree. A full circle contains 360 degrees, written $360^{\circ}$.

The instrument used to measure angles is called a protractor. When you use a protractor, the units commonly used to measure the size of an angle are degrees as described above.

Protractors have degree markings along the outside of the curved edge. To measure an angle, place the vertex at the center of the semicircle so that one ray passes through $0^{\circ}$ or $180^{\circ}$ and the other ray passes through a mark on the curved edge. If necessary, extend the other ray so that it falls on a mark along the curved edge. The degree at this mark is the measure of the angle, or its supplement, which we will define later in this section.


If two rays with a common endpoint form a straight line, the angle they form has a measure of 180 degrees, or $180^{\circ}$. This is called a straight angle.

## EXPLORATION

Teachers: Measuring of angles with protractors is optional. Enlargement of angles is at the end of this section.

1. a. $\angle J K L$ has measure 65 degrees.
b. $\angle X Y Z$ has measure 90 degrees.
c. $\angle \mathrm{MNO}$ has measure 170 degrees.
d. $\angle \mathrm{UVW}$ has measure 25 degrees.
e. $\angle Q P R$ has measure 160 degrees.
2. $\angle Q P R$ or $\angle R P Q$ or $\angle P$, etc.
3. The sum of the measures should be 180 degrees.

## EXPLORATION

1. Estimate the measure of each angle below and then use a protractor to check your answers.
a.

d.

b.


c.

2. Consider the rays and points above. Write each angle in two ways.
3. Divide a straight angle into two parts. Describe how you constructed it to another student or your teacher. Measure each angle using your protractor.
4. Use your protractor to draw rays making the following angles: $35^{\circ}, 80^{\circ}$, and $100^{\circ}$.
5. $m(\angle Q P R)=64^{\circ}$

How did you construct the angles in question 4? Here is one approach to construct an angle with a given measure.

1. Draw an initial ray and label it $P R$. The initial ray is usually, but not necessarily, horizontal.
2. Place the center of the semicircle of the protractor on top of the point $P$, with the ray passing through $0^{\circ}$.
3. Find the place along the curved edge of the protractor that corresponds to the degree measure you are constructing, and mark it with a new point $Q$.
4. Draw a line connecting the point $P$, which is the vertex of the angle, to the new point $Q$ with a straight edge to obtain an angle with a given measure.
5. What is the measure of the angle QPR below?


## PROBLEM 1

Construct an angle starting at $0^{\circ}$ with measure $45^{\circ}$.

## PROBLEM 2

Construct an angle starting at $20^{\circ}$ with measure $45^{\circ}$.


Mathematicians use the notation $\mathrm{m}(\angle Q P R)$ to mean the measure of angle $Q P R$. Notice that $\angle Q P R$ and $\angle R P S$ divide the $\angle Q P S$ into two parts. The measure of each angle is a number, and we can add these numbers together to get the equations below:

$$
\mathrm{m}(\angle Q P R)+\mathrm{m}(\angle R P S)=\mathrm{m}(\angle Q P S)
$$

$\overrightarrow{Q S}$ is a line, so $m(\angle Q P S)=180^{\circ}$. Then we have

$$
\mathrm{m}(\angle Q P R)+\mathrm{m}(\angle R P S)=180^{\circ} .
$$

## DEFINITION 11.2: SUPPLEMENTARY

Two angles are supplementary if the sum of their measures totals $180^{\circ}$.

In the example above, $\angle Q P R$ and $\angle R P S$ are supplementary angles. This means $\angle R P S$ is the supplement of $\angle Q P R$, and $\angle Q P R$ is the supplement of $\angle R P S$.

Now divide a straight angle in half. Each angle formed is called a right angle and measures $90^{\circ}$ because $\frac{1}{2}$ of $180^{\circ}$ is $90^{\circ}$. We label the right angle with a to indicate that it is $90^{\circ}$. When two lines meet and form a right angle, they are perpendicular to each other.


When two rays meet to form a right angle, they are perpendicular rays.


Next, divide the right angle $P R Q$ above into two parts:


Because $\angle P R S$ and $\angle S R Q$ divide $\angle P R Q$,

$$
\begin{aligned}
& \mathrm{m}(\angle P R S)+\mathrm{m}(\angle S R Q)=\mathrm{m}(\angle P R Q) \text { and } \\
& \mathrm{m}(\angle P R S)+\mathrm{m}(\angle S R Q)=90^{\circ} .
\end{aligned}
$$

## EXERCISES

1. a. Answer will vary. Possible answers: $\angle \mathrm{AGB}, \angle \mathrm{EGF}$
b. Answer will vary. Possible answers: $\angle D G F, \angle A G C$
c. $\angle \mathrm{EGD}, \angle \mathrm{AGE}$
d. $\angle \mathrm{AGB}$ and $\angle \mathrm{BGD}, \angle \mathrm{AGC}$ and $\angle \mathrm{CGD}, \angle \mathrm{AGF}$ and $\angle \mathrm{FGD}, \angle \mathrm{DGE}$ and $\angle \mathrm{EGA}$
e. $\angle A G F$ and $\angle F G E$
2. $140^{\circ}$. Check for accuracy.

## DEFINITION 11.3: COMPLEMENTARY

Two angles are complementary if the sum of their measures totals $90^{\circ}$.

In the example above, $\angle P R S$ and $\angle S R Q$ are complementary angles. This means $\angle S R Q$ is the complement of $\angle P R S$, and $\angle P R S$ is the complement of $\angle S R Q$.

Angles that have a measure between $0^{\circ}$ and $90^{\circ}$ are called acute angles. Angles that have a measure greater than $90^{\circ}$ but less than $180^{\circ}$ are called obtuse angles. Using a protractor, construct and label a straight angle, a right angle, an acute angle, and an obtuse angle.

## EXERCISES

1. Use the drawing to answer the following questions:

a. Name 2 acute angles.
b. Name 2 obtuse angles.
c. Name all right angles.
d. Find all supplementary angle pairs.
e. Find all complementary angle pairs.
2. If $\angle Q P R$ has measure $40^{\circ}$, what is the measure of the angle that is supplementary to $\angle Q P R$ ? Construct the two angles and measure them using your protractor. Do your answers agree?
3. $x+75^{\circ}=90^{\circ} \cdot x=15^{\circ}$
4. $x+20^{\circ}=90^{\circ} \cdot x=70^{\circ}$
5. a. $x+x=180^{\circ} . x=90$
6. $x=$ measure of the other angle
$x+\left(x+10^{\circ}\right)=180^{\circ} .2 x+10^{\circ}=180^{\circ} . x=85^{\circ}$ and $x+10^{\circ}=95^{\circ}$.
7. $x=$ measure of the smaller angle
$x+2 x=90^{\circ} . x=30^{\circ}$ and $2 x=60^{\circ}$.
8. $x+2 x=180^{\circ} . x=60^{\circ}$ and $2 x=120^{\circ}$.
9. $80^{\circ}<x<90^{\circ}$. If you have time, this discussion leads to the density of real numbers. Right now, if your students give you an answer like 9 or 10 , you might ask if an angle like $81.234^{\circ}$ exists. To show any doubting student this fact, ask your students to tell you two angles that are next to each other. Then choose an angle in between them, probably exactly half-way, although that's not necessary. This could go on forever.
10. D
11. A certain angle measures $75^{\circ}$. What is the measure of its complement? Use your protractor to draw angles with these measures.
12. Draw an angle that measures $20^{\circ}$. What is the measure of the complementary angle to the original angle?
13. An angle and its supplement have equal measures.
a. Find their measures.
b. Draw these angles.
14. Two angles are supplementary, and one angle has a measure that is $10^{\circ}$ more than the other angle. What is the measure of each of the angles? (Hint: Call the measure of the original angle $x$; then write an equation involving $x$ and solve the equation.)
15. Two angles are complementary, and one of the angles has a measure that is twice as large as the measure of the other angle. What is the measure of each of the angles? (Hint: Call the measure of your original angle $x$. Write an equation involving $x$ and solve the equation.)
16. An angle's measure is twice its supplement's. What is the measure of the two angles?
17. How many acute angles can you find with measures between $80^{\circ}$ and $90^{\circ}$ ?
18. If $L_{1}$ and $L_{2}$ are parallel, which of the following angles are supplementary to angle A?
a. $\angle 1$
b. $\angle 2$
c. $\angle 3$
d. $\angle 4$


## Ingenuity

11. Discuss pitch with the students. Make sure the students have a picture of the horizontal and vertical parts used to define pitch. Note that the students may use any units of measure. They may observe similar triangles, depending on what units they choose.
a. About $27^{\circ}$
b. About $30^{\circ}$. A $\frac{7}{12}$ pitch is steeper than a $\frac{6}{12}$ pitch.
c. About $14^{\circ}, 18^{\circ}$, and $2^{\circ}$. Draw sides but note units can be anything due to similar triangles.
d. The largest pitch is $\frac{5}{12}$, and the smallest pitch is $\frac{3}{12}$. The angles increase as the pitch ratio increases.

## Investigation

12. This is a historical question that your students can solve most easily on the internet. You might team with the history or science teacher in a joint assignment of these questions.
In ancient Mesopotamia, where writing was invented about 4000 years ago, they counted about 360 days in a year. They added several "leap days" to set the seasons to come out right. In the course of a year, the stars made a full circle of the heavens, so it was natural to break a circle into 360 pieces. This led naturally to 360 degrees and motivated their "base sixty" system of numbers.

## 11. Ingenuity:

The pitch of the roof is a ratio used in place of the angle that the roof makes with a horizontal ray. A common pitch on a roof is given by a right triangle with horizontal side 12 and vertical side 6 . This roof has pitch $\frac{6}{12}$. In fact, $\frac{6}{12}$ gives a measure for how much the roof is sloped.
a. Draw a scale model of a roof with this pitch, and measure the angle of the roof.
b. If the vertical side is 7 , will the roof be steeper or less steep? What is the angle of a roof with a $\frac{7}{12}$ pitch?
c. Draw angles $A, B$, and $C$ that correspond to roofs with pitches $\frac{3}{12}, \frac{4}{12}$ , and $\frac{5}{12}$. Measure each angle.
d. Which pitch corresponds to the largest angle? Which pitch corresponds to the smallest angle? What do you notice about the angles as the pitch ratio increases?

## 12. Investigation:

Do you know historically why there are $360^{\circ}$ in a full circle? Who first used the number 360 ? Why didn't they choose 300 or 400 or 500 degrees to make a circle?

## What's Your Angle?



Objective: To have the students use their protractors and their prior knowledge about angles to construct a picture with exact angle measurements.

## Materials:

Protractor
White, unlined paper
Markers, crayons, or colored pencils

## Activity Instructions:

With their protractors, your students will create a design or drawing that contains EXACTLY the number of each type of angle listed below:

10 Right Angles
15 Acute Angles
8 Obtuse Angles

Once the students have created their drawings, they label their angles appropriately with an R for right angles, A for acute angles, and an O for obtuse angles.

For a challenge, you can increase any number of the angles, or you could add a set number of straight angles to the list.

You might show struggling students some pictures in books or magazines and have them count the number of angles in the pictures.

## Measuring Angles

## Classifying Angles:

A ray is part of a line. Rays have one endpoint and continue forever in one direction.
Angles are formed when two rays meet at a vertex.

## Types of Angles

| Angle Type | Right Angle | Acute Angle | Obtuse Angle | Straight Angle |
| :---: | :---: | :---: | :---: | :---: |
| Draw |  |  |  |  |
|  |  |  |  |  |
| D E |  |  |  |  |
| F I |  |  |  |  |
| N E |  |  |  |  |

## Measuring Angles:

Angles are measured as part of a circle. From its center, a circle is divided into 360 equal parts. Each part is one degree. A full circle contains 360 degrees.

The instrument used to measure angles is called a protractor.


To measure an angle, place the vertex of the angle on the origin (0) and line up one of the angles with the straight edge, or base line, of your protractor.
How do you measure an angle whose rays are not on the baseline?
To identify an angle, letters are used to identify the 2 rays and the vertex that forms the angle. The middle letter identifies the VERTEX. ex. < XYZ ( X and Z are the rays, Y is the vertex).

## Lesson Review

Supplementary and Complementary Angles, See page 311-312 in Math Exploration Book.
Angles are part of a $\qquad$ . A circle measures $\qquad$ _.

A $\qquad$ is an instrument used to measure angles.

An angle is made up of two $\qquad$ .

They share a common $\qquad$ .

An ACUTE angle measures $\qquad$ .

An OBTUSE angle measures $\qquad$ .

A RIGHT angle measures $\qquad$ .

Practice reading and writing angle measures using the model below:

What is the measure of <UST? $\qquad$
What is the measure of $<$ RSV ? $\qquad$

What is the measure of <USV? $\qquad$

Identify two ACUTE angles $\qquad$ ,

Identify a STRAIGHT angle $\qquad$
Identify an OBTUSE angle $\qquad$

Identify two RIGHT angles $\qquad$
$\qquad$


## Angles All Around Us

Find 3 different types of angles in your household. Determine the angle's function, and explain how dysfunctional it would be if it were any other type of angle.

Angle 1: $\qquad$

Description: $\qquad$

Function: $\qquad$

Angle 2: $\qquad$

Description: $\qquad$

Function: $\qquad$

Angle 3: $\qquad$

Description: $\qquad$

Function: $\qquad$

## Protractor Practice



Angle $=$ $\qquad$


Angle $=$ $\qquad$


Angle $=$ $\qquad$


Angle =
a.

b.

c.


1192

## Section 11.2-Angles in a Triangle

## Big Ideas:

Discovering relationships between special angles

## Key Objectives:

- Discover relationship between vertical angles: Vertical Angle Theorem.
- Recognize parallel lines and their transversal.
- Discover relationship between angles created when 2 parallel lines are cut by a transversal: Corresponding Angle Postulate.
- Discover relationship between angles in a triangle: Triangle Sum Theorem.
- Understand the basic vocabulary of proofs.
- Use algebra to discover missing information about the angles in triangles.


## Materials:

- Straight edge, Protractor


## Pedagogical/Orchestration:

- This section lays the groundwork for high school geometry, so inform your students that they will see this again. Although the discoveries in this book are shown and not proved, this section gives students a taste of geometric proof.
- Encourage your students to develop strong habits in constructing accurate geometric models. Attention to detail and resistance to sloppy habits will reward students handsomely.


## Activity:

"Tessellation Project" on CD and at end of section

## Vocabulary:

triangle, proof, vertical angles, parallel lines, transversal, postulate, axiom, congruent, corresponding angle, theorems, conjecture, tessellation, trisected, intersecting lines, right triangle, polygon, quadrilateral

## TEKS:

6.13(B);
7.1(A); 7.5(A)(B);
7.6(A); 7.13(A)(B)(C);
7.14(A);
7.15(A)(B);
8.5(A);
8.9(B);
8.14(A);
8.15(A);
8.16(A)

NOTE TO TEACHER: Make sure students have completed their 8-10 triangles in previous day's homework in order to have a successful launch.

## WARM-UPS for Section 11.2 (Angles in a Triangle)

1. Which of the following groups of numbers is not the lengths of a triangle?
a. $\{6 \mathrm{in}, 7 \mathrm{in}, 12 \mathrm{in}\}$
b. $\{2 \mathrm{in}, 7 \mathrm{in}, 7 \mathrm{in}\}$
c. $\{6 \mathrm{in}, 7 \mathrm{in}, 14 \mathrm{in}\}$
d. $\{2 \mathrm{in}, 7 \mathrm{in}, 5 \mathrm{in}\}$

Ans: (c) because the lengths of any two sides must be greater than the length of the third side.
2. In a bag of marbles, the ratio of green marbles to red marbles is $5: 3$ and the ratio of blue marbles to red marbles is $3: 4$. What is the ratio of green marbles to blue marbles?
a. $20: 9$
b. 5:4
c. 8:7
d. 15:12
Ans: (a) because $\left(\frac{5 \text { green }}{3 \text { red }}\right)\left(\frac{4 \text { red }}{3 \text { blue }}\right)=\frac{20 \text { green }}{9 \text { blue }}$, so the ratio of green to blue is 20:9.

## Launch for Section 11.2:

Assign Launch as homework:
Working in groups, have students draw a triangle on $\frac{1}{4}$ of a page (on tag board if possible). Once 1 triangle has been drawn, have students cut and carefully trace 8-10 more. Students are to take this home, cut them out carefully, and bring them back to class. Have students take out their triangles. Using tessellation, set up their triangles to create a straight angle and/or a circle. Lead the class in the discussion of degrees in a straight angle and a circle.

Key Concepts: Triangle sum theorem; Use tessellation to prove theorem.

## EXPLORATION 1

Make sure your students use a straight edge to draw the lines. Otherwise, they will not be able to measure the angles correctly.

The sum of the angles on the same side of the line is $180^{\circ}$. The opposite angles have the same measure. They are called vertical angles. The sum of the 4 angles formed is $360^{\circ}$. There are 4 pairs of supplementary angles from the first observation.

Review supplementary angles from the previous section.

## SECTION 11.2 PARALLEL LINES AND ANGLES IN A TRIANGLE

In the previous section you measured angles and worked with complementary and supplementary angles. These ideas are very useful for studying triangles and other shapes in geometry.

## EXPLORATION 1

Draw two straight lines that intersect at a point $P$. Label several other points on both lines, and use these to name all four of the angles formed by the two lines. Measure the angles using your protractor. What do you notice about the angles? What do you notice about the sums of the angles?

Repeat the Exploration using a different pair of intersecting lines. What do you notice about the measures of the angles? Write a sentence or equation that describes what you have discovered about the angles made by intersecting lines.

Consider the four labeled angles formed by the two intersecting lines:


If two of these angles share a common ray between them, then the angles are said to be adjacent. Note that two angles might share a common ray but not be adjacent if the ray is not between the two angles. In the figure above, $\angle 1$ and $\angle 2$ are adjacent, and similarly, $\angle 1$ and $\angle 4$ are also adjacent. What other pairs of adjacent angles can you find?

If the two adjacent angles have their non-adjacent rays lying on a straight line, then the two adjacent angles will have measures that add up to $180^{\circ}$. So, $\mathrm{m}(\angle 1)+\mathrm{m}(\angle 2)=180^{\circ}$. Do you see why this is true?

From the same figure, consider a pair of angles that are not adjacent. These angles form a pair of vertical angles.

In the figure, $\angle 1$ and $\angle 3$ are vertical angles, and $\angle 2$ and $\angle 4$ are vertical angles. This is described more precisely in the following definition:

## DEFINITION 11.4: VERTICAL ANGLES

If two straight lines intersect at a point, then each line is divided into two rays. The angles formed by using the opposite rays from each line segment are called vertical angles.

Note: Opposite rays are two rays with a common endpoint that form a straight line.
When you measure each pair of vertical angles, do you always get the same answer? It seems to be the case that any two vertical angles will always have the same measure. Let's explore this in the example that follows.

## EXAMPLE 1

For each pair of supplementary angles, write down a corresponding equation that expresses the relationship between the two supplementary angles.


## SOLUTION

From the picture, $\angle 1$ and $\angle 2$ are supplementary because their sum totals $180^{\circ}$, and they lie on line $m$. Also $\angle 1$ and $\angle 4$ are supplementary because together they make a $180^{\circ}$ angle, and they lie on line $n$. So,

$$
m(\angle 1)+m(\angle 2)=180^{\circ} \text {, and } m(\angle 1)+m(\angle 4)=180^{\circ} \text {. }
$$

From these two equations, you can substitute for $180^{\circ}$ and obtain

Ask your students if they believe lines $L$ and $M$ are parallel. How do they know whether the lines are parallel?

$$
m(\angle 1)+m(\angle 2)=m(\angle 1)+m(\angle 4) .
$$

Subtracting $m(\angle 1)$ from both sides of this equation, you get that

$$
m(\angle 2)=m(\angle 4) .
$$

This is a proof of your observation that vertical angles always have the same measure! This is a famous theorem, called the Vertical Angle Theorem:

THEOREM 11.1: VERTICAL ANGLE THEOREM
If two lines intersect at a point $P$, then the vertical angles will have the same measure.

You have begun the process of exploring problems in geometry. The fun part is seeing why observations that you make always hold true. Let's think carefully about what seems like a simple concept, the idea of "parallel" lines. The question is how you decide whether two lines are actually parallel. In fact, what does it mean to say that they are parallel in the first place?

One way to describe parallel lines is to say that two lines in a plane are parallel if they never intersect, even if they are extended forever in both directions. The question is how to decide whether the lines will have this property or not.

This is a problem studied by the ancient Greeks. One approach they took was to begin by adding in a third line called a transversal. A transversal is a line that intersects the pair of lines that you begin with. For instance, in the drawing, line $N$ transverses, or goes across, lines $L$ and $M$. The transversal cuts the two lines $L$ and $M$. What they observed was the following:

1. Begin with two original lines.
2. Cut these with a transversal.

3. Form a pair of corresponding angles using part of the transversal as one ray and the part of the lines on the same side of the transversal as the other ray. (See picture.) Note that $\angle 1$ and $\angle 2$ are corresponding angles, and $\angle 3$ and $\angle 4$ are also corresponding angles. So, corresponding angles are pairs of angles on the same side of the transversal.
4. Observe that $\mathrm{m}(\angle 1)=\mathrm{m}(\angle 2)$, and $\mathrm{m}(\angle 3)=\mathrm{m}(\angle 4)$.

The Corresponding Angle Postulate of Euclid says the following:

## THEOREM 11.2: CORRESPONDING ANGLE POSTULATE

If two parallel lines are cut by a transversal, then the corresponding angles have the same measure, and if the two lines are cut by a transversal so that the corresponding angles have the same measure, then the two lines are parallel.

What this says is that two lines will be parallel precisely when the corresponding angles from a transversal are equal!

## EXAMPLE 2

Lines M and N are parallel, and lines O and P are transversals. What are the measures of angles A and B ?


## SOLUTION

Using the corresponding angle postulate, we know that $\mathrm{m}(\angle \mathrm{D})+70^{\circ}=135^{\circ}$ because line 0 is a transversal of lines $M$ and $N$. Therefore, the $m(\angle D)=135^{\circ}-70^{\circ}$ which equals $65^{\circ}$. Since angles $D$ and $B$ are vertical angles, and the $m(\angle D)=65^{\circ}$, then the $m(\angle B)=65^{\circ}$.

Using the definition of supplementary angles, $m(\angle C)+135^{\circ}=180^{\circ}$. Therefore, the $\mathrm{m}(\angle \mathrm{C})=180^{\circ}-135^{\circ}$, which equals $45^{\circ}$.

Since line $O$ is a transversal of lines $M$ and $N$, then angles $E$ and $C$ are corresponding. Since the $m(\angle C)=45^{\circ}$, then the $m(\angle E)=45^{\circ}$.
Using the corresponding angle postulate, we know that $m(\angle D)+m(\angle E)=m(\angle A)$ because line $P$ is a transversal of lines $M$ and $N$. Since the $m(\angle D)=65^{\circ}$ and the $m(\angle E)=45^{\circ}$, then the $m(\angle A)=65^{\circ}+45^{\circ}=110^{\circ}$.

## EXAMPLE 3

In a triangle $A B C, m(\angle C)=108^{\circ}$ and the measure of $\angle \mathrm{A}$ is twice the measure of $\angle B$. What are the measures of angles $\angle A$ and $\angle B$ ?

## SOLUTION

Call the measure of $\angle A=x$ and the measure of $\angle B=2 x$. By the Triangle Sum Theorem,

$$
\begin{gathered}
m(\angle \mathrm{~A})+m(\angle \mathrm{~B})+m(\angle \mathrm{C})=180^{\circ} \\
x+2 x+108=180^{\circ} \\
3 x+108^{\circ}=180^{\circ} \\
3 x=180^{\circ}-108^{\circ}=72^{\circ} \\
3 x=72^{\circ} \\
x=\frac{72}{3}=24^{\circ}, \text { so } A=24^{\circ} \text { and } B=48^{\circ} .
\end{gathered}
$$

## EXPLORATION 2

Each group should do this activity and report their results. This should produce a variety of triangles to compare. The rule they will notice is that the sum of the measures of the angles in a triangle is always close to $180^{\circ}$ (it might not be exactly $180^{\circ}$ due to slightly inaccurate measurements and rounding errors).

Make sure your students color or label inside the triangle so that, when they cut it apart, they can identify the original angles.

## EXPLORATION 3

Use the lined paper in the CD, if you want.

## EXPLORATION 2

Draw a large triangle on a sheet of paper using a straight edge. Color or label the three angles of the triangle with different colors. Carefully cut out the triangle. Next, cut the triangle into 3 triangular pieces, each including one angle from the triangle. Put the angles together. What is the sum of the three angles of the triangle? Compare your result with others.

In each case, the sum of the measures of the angles in the triangles appears to be $180^{\circ}$. This leads to a conjecture: "The sum of the measures of the angles in any triangle is $180^{\circ}$." This is a conjecture because it is a statement we think might be true based on our observations, but we have not yet proved it is always true. Is there a way to give a convincing argument or proof that the sum of the measures of the angles in any triangle is $180^{\circ}$ ? To answer that, we investigate further with the next explorations.

## EXPLORATION 3

Divide the class into groups. In each group, make a small triangle and several copies of it on lined paper using a straight edge. Be as exact in your work as possible. Use these copies to tessellate the paper, or plane. A tessellation, or tiling of the plane with some shape, is a way of covering the plane with that shape with no gaps. This tessellation can be used to show that the sum of the measures of the angles of any triangle adds up to $180^{\circ}$.

To do this, begin by putting your triangles together to make a series of equal four-sided figures whose opposite sides are parallel. Use these to cover the plane. Label one of the triangles $A B C$. Place it in the middle of a sheet of paper and, using a straight edge, draw lines parallel to the three sides. Then draw three sets of equally-spaced parallel lines like the example.


## EXERCISES

1. a. They are vertical angles.
b. $m(\angle 2)=105^{\circ}, m(\angle 3)=75^{\circ}$, and $m(\angle 4)=75^{\circ}$.
c. $\mathrm{m}(\angle 2)=x, \mathrm{~m}(\angle 3)=180^{\circ}-x$, and $\mathrm{m}(\angle 4)=180^{\circ}-x$.

This forms tessellated tiles in which each tiling piece is a triangle congruent to triangle $A B C$. Congruent means that all the tiling pieces, or triangles, have exactly the same size and shape. Label each of the angles in the picture $A, B$, or C. One way to see that an angle with measure $A$ appears at different places is to use the Corresponding Angle Postulate and the Vertical Angle Theorem. It is now easy to see something quite remarkable: the measures of angles $A, B$, and $C$ sum to a straight angle. Explain why.

We can now state the Triangle Sum Theorem:

## POSTULATE 11.2: TRIANGLE SUM THEOREM

The sum of the measures of the angles in any triangle equals $180^{\circ}$.

The tessellation is a sketch of the proof that the sum of the measures of the angles in a triangle always adds up to $180^{\circ}$. In geometry, you will learn how to prove many interesting properties of geometric shapes using only the basic ideas above.

Much of the geometry that you study in middle school comes from the studies developed by Euclid several thousand years ago. He based the study of geometry on the foundations of axioms, postulates, and theorems. Part of the excitement of mathematics involves seeing new relationships based on simple ideas, like corresponding angles.

## EXERCISES

1. Answer parts $\mathrm{a}, \mathrm{b}$, and c based on the figure below.

2. a. It is a transversal.
b. They are corresponding angles. $\quad \mathrm{m}(\angle 1)=\mathrm{m}(\angle 2)$
c. They are parallel.
3. $30^{\circ}$
4. $60^{\circ}$
5. $45^{\circ}$
6. $60^{\circ}$
a. Write an equation that describes the relationship between $\angle 1$ and $\angle 2$ and between $\angle 3$ and $\angle 4$ ?
b. If $m(\angle 1)=105^{\circ}$, what are the measures of the other angles?
c. Write equations that express the measures of $\angle 2, \angle 3$, and $\angle 4$ if the measure of $\angle 1=x$.
7. Given that lines $M$ and $N$ are parallel, answer parts $a, b$, and $c$ based on the figure below.

a. What term describes line $L$, which crosses line $M$ and line $N$ ?
b. Write an equation to describe the relationship between $\angle 1$ and $\angle 2$ ?
c. If you are only given that $\mathrm{m}(\angle 1)=\mathrm{m}(\angle 2)$, what can you conclude about line $M$ and line $N$ ?

Find the measure of the angle missing in each of the triangles in Exercises $3-6$.
3.

5.

4.

6.

7. $75^{\circ}$
8. $45^{\circ}, 90^{\circ}$
9. $60^{\circ}$
10. $30^{\circ}$
11. $x=30^{\circ}, 2 x=60^{\circ}, 3 x=90^{\circ}$
12. a. $145^{\circ} \quad$ b. $35^{\circ}$
13. a. $70^{\circ}$
b. $110^{\circ}$
c. $115^{\circ}$
7. A triangle has two angles whose measures are $45^{\circ}$ and $60^{\circ}$. Find the measure of the third angle.
8. A right triangle is a triangle where one of the angles has measure $90^{\circ}$. If a right triangle has another angle whose measure is $45^{\circ}$, find the measure of the third angle.
9. All of the angles in $\triangle A B C$ have the same measure. What is the measure of each of the angles?
10. A $90^{\circ}$ angle is trisected, or divided into three equal angles. What is the measure of each of these angles?
11. A triangle has three angles with measures $x, 2 x$, and $3 x$. Find the measure of each of these angles.
12. Lines J and K are parallel. Lines L and M are transversals of lines J and K .

a. Calculate the measure of angle A.
b. Calculate the measure of angle $B$.
13. Lines $R$ and $Q$ are parallel. Lines $S$ and $T$ are transversals of lines $R$ and $Q$.


## Ingenuity

14. The sum of the measures of the angles in any quadrilateral is $360^{\circ}$. To see this, divide the quadrilateral into two triangles and use the fact that the sum of the measures of the angles in any triangle is $180^{\circ}$.

A pentagon can be divided into 3 triangles. A hexagon can be divided into 4 triangles. In general, an $n$-gon can be divided into $(n-2)$ triangles. Generalizing, the sum of the angles in an $n$-gon is $(n-2) 180^{\circ}$. Start by making an organized chart, recording number of sides and sums. You can make this easier if you can construct and measure regular polygons, like equilateral triangles and squares, whose sides are all equal. Add the measure of each angle to your chart and look for patterns.
After you have found the formula, you can prove that you are right by constructing triangles inside the other polygons. A student might ask how you can do this. If you choose one of the vertices of the n-gon, and then connect each other vertex with this point by a segment, there will be 2 fewer triangles than there are number of sides in the polygon.
a. Calculate the measure of angle A.
b. Calculate the measure of angle B.
c. Calculate the measure of angle C .
14. Investigation:

The word polygon comes from the Greek words poly, meaning "many," and gon, meaning "angles." A polygon is made by joining a number of line segments to make a closed shape. Each polygon can have many sides. The simplest polygon is a triangle. A quadrilateral is a polygon with "quad" or 4 sides.
a. Show that the sum of the angles of every quadrilateral is the same. What is this sum?
b. Show that the sum of the angles of every pentagon is the same. What is the sum?
c. Find the sum of the angles in an $n$-gon, a polygon with $n$ sides.


Objective: The students will learn about M.C. Escher and his marvelous mathematical and artistic work with tessellations. After previewing the works of Escher, the students will use their mathematical skills to make their own tessellated creations.

## Materials:

Books from the library on M.C. Escher. If you are able to display images from your computer, you can find numerous websites with his works, as well as student designed tessellations.
Cardstock (cut into 4"x4" squares)
Scissors
Tape
White, unlined paper for rough drafts
White or lightly colored poster paper. Lightly colored construction paper works well, too.
Markers, crayons, or colored pencils

## Activity Instructions:

1) Spend some time sharing with your class some background information about the mathematician M.C. Escher. You can find books in your library or information on the web site that your students find interesting. If possible, try to find as many examples of tessellations, created by Escher and others to share with the class. After your students have seen enough examples to have some idea what a tessellation looks like, they are ready to begin the activity. Distribute a 4"x4" square of cardstock and a pair of scissors to each student.
2) Demonstrate for the class how to make two cuts, either curvy or jagged, on the $4 \times 4$ " square. One cut is on a vertical edge, and the other cut is on a horizontal edge. After making the first cut, slide, or translate, that cut piece to the other parallel side of the 4 " $\times 4$ " square, and tape it back on to make sure that the cut pieces will fit well. Follow the same instructions for the second cut.
3) When finished, the students will no longer have a square, but instead a new shape that resembles a puzzle piece. Hand each student a sheet of rough draft paper. Have her set her puzzle piece directly in the middle of this sheet of paper and trace it She will then continue tracing her puzzle piece until the entire sheet of paper is covered with the tracings. There should be no gaps, spaces, or overlapping anywhere on the paper.
4) This is when things get fun! Now the students try to take this plain white piece of paper and turn it into something original and creative. Sometimes their tracings will resemble a car, duck, person, star.... The sky's the limit. Once their creative minds envision a picture in their design, they can start drawing and coloring.
5) When they have finished their rough drafts, they will bring them to you and you will check over their work. If you have any suggestions or comments, please share them at this time. Then, they are ready for their poster paper so they can redo the whole thing again as a final draft.

Just a couple of tips:
*I have done this project for several years, and the students always enjoy it. Some students will finish very quickly and want to do another poster. Instead, I suggest you try to pair them with a student who is struggling to complete his or her project.
*Some students need help turning their rough draft tracings into a design. Encourage them to turn their paper in several directions. Sometimes the new perspective will help inspire them to see a design. If not, put them on a computer where they can research examples of tessellations. Sometimes examples help stimulate their creativity. Another idea is to allow them to walk around the room with their design and ask for input from their peers. It's amazing how creative students can be.
*Please don't let your students be sloppy. If they work carefully and take pride in their work, they will create something wonderful. It can hang in your room or on the walls in the hallway. Some exceptional pieces might even be framed and hung in the office or library.

## Section 11.3 - Two-Dimensional Figures

## Big Idea:

Discovering properties of certain two-dimensional figures

## Key Objectives:

- Recognize different types of triangles from their properties.
- Explore the properties of a parallelogram.
- Derive the formulas for the areas of a triangle and parallelogram.
- Apply the formulas to everyday situations.


## Materials:

- Grid paper, Protractor, Straight edge


## Pedagogical/Orchestration:

- In a departure from regular introductions to triangles, this section introduces triangles with parallelograms. The book's treatment is logical and leads to a discovery of the area formulas for both triangles and parallelograms.
- Although most of the time this book demonstrates patterns that lead to mathematical insight, this section introduces middle school students to formal proof, without the rigor of a traditional geometry course.


## Internet Resource:

Rags to Riches: Perimeter and Area- http://www.quia.com/rr/91670.html

## Activity:

"Triangle Picture Book;" "Formula Supplement;" "Quadrilateral Angles Puzzle;" "Polygon Practice;" "Triangle Angles Puzzle;" "Polygon Riddles;" and "Two-Dimensional Shapes" on CD and at the end of the section

## Exercises:

In the exercises, students will explore how to compute the areas and perimeters of different shapes. Encourage them to divide the shapes into triangles, parallelograms, and rectangles and find the total area by adding up the sums of the areas of the parts.

## Vocabulary:

area, perimeter, attribute, equilateral triangle, isosceles triangle, scalene triangle, right triangle, hypotenuse, opposite, legs, acute triangle, obtuse triangle, congruent triangles, parallelogram, base, height, altitude, (See CD: quadrilateral, square, rectangle, polygon, regular polygon, pentagon, hexagon, octagon).

## TEKS:

6.6(B); 6.8(B); 6.13(B); 7.4(A); 7.6(B); 7.9(A); 7.13(A,B,C); 7.14(A); 7.15(A,B); 8.4(A); 8.9(B); 8.14(A); 8.15(A); 8.16(A)

## WARM-UPS for Section 11.3 (Two-Dimensional Figures)

1. If the right triangle $A B C$ has legs of lengths 4 and 7 centimeters, which of the following is the area of this triangle?
a. $22 \mathrm{~cm}^{2}$
b. $28 \mathrm{~cm}^{2}$
C. $14 \mathrm{~cm}^{2}$
d. $11 \mathrm{~cm}^{2}$
2. Suppose the following angles have the given measures. Which three angles could be the angles of a triangle?

| Angle | Measure |
| :---: | :---: |
| A | $37^{\circ}$ |
| B | $44^{\circ}$ |
| C | $52^{\circ}$ |
| D | $89^{\circ}$ |
| E | $105^{\circ}$ |

Ans: Angles A, C and D because the sum of their measures is $180^{\circ}$.

## Launch for Section 11.3:

Ask students, "What is a polygon?" Lead the class in a discussion of what a polygon is. Then draw a circle and ask "Is this a polygon?" Lead the class in a discussion. Draw several examples of polygons and non-polygons. Make sure to include the following pictures:


Have students discuss in groups the figures' similarities and differences. Have students share with the class their ideas of polygons and their characteristics. Lead students in a discussion and creation of a table into different types of polygons, their names, and number of sides. (See end of section for copy of handout.)

## SECTION 11.3 TWO-DIMENSIONAL FIGURES

The word polygon comes from the Greek words poly, meaning "many", and gon, meaning "angles." A polygon is made by joining a finite number of line segments to make a closed shape. Each polygon can have many sides. The simplest polygon is a triangle. A quadrilateral is a polygon with "quad" or four sides.

## DEFINITION 11.5: POLYGON

A polygon is a simple, closed, plane figure formed by 3 or more line segments.

In a regular polygon, the line segments are equal, and the interior angles are congruent. An irregular polygon is a polygon that is not a regular polygon.

Look at the table below:

| \# of sides | Name | Regular Polygon | Irregular Polygon |
| :---: | :---: | :---: | :---: |
| 3 | Triangle |  |  |
| 4 | Quadriateral | $\square$ |  |
| 5 | Pentagon |  |  |
| 6 | Hexagon |  |  |
| 7 | Heptagon |  |  |
| 8 | Octagon |  |  |
| 10 | Denagon |  |  |
| 11 | Undecagon |  |  |
| 12 | Dodecagon |  |  |

Have students discuss and present different ways to compute the perimeter: $L+W+L+W$, or $2 L+2 W$, or $2(L+W)$.

## EXPLORATION 1

Materials: Ruler and protractor.
a. Students should notice that the sum of any two sides is greater than the third side.

| \# of sides | Name | Regular Polygon | Irregular Polygon |
| :---: | :---: | :---: | :---: |
| n | n -gon |  |  |

A polygon with 21 sides is called a 21 -gon. In this section, we will focus on 3 and 4 sided polygons. We will study different attributes of these figures, such as perimeter and area.

What is the definition of perimeter? What is the perimeter of a $3 \times 4$ rectangle? In general, if a rectangle has length $L$ and width $W$, what is the perimeter of the rectangle?

Just as you found the perimeter of a rectangle by adding the lengths of the four sides, you can find the perimeter of any polygon.

## DEFINITION 11.6: PERIMETER

The perimeter of a polygon is the sum of the lengths of all of its sides.

If the units of the length and width are inches, then the perimeter is measured in inches, but the area is measured in square inches. Each square inch corresponds to a unit square.

Next, look at other shapes.
Triangles can be classified by different properties - their size, shapes, and angles. In Exploration 1, you may discover different properties of triangles from making measurements of some of these properties.

## EXPLORATION 1

a. Draw a line segment that is five units long. Now draw two other line segments to complete a triangle. Repeat this process several times. What do you notice about the sum of the lengths of the other two sides relative to the length of the original side?
b. The angles should be equal, depending on how accurately the students drew the triangle. Drawing an equilateral triangle can be difficult with only a protractor and a ruler, in particularl, if they have not discovered that the angles all must be equal.
c. The students should notice that the base angles in an isosceles triangle are equal. This foreshadows Problem 1 and 2.
b. Make a triangle with all three sides of equal length. In groups, reflect on what you notice about each of these triangles. Measure and then make a rule about the angles of a triangle with all three sides of equal length.
c. Make a triangle with two of the sides of equal length. In groups, reflect on what you notice about each of these triangles. Measure and then make a rule about the angles of a triangle having two sides of equal length.
d. Make up three different triangles whose sides are different lengths. Measure the angles opposite each of the sides. What do you notice about the measures of the angles opposite the larger sides?

A triangle in which all three sides have the same length is called an equilateral triangle. This term comes from the Greek equi, meaning "the same," and Latin latus, meaning "side." You have probably discovered in the Exploration that all the angles of an equilateral triangle have the same measure.


## DEFINITION 11.7: EQUILATERAL TRIANGLE

An equilateral triangle is a triangle with three congruent sides. An equilateral triangle also has three congruent angles and is also called an equiangular triangle.

From previous years, you may recall another type of triangle with two equal sides.


PROBLEM 1
Students should generalize that isosceles triangles have at least two congruent angles. These congruent angles are the angles opposite the congruent sides.

## DEFINITION 11.8: ISOSCELES TRIANGLE

A triangle with at least two sides of equal length is called an isosceles triangle.

## EXPLORATION 2

Draw two different isosceles triangles with two equal sides of length 1 inch. Measure each of the angles.

## PROBLEM 1

Draw two isosceles triangles with two equal sides of length 2 inches but different lengths for the third sides. Measure each of the angles.

In each of the problems above, did you notice that the angles opposite the equal sides are also equal? This is actually one of the properties of all isosceles triangles. In an isosceles triangle, the angles opposite the equal sides are always equal. Conversely, if two of the angles in a triangle are equal, then the sides opposite these equal angles will be equal, and the triangle will be isosceles. These are properties that you will learn when you study geometry. Do you see why they might be true?

It is possible that all three sides of a triangle have different lengths. We call this type of triangle a scalene triangle. In a scalene triangle, the three angles will all have different measures.


## DEFINITION 11.9: SCALENE TRIANGLE

A triangle with all three sides of different lengths is called a scalene triangle.

One reason that the sum of the two legs of a right triangle is always greater than the hypotenuse is that the shortest distance between two points is a straight line. If the sum of the two legs was less than the hypotenuse, that sum would be a shorter way to go from the end two points of the two legs than the straight line, but that is the hypotenuse.

After asking these questions and having a class discussion, you can pass out the Classifying Triangles Chart.

Categorizing the triangles by angles rather than sides, one kind of triangle is a right triangle. A right triangle is a triangle with a right angle, an angle whose measure is $90^{\circ}$.


The longest side of a right triangle is called the hypotenuse. The right angle is opposite the hypotenuse. The two shorter sides are called the legs of the right triangle.


You will learn a special theorem that relates the lengths of the legs of a right triangle to the length of the hypotenuse. This theorem, the Pythagorean Theorem, enables you to find the length of any side of a right triangle if you are given the lengths of the other two sides.

In addition to right triangles, there are other ways to classify triangles by their angles. If all three angles of a triangle are acute, or less than $90^{\circ}$, the triangle is called an acute triangle. If one of the angles is larger than $90^{\circ}$, the triangle is called an obtuse triangle. Is it possible for a triangle to have two angles larger than $90^{\circ}$ ? Explain.

## CLASSIFYING TRIANGLES ACTIVITY:

1. Can an equilateral triangle be a right triangle? Justify your answer.
2. Is it possible for a scalene triangle to be an acute triangle? Justify your answer.
3. Can you draw a triangle that is both obtuse and isosceles? Justify your answer.
4. Is it possible for an obtuse triangle to also be equilateral? Justify your answer.

## ACTIVITY

Name all the different kinds of quadrilaterals that you can. Name the attributes that distinguish each type of quadrilateral from others.

Brainstorm in groups and make a list of these on the board. List conditions on sides and angles that distinguish each type, i.e, a parallelogram has two pairs of opposite parallel congruent sides, and the opposite angles are congruent. Eventually, use power point from CD.

## QUADRILATERALS

A four-sided polygon is called a quadrilateral.


## CLASSIFYING QUADRILATERALS ACTIVITY

1. Is every rectangle a parallelogram? Justify your answer.
2. Is a trapezoid a parallelogram? Justify your answer.
3. Is every square a rectangle? Justify your answer.

## AREA OF POLYGONS

We will now discuss the area of a parallelogram.

## DEFINITION 11.10: PARALLELOGRAM

A parallelogram is a four-sided figure with opposite parallel sides.


## EXPLORATION 3

Draw four parallelograms using grid paper. For this Exploration, make sure the longest side is on one of the grid lines.
a. Measure the length of each of the sides and the measure of each angle. What do you observe?
b. Find the area of one of the parallelograms by cutting the parallelogram apart, as illustrated below, and reassembling it to make a rectangle.


Label one of the horizontal parallel sides of the parallelogram the base, with length b. To find the height, draw a line segment between the two bases, perpendicular to each base. The height, $h$, is the length of the perpendicular distance between the two bases. Notice that the height is not the same as the length of either of the two non-horizontal sides. What is the formula for the area of the new figure?


When reassembled, the parallelogram creates a rectangle with length, or base, $b$, and width, or height, $h$. That means the formula for the area $A$ of a parallelogram is:

FORMULA 11.1 AREA OF A PARALLELOGRAM

$$
A=b \cdot h \text { or } A=b h .
$$

However, what happens if you have a long, skinny parallelogram, $A B C D$ ? IN this case, in order to find the height, you will have to extend the base at the bottom. The height is the length of a perpendicular from the top to the bottom. Does our formula still work? In order to investigate this situation, we enclose our original parallelogram in a bigger rectangle AEDF as shown.

EXAMPLE 1


For shorthand, we label the area of $A B C D$ as $[A B C D]$ and the area of triangle $A D F$ as $[A D F]$. If we have a figure with vertices $P Q R$, we will put square brackets around these vertices to indicate the area of the figure.
a. What is the width of the rectangle $A E C F$, i.e. what is the length of side $A E$ ?
b. What is the length of the rectangle $A E C F$, i.e. what is the height of the rectangle?
c. What is the total area of the rectangle $A E C F,[A E C F]$ ?
d. What is the sum of the areas of the two triangles $A D F$ and $B E C$ ?
e. What is the area of parallelogram $A B C D$ ?

## SOLUTION

a. The width of the rectangle is $A B+B E=b+x$.
b. The height of the rectangle is $h$.
c. The total area of rectangle $A E C F$ is $[A E C F]=(b+x) \bullet h=b h+x h$ (by the distributive property)
d. The two right triangles $A D F$ and $B E C$ can be put together to form a rectangle. This rectangle has area $h x$. Each triangle has area $[A D F]=[B E C]=$ $\frac{1}{2}(h x)$
e. The Sum of areas of the two triangles and the area of the parallelogram equals the total area of the rectangle.

$$
\begin{gathered}
[A D F]]+[B E C]+[A B C D]=[A E C F] \\
\left(\frac{1}{2}\right) h x+\frac{1}{2}(h x)+[A B C D]=b h+h x \\
h x+[A B C D]=b h+h x \\
{[A B C D]=b h, \text { exactly as before! }}
\end{gathered}
$$

Rectangles are special parallelograms with four right angles. In this case, the height of the parallelogram is nothing more than the width of the rectangle, which produces the original formula for the area of a rectangle.


## EXPLORATION 4

Using the grid below, find the area of each triangle.
Calculate the area of the triangles below. How does the area of each rectangle relate to the area of the triangle inside it?


## EXPLORATION 4 (See worksheet at the end of the section.)

a. Expect students to cut and piece together unit squares to find the total area. The ways in which they do this vary and will indicate answers for part (b).
b. They might see that each triangle is half of a rectangle with the length of the rectangle being the base of the triangle and the width of the rectangle being the height of the triangle. If they don't discover this, move on to part (c).
c. The students might discover and need to name the "height" of the triangle. Let them discover that this measurement is useful. Help them discover the connection between the rectangle with area $L \cdot W=$ base $\cdot$ height $=b \cdot h$.

a. How did you compute the areas of each triangle?
b. What patterns did you notice? Explain.
c. Using the triangles above, make a copy of each triangle and paste it together with the original triangle. What shape do you get? Use this process to find a rule for the area of these triangles.

In the exploration above, you were able to put together two triangles of any shape to form a parallelogram. Another way of saying this is that you can decompose a parallelogram into 2 congruent triangles.


Note that for an obtuse triangle, the height is outside the triangle. It should be pointed out that the length of the base does not change despite the extension of the base used to indicate the height.

$$
\begin{gathered}
\text { So }[A B D C]=[A B C]+[B C D]=2 \bullet[A B C] \\
\frac{1}{2}[A B D C]=[A B C]
\end{gathered}
$$

So, what is the area of a triangle, and is there a formula to compute this area? You have seen that taking any triangle, copying it exactly, and putting the two triangles together creates a parallelogram. Use the formula for the area of the parallelogram and take one-half of it to compute the area of the triangle. Each of the triangles will have area that equals $1 / 2$ the area of the parallelogram. Be careful in identifying the base and the height of the triangle. The base, $b$, must be a side of the triangle, and the height, or altitude, $h$, must be perpendicular to the base, or an extension of the base, and be drawn from the vertex opposite the base. With those restrictions, the formula for the area $A$ of a triangle is:

FORMULA 11.2: AREA OF A TRIANGLE

$$
A=\frac{1}{2} b \cdot h \text { or } A=\frac{1}{2} b h \text { or } A=\frac{b h}{2} \text {. }
$$



## EXPLORATION 5

For each of the trapezoids below, make two copies on a grid paper. Cut them out and put them together to form a parallelogram. Use a strategy similar to that developed for triangles to compute the area of each trapezoid.


The parallel sides of the trapezoids are referred to as the bases of the trapezoid and can be labeled base 1 and base 2 , with base lengths $b_{1}$ and $b_{2}$. The height, $h$, of the trapezoid is the length of a line segment between the two bases that is perpendicular to the bases. As with the parallelogram, notice that the height of the trapezoid is not usually the same length as either side of the trapezoid.

We can follow a similar strategy used to determine the area of triangles to compute the area of trapezoids. Starting with trapezoid A, label one of the horizontal, parallel sides of the trapezoid $b_{1}$, for base 1 , and the other $b_{2}$, for base 2. Label the corresponding sides of the copy of trapezoid A with identical labels. Flipping the copy over, the two trapezoids can be combined to form a parallelogram with a new base of length $b_{1}+b_{2}$.


3
7


Total new base length $=b_{1}+b_{2}=10$

The area for the created parallelogram from two trapezoids put together is $A=b \cdot h$, where $b=b_{1}+b_{2}$, or $A=\left(b_{1}+b_{2}\right) \cdot h$. Therefore, the area of the one trapezoid is half of the area of the parallelogram. We have the formula.

FORMULA 11.3: AREA OF A TRAPEZOID
The area A of a trapezoid is given by
$A=\frac{1}{2}\left(b_{1}+b_{2}\right) h$
where $b_{1}$ and $b_{2}$ are the length of the parallel sides, and $\mathbf{h}$ is the height.

## EXAMPLE 2

Find the area of the following polygons, whose sides are the rational numbers as indicated.




SOLUTION
a. $A=\frac{b h}{2}=\frac{\left(5 \frac{1}{3}\right)\left(3 \frac{1}{5}\right)}{2}=\left(\frac{16}{3}\right)\left(\frac{16}{5}\right) \cdot \frac{1}{2}=\left(\frac{16}{3}\right)\left(\frac{8}{5}\right)=\frac{128}{15}$

$$
=8 \frac{8}{15} \mathrm{sq} . \mathrm{cm}
$$

b. $A=\left(8 \frac{2}{3}\right)\left(6 \frac{1}{2}\right)=\left(\frac{26}{3}\right)\left(\frac{13}{2}\right)=\frac{13}{3} \cdot \frac{13}{1}=\frac{169}{3}=56 \frac{1}{3}$ sq. in.
c. $\quad A=b h=\operatorname{lw}=\left(7 \frac{3}{5}\right)\left(10 \frac{1}{2}\right)=\left(\frac{38}{5}\right)\left(\frac{21}{2}\right)=\left(\frac{19}{5}\right)(21)=\frac{399}{5}$

$$
=79 \frac{4}{5} \text { sq. m. }
$$

d. $A=\frac{1}{2}\left(b_{1}+b_{2}\right) h=\frac{1}{2}\left(6+8 \frac{5}{7}\right) \cdot 4 \frac{1}{2}=\frac{1}{2}\left(14 \frac{5}{7}\right) \cdot\left(4 \frac{1}{2}\right)=\frac{1}{2}\left(\frac{103}{7}\right)\left(\frac{9}{2}\right)$

$$
=\frac{927}{28}=\frac{3}{28} \text { sq. ft. }
$$

## SOLUTION

a. By the Triangle Sum Theorem, the sum of the measure of the angles is $180^{\circ}$.
b. Each angle is a right angle. There are 4 right angles. Hence, $4(90)=m 360^{\circ}$.
c. The sum of the measures of angles $A, B, C$ is $180^{\circ}$. The sum of the
 measures of angles $X, Y, Z$ is $180^{\circ}$. The total sum is $360^{\circ}$.

## Problem 1 Solution

a. $\quad 360^{\circ}$, quadrilateral, Trapezoid
b. $360^{\circ}$, quadrilateral, parallelogram
c. $\quad 540^{\circ}$, pentagon
d. $\quad 540^{\circ}$ pentagon
e. $\quad 720^{\circ}$, hexagon

PROBLEM 2:

| Polygon | Number of Sides | Sum of the measures of <br> all interior angles |
| :---: | :---: | :---: |
| Triangle | 3 | $180^{\circ}$ |
| Quadrilateral | 4 | $180+180=360$ |
| Pentagon | 5 | $3(180)=540^{\circ}$ |
| Hexagon | 6 | $4(180)=720$ |
| Septagon | 7 | $5(180)=900$ |
| Octagon | 8 | $6(180)=1080$ |
| n-gon | n | $(\mathrm{n}-2)(180)$ |

## EXAMPLE 3

What is the sum of the measures of the angles inside of each of the following polygons?

b.

c.


## PROBLEM 2

What is the sum of the measures of the angles for each of the following polygons? Write the name of each polygon. For example, a pentagon or a quadrilateral.


## PROBLEM 3

| Polygon | Number of Sides | Sum of the measures of <br> all interior angles |
| :---: | :---: | :---: |
| Triangle |  |  |
| Quadrilateral |  |  |
| Pentagon |  |  |
| Hexagon |  |  |
| Septagon |  |  |
| Octagon |  |  |
| n-gon |  |  |

## EXERCISES

1. a. Area $=b h=(18) \cdot(8)=144$ square inches; Perimeter $=56$ inches.
b. Area $=b h=(21) \cdot(12)=252$ square inches; Perimeter $=68$ inches.
c. Area $=b h=(5) \cdot(4)=20$ square inches; Perimeter $=20$ inches.
2. Row 1: $7.5,10,9$ sq units

Row 2: $\quad 17.5,10,9$ sq units
Row 3: $\quad 3,4$
3. Area $=52(15)-5(5)+\frac{1}{2}(5)(5)=780-25+12.5=767.5$ sq. ft.

## EXERCISES

1. Find the area and perimeter of the following parallelograms:

a.

2. Calculate the areas of the triangles below using the area formula for a triangle:

3. A rectangular house has a porch on the rear of the house as shown. Find the area of the house and the porch.
4. Ms. Garcia will need to know the area as well as the dimensions of the room. The dimensions will be helpful because carpet comes in fixed widths, and she needs to consider the most efficient way to use the carpet to cover the room.
5. Let $w=$ width of the garden in feet. $2 w=$ length, and $w+w+2 w+2 w=210$. Solving for $w, w=35$ feet, and the area is $35 \cdot 70=2450 \mathrm{sq} \mathrm{ft}$.
6. 

a. $360^{\circ}$
b. $\quad 360^{\circ}$
c. $540^{\circ}$
d. $\quad 5(180)=900^{\circ}$

52 ft

4. Find the area of the following trapezoids in several ways.
a.

b.

c.

a. Find the area by decomposing the trapezoid into triangles and rectangles, and finding the area of each.
b. Find the area by putting two trapezoids together to form a parallelogram, and then rearranging the shapes to form a rectangle.
c. Find the area by using the formula for the area of a trapezoid to check your answers.
5. Ms. Garcia wants to buy new carpet for her living room. What information about the room will she need when she talks to the salesman about the cost of the carpet?
6. A gardener wants to put a fence around his garden. The garden is twice as long as it is wide. If he uses 210 feet of fencing, what is the area of the garden?
7. Compute the sum of the measures of the angles for each of the following polygons.
a.

b.



8. a. Area $=\frac{1}{2} b h=(0.5) \cdot(16) \cdot(6.2)=49.6 \mathrm{sq} \mathrm{ft}$; Perimeter $=11.8+8.6+16=36.4 \mathrm{ft}$.
b. $\quad$ Area $=\frac{1}{2} b h=(0.5) \cdot(8.5) \cdot(12.2)=51.85 \mathrm{sq} \mathrm{mm} ;$ Perimeter $=14.9+12.2+8.5=35.6 \mathrm{~mm}$.
c. Area $=\frac{1}{2} b h=(0.5) \cdot(9.6) \cdot(13.9)=66.72 \mathrm{sq} \mathrm{m} ;$ Perimeter $=14.7+14.7+9.6=39 \mathrm{~m}$.
8. Write equations and solve to find the area and perimeter of the following triangles.

9. Write an equation that represents the area and perimeter of the following figures. Use this equation to find the area and perimeter.

10. The answer is $360^{\circ}$. You can see this by dividing the quadrilateral into two triangles by drawing one line to form two interior triangles.
11. a. Draw a triangle with a small vertex or with a vertex that is slightly smaller than $180^{\circ}$ to make the area less than 1 square inch.
b. Answers will vary. One answer might be a right triangle that has area 8 square inches.
12. Although not drawn to scale, students may decompose the figure into shapes that they can find the areas of and add together these areas. $(45)(82)+\frac{1}{2}(3)(7)+(8)(7)=3690+10.5+56=3756.5 \mathrm{sq} \mathrm{ft}$
10. For each triangle above, decide whether the triangle is a right triangle, an acute triangle, or an obtuse triangle. Explain your decision.
11. Is it possible to have a triangles with sides of lengths:
a. $10,11,12$
b. $8,10,15$
c. $10,20,50$
d. $5,20,30$
e. 1,2,3
f. $2,4,8$
12. Draw a triangle with sides of lengths $a=6, b=8, c=9$. Label the angles opposite these sides as angles $\mathrm{A}, \mathrm{B}, \mathrm{C}$. Which of these angles has the largest measure? The smallest measure?
13. Draw 3 different shaped quadrilaterals. Measure the interior angles of each, and find the sum of these angles. Make a conjecture about what the sum is of the interior angles of a quadrilateral.
14. Use a protractor and ruler to draw an isosceles triangle with two sides 4 inches long and having an area that is
a. less than 1 square inch.
b. more than 2 square inches.
15. A rectangular house has a porch on the rear of the house as shown. What is the area of the house and porch combined? Note, this picture is not drawn to scale.

13. a. The width of the figure is 11 feet, and the height is 12 feet. Then the area of the rectangle the room fits into is 132 square feet. Now we need to subtract the triangle in the upper left corner ( 3.5 sq ft ), the rectangle in the upper right corner ( 5 sq ft ), and the triangle in the lower right corner ( 5 sq ft ). The total area is $132-13.5=118.5 \mathrm{sq} \mathrm{ft}$. Alternatively: $85+10+15+5+3.5=118.5 \mathrm{sq} \mathrm{ft}$ if you subdivide the figure into three rectangles and two triangles.
b. $A=344 \mathrm{sq} \mathrm{ft}$
c. $\quad(344 \mathrm{sq} \mathrm{ft})(1$ gallon $/ 200 \mathrm{sq} \mathrm{ft})=1.72$ gallons
14. a. $A=\frac{1}{2}(12)(5)+\frac{1}{2}(8)(5)+\frac{1}{2}(4)(3)+(3)(5)=(6)(5)+(4)(5)+(2)(3)+(3)(5)=30+20+6+15=71$ sq inches
b. $P=13+1+5+5+3+1+10+4=42$ inches
c. $\quad A=(13)(3)+(4)(3)+(10)(3)+(1)(3)+(3)(3)+(5)(3)+(5)(3)+(1)(3)=39+12+30+3+9+15+$ $15+3=126$ sq inches. Or $A=(42)(3)=126$ sq inches.
16. A room has the following floor plan and dimensions.

a. Find the area of the room.
b. If the approximate perimeter is 43 feet and the walls in the room are 8 feet high, what is the total area of the walls?
c. It takes one gallon of paint to cover 200 square feet of wall. How many gallons will it take to paint the room?
17. Rhonda baked a cake shaped like a sailboat for her nephew's birthday, as shown below.

a. What is the area of the top surface of the cake?
b. What is the perimeter of the top surface of the cake?

## Investigation

17. There are many equal angles. In the figure, the students should find all pairs of vertical angles and measure them.
They should find all corresponding angles, coming from transversals to parallel lines. Using a protractor, what are their measures?
Your students should notice the two triangles ABC and DCE appear to have the same shape. This is a foreshadowing of the idea of similarity between triangles.
18. On a coordinate plane, plot the points $A(0,0), B(5,0)$ and $C(4,3)$.
a. Draw line segments connecting these points in alphabetical order.
b. Double each coordinate of the points given in part a. Plot these points and draw line segments connecting the points as you did in part a.
c. On a coordinate plane, plot the points $D(2,1), E(7,1)$ and $F(6,4)$. Draw line segments connecting these points in alphabetical order.
d. Double each coordinate of the points given in part c. Plot these points and draw line segments connecting the points as you did in part c.
e. What do you notice about the figures in parts a and c? Parts a and b? Parts c and d ? Parts b and d ? Do you see a scale factor between any of these figures? Explain.
19. Explain how to find the area of a triangle, parallelogram, and trapezoid by decomposing and rearranging into simpler shapes. Use this method to check your answers on exercises 8 and 9 .

## 20. Ingenuity:

Draw five different triangles with the same area and the same base.

## 21. Investigation:

Two parallel lines are $\overleftrightarrow{D} \overleftrightarrow{E}$ and $\overleftrightarrow{A B}$ cut by two transversals. Discuss everything you notice about the triangles $C D E$ and $C B A$.

| \# of Sides | Name of Polygon |  |
| :--- | :--- | :--- |
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|  | ANGLES |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  |  | Right | Obtuse | Acute |
| $\begin{aligned} & \stackrel{\sim}{山} \\ & \stackrel{\rightharpoonup}{二} \end{aligned}$ |  | Impossible | Impossible |  |
|  | $\begin{aligned} & \stackrel{\ddot{0}}{\stackrel{u}{0}} \\ & \underline{0} \end{aligned}$ |  |  |  |
|  | $\frac{\stackrel{巳}{0}}{\frac{\stackrel{0}{0}}{\sim}}$ |  | $\mathrm{C}^{9}$ | $\pm$ |



Objective: To have students recognize and classify different types of triangles in real world objects.

## Materials:

Construction Paper
Markers, colored pencils, or crayons
Magazines
Newspapers
Scissors
Glue
Access to computer with color printer - optional

## Activity Instructions:

1) Explain to your students that they will create a picture book organized as follows:

Front - Cover Page
Pg. 1 - Table of Contents
Pg. 2 -- Equilateral Triangles
Pg. 3 -- Scalene Triangles
Pg. 4 - Isosceles Triangles
Pg. 5 -Acute Triangles
Pg. 6 - Obtuse Triangles
Pg. 7 - Right Triangles
Pg. 8 - Credits, where they name their sources
2) The students first need to find several pictures of triangles using magazines, newspapers, and/ or computer printouts. Be very specific that all pictures must be REAL WORLD examples. It is not acceptable for a student to simply find a picture of a right triangle and put this in his book. The pictures that make it into the book must be objects that are examples of different types of triangles. It might be helpful to your students if you can provide some examples of pictures that you have previously found.
3) As your students cut out or print their pictures, remind them that they need to remember to name their sources. Have them keep their pictures in little envelopes with the name of their sources printed on the outside. For example, if a student finds 5 pictures in Architectural Digest, put the magazine title and issue month and year on the credits page. If a student finds a picture on the internet, name the website where he or she found the picture.
4) After your students have searched for their pictures, they begin creating their books in the format listed above. Encourage the students to make their books neat and colorful. Have them lay their pages out before they start attaching the pictures. They should first title and number their pages in pencil, then go back and rewrite in marker or crayon.
5) When finished, display the books in the classroom so that all students can have an opportunity to see their peers' books. If there are any that are particularly good, it might be fun to have the student share it with the whole class.

## Two-Dimensional Shapes

(Suggestion: Make "Magic Strip" foldable for vocabulary)

## POLYGONS and REGULAR POLYGONS

The word POLYGON comes from the Greek words poly, meaning "many," and gon meaning "angles." A polygon is made by joining a number of line segments to make a closed shape. Each polygon can have many sides. The simplest polygon is a triangle. A quadrilateral is a polygon with "quad" or 4 sides. A polygon is named by the number of sides it has. They have the same number of angles as sides.

Regular Polygons have equal sides and equal angles.
Here are some common polygons:

| Name of Polygon | Number of Sides | Sum of the Measure of All the <br> Angles |
| :--- | :---: | :---: |
| Triangle | 3 | 180 |
| Quadrilateral | 4 | 360 |
| Pentagon | 5 | 540 |
| Hexagon | 6 | 720 |
| Octagon | 8 | 1440 |

Let's explore the characteristics of the following polygons:

1. Quadrilaterals:
a. Rectangle
b. Square
c. Trapezoid
d. Parallelogram
e. Rhombus
2. Triangles:
a. Characterized by their angles:

Right Triangle
Obtuse Triangle
Acute Triangle
b. Characterized by their sides:

Scalene Triangle
Isosceles Triangle
Equilateral Triangle
3. Other Polygons:

Hexagon
Pentagon
Octagon

## Polygon Riddles

Use the GEO-pieces (found on the next page) and your notes to answer the riddles.

1. I am a quadrilateral with one pair of parallel sides.
2. I am a triangle with no equal sides.
3. I am a quadrilateral with 2 pairs of parallel sides.
4. I am a three-sided polygon with one right angle.
5. I am a parallelogram with four equal sides, and my opposite angles are congruent.
6. I am a polygon with four sides, four right angles, and the sum of my four angles is 360 degrees.
7. I am a polygon with three sides, and the sum of my angles is 180 degrees.
8. I am a polygon with one obtuse angle.
9. I am a 5-sided polygon.
10. I am an 8 -sided polygon.
11. I am a three-sided polygon with two congruent sides.
12. I am a triangle with equal sides and equal angles.
13. I am a 3-sided polygon with 3 acute angles.
14. I am a rectangle, but a rectangle is not the same as me.
15. I am a parallelogram with only equal sides.

- -■ GEO-pieces $\boldsymbol{\Delta}$-■


1268

## Triangle Angles Puzzle

Objective: Students will prove that the sum of the interior angles of a triangle is equal to 180 degrees. They can think of this like putting a puzzle together.

## Materials:

Multi-color sheets of construction paper
Glue sticks
Scissors
Protractors
Rulers

## Activity Instructions:

Teacher provides each student with $1 / 2$ of sheet construction paper in two different colors. Each student takes one of the pieces of construction paper, draws a medium-size triangle of any type (isosceles, equilateral, right, etc.), and labels each angle inside with numbers: $1,2,3$. Students cut out their respective triangles.

Using their rulers, students draw a straight line in the middle of the remaining $1 / 2$ of construction paper. Teacher shows students how to tear off each corner of the triangle. Students collect all three corners and glue them together on top of the straight line drawn on their construction paper. Hence, they put the puzzle together!!

Students use their protractors to measure and prove that the sum of the three angles glued on the paper equals 180 degrees.

At the end, students will have a clear, concrete picture of how the three angles equal to the sum of 180 degrees.

## Quadrilateral Angles Puzzle

Objective: Students will prove that the sum of the interior angles of a quadrilateral is equal to 360 degrees.

Materials:
Multi-color sheets of construction paper
Glue sticks
Scissors
Protractors
Rulers

## Activity Instructions:

Teacher provides each student with $1 / 2$ sheet of construction paper in two different colors. Each student starts by taking one of the pieces of construction paper and draws a medium-size quadrilateral of any type (square, rectangle, parallelogram, trapezoid, etc.) and labels each angle or corner with numbers: $1,2,3,4$. Students cut out their respective quadrilateral.

Using their rulers, students draw a straight line in the middle of the remaining $1 / 2$ of construction paper. Then, the teacher shows students how to tear off each corner of the quadrilateral. Students collect all four corners and glue them together on top of the straight line drawn on their construction paper. This is like putting the quadrilateral puzzle together!! The piece of the quadrilateral that is left over once the four corners are cut off may be glued next to the number line as a separate piece of the puzzle.

Students take time to use their protractors to measure the sum of the four angles glued on the paper to prove that the sum is indeed equal to 360 degrees.

At the end, students will have a clear, concrete picture of how the four angles equal to the sum of 360 degrees.

## Polygon Practice

Practice finding the angle measures of the following polygons:

| 1. What is the sum of the angle measures of a triangle? | 2. What is the sum of the angle measures of a quadrilateral? | 3. A quadrilateral has angles that measure 90 degrees, 100 degrees, and 120 degrees. What is the measurement of the fourth angle? |
| :---: | :---: | :---: |
| 4. A parallelogram has opposite angles that measure 100 degrees. What is the measurement of the other angles? | 5. A triangle has angles that measure 40 degrees and 90 degrees. What is the measurement of the third angle? | 6. A trapezoid has an angle that measures 45 degrees. What are the measurements of the other three angles? |
| 7. What are the angle measures of a rhombus if one of its angles is 25 degrees? | 8. The sum of the angle measures of an equilateral triangle is 180 degrees. What is the measure of each angle? | 9. The sum of the angle measures of a regular hexagon is 720 degrees. What is the measure of each angle? |

10a. Suppose four angle measures are 25 degrees, 75 degrees, 50 degrees, and 30 degrees. Can the measures of these three angles be used to form a quadrilateral? Justify your answer.

10b. Write an equation to prove that the sum of the angle measurement is 180 . Use a variable to represent the unknown angle measurement.

## Formula Supplement

The following formulas and problems are meant to Supplement sections 11.3 and 11.6 to meet the needs of $6{ }^{\text {th }}$ grade students.

## Perimeter

Square $\quad P=4 s$
Rectangle $\quad P=2(\mid+w)=2 l+2 w$

## Area

a. Square $\quad A=s^{2}$
b. Rectangle $\quad A=l w=b h$
c. Parallelogram $A=b h$
d. Trapezoid $\quad A=\frac{1}{2} h\left(b_{1}+b_{2}\right)$
e. Triangle $\quad A=\frac{1}{2} b h=\frac{b h}{2}$
f. Circle $\quad A=\pi r^{2}$

## Circumference

$$
\text { Circle } \quad \mathrm{C}=2 \pi r=\pi \mathrm{d}
$$

## Volume

$$
\begin{array}{ll}
\text { Cube } & V=s^{3} \\
\text { Rectangular Prism } & V=l w h
\end{array}
$$

## Perimeter:

1. What is the perimeter of a polygon with side lengths of $6 \mathrm{~cm}, 5 \mathrm{~cm}, 7 \mathrm{~cm}, 6 \mathrm{~cm}, 9 \mathrm{~cm}$, and 8 cm ? $\qquad$ Can you name the type of this polygon? $\qquad$
2. What is the perimeter of a trapezoid with side lengths of $11 \mathrm{ft}, 9 \mathrm{ft}, 3 \mathrm{ft}$, and 5 ft ?
3. The width of a rectangle is 20 ft . What is the perimeter of the rectangle if the length is 5 ft . longer than the width?
4. The perimeter of a regular octagon is 20 m . What is the length of one side of the octagon?
5. The Parker's pool measures 1.5 kilometers on each side. How much fencing is needed to enclose the pool with a safety fence?

## Area: (square, rectangle, parallelogram, triangle, trapezoid, circle)

1. What is the area of a rectangle that measures 6 inches $x 4$ inches?
2. Marco's bedroom measures 15 feet long by 15 feet wide. He wants to remodel and expand each side by two feet. What does the area of his new bedroom measure?
3. Thomas has a swimming pool that has a 10 ft . radius. He wants to place a solar cover over the pool and needs to know the area of the swimming pool. What is the area of the swimming pool rounded to the nearest whole number?

## Circumference:

1. If the radius of a circle is equal to 14 cm , what is a good estimate of the circumference of the circle?
2. If the diameter of the pond is 5 yards, what is the circumference?
3. If the radius of the garden is 7 yards, what is the circumference?
4. A Frisbee has a diameter of 10 inches. What is the circumference of the Frisbee rounded to the nearest inch?

## Volume:

1. A cube measures 6 centimeters in width. What is the volume of the cube?
2. What is the volume of a cube that is 8 inches wide?
3. A box is a rectangular prism and measures 2 feet by 4 feet by 4 feet. What is the volume of the box?
4. What is the volume of a rectangular prism that measures $10 \mathrm{~cm} \times 4 \mathrm{~cm} \times 14 \mathrm{~cm}$ ?

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## Section 11.4 - The Pythagorean Theorem

## Big Idea:

Discovering the Pythagorean Theorem

## Key Objectives:

- Derive the formula for the Pythagorean Theorem.
- Apply the Pythagorean Theorem to everyday situations.
- Develop idea of square roots and estimate square roots.
- Discuss irrational numbers.
- Understand the definition of a trapezoid and derive a formula for its area.


## Materials:

Right Triangle worksheet for Exploration 1 (from CD), Calculator, Straight edge

## Pedagogical/Orchestration:

- In Texas, the Pythagorean Theorem is one of the eighth grade TEKS. The theorem involves irrational numbers in the form of square roots.
- In the spirit of the book, students who study 11.4 discover the Pythagorean Theorem. This is a great exercise in non-trivial pattern recognition.
- For Exploration 1, if students are having trouble measuring the hypotenuse, have them mark off the grid intervals on a separate sheet of paper to create their own "ruler" to be used with that particular grid. Then students can use their rulers to measure the hypotenuse. An alternative method is to have students make the drawings on centimeter grid paper and use a centimeter ruler to measure the hypotenuse. If they make the drawings on giant grid paper, they can use an inch ruler since most giant grid paper is made of 1 -inch squares.


## Activities:

"Measuring the Diagonal of a $1 \times 1$ Square" on CD

## Supplemental Activity for Exploration 1:

Assign each group of students a right triangle, like a right triangle with legs of 1 and 4 , and have the students draw the squares on the legs and hypotenuse on giant grid paper. Each group is assigned a different right triangle and then displays their posters on the wall. A discussion ensues and the teacher can make a table with the areas of the leg squares and hypotenuse squares for each of the group's triangles. Allow students to discover that the sum of the leg square areas equals the hypotenuse square area.

## Vocabulary:

Pythagorean Theorem, square root, irrational numbers, radical (optional), trapezoid, bases

TEKS:
7.1(C); 7.2(B)(E); 7.4(B); 7.6(B); 7.7(A); 7.9(A); 7.13(A)(B)(C); 7.14(A); 7.15(A)(B)

## WARM-UPS for Section 11.4 (Pythagorean Theorem)

1. What is the sum of the interior angles for the figure below? Explain why your answer is true.


Ans: The sum of the angles is $(8-2)(180)=1080^{\circ}$. Cut the octagon into triangles and show that there are 6 triangles.
2. In a right triangle, the length of one leg is 5 units and the length of the hypotenuse is 11 units. Which of the following describes the other leg of the right triangle?
a. has length between 7 and 8 units
b. has length between 8 and 9 units
c. has length between 9 and 10 units
d. has length between 6 and 7 units

Ans: (c) because the length is $\sqrt{96}$.

## Launch for Section 11.4:

The following launch is a brief history of Pythagoras, and may whet the appetite of your students to find out more information on this philosopher/mathematician. Tell your students, "Today we will learn about a mathematician named Pythagoras and the famous theorem The Pythagorean Theorem. Pythagoras lived around 530 B.C.E., and founded a philosophical school we will call the Pythagorean society. The sect was very strict. In order to join, members had to give up all their possessions, swear a vow of silence for five years, become vegetarians, and live in caves. Pythagoras was known as The Master and his authority was absolute, no student could question anything he said, and in fact could not say anything until they had listened to his teachings for five years. Pythagoras taught his followers geometry, music and astronomy which he considered to be the triangular foundation of all arts and sciences. The Pythagorean society did not discriminate, allowing men and women to join, and members of all races, religions and social standings. Although the Pythagorean society was widely persecuted and many of the teachings lost, The Pythagorean Theorem discovered by the Pythagorean society remains and has been called the root of all geometry and the cornerstone of mathematics. Pythagoras is also credited with creating the word 'philosopher' which means 'one who is attempting to find out.' So in our study of the Pythagorean Theorem we shall join the Pythagorean society in becoming philosophers as we attempt to discover the secrets of this important theorem."

The following symbol was very important to the Pythagoreans and is a version of the tetractys of Pythagoras. Each mathematical shape within the symbol signified different mysteries of the universe. Ask your students to see how many mathematical shapes they can find within the symbol. Can they see the cube hidden within?

"Let no one ignorant of Geometry enter here."

We plan to help the students discover the Pythagorean Theorem. The students will need a calculator and a ruler.

## SECTION 11.4 THE PYTHAGOREAN THEOREM

The previous section involved the area and perimeter of rectangles, parallelograms and triangles. To find the perimeter, find the lengths of each of the sides and add them together. This is certainly possible if you know the lengths of all three sides, but what if you only know the lengths of two of the sides in the triangle? Is it possible to find the length of the third side?

Start exploring this question with right triangles. Look at each of the right triangles below and measure the lengths of the sides.


You already know that the hypotenuse is the longest side and that the sum of the lengths of the legs is more than the length of the hypotenuse. Is there another way to find the lengths of the sides of a triangle without using a ruler?

Sometimes it is difficult to see a pattern with only three examples. In the following diagram the squares attached to the triangle have been drawn. Draw a square off of each side of the right triangle as shown below so that the base of the right triangle is one side of the square. Remember to look for a pattern involving the lengths of the sides of right triangles.

## EXPLORATION 1

The students may have some difficulty measuring the hypotenuse. Have them be as precise as possible in their measuring. This will help them to see the relationship among the areas of the squares and grounds them with a numerical understanding of the Pythagorean Theorem.

| Length of <br> Vertical leg | Length of <br> Horizontal Leg | Length of <br> Hypotenuse | Area of Square 1 | Area of Square 2 | Area of Square 3 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 12 | 13 | 25 | 144 | 169 |
| 3 | 4 | 5 | 9 | 16 | 25 |
| 6 | 8 | 10 | 36 | 64 | 100 |

## EXPLORATION 1

Your teacher will give you copies of the following right triangles to measure.


Copy and fill out the table below to record the lengths of the sides and the areas of the attached squares.

| Length of <br> Vertical Leg | Length of <br> Horizontal Leg | Length of <br> Hypotenuse | Area of <br> Square 1 | Area of <br> Square 2 | Area of <br> Square 3 |
| :--- | :--- | :--- | :--- | :--- | :--- |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |

Remind your students that we call the product $a \cdot a=a^{2}$ and we say "a squared."
The difference between a pattern and a theorem is the difference between several specific examples and a general example that will work, no matter what numbers are represented by the variables $a, b$ and $c$. The result is the Pythagorean Theorem: $c^{2}=a^{2}+b^{2}$.

The difference between a pattern and a theorem is the difference between several specific examples and a general example that will work, no matter what numbers are represented by the variables $\mathrm{a}, \mathrm{b}$ and c

Looking at the triangle below, what is the area of each square?


The area of the square with side length $a$ is $a \cdot a=a^{2}$. The areas of the other two squares are $b^{2}$ and $c^{2}$.

You might have noticed that the area of the square attached to the hypotenuse is equal to sum of the areas of the other two squares. This is usually written $a^{2}+b^{2}=c^{2}$, where $c$ is the length of the hypotenuse, and $a$ and $b$ are the lengths of the legs. Check that this formula works for each of the right triangles in the exploration. The formula is called the Pythagorean formula or the Pythagorean Theorem.

## THEOREM 11.3: PYTHAGOREAN THEOREM

If $a$ and $b$ are the lengths of the legs of a right triangle and $c$ is the length of the hypotenuse, then

$$
a^{2}+b^{2}=c^{2}
$$

You discovered the pattern by looking at examples and computing areas. But how do you know that it is always true for any right triangle with legs of length a and $b$ and hypotenuse of length $c$ ? There are many different proofs of this ancient

In the left square, we can write the area in two ways: $(a+b)^{2}=4\left(\frac{1}{2} a b\right)+c^{2}$. In the right square, we can write the area in two ways: $(a+b)^{2}=a^{2}+b^{2}+2 a b$. If we equate the right sides of both equations we have $4\left(\frac{1}{2} a b\right)$ $+c^{2}=a^{2}+b^{2}+2 a b$. This gives us $c^{2}=a^{2}+b^{2}$ and the result of the Pythagorean Theorem.

## EXPLORATION 2

## Extension:

## Optional Square Root Estimating Activity to be used after Exploration 2:

Draw a square on the board and mention that it has an area of 51 square inches. What is the length of the side? Students are to guess lengths and zero in on the right answer. Students can use calculators but do not mention the square root button. Students may start by saying 7.5 by 7.5 - too big. 7.2 - too big. 7.1 - too small, 7.15 - too big, and so on. After further refinement, let students know that this number will never end or repeat and is called an irrational number. More examples could be done.
theorem. One of the most beautiful proofs can be seen from the following two pictures. In each picture there are four copies of the original right triangle, with side lengths $a, b, c$.


Each of the pictures represents a square with side length $a+b$. On the left, there are four triangles and one square with side length $c$. On the right, there are the same four triangles and two squares, with side lengths $a$ and $b$. Copy and label all the sides in the two pictures to check the statements. Why does the picture prove that the area of the large square is the sum of the areas of the two smaller squares? The result is the Pythagorean Theorem: $a^{2}+b^{2}=c^{2}$.

## EXPLORATION 2

On the floor, table or board, draw a square with sides 1 meter long. Next, draw a diagonal line from one corner to the opposite corner. Your challenge is to find the length of the diagonal.
First, estimate the length of the diagonal.
Next, measure the length of the diagonal. How precise an estimate is possible using a meter or yardstick?

Now, use the Pythagorean formula to find the length, $c$, of the diagonal. What is the value of $c$ ?

Use the Pythagorean formula, $1^{2}+1^{2}=2=c^{2}$. This means that $c^{2}$ or " $c$ squared" is 2 . Soc is a number that when multiplied by itself equals 2 . The square root notation $\sqrt{n}$ denotes the positive number whose square is $n$. The symbol $\sqrt{ }$ is called the square root symbol. So $\sqrt{25}=5$ because 5 is a number whose

1. a. 1.732 b. 2.000 c. 3.162 d. 6.481
e. 3
f.
13 g.
7
e. $\quad 100$

This is guess and check. It is good to use the number line to help locate the next guess (as in activity) but not necessary. For instance, $\sqrt{3}$ is between 1 and 2 because 3 is between $1^{2}$ and $2^{2}$, so the first digit of $\sqrt{3}$ is 1 .
2. $-5,4$ and -4
$16=4^{2}$ or $(-4)^{2}$
3. $5 \mathrm{in}, 6 \mathrm{in}^{2}$
4. $8 \mathrm{in} ; 24 \mathrm{in}^{2}, 24 \mathrm{in}$
5. $c=\sqrt{a^{2}+b^{2}}$
6. 13 cm
7. $\sqrt{130}$ which approximates 11.4 cm
8. 6.00

Distance traveled on road $=$ length of hypotenuse $=(1.25$ miles $)(5280$ feet per mile $)=6600$ feet.
Let the horizontal distance $=x$, then $x^{2}+395=6600^{2}, x=\sqrt{6600^{2}-395^{2}} \approx 6588$.
So the grade is $\frac{(395)(100)}{6588} \approx 6.00$.
square is 25 . The square is a function of $x$ : $f(x)=x^{2}$. The square root is called the inverse function of the square because taking the square root of the square of a number gives the number. For instance, $4^{2}=16$ and $\sqrt{16}=4$, so $\sqrt{4^{2}}=4$.

## EXERCISES

1. Using a calculator and the same technique as in Exploration 2, estimate the exact values for the following square roots to four decimal places. Then use the "square root" button on the calculator to check your answer.
a. $\sqrt{3}$
b. $\sqrt{4}$
c. $\sqrt{10}$
d. $\sqrt{42}$
e. $\sqrt{9}$
f. $\sqrt{169}$
g. $\sqrt{49}$
h. $\sqrt{10000}$
2. As you know, $\sqrt{25}=5$. Are there other numbers whose square is 25 ? What are all the numbers whose square is 16 ?
3. A right triangle has legs 3 inches and 4 inches long. Find the length of the hypotenuse and the area of the triangle. Confirm the your answer using a lined paper and a ruler.
4. A right triangle has hypotenuse 10 inches long, and one leg 6 inches long. Find the length of the other leg. What are the area and perimeter of the triangle?
5. Call the lengths of the sides of a right triangle $a$ and $b$, and the length of the hypotenuse $c$. By the Pythagorean Theorem, $a^{2}+b^{2}=c^{2}$. Find $c$ using square root notation.
6. Draw a right triangle with legs of length 5 cm and 12 cm . Measure the length of the hypotenuse, then calculate the exact length of the hypotenuse.
7. Draw a right triangle with legs of length 7 cm and 9 cm . Measure the hypotenuse to estimate its length. Then use the Pythagorean Theorem to find the exact length of the hypotenuse, as well as a better decimal approximation.
8. The grade of a road is the change in elevation times 100 divided by the horizontal distance traveled. Sam traveled 1.25 miles along a local road and the change in elevation was 395 feet. What is the grade of the road?
9. Yes. The door is a rectangle with diagonal $\sqrt{49+16}=\sqrt{65}>8$, so the glass will fit.
10. $8 \mathrm{ft} \times 16 \mathrm{ft}$

Let $x=$ width and $2 x=$ length. By the Pythagorean theorem, $x^{2}+(2 x)^{2}=320$, so $5 x^{2}=320, x^{2}=64, x=8$

## Ingenuity

11. Use the theorems and postulates in the first sections of this chapter. Also use the idea that if two figures' angles are congruent, the figures have the same shape
a. $\angle A=\angle C A B$
$m(\angle C)=90^{\circ}$
$m(\angle A)+m(\angle B)+m(\angle C)=180^{\circ}$
So $m(\angle B)+m(\angle C)=90^{\circ}$
Similarly, $m(\angle B C P)+m(\angle B)=90^{\circ}$
Thus, $m(\angle A)=m(\angle B C P)$
b. Since $\triangle A B C$ and $\triangle P B C$ have the same angles, the triangles are similar, i.e. they have the same shape. c. $\triangle A B C$ is also similar to $\triangle A P C$ since they have the same angles.
12. Mike and Larry are carrying a square 8 ft by 8 ft piece of glass through a 7 ft by 4 ft doorway. Will the glass fit through the doorway? Explain.
13. In rectangle $A B C D$, the diagonal is $\sqrt{320} \mathrm{~cm}$ and the length $A B$ is twice as long as the width BC . Find the dimensions of the rectangle.
14. A 10 -foot ladder is leaning against a wall. The bottom of the ladder is six feet from the wall. Where will the top of the ladder touch the wall?

## 11. Ingenuity:

Suppose $\triangle A B C$ is a right triangle $m(\angle A C B)=90^{\circ}$ and with base $\overline{A B}$ as pictured. Draw altitude $\overline{C P}$. Draw a line through $C$ that is parallel to the base $\overline{A B}$. Find the area of $\triangle A B C$.

a. Show that $\angle C A B$ has the same measure as $\angle B C P$.
b. Show that $\triangle A B C$ has the same angles as $\triangle P B C$.
c. Show that $\triangle A B C$ has the same angles as $\triangle A P C$.

## Section 11.5 - Circles

## Big Idea:

Discovering properties of circles

## Key Objectives:

- Learn the vocabulary of circles.
- Understand the constant $\pi$.
- Develop the formula for the circumference and area of a circle.
- Understand the terms constant and coefficient.
- Use properties of circles to solve everyday problems.


## Materials:

Mathematical compass or string with pencil attached, Tape, Ruler, Grid paper

## Pedagogical/Orchestration:

- This lesson develops circumference and area of a circle.
- Circle measurement, both one- and two-dimensionally, is carefully and actively developed in this section. Encourage your students to cut and measure to prove to themselves that the formulas work. Take time to make sure your students know the difference between a symbol that is a variable and a symbol that is a constant.
- If, as many of us believe, discovering a fact for yourself makes it yours, this section will be a benefit to your students in remembering the formulas associated with circles and working with those formulas.


## Activities:

"Circles, Circles, Circles" on CD and at end of section.
"Discovering Pi" on CD and at end of section.

## Exercises:

The beautiful designs in these exercises should draw students into working them.

## Vocabulary:

center, circle, radius, diameter, circumference, constant, coefficient

## TEKS:

6.6(C); 6.8(B); 6.10(C); 7.1(A); 7.5(A,B); 7.6(A); 7.9(A); 7.13(A,B,C); 7.14(A); 7.15(A,B);
8.1(C); 8.2(A,B,C); 8.5(A,B); 8.14(A); 8.15(A); 8.16(A)

## WARM-UPS for Section 11.5 (Circles)

1. An 8 ft tall ladder leaning against a wall. The point where it touches the wall is a height of 5 feet. How far is the bottom of the ladder from the base of the wall?

## Ans: $\sqrt{39}$ feet which is approximately 6.24 feet.

2. What is the area of a circle with diameter 7 ?
a. $12.25 \pi$
b. $7 \pi$
c. $49 \pi$
d. $3.5 \pi$

Ans: (a) because area is $\pi r^{2}=\pi(3.5)^{2}$

## Launch for Section 11.5:

Lead students through the questions in the first paragraph of Section 10.4. "What is the definition of a circle? How do you draw a circle? Try to describe a circle to someone without using the word 'circle.'" Let students struggle with this challenge and let students share good attempts. Go through Exploration 1 examining the definitions given. (For more on the discovery of the relationship of the diameter to the circumference see the Circles, Circles, Circles Activity for 11.6 on the CD.) Tell your students, "Today we will learn how pi is related to circles, and make many important discoveries including how to find the area and circumference of any circle."

## EXPLORATION 1

Have a variety of items that can be used like string, yarn, ruler, but no compass yet because we want the students to be resourceful about what characteristics are important in a circle.

A compass may be used to draw a circle, but the string engages the student to think about how important the constant distance is. It is a good idea to provide the compass after the student understands the key elements of circles.

## SECTION 11.5 CIRCLES

Everyone has seen circles of various sizes, but what is the definition of a circle? How do you draw a circle? Try to describe a circle to someone without using the word "circle."

## EXPLORATION 1

How do you draw a circle? Once you have drawn a circle, write directions someone could use to draw a circle. Then state your definition for a circle.

In general, one way to draw a circle is by marking a point $P$, called the center of the circle. Then take a length of string, $r$ units long, place one end of the string at point $P$ and attach a pencil to the other end. Stretch the string to its full length and draw the circle with the pencil. Each point on the circle is $r$ units from $P$. The circle is made up of all points that are distance $r$ from $P$. The fixed distance $r$ from the center $P$ to the edge of the circle is called the radius of the circle. A straight line connecting two points on a circle is called a chord. A straight line connecting two points on the circle and passing through the center $P$ is called a diameter. The length of the diameter is equal to the length of 2 radii. Radii is the plural of radius. The distance around the circle is called the circumference and is like the perimeter of a polygon.


EXPLORATION 2-optional for 7th graders
Give each student a ruler and some string or a tape ruler, as time permits. If not, teacher will make copies of varying sizes of circles on graph paper to pass out to students.

Note: Answers may vary depending on units used, size of grid paper, and slight resizing at print time.
Lead students to circles A, C, and E if they are having difficulty discovering a pattern. Make sure in the discussion that the students can justify that all circles are similar because the ratio of the circumference to radius is the same for any 2 circles.

| Circle | Radius | Diameter | Circumference | Area |
| :--- | :--- | :--- | :--- | :--- |
| A | $\frac{1}{4}=0.25$ | $\frac{1}{2}=0.5$ | 1.5 | 0.2 |
| B | $\frac{3}{8}=0.375$ | $\frac{3}{8}=0.75$ | 2.4 | 0.4 |
| C | $\frac{1}{2}=0.5$ | 1 | 3.1 | 0.8 |
| D | $\frac{9}{16}=0.5625$ | $1 \frac{1}{8}=1.125$ | 3.5 | 1.8 |
| E | $\frac{11}{16}=0.6875$ | $1 \frac{3}{8}=1.375$ | 4.3 | 2.1 |

Diameter is twice the length of the radius. $d=2 r$.

## EXPLORATION 2

Use a variety of different size circles. Using a piece of string, carefully measure the radius and circumference. Place the circle on grid paper and estimate the area, then complete the table below:


| Circle | Radius | Diameter | Circumference | Area |
| :--- | :--- | :--- | :--- | :--- |
| $A$ |  |  |  |  |
| $B$ |  |  |  |  |
| $C$ |  |  |  |  |
| $D$ |  |  |  |  |
| $E$ |  |  |  |  |

Looking at the table, what patterns do you notice?

Do you notice a relationship between the radius and the diameter? Using the variable $d$ to denote the length of the diameter, express the diameter in terms of the radius $r$.

Get an average of the class' individual ratios. The ratio should approximate 3. The inaccuracies of individual measures account for the error. Have your students recheck their measurements. Encourage greater accuracy. They might have less variation this time and the average might get even closer to 3.14 or approximately $\pi$.

You will have some students who are captivated by $\pi$. Encourage any research about pi's history. You might assign a willing student to look up the beginning of the exact value, say to 100 digits.

What is the relationship between the circumference of a circle and its diameter? Compute the ratio of the circumference to its diameter for the five circles. What do you notice about the ratios? The ratio you computed approximates the exact ratio of the circle's circumference to its diameter, the number pi, written as the Greek letter $\pi$. This ratio is the same regardless of the size of the circle.

Look at a circle with diameter 1 unit. Remember, a unit can be any length you choose. Take out a string or tape ruler and measure the circumference. The circumference has a length of $\pi$ units. The number $\pi$ is approximately equal to the fraction $\frac{22}{7}$ or the decimal 3.14. These two approximations are not exactly equal to $\pi$. However, the two approximate values are very close to the actual value of $\pi$, which begins 3.1415926....

## DEFINITION 11.11: PI

The ratio of the circumference to the diameter of any circle, represented either by the symbol $\boldsymbol{\pi}$, or the approximation $\frac{22}{7}$ or 3.1415926.

What happens to the circumference when the radius doubles? What pattern do you notice when you measure the scaled circumference and compare it to the original circumference? Just as with a square, when scaled by a factor of 2 , the perimeter, or circumference $C$, doubles. The ratio $\frac{C}{d}$ of the circumference to the diameter remains the same, $\boldsymbol{\pi}$.

In summary, call $C$ the circumference of a circle, $d$ the diameter, and $r$ the radius. Then,

$$
\begin{aligned}
d & =2 \cdot r & & \text { and } & C & =\pi \cdot d \\
& =2 r & \text { and } & & \text { or } & C=2 \cdot \pi \cdot r
\end{aligned}
$$

## EXPLORATION 3

What is the area $A$ of a circle whose radius is 1 ? Draw a circle with radius 1 and circumference $2 \pi$ and cut it in half. Then cut each half into many small pie slices:


Take the slices from one half of the circle and lay the points of the slices along a line:


Do the same with the bottom half of the circle, filling in the spaces:


The shape looks a little like a rectangle. The more slices, the closer the shape is to a rectangle. If this cutting process continued infinitely, the area of the circle with radius 1 would approximate the area of the rectangle with length $\pi$ and width 1 . The area $A=\pi \cdot 1=\pi$ square units.

What happens to the area of the circle when its radius is a number $r$ ? One way to visualize this is to create slices in the circle with radius $r$, like the process with radius 1.


Cut the circle into two equal semicircles as you did in the unit circle and fit one semicircle into the other semicircle.


What is the length of this rectangular shape? What is its width? What is the area of the rectangle?

In this rectangle the length is half the circumference $2 \pi r$, or $\pi r$, and the width is $r$. The area of the rectangle is length times width or $\pi r \cdot r$ or $\pi r^{2}$. Any area is measured in square units. So if $r$ is measured in inches, $r \cdot r$, or $r^{2}$, is measured in square inches. To summarize:

## FORMULA 11.4: AREA OF A CIRCLE

The area of a circle with radius $r$ is $A=\pi r^{2}$ square units.

## EXAMPLE 1

In general, exact answers here will have $\pi$ in its answer while an approximation uses an approximate value of $\pi$ such as 3.14 or $\frac{22}{7}$.
a. $8 \pi$ inches
b. 25.1 inches
c. $16 \pi$ square inches
d. 50.24 square inches

## SOLUTION

Lead students to the realization that although pi is the ratio of the circumference to diameter of any circle, it is not the ratio of any two integers. It is not a rational number and so it is a non repeating decimal number.

EXAMPLE 2
diameter $=\frac{37.68}{\pi}$ or approximately 11.9939167...
radius $=\frac{3768}{2 \pi}$ or approx. 5.9969583...
area $=\pi\left(\frac{37.68}{2 \pi}\right)^{2}$ or approx. 112.9826932...

## EXAMPLE 1

A circle has radius 4 inches.
a. Find the exact circumference of the circle.
b. Approximate the circumference to the nearest tenth of an inch.
c. Find the exact area of the circle.
d. Approximate the area to the nearest hundredth of an inch.

## SOLUTION

Apply the above formulas.
a. The exact circumference needs to be written as $8 \pi$ inches because $\pi$ is a non repeating decimal number.
b. 25.1 inches
c. $\quad 16 \pi$ square inches because $\pi$ is a non repeating decimal.
d. 50.24 square inches

The circumference is $C=2 \pi \cdot 4$ inches $=8 \pi$ inches.
The area is $A=\pi \cdot(4 \text { inches })^{2}=16 \pi$ square inches $=16 \pi \mathrm{in}^{2}$.

## EXAMPLE 2

A circle has the circumference of 37.68 ft . What is the circle's diameter? Find the radius and its area.

In mathematics, a product that includes a constant, or number, times a variable is written with the constant first, like $2 x$. In the product of a constant and a variable, the constant 2 in this example is called the coefficient of the product $2 x$. However, even though $\pi$ is a constant, not a variable, the product of $\pi$ and a constant like 16 is usually written $16 \pi$.

## EXERCISES

Explain to students that even when they use the $\pi$ key on the graphing utility, though it looks like it's really precise, they are still just getting an approximation. The only way to get an exact answer is to leave the answer in terms of $\pi$.

1. a. $A=4 \pi$
$C=4 \pi$
c. $A=56.25 \pi$
$C=15 \pi$
b. $A=49 \pi$
$C=14 \pi$
d. $A=81 \pi$
$C=18 \pi$
2. Circumference: $10 \pi \mathrm{~cm}$, Area: $25 \pi$ square cm
3. Circumference: $40 \pi$ miles, $400 \pi$ square miles
4. $d=31.8 \mathrm{in} ., r=15.9 \mathrm{in}$.
5. 20 in. $100 \pi \mathrm{in}^{2}, 314 \mathrm{in}^{2}$

## EXERCISES

1. Find a circle large enough to measure easily in your home or school. Use a measuring tape or string to mark the length of the circumference of each circular object. Use the tape or string to measure the diameter of the circle. Calculate the approximate ratio of circumference to diameter? Explain why this result makes sense.
2. Find the area and circumference of the circles below. Use $\pi, 3.14$, or $\frac{22}{7}$.
a.

c.
b.

d.

3. Use the EXPLORATION 3 in this section to explain to someone at home or someone in class how to discover the formula for the area of a circle. In particular, explain how the number $\bar{\sigma}$ is used from the beginning of the activity.
4. The diameter of a circle is 10 centimeters. What is the circumference of the circle? What is the area of the circle?
5. A circle has a radius of 20 miles. Find its circumference. Find its area.
6. The circumference of a circle is 100 inches. To the nearest tenth of an inch, what is the circle's diameter? Its radius? Use $\pi=3.14$.
7. The radius of a circle is 10 inches. What is the circle's exact diameter? Its exact area? Its approximate area, to the nearest inch?
8. $357 \mathrm{yd}, 6963 \mathrm{yd}^{2}$
$\mathrm{P}=200+2 \pi(25)$ which is approximately 357 yd
A $=5000+\pi(252)$ which is approximately $6973 \mathrm{yd}^{2}$
9. 180 square units
10. $4 r^{2}-\pi r^{2}$ or $(4-\pi) r^{2}$
11. $3 \pi r^{2}$

$$
\pi(2 r)^{2}-\pi r^{2}=3 \pi r^{2}
$$

8. A track is shaped like a rectangle with semi-circles at each end. The track is 100 yards long, and 50 yards wide. What is its perimeter to the nearest yard? Its area to the nearest square yard?

9. A circle has area 20 square units. Each dimension is scaled by a factor of 3 to make a new circle. What is the area of the new circle?
10. A circle with radius $r$ lies inside a square with each side $2 r$ long.


Find the area inside the box but outside the circle.
11. A circle with radius $r$ is contained in a larger circle with radius $2 r$ touching at the bottom.


What is the area outside the smaller circle and inside the larger circle?
10. The same

Let $r$ be the radius of each small circle and $R$ the radius of the large circle.
$R=2 r$ The area of the large circle is the same as the area of the 4 small circles because $\pi(2 r)^{2}=4\left(\pi r^{2}\right)=$ $4 \pi r^{2}$.
12. Angelique wants to water her front lawn with sprinklers that cover a circular area. Would a giant sprinkler cover more area or would 4 smaller sprinklers cover more area? Explain.

13. In the figure below, five circles are nested inside a larger circle. Consider the region in the upper right of the large circle between the top circle, the circle on the right, the center circle, and the circumference of the large circle. Is the area of this region greater than, less than, or equal to the area of one of the smaller circles? First make a guess, and then use geometry to check your answer.

14. Barry is making a stained glass window for a client. The window has an arch on the top, like the picture below. How many square feet of glass does he need to make the window? use $\pi$ as 3.14, and round your answer to the nearest tenth.

## 12. 4.1 square feet

13. The area of the circle is given by $A=\pi \cdot r^{2}=\pi \cdot 4^{2}=\pi \cdot 16$. Since we want only a fraction of the circle, we see that the area of the sector is to the area of the circle as the angle 60 degrees is to 360 degrees. 60 is $\frac{1}{6}$ of 360. The area of the sector must be $\frac{1}{6}$ the area of the circle so $\frac{16}{6} \pi=\frac{8}{3} \pi$ square inches. This approximately 8.4 square inches.

## Investigation

14. The most demanding part of this Investigation is that there is not graphic. Encourage your students to recreate the figures described. After that, they simply look for congruent angles, corresponding sides and similar triangles.
a. The two triangles are congruent.



36 in

## 15. Ingenuity:

A sector of a circle is the part of the interior of the circle between two radii, like a slice of pie. A circle has radius 4 inches, and two radii make a sector with a $60^{\circ}$ angle. Find the exact area of the sector these radii enclose.


## 16. Investigation:

a. Draw a circle with radius $r$ and center $O$. Pick two points $A$ and $B$ on the circle that do not lie on a common diameter.
b. Draw the diameter that goes through point $A$. It intersects the circle at point $A$ and another point, $A^{\prime}$. The diameter that goes through point $B$ likewise intersects the circle at another point $B^{\prime}$. In what way are triangles $A B O$ and $A^{\prime} B^{\prime} O$ related? Why?

## Circles, Circles, Circles <br> 

Objective: The class will approximate the value of $\pi$ by measuring the diameters and circumferences of various circles and finding the ratio of C to d by graphing the class data.

## Materials:

Lots of objects, of various sizes, with circular bases
Measuring tape
String is not necessary, but some students may find it helpful for measuring
Graph Paper
Notebook paper

This activity is best done in groups of 3 to 4. Each group is given 3 to 5 objects to measure. After each group has completed the measurements and graphs, the data should be compiled on a large class coordinate plane to discuss the results. Ask probing questions to lead the students to find relevant patterns in the data and ultimately find the relationship between the diameter and circumference of all circles.

## Activity Instructions:

Mathematicians have found a relationship between the diameter and circumference of a circle. Let's see if you can discover this relationship too. We will begin by measuring the circular bases of the objects on our tables and look for patterns.

1) Use a tape measure to find the diameter and circumference of each object. Record your results in a table with these column headings.

Object
Diameter
Circumference
2) Make a coordinate graph of your data. Use the horizontal axis for diameter and the vertical axis for circumference.
3) Add your results to the class graph. Draw a line through the origin and as close to as many points as possible. Try to find the ratio of the vertical values and the horizontal values.
Teacher Edition
Section 11.5
Circles


## Discovering Pi is Easy as p

Objective: The students will approximate the value of $\pi$ by measuring the diameters and circumferences of various size circular lids. Students will discover that there are about 3 diameters in every circumference, regardless of the size of the circular lids.

## Materials:

Lids of various sizes (at least 3 different size lids, per group)
Yarn
Ruler
Large piece of poster paper (one per group)

## Activity Instructions:

1) Divide your class up into groups of 3 to 4 . Make sure that each group has at least three lids of different sizes, poster paper, yarn and a ruler.
2) Students will trace each of their lids onto their poster paper. Using the yarn and the ruler, students will explore the size of the circles to find relationships between the radius, diameter, and circumference of each circle. This information should be organized and recorded on a piece of paper, to be shared and compared with the rest of the class when all groups are finished measuring.
3) During the sharing and comparing stage of this activity, the students should discover that regardless of the size of the circles, each radius is $1 / 2$ of each diameter, each diameter is twice the size of each radius, each circumference is equal to approximately 3 diameters, and each circumference is also equal to approximately 6 radii.
4) Once your class finds these discoveries, ask the class to explain to you and to each other how this relates to the two formulas on their formula chart for circumference and the number $\pi$.

## Extension:

Students will find 3 objects from home in which understanding the circumference, diameter, and radius of a circle was useful.

Literary Extension:
Read Sir Cumference and the First Round Table by Cindy Neuschwander and Wayne Geehan.

## Section 11.6 - Three-Dimensional Shapes

## Big Idea:

Discovering volume of prisms and cylinders and exploring the effect on volume when scaling a 3-dimensional figure;
Classifying 3-dimensional shapes

## Key Objectives:

- Learn the vocabulary associated with volume of three-dimensional figures.
- Develop the formula for the volume of a rectangular prism.
- Generalize the volume formula for other types of prisms and cylinders.
- Extend scaling to three dimensions and explore the effect on volume.
- Use the math learned in this section to solve everyday problems, including conversions.


## Materials:

Poster board, Ruler, Tape, Unit cubes, Rice or modeling clay, Boxes and cylinders
Solid Power Point from the CD

## Pedagogical/Orchestration:

- This lesson sets the groundwork for three-dimensional measurement.
- The section starts by using volume to discover conversion formulas. It then moves to discovering the formula for the volume of a rectangular solid. It concludes by developing the general formula for solids like cylinders and prisms.


## Activities:

"Guess My Solid"; "Figure Out My Volume" and "How Many Ice Cubes?" on CD and at end of section-nets can be found on website: http://www.learner.org/interactives/geometry/area_volume.html

## Exercises:

Make sure students understand that volume is measured in cubic units, and that their answers to the exercises need to include correct units.

## Vocabulary:

cube, volume, polyhedron, face, vertices, edges, prism, rectangular prism, regular, cylinder, space diagonal, pyramid, cone (See CD: cone, pyramid, solid figure, right triangular prism)

## TEKS:

6.4(B); 6.12(A); 7.2(D); 7.4(A); 7.6(A,C,D); 7.8(A); 7.9(A); 7.13(A,B,C); 7.14(A); 7.15(A,B);
$7.9(\mathrm{~B}, \mathrm{C}) ; 7.8(\mathrm{C}) ; 8.2(\mathrm{D}) ; 8.4(\mathrm{~A}) ; 8.8(\mathrm{~B}) ; 8.9(\mathrm{~B}) ; 8.10(\mathrm{~B}) ; 8.14(\mathrm{~A}) ; 8.15(\mathrm{~A}) ; 8.16(\mathrm{~A})$

## WARM-UPS for Section 11.6 (Three-Dimensional Shapes)

1. What is the area of the shaded region if the square is a 5 cm by 5 cm ?

a. $\left[25-\left(\frac{25}{4}\right) \pi\right] \div 4$
b. $\quad\left[25-\left(\frac{25}{4}\right) \pi\right]$
c. $\frac{25}{4}-\left(\frac{25}{4}\right) \pi$
d. $[25-25 \pi] \div 4$

Ans: (a) You compute area of square, 25 , then subtract area of circle, $25-\left(\frac{25}{4}\right)$ $\pi$ and then divide by $4,\left[25-\left(\frac{25}{4}\right) \pi\right] \div 4$.
2. What is the area of the figure below? Explain how you compute the area.


## Launch for Section 11.6

Brainstorm 3-dimensional shapes and their names. Students may use informal names such as box, pyramid, ball. Give formal names such as prisms, pyramids, spheres. Classify the characteristics of the mentioned 3-dimensional shapes. Name objects in real-life with such shapes. Also, have them draw these shapes. They may note that drawing 3 -dimensional figures in 2-dimensions is challenging.

## Alternate Launch for Section 11.6:

One of the most useful tools for a geometry teacher is a model of a cubic foot. This would come in handy both for this Launch and the Exploration 2 from this section. Exploration 2 asks students to determine how many cubic inches are in one cubic foot. Today's Launch starts off simpler. Ask students to discover how many cubic feet are in a cubic yard. Tell them they can make a guess, but they must actually demonstrate their answer. Have plenty of yardsticks so students can fashion a cubic yard. They can measure a square yard on the floor and outline it with tape. Then different students can hold up four yard sticks on each corner. Another student can then take the cubic foot and move it to different locations within the cubic yard to determine how many cubic feet it would take to fill the cubic yard. Ask students if their initial guess was correct. This is a case when our intuition sometimes lets us down, as often times students will guess that there are 3 or 9 cubic feet in a cubic yard when there are actually 27 . Let students know, "Today is all about 3-dimensional shapes and discovering their properties."

## SECTION 11.6 THREE-DIMENSIONAL SHAPES

In the previous sections, you studied shapes in two dimensions: triangles, squares, rectangles, parallelograms, trapezoids, and circles. In this section, you will learn about three-dimensional shapes. Some of these shapes appear as familiar objects like beach balls, blocks, paper towel rolls or cardboard boxes. In this section, you will learn some mathematical terminology and ways to measure volume.

A basic kind of three-dimensional figure is called a polyhedron. This word comes from the Greek words poly, meaning "many," and hedra meaning "faces." So a polyhedron is a three-dimensional figure with many faces. Each face of a polyhedron is a polygon. The vertices of the polygons are the vertices of the polyhedron. The edges are the borders of the faces that are also the line regments that join the vertices.


A box shape is an example of the most common type of polyhedron called a prism. In a prism, two of the faces, called bases, are parallel and congruent. Prisms are named by their bases. In the case of a box, the polyhedron is a rectangular prism, because the bases are rectangles. The lateral surfaces are the faces of a geometric figure, excluding the bases. In a prism, the lateral surfaces are always rectangles.


PROBLEM 1
In a rectangular prism, there are 6 faces, 8 vertices, and 12 edges.
4 faces are lateral surfaces and 2 are bases.
In a triangular prism, there are 5 faces, 6 vertices, and 9 edges. There are 3 lateral surfaces and 2 bases.

Teachers: There is a blank copy of this table at the end of this section that you can use as a worksheet with your class.

## PROBLEM 1

How many faces does a rectangular prism have? How many are bases? How many are lateral surfaces? How many vertices? How many edges? How does this change for a triangular prism?

A cube is a regular rectangular prism. A cube is regular because each of its faces has equal sides and angles. All the cube's faces are squares.

## EXAMPLE 1

Identify the 5 prisms below. Determine the number of vertices, faces, and edges for each.


## SOLUTION

| Prism | Name | Vertices | Faces | Edges |
| :---: | :---: | :---: | :---: | :---: |
|  | Triangular <br> Prism | 6 | 5 | 9 |
|  | Rectangular <br> Prism | 8 | 6 | 12 |
|  | Pentagonal <br> Prism | 10 | 7 | 15 |
|  | Hexagonal <br> Prism | 12 | 8 | 18 |

Note that in the technical definition, cylinders need not have bases as circles. Most middle school grades assume that cylinders are right circular ones.

A pyramid is a 3-dimensional figure with one polygonal base that connects to a point called the apex. The lateral surfaces formed when the base is connected to the apex are all triangles.


The pyramids are named for their bases.

Let's consider another three-dimensional figure called the cylinder. Like a prism, it has two congruent and parallel bases. It cannot be classified as a prism because the bases are not polygons but are circles.


A circular cone is a 3-dimensional figure with one circular base that connects to a point called the apex.


The simplest three-dimensional shape to measure is a cube, two parallel congruent square bases connected by four perpendicular congruent squares.

## EXPLORATION 1

Nets for blocks can be found at:
http://www.learner.org/interactives/geometry/area_volume.html
Student groups will need 27 cubes each..


A cube one unit long, one unit wide and one unit high has a volume of one cubic unit. Recall that the area of a two-dimensional figure is measured by the number of unit squares needed to cover it. The volume of a three-dimensional shape is measured by the number of unit cubes needed to fill it. For example, if each side of a cube is 1 foot long, the volume of the cube is 1 cubic foot, written 1 cu . ft . or $1 \mathrm{ft}^{3}$.

## EXPLORATION 1

How many inch cubes (also called cubic inches) are there in a cube that is 2 inches long on each side? How many cubic inches are there in a cube that is 3 inches long on each side?

Materials: You will need approximately 30 cubes of the same size for this activity.
In groups, make a cubic box that is two units long on each side. Fill this box using one unit cubes. How many one unit cubes does it take to fill the box that is 2 units long on each side? A box that is 3 units long on each side?

Are you surprised that the two unit cube has volume 8 cubic units, while the three unit cube has volume 27 cubic units? In general, if a cube has side length $s$ units, what is the volume? You should get the formula:

## FORMULA 11.5: VOLUME OF A CUBE

The volume of a cube with each side of length $s$ units is $s^{3}$ cubic
units. $V=s^{3}$ or $V=B \cdot h$
where $B$ is the area of the base of a 3 -dimensional figure.

## EXPLORATION 2

How many cubic inches are there in one cubic foot?
In order to think about this problem, let's begin by reviewing how to change units in computing areas. A square that is one foot long on each side has an area of one square foot. Thinking in terms of smaller units, each side of the square foot is 12 inches long. Using this ratio of feet to inches,

1 square foot $=(1$ foot $)(1$ foot $)=(12$ inches $)(12$ inches $)=144$ square inches.
Using the same pattern,

$$
\begin{aligned}
1 \text { cubic foot } & =(1 \text { foot })(1 \text { foot })(1 \text { foot }) \\
& =(12 \text { inches })(12 \text { inches })(12 \text { inches }) \\
& =1728 \text { cubic inches. }
\end{aligned}
$$

Another way to compute the volume of this cube is to use the formula you just learned. Since one foot $=12$ inches, each side of the cube is 12 inches long. So the volume of your cube is $12^{3}=(12)(12)(12)=1728$ cubic inches.

In three-dimensions, conversions to smaller units make volumes seem much larger even though the shape and size have not changed at all!

## EXAMPLE 2

Draw a rectangular prism with edges that are 2,3 and 4 units long.
a. Find its volume.
b. Now scale the prism using a scale factor of 2 , then a scale factor of 3 . What are the new dimensions with each scale factor? What is the new volume in each case?

## SOLUTION

The first step is to draw the two base rectangles. For example, make the bases $2 \times 3$ rectangles. Place these rectangles 4 units apart to make the height of the box.

a. How many unit cubes does it take to fill the box? Using the two-dimensional pattern of cutting a $2 \times 3$ rectangle into 6 unit squares, cut the box into $6 \cdot 4=24$ unit cubes. Notice that each layer has the same number of cubes: There are 6 cubes in the first layer, 6 cubes in the second layer, 6 in the third layer and 6 in the fourth layer.
b. When you scale with a scale factor of 2 , the new edges will be 4,6 , and 8 units. The volume will then be:
$4 \cdot 6 \cdot 8=(2 \cdot 3) \bullet(2 \cdot 3) \bullet(2 \bullet 4)=8 \cdot 24=192$ cubic units
When you scale with a scale factor of 3 , the new edges will be 6,9 , and 12 units. The volume will then be

$$
6 \cdot 9 \cdot 12=(3 \cdot 3)(3 \cdot 3)(3 \bullet 4)=27 \bullet 24=648 \text { cubic units }
$$

Do you see a relation between the scale factor and the new volume?
In each case, to find the volume multiply the area of the base by the height to get the volume.

In general the volume of a prism is equal to the area of the base times the height. This formula is often written as $V=B \cdot h=B h$. The variable $B$ is the area of the base. This general formula is true for any prism, regardless of the shape of the base, whether it is a rectangle, a triangle, a hexagon or any polygon.

## FORMULA 11.6: VOLUME OF A PRISM

The volume of a prism is the area of the base of the threedimensional figure times the height of the prism.

$$
V=B \cdot h
$$

where $B$ is the area of the base of the 3 -dimensional figure

PROBLEM 2
a. $108 \mathrm{~cm}^{3}$
b. $216 \mathrm{~m}^{3}$
c. $64 \mathrm{in}^{3}$
d. $63 \mathrm{ft}^{3}$
e. 66.99 units $^{3}$

## EXAMPLE 3

Using the triangular prism below, which has a height of 7 units, a triangle base of 4 units and a height of 5 units, determine the volume of the prism.


## SOLUTION

Begin with the area of the base, since the base is a triangle. Use the formula $A=\frac{1}{2} \bullet b \bullet h$, where $h$ is 5 the height of the triangular base.

Area of the Triangle:
$A=\frac{1}{2} \bullet b \bullet h \quad A=\frac{1}{2}(4)(5) \quad A=\frac{1}{2}(20) \quad A=10$ units $^{2}$
After you find the area of the triangle, which in the volume formula is $B$, you need to "stack" the area to determine the volume of the prism. Do this by multiplying the area of the triangle by the height of the prism.

So, $V=B \cdot h \quad V=10$ units $^{2} \cdot(7$ units $)=70$ units $^{3}$

## PROBLEM 2

Write equations to represent the volume, and use these equations to calculate the volume of the following prisms:
a.

b.


## EXPLORATION 3

Students can use rice or any other material that will take the shape and volume of the inside of the roll. Measurements are easier if you have a graduated liter measure and a centimeter ruler.

Make sure your students measure the height of the cylinder and the radius of the circle to make the connection between the two parameters and the volume. Use several measures to get an average. Review the definition and approximate value of pi.
c.
e.

d.


Remember, the volume in cubic inches of three-dimensional shapes is the number of one-inch cubes it takes to fill the shape exactly. Because some shapes cannot be easily filled with one-inch cubes, the volume might be a fraction or a decimal part of a cube. As in the case of prisms, you can examine volumes and arrive at formulas that will make the computation much easier than counting blocks every time.

## EXPLORATION 3

Find a hollow cylinder, like a paper towel roll or an empty can. Measure the volume inside the cylinder.

If possible, fill the inside space with something such as a non-drying clay. Remove the play dough and form a rectangular prism. Measure its dimensions and determine its volume. Measure and record the dimensions of the cylinder using a ruler. Use the radius of the circular base and the height of the cylinder to compute the volume of the cylinder. Use several measurements to get an average. Review the definition and approximate value of $\pi$.

Compare the two computed volumes with the play dough or with the ruler to the cylinder's volume.

Adapting the prism volume formula, the volume of a cylinder is the area of the base times the height. Because the base is a circle, its area is $\pi r^{2}$. Therefore, the formula for the volume of a cylinder is $V=\pi r^{2} h$.

## EXAMPLE 4

Determine the volume of the cylinder below.


## SOLUTION

In order to calculate the volume of a cylinder, we can use the formula $\mathrm{V}=\mathrm{B} \cdot \mathrm{h}$ (area of the base of the figure times the height of the figure). Since the base is a circle, the formula to determine the area is $\pi r^{2}$.
$A=\pi r^{2}=3.14 \cdot 7 \cdot 7$
$A=153.86$ square inches

The next step is to multiply the area of the base times the height of the cylinder.
$V=B \cdot h$
$V=153.86 \cdot 10$
$V=1538.6$ square inches
Imagine a pyramid and a prism like the ones below with congruent bases and the same height.


PROBLEM 3
$V=\frac{1}{3} B h$
$B=\frac{1}{2}(4)(5)=10$ square units
$V=\left(\frac{1}{3}\right)(10)(7)=23 \frac{1}{3}$ cubic units

1. $30 \mathrm{ft}^{3}$
2. $64 \mathrm{in}^{3}$

How do you suppose the volume of the pyramid is related to the volume of the prism? Do you think it would be $\frac{1}{2}$ as much? $\frac{1}{4}$ as much? If we were to have two paper models like the ones in the picture above, we could calculate the relationship using rice or sand to fill the prism and see how many times the amount of material would fill the pyramid. In fact, the volume of a prism is equal to 3 times the volume of a pyramid with congruent bases and the same heights. Alternately, the volume of the pyramid is $\frac{1}{3}$ the volume of the prism.
The formula for the volume of a pyramid is often written as $V=\frac{1}{3} B h$, where $B$ is the area of the base, and $h$ is the height of the pyramid. As with prisms, this formula is true for all pyramids, regardless of the shape of their base, whether it is a rectangle, a triangle or any polygon.

## FORMULA 11.7: VOLUME OF A PYRAMID

The volume of a pyramid is equal to $\frac{1}{3}$ the area of the base, B , times the height of the pyramid.

$$
V=\frac{1}{3} B h
$$

## PROBLEM 3

Calculate the volume of a triangular pyramid, in which the base is identical to the triangular base in Example 3, with a height of 7 units.

## EXERCISES

1. What is the volume of a rectangular prism with edges $2 \mathrm{ft}, 3 \mathrm{ft}$ and 5 ft ?
2. Sketch a rectangular prism with dimensions 2 by 6 by 8 in three different ways by switching the dimensions for each sketch.
a. Using each prism, inscribe a rectangular pyramid inside the prism with the same base and height as the original prism. Compare the shapes of the three pyramids. Can you see that each pyramid has a different shape?
b. Compute the volume for each of these inscribed pyramids.
3. What is the volume of a cube with side lengths of 4 inches? Now scale using a scale factor 5 .
a. What is the new volume?
4. $20 \pi \mathrm{in}^{3}$
5. a. $160 \pi \mathrm{in}^{3}$
b. The volume is multiplied by 8 .
c. The volume is multiplied by 4 .
d. The volume is multiplied by 2 .
6. The base is a right triangle, so the volume is $B h=\frac{1}{2}(6)(8)(12)=288 \mathrm{~cm}^{3}$.
7. $V=B h=\pi(0.6)^{2}(6)=2.16 \mathrm{in}^{3}$
8. $600 \mathrm{~cm}^{3}$
9. $\mathrm{V}=234 \mathrm{in}^{3}=(x)(4)(5) . x=11.7 \mathrm{in}$.
10. Note the 10 pounds is unnecessary information since one can solve the problem without using it. Compute each cube volume and use proportion with cost as follows. Volume of the 2 inch cube is 8 cu in and the volume of the 3 inch cube is 27 cu in, so the ratio is $\frac{27}{8}$-i.e. increasing the side of the cube from 2 inches to 3 inches multiplies the volume by $\left(\frac{3}{2}\right)^{2}=\frac{27}{8}$. We want to know the cost of the 3 inch cube given the 2 inch cube is $\$ 144$. The proportion is $x / \$ 144=\frac{87}{8}$ or $x=\frac{27}{8}(\$ 144)=\$ 486$.
11. The volume of the tank is $36 \pi(15)=540 \pi=1695.6 \mathrm{ft}^{3}$. Because the tank is being filled at a rate of $1.5 \mathrm{ft}^{3} /$ second, it will take $\frac{1695.6}{1.5}=1130.4 \mathrm{sec}=18 \mathrm{~min} 50.4 \mathrm{sec}=$ about 19 min .
12. $V_{\text {pism }}=72$ cubic inches
$V_{\text {pyamid }}=24$ cubic inches
b. How does the volume of the original prism change if you scale using a scale factor of 10 ?
c. Do you notice a pattern between the scale factor and the volume?
13. Make a scaled sketch of a cylinder whose base has radius 2 inches and whose height is 5 inches, and label the dimensions. What is the volume of the cylinder?
14. A cylinder has bases with radius 4 inches and height 10 inches.
a. What is its volume?
b. What happens to the volume if both the radius and height are doubled?
c. What happens to the volume if only the radius is doubled?
d. What happens to the volume if only the height is doubled?
15. Draw a triangular prism whose base is a right triangle with sides $6 \mathrm{~cm}, 8 \mathrm{~cm}$ and 10 cm and whose height is 12 cm . Find the volume of the triangular prism.
16. What is the volume of a cylinder that has base radius 0.6 in and height 6 in?
17. The height of a triangular prism is 20 cm . The triangular base has height 12 cm , and the length of the base of the triangle is 5 cm . What is the volume of the prism?
18. A rectangular prism has volume $234 \mathrm{in}^{3}$ and edges with lengths $4 \mathrm{in}, 5$ in and $x$ in. What is the value of $x$ ?
19. A solid two-inch cube weighs 10 pounds and is worth $\$ 144$. How much is a solid three-inch cube made of the same material worth?
20. A cylindrical tank 12 feet in diameter and 15 feet high fills with water at the rate of 1.5 cubic feet per second. How long does it take to fill the tank?
21. Calculate the volumes of a pyramid and a prism, in which the base is $3^{\prime \prime} \times 4$ " rectangle, and the height is $6^{\prime \prime}$.
22. Ingenuity
a. $40 \mathrm{in}^{3}$ less
b. $28 \mathrm{in}^{3}$ less
c. 4 less. Check the answers to $a$ and $b$ to see if they agree with $c$.

Investigation
13. a. $V=\pi r^{2} h=\pi r^{2}=\pi$ so the radius $r=1$ in
b. $\quad \pi r^{2} \pi=\pi^{2} r^{2}=1$, so $r=\frac{1}{\pi}$ in
c. $\pi r^{2} h=\pi r^{2}=1$, so $r=\frac{1}{\sqrt{\pi}}$ in.

Remember, $\pi$ is just a number, like 3 , so treat it that way.
13. The volume of a rectangular pyramid that is 8 inches high is 96 cubic inches $A$ rectangular prism has the same base and height as the rectangular pyramid. What are some possible dimensions for the base of both the rectangular prism and pyramid?
14. The base of a triangular prism is a right triangle with dimensions of 6,8 , and 10 units. The height of the prism is 5 units.
a. What is the volume of the prism?
b. Sketch three possible triangular pyramids that have the same base triangle and the same height as the original triangular prism.
c. Calculate the volume of these three pyramids. What do you notice about your answers?

## 15. Ingenuity:

A rectangular prism has a square base. Then one pair of opposite sides of the base of the prism are each increased by two inches. The other pair of opposite sides of the base are decreased by two inches.
a. What is the change in volume if the height of the prism is 10 inches and the original base had side length 4 inches?
b. What is the change in volume if the height of the prism is 7 inches and the original base had side length 5 inches?
c. What is the change in volume if the height of the prism is $h$ inches and the original base had side length $s$ inches?

## 16. Investigation:

a. What is the radius of a cylinder that has height 1 in and volume $\pi$ in $^{3}$ ?
b. What is the radius of a cylinder that has height $\pi$ in and volume $1 \mathrm{in}^{3}$ ?
c. What is the radius of a cylinder that has height 1 in and volume 1 in $^{3}$ ?


## Comparing the Volume of Pyramids and Prisms

Objective: This activity will model the relationship between the volume of pyramids and prisms with congruent bases and heights. The teacher can demonstrate the concept to the class, or the students can participate by creating their own models and determining this relationship on their own.

## Materials:

Templates for the net of an open-ended pyramid and an open-ended prism with congruent bases and heights (included in TE; it is best to reprint the templates on card stock, as you will need them to be sturdy). The height and base should be identical for the two objects.
Tape to secure nets/templates into 3D forms
Rice or sand to fill the forms

## Activity Instructions:

1) Cut out and tape the edges of the templates for the open-ended prism and open-ended pyramid.
2) Measure the height and base of each object, and calculate the area of the base. They should be identical.
3) Use sand or rice to fill the pyramid.
4) Dump the sand out of the pyramid into the prism. Repeat this until you have determined how many pyramids worth sand are needed to fill the prism.

Prism Worksheet

| Prism | Name | Vertices | Faces | Edges |
| :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |

## How Many Ice Cubes?

Objective: The students will use their knowledge of volume to design a cooler that will hold ice for a family picnic. The problem is open-ended and encourages students to use their math skills and their creativity.

## Materials:

Copy of "How Many Ice Cubes?" worksheet, one per student
Access to centimeter cubes, if they want them
Rulers, or any straight edge

## Activity Instructions:

1) Distribute a copy of the worksheet, one per student.
2) Encourage students to work alone, but allow them to collaborate with peers if they need a little help or just want some reassurance that their ideas are working.
3) If students are struggling to find the necessary volumes, encourage them to use the centimeter cubes to model the problem.
4) It is important to stress to your students that there is not ONE correct answer to this problem. You are interested in them using correct math, but also interested in their creative ideas in the design of their cooler.
5) When checking students' work, remember that they are looking for a cooler size that will maximize space for cubes that are 3 centimeters on each edge. The cooler should be large, but not too large. It is my belief that a cooler in the shape of a rectangular prism would work best, but don't discourage students from using other shapes if they have a valid reason why they think it will work best.
6) It would be fun to have students share their designs with the class when finished. If you don't have time for all students to share, maybe you could pick your top 5 favorites and let them do a little presentation in front of their peers.

Name $\qquad$

## How Many Ice Cubes?

Joselyn's family is going on a picnic and wants to pack one of their coolers full of ice. This cooler will only contain ice, as they will be outside all day long and it is expected to be very hot. Her family wants to make sure that they have plenty of ice to keep their drinks cold all day long. Joselyn prefers ice that is shaped in cubes, and the ice trays at her house make perfect cubes that are 3 centimeters on each edge. In the space below, design a cooler that you would recommend for Joselyn and her family to take on this picnic for their ice. After you design the cooler, be sure to label its dimensions and find its volume. Then, in the space at the bottom of the page (or on the back of this sheet), explain in detail why you think your cooler will work best for this situation. In your explanation, please include the capacity (volume) of your cooler and the number of ice cubes that your cooler will allow them to take with them on their picnic.

## Figure Out My Volume

## Objective:

Students will use the formula to find the volume of rectangular prisms and cubes.

## Materials:

Index cards with dimensions (length, width, height) of rectangular prisms and Cubes
Numbered cube Timer/stop watch
Unifix cubes
Chart with formulas (optional)

## Activity Instructions:

1) Students throw a number cube to figure out who goes first, etc.
2) Teacher prepares index cards with $L, W, H$ (dimensions in inches, feet, cm , etc.)
3) Students shuffle the dimensions index cards and leave them in a pile.
4) Player1 draws one card from the top of the pile. Player1 writes the dimensions given in his card to figure out the volume of the 3-D shape.
5) The other players check Player1's answer
6) If Player 1 figures out the volume correctly, he/she gets 5 points. Then he/she gets 1 min . to build his 3 -shape using unifix cubes. If the shape is done on time, Player1 gets 5 more points. Players keep their score on a scoring sheet (plain sheet of paper).
7) If Player1 figures out the volume incorrectly, he/she continues to step8.
8) Player2 gets a turn and repeat Steps 4-7
9) Activity continues until all the cards have been played.
10) The player with the most points wins.

## Guess My Solid

## Objective:

Students will identify, describe, and classify attributes and properties of three-dimensional figures.

## Materials:

3-D models (square pyramid, triangular prism, hexagonal prism, cylinder, cone, cube of different colors) Brown paper bag
Numbered cube

## Activity Instructions:

1) Students use a numbered cube to figure out who goes first, etc.
2) Each player1 pulls out one 3-D solid from brown bag without letting others see it.
3) The other players try to guess both the solid and the color of the 3-D shape drawn by player 1 .
4) The player who thinks he/she knows the shape and color, takes a guess.

If correct, that player keeps the 3-D solid.
5) Players continue taking turns, asking each other one question at a time about the mystery 3-D solid to try to guess (and keep) their opponent's solid.
6) The game continues until the first player who gets 5 solids wins the game.

## Section 11.7 - Surface Area and Nets

## Big Idea:

Discovering surface area of prisms and cylinders using nets, and exploring different views of 3-dimensional solids

## Key Objectives:

- Learn the vocabulary associated with surface area of 3-dimensional figures.
- Given a solid, sketch its net.
- Sketch the solid represented by a net.
- Find the surface area of a solid, given its net.
- Construct a solid, given two-dimensional views of its sides and base.


## Materials:

Small cereal boxes, Unit cubes, Straight edge, Protractor, Grid paper, Scissors, Tape, Nets from the CD or the end of this section

## Pedagogical/Orchestration:

- This section connects two dimensions to three dimensions with an exploration of surface area. Students who have a hard time with numbers but can visualize geometric shapes and forms might love this section.
- The ability to see how a net forms a 3-dimensional figure, or how a solid can be created by cutting and folding a net is key to this section. Active practice is the way that most students will attain this important skill.


## Activity:

"Diamond" Activity at the end of the section and on CD

## Internet Resource:

Rags to Riches: Geometry Review- http://www.quia.com/rr/237636.htm|

## Vocabulary:

surface area, nets, views, lateral surface area

## TEKS:

$7.8(\mathrm{~A}, \mathrm{~B}, \mathrm{C}) ; \quad 7.13(\mathrm{~B}, \mathrm{C}) ; \quad 7.14(\mathrm{~A}) ; \quad 8.7(\mathrm{~A}, \mathrm{~B}) ; \quad 8.8(\mathrm{~A}, \mathrm{C}) ; \quad 8.15(\mathrm{~A})$

## WARM-UPS for Section 11.7 (Surface Area and Nets)

1. Suppose the two triangles below, $\triangle A B C$ and $\triangle A^{\prime} B^{\prime} C^{\prime}$, are similar. If $a=4$ and $b=10$ then which of the following lengths of $a^{\prime}$ and $b^{\prime}$ are possible?

a. $\mathrm{a}^{\prime}=8$ and $\mathrm{b}^{\prime}=14$
b. $\mathrm{a}^{\prime}=12$ and $\mathrm{b}^{\prime}=18$
c. $a^{\prime}=5$ and $b^{\prime}=11$
d. $a^{\prime}=10$ and $b^{\prime}=25$

Ans: (d)
2. A group of scientists are testing 4 different hybrid cars for their gas mileage. If the chart below is a record of their preliminary trails, which car has the best gas mileage?

| Hybrid Car | Amount of Gasoline | Miles Driven |
| :---: | :---: | :---: |
| A | 11 | 484 |
| B | 12 | 504 |
| C | 10 | 410 |
| D | 13 | 520 |

a. Car A
b. Car B
C. Car C
d. Car D

## Ans: (a) Car A gets 44 mi/gal, Car B gets 42 mi/gal, Car C gets 41 mi/gal and Car D gets 40 mi/gal

## Launch for Section 11.7:

Have students bring small one-serving cereal boxes or provide the boxes for them. Today's launch incorporates the first two paragraphs of the lesson. Hand out the cereal boxes to each student and ask them what the measurement is called that tells us how much the box will hold. Students should answer "volume." Then ask the students what unit of measurement is used to measure volume. The answer is cubic units but students may say cubic feet, or cubic inches which are more specific. Ask students when it would be best to use cubic feet and when it would be best to use cubic inches. For instance, if we are measuring how much space is inside a refrigerator, which unit should we use? In fact, refrigerators are measured in cubic feet. Hold up the cereal box again and ask what units would be best to measure the space inside of it. Cubic inches or cubic centimeters would work best for the cereal box. Tell your students, "Now there are other interesting aspects of the cereal box to explore. We are no longer going to talk about the inside of the box, but the surface of the box, instead. For example, how much cardboard is needed to make a cereal box, ignoring the flaps?" Instruct your students to take apart their boxes and flatten them to make flat patterns, otherwise known as nets. Ask them how many rectangles it takes to make the box, and to find the area of each rectangle. Once they have added the areas of the 6 rectangles, tell them they have found the "surface

## LAUNCH CONTINUED

area" of the box. Ask them what units they think the surface area will use. Students often mistakenly use cubic units for this, but remind them that the flat pattern is 2-dimensional and so square units will be used for the surface area. Another way to ask about surface area is "How many square inches can be drawn on the surface of the box?" Tell your students, "Today we will be exercising our ability to see what happens when a three-dimensional solid is flattened and how this relates to the surface area of a solid."

This is a great opportunity to do a class activity on folding nets using the nets provided at the end of the section in the TE.

1. Give each group 2 copies of each net: Rectangular Prism, Cube, Triangular Prism, Pentagonal Prism, Hexagonal Prism, Triangular Pyramid, Rectangular Pyramid, Cone, and Cylinder.
2. Examine each net and speculate what shape they will form when folded.
3. Cut out each net and fold. Attach each figure to a butcher paper using glue or tape.
4. Label each figure.

## SECTION 11.7 SURFACE AREA AND NETS

In the previous section, you explored finding the volume of three-dimensional shapes. Finding volume involves measuring the space inside shapes. You can measure the volume of a cereal box. But there are other interesting aspects of the cereal box to explore. For example, how much cardboard is needed to make a cereal box, ignoring the flaps? To find out, flatten the box and measure the area of the cardboard.

The surface area of a three-dimensional figure is the area needed to form its exterior. One way to see surface area of a three-dimensional figure is to cut the figure along its edges and "flatten" its exterior on a surface to make a twodimensional shape. The flattened exteriors are called nets. The shapes that make up the nets are often rectangles, triangles and circles. The net of a cereal box might look like this:


Multiple nets can be drawn for the same geometric solid depending on which edges are cut. The figures below are two additional nets of the same prism that is deconstructed above.


Make sure your students understand that " $2 \times 3 \times 4$ prism" means a prism that has a length, width and height of 2,3 and 4 units, in any order. If students are confused, it might help to point out that for a rectangular prism, we can think of any of the opposite sides as the bases: it's just a matter of perspective.

This is NOT the time for passive learning. Ask your students to bring some box that they can cut up to class. Provide scissors and ask them to make nets of their boxes. Then, have them look at each other's boxes. There should be some variation in the shapes for the class to discuss.

## EXAMPLE 1

Find the surface area of the net of a rectangular $2 \times 3 \times 4$ prism.


## SOLUTION

To find the surface area, construct a net from your figure and place it on a flat surface. What does it look like? Compare various nets formed from the same prism. For one possible net, the top and bottom, or the bases, each have area $3 \times 4$. Two of the opposite sides have area $2 \times 3$, and the other two sides have area $2 \times 4$.. One way to cut this apart is shown below:


The total surface area is $6+6+12+12+8+8=52$ square units. Can you identify from where each term in this sum came? Remember, a net or surface area is measured in square units, like all two-dimensional areas.

Is there a more efficient way to find the surface area of a rectangular prism? The prism below has dimensions $I \times w \times h$, where $I=$ length, $w=$ width, and $h=$ height of the rectangular prism.


What are the areas of the two congruent bases and opposite sides? Each base has area $B=I \times w=\mid \cdot w$. Two of the parallel sides have equal area $h \times I$. The other two parallel sides have equal area $h \times w$. Notice that all four sides have equal height $h$. The sum of the areas of each of the four sides, also called the Lateral Surface Area, is

$$
(l \cdot h)+(w \cdot h)+(l \cdot h)+(w \cdot h)=h(l+w+l+w)=(2 l+2 w) h .
$$

The common factor $h$ is the height of the prism, and the sum $(2 l+2 w)$ is the perimeter, $P$, of the base rectangle. The sum of the areas of each surface, which includes all lateral surfaces and the bases, is known as the Total Surface Area. From the figure below, you can see that $S=2 / h+2 w h+2 / w$, where $S$ is the total surface area of the rectangular prism.


Since $S=2 l h+2 w h+2 / w=(2 l+2 w) h+2 / w$, this observation leads to a formula for the total surface area of a rectangular prism, $S=2 B+P h$, where $S$ is the total surface area of the rectangular prism, $B$ is the area of the base, $P$ is the perimeter of the base rectangle and $h$ is the height of the prism.

## PROBLEM 2

There are 11 different nets that form a cubesx.

## FORMULA 11.7: SURFACE AREA OF A RECTANGULAR PRISM

The surface area of a rectangular prism, S, is given by the formula

$$
S=2 B+P h
$$

where $B$ is the area of the base, $P$ is the perimeter of the base rectangle, and $h$ is the height of the prism.

## PROBLEM 1

Use a similar process to find the surface area of a cube. Sketch a net to explain how you found the formula.

FORMULA 11.8: SURFACE AREA OF A CUBE
The surface area, S , of a cube is given by the formula

$$
S=6 \cdot s^{2}=6 B
$$

where $s$ is the length of a side, and $B$ is the area of the base.

## PROBLEM 2

a. There are many ways to draw a net for a cube. Draw as many nets as you can that form a cube. What would be an example of one that would not work? Explain why.
b. Draw a net for a triangular prism with side lengths of 3,5 , and 6 and height 10. Compute the lateral surface area of this prism.

Find an oats container that you can cut apart. Cut it so the class can see what it looks like.

## EXPLORATION 2

Find the surface area of the cylinder by examining the net for the cylinder.


The net for a cylinder has a middle section that is a rectangle representing the lateral surface. This rectangle has width $h$ equal to the cylinder's height, and length / equal to the circle's circumference: $I=2 \pi r$. The area of the lateral surface of a cylinder is the product $2 \pi r h$. The two bases have a combined area of $2 \pi r^{2}$.

FORMULA 11.9: SURFACE AREA OF A CYLINDER
The total surface area, S , of a cylinder is the sum of the areas of the bases and the latreal surface,

$$
S=2 \pi r^{2}+2 \pi r h,
$$

where $r$ is the radius of the base and $h$ is the height of the cylinder

## PROBLEM 3

Draw a net for a cylinder with a radius of 4 cm and a height of 6 cm . Using $\pi=3.14$, calculate the surface area of the cylinder.

## EXPLORATION 3

a. The pyramid has only 1 base. Its lateral surfaces are triangular.
b. The nets have the same base shape and base area.
c. Remove one of the bases. Change the lateral surfaces into triangles.
d. Take away 1 base, i.e. $2 B-B$. Divide the lateral surfaces by 2 or multiply the lateral surfaces by $\frac{1}{2}$, i.e. $\frac{1}{2} P h$. Note: You may need to demonstrate to the students that changing a rectangle or a square into a triangle is the same as dividing the area of the square or rectangle by 2 . For example:


If the area of a rectangle is $(4)(5)=20$
Then the area of a rectangle inscribed within the rectangle is $\frac{1}{2}(4)(5)=10$
The area of the rectangle minus the area of the triangle is the area of the shaded region. So, changing a rectangle or square into a triangle is the same as dividing by 2 .

The students should now have changed the formula for a prism into $S=B+\frac{1}{2} P h$. Here is where you must explain that we no longer call the height of the pyramid $h$. Calling it $h$ is implying that it is the height of the pyramid. It was the height of the prism, but now it is no longer perpendicular with the base. We change the $h$ and call it $l$, which is the slant height. So, the formula is $S=B+\frac{1}{2} \mathrm{Pl}$.

## EXPLORATION 3

Below are two nets, a net of a cube and a net of a square pyramid. Using the formula for the total surface area of a prism, which is $S=2 B+P h$, can you create a formula to calculate the surface area of a pyramid?

a. What is different about the two nets?
b. What is similar about the two nets?
c. What should be done to the net of the cube to make it resemble the net of a pyramid?

d. What should be done to the formula for the surface area of a cube for it to resemble the changes in question c?
$H$ is important to distinguish between the height of the pyramid $(h)$ and the slant height ( $/$ ).

In a pyramid the surface area is equal to the area of the base plus the areas of each triangular lateral side. It is important when determining the areas for the triangular sides to distinguish between the height of the pyramid $h$, and the height of the individual sides, also known as the slant height /.

In the picture below, the height of the pyramid is 15 cm , the slant height is equal to 17 cm , and the dimensions of the base are $16 \mathrm{~cm} \times 16 \mathrm{~cm}$.


16 cm
The area of the base is $B=16 \mathrm{~cm} \times 16 \mathrm{~cm}=256 \mathrm{~cm}^{2}$. The surface area of each lateral side $S=\frac{1}{2} 16 \mathrm{~cm} \times 17 \mathrm{~cm}=136 \mathrm{~cm}^{2}$. The total surface area is equal to the sum of the area of the base and each lateral side, $\mathrm{S}=256 \mathrm{~cm}^{2}+4\left(136 \mathrm{~cm}^{2}\right)$ $=800 \mathrm{~cm}^{2}$. This can be written as $S=B+\frac{1}{2} P 1$, since the combined surface area of all the lateral sides is equal to $\frac{1}{2}$ the perimeter times the slant height.

## FORMULA 11.10: SURFACE AREA OF A PYRAMID

The total surface area, $S$, of a square pyramid is the sum of the area of the base and the area of the lateral surfaces,

$$
S=B+\frac{1}{2} P h,
$$

where $B=x^{2}$ is the area of the $x$-by- $x$ square base, $P=4 x$ is the perimeter of the base and $h$ is the slant height of each of the triangles that form the sides of the pyramid. Note: the slant height of each side is not the length of the edge. It is the perpendicular distance from the apex of the pyramid to the base of each side.

## PROBLEM 4

Answer: $111 \mathrm{in}^{2}$.

## EXPLORATION 4

Isometric dot paper may be used to help visualize and create pictures of 3-dimensional figures. Isometric paper can be found on this website: http://illuminations.nctm.org/activitydetail.aspx?ed=125

Talk with Art teacher and Technology teacher about enrichment and interdisciplinary units of study.

Additional Resources: http://members.westnet.com/au/molinasantos/strands/space/nelsonbook13-3.pdf

## PROBLEM 4

The picture below is the net of a rectangular pyramid.

a. Calculate the lateral area using the net that is labeled below.
b. Calculate the volume of this pyramid.
c. Write a general formula for the lateral area of a rectangular pyramid

## PROBLEM 5

Johnny has a toy in the shape of a tetrahedron, which is a triangular pyramid of which each side is an equilateral triangle with sides of length 10 cm . He measures the height of each triangular side (they are all congruent) to be approximately 8.7 centimeters.
a. Sketch a net for this triangular pyramid.
b. Compute the approximate lateral surface area of this pyramid.
c. Compute the approximate total surface area of this pyramid.

## PROBLEM 6

Now scale the figure using a scale factor of 2 . What is the new surface area?

## EXPLORATION 4: DIFFERING VIEWS

Work with a partner. Each of you should take 12 unifix cubes, unit base-ten blocks or some kind of cubes and construct a three-dimensional shape. Each face must fully touch another face or touch nothing. Declare one side the front. Now change places with your partner and draw on grid paper the front, left, right and top view of your partner's three-dimensional shape. After each of you has finished drawing your two-dimensional views of your partner's shape, discuss your results with your partner.

PROBLEM 7


Use physical blocks to demonstrate the different views. Use the following link to an online 3D view of these objects. http://www.fi.uu.nl/toepassingen/02015/toepassing_rekenweb.xml?style=rekenweb\&language=en\&use=game
1.
a.

b.

c. Answers may vary.
2. Pentagonal Prism

## PROBLEM 7

Consider the following two-dimensional views of a three-dimensional solid. Create the three-dimensional figure that corresponds to the three views. Is there only one such figure? Could there be more?


## PROBLEM 8

Consider the irregularly shaped object: A rectangle 10 feet by 15 feet with a right triangle with its hypotenuse on one side, and a semicircle on the opposite side. The height of the object is 10 feet.
a. What is the are of the base of the object?
b. What is the lateral area?
c. What is the volume of the object?

## EXERCISES

1. Sketch the following nets:
a. triangular prism
b. triangular pyramid
c. Explain the differences in the nets from $a$ and $b$.
2. Name the three-dimensional figure represented by the following net.

3. 


4. $(2)(4)(5)+(2)(4)(7)+(2)(5)(7)=166 \mathrm{~m}^{2}$
5. $(2)(5)(3)+(2)(5)(6)+(2)(3)(6)=126 \mathrm{ft}^{2}$
6. By Pythagorean Theorem, $3^{2}+4^{2}=c^{2} . c=5$.
(2) $\frac{1}{2}(3)(4)+(3)(8)+(4)(8)+(5)(8)=108 \mathrm{~cm}^{2}$
3. Use the following three-dimensional solid to answer a-c.

a. Draw the top view of this solid.
b. Draw the front view of this solid.
c. Draw the side view of this solid.
4. Draw a scaled net of a rectangular prism with edges of lengths 4 meters, 5 meters, and 7 meters. What is the surface area of the prism?
5. Draw a net of the rectangular prism below. What is the prism's surface area? Shade the lateral area of the net, excluding the bases. What is the lateral area?

6. Draw a net for the triangular prism and compute the lateral area. Compute the total surface area of this prism.

7. A triangular prism has sides of length $a, b$, and $c$ and $a$ height of $h$. Write $a$ formula for the lateral area for this prism.
8. Answer: c
9. $2 \pi 3^{2}+2 \pi 3(4)=42 \pi$
8. Which of the following three-dimensional solids correspond to the following top, front, and side views?


d.

9. Sketch a net of a cylinder of height 4 and base radius 3 .
a. What is cylinder's surface area?
b. Scale the cylinder by multiplying each dimension by 3 . What is the new surface area?
10. $\mathrm{V}=(2)(2)(5)=20 \mathrm{in}^{3} ; \mathrm{SA}=(2)(2)(2)+(4)(5)(2)=48 \mathrm{in}^{2}$
11. The original surface area of a cube with side $s=6 s^{2}$. If $s$ is decreased by $20 \%$, then $80 \%$ remains and the surface area of the smaller cube $=6(0.8 s)^{2}$. The decrease amount is given by $6 s^{2}-6(0.64) s^{2}=6(0.36) s^{2}$. The percent decrease $=$ the difference over the original $=6(0.36) s^{2} / 6 s^{2}=0.36=36 \%$ decrease .
12. $6 s^{2}=864$, so $s=\sqrt{864 / 6}=12 \mathrm{~cm}$
13. $2\left(\frac{3}{5} \cdot \frac{7}{4}+\frac{3}{5} \cdot \frac{1}{3}+\frac{1}{3} \cdot \frac{7}{4}\right)=\frac{11}{3} \mathrm{ft}^{2}$
14. $36.5 \mathrm{ft}^{2}$
15. $42 \mathrm{ft}^{2}$
16. $2 \pi(0.6)(0.35)+2 \pi(0.6)^{2}=1.14 \pi \mathrm{~mm}^{2}$
10. A prism 5 inches high has parallel square bases 2 inches long.
a. What is its volume? What is its surface area?
b. Scale the cylinder by multiplying each dimension by 5 . What is the new surface are? What is the new volume? What pattern do you notice between the scale factor of the cylinder and the surface area? The volume?
11. The length of each side of a cube decreases by $20 \%$. What is the percent decrease in surface area of the cube?
12. The surface area of a cube is $864 \mathrm{sq} . \mathrm{cm}$. What is the length of the each side of the cube?
13. What is the surface area of a rectangular prism with side lengths $\frac{3}{5} \mathrm{ft}, \frac{7}{4} \mathrm{ft}$ and $\frac{1}{3} \mathrm{ft}$ ?
14. Determine the lateral surface area and the total surface area of the rectangular pyramid given by the net below.

15. A triangular pyramid has an equilateral triangle as its base with side length 6 cm . Each lateral side is an isosceles triangle with slant height of 9 cm .
a. Sketch a net for this triangular pyramid.
b. Compute the lateral surface area of this pyramid
16. What is the surface area of a cylinder with height 0.35 mm and base radius 0.6 mm ?
17. $143.18 \mathrm{~cm}^{2}$
18. $\begin{array}{lll}\text { a. } 50.24 \mathrm{in}^{2} & \text { b. } 251.2 \mathrm{in}^{2}\end{array}$
19. $1584 \pi \mathrm{in}^{2}$ or $11 \pi \mathrm{ft}^{2}$
21. $\frac{(24)(24)(6)}{(2)(2)(6)}=\frac{144}{1}$ or $144: 1$

Ingenuity
22. The key to this solution is to think about the cubes that are not painted, then subtract the inner cubes from the total number. The total number of cubes is the volume, 512 . The cubes not painted are on the inside, $(8-2)^{3}$ $=6^{3}=216$. So the number of painted cubes is $512-216=296$.

Investigation
23. Use the formula for the area of a trapezoid. Use the generalized formula for the volume of a prism, substitute, and solve the equation. Let $h=$ height of the prism. The altitude of the base $a=\sqrt{13^{2}-(0.5(25-15))^{2}}=$ $\sqrt{169-25}=12 \mathrm{SA}=13 h+13 h+15 h+25 h+2\left(\frac{1}{2}(15+25) 12\right)=66 h+480=1998$.
So $h=\frac{1998-480}{66}=23 \mathrm{~cm}$.
Note the use of the Pythagorean Theorem to determine the height of the trapezoid, which is 12 cm .
17. A cylindrical soup can has a radius of 3.8 cm and a height of 6 cm . The soup company needs to determine how much paper is required to label each can. Find the lateral surface area of the soup can.
18. William is painting a wall with a cylindrical paint roller. The diameter of the base is 2 inches and the distance between the bases is 8 inches.
a. What is the lateral surface area of the paint roller?
b. What is the area William will paint in 5 revolutions?
19. A glass company is designing a cylindrical fish tank for an aquarium-themed restaurant. The fish tank needs to be open at the top. The restaurant wants the radius of the tank to be 12 inches and the height to be 5 feet. What is the surface area of the aquarium in terms of $\pi$ ?
20. A rectangular swimming pool is 20 feet wide and 50 feet long. THere is a semi-circular extension at the endo $f$ the pool, with a 20 -foot diameter. The pool is 10 feet deep.
a. What is the are of the base of the object?
b. What is the lateral are?
c. What is the volume of the object?
21. What is the ratio between the surface area of a 24 in $\times 24$ in $\times 24$ in cube in square inches and the surface area in square feet?
22. Ingenuity:

An $8 \times 8 \times 8$ cube is made of 512 unit cubes glued together. If the large cube is dipped in paint, how many of the unit cubes are painted?

## 23. Investigation:

The following diagram shows the base of a trapezoidal prism:


Draw the prism and its net. The prism has surface area $1998 \mathrm{~cm}^{2}$. What is the measure of its height?

## Rectangular Prism Net

*Note: these nets can be found and printed at: www.senteacher.org/wk/3dshape.php


## Square Pyramid Net



Triangular Prism Net


## Pentagonal Prism Net



Pyramid Net


## Square Pyramid Net



Cylinder Net


## Cone Net



## DIAMOND

Objective: Students will practice the skill of matching nets to their respective shapes. Students will also practice the skill of recognizing the two-dimensional faces that make up the net. Lastly, students will also practice measuring and finding the surface area of each of the nets.

## Materials:

Copy of each of the 6 Nets (be sure to shuffle them up before handing them out)
Tape
Scissors
Copy of Crystal Worksheet (one per group, best if copied 1 -sided to 2 -sided)
Access to a computer (or books) to research characteristics of the crystals
Rulers (to measure in centimeters)
Marker, crayons, or colored pencils

## Activity Instructions:

Divide your students into groups of approximately 3 to 4 . Make sure that each group gets one copy of each of the 6 nets, one copy of the Crystal Worksheet, and has access to scissors, tape, and a ruler.

Students will first try to match each of the 6 nets to one of the 6 crystals, using the descriptions of the crystal's faces. Students will record their guesses on the Crystal Worksheet. Next, students will measure the edges of the faces on each net, and compute the surface area of each. Students will record the surface areas on the Crystal Worksheet. Each group can divide up this work however they see fit. Also, remember that due to reproduction issues, some measurements may differ by 1 or 2 cm from the solutions on the answer key. Remind students, however, that they should be as precise as possible when measuring.

Once the Crystal Worksheet is complete, the groups now have two more missions to accomplish. The group will need to research each of the crystal names and try to find distinguishing characteristics about each of the 6 crystals, other than their "system type" which is already given. Student's research should provide them with enough information to be able to decorate their nets to look like the actual crystals. This research is important because the student's next mission is to assemble the net into a three-dimensional shape that represents the actual crystal. It is probably best if they do their research before their assembly, as it is easier to color the shapes when they are in their 2-dimensional form. All finished crystals should look like the actual crystals found in the world.

When finished, students will tape their assembled crystals in the final empty boxes on their Crystal Worksheet. Each group member will record their name at the top, and the worksheets will be turned in to be graded.
*(Another possible idea may be to have students record the information from the Crystal Worksheet (name and surface area) onto an index card and somehow attach the index card to the three dimensional shape. Using string, each crystal could then be attached to a wire hanger, to make a mobile out of the finished crystals. Finished mobiles could be hung in your classroom to display the students hard work.)

## Solutions:

Cubic $=$ Natural Diamond $=S A \approx 73.5 \mathrm{~cm} 2$
Tetragonal $=$ Zircon $=S A \approx 78 \mathrm{~cm} 2$
Orthorhombic $=$ Topaz $=$ SA $\approx 86.46 \mathrm{~cm} 2$
Rhombohedral $=$ Calcite $=S A \approx 71.04 \mathrm{~cm} 2$
Monoclinic $=$ Gypsum $=$ SA $\approx 112.2 \mathrm{~cm} 2$
Triclinic $=$ Turquoise $=S A \approx 129.6 \mathrm{~cm} 2$

Natural Diamond


Zircon


Topaz



Gypsum


Turquoise

Group Members:


|  |  |  |  |
| :---: | :---: | :---: | :---: |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  | $\bigcirc$ | $\sigma$ | $\sigma$ |
|  | $\frac{0}{3}$ |  | 0 0 0 0 0 0 0 0 |

Group Members:

| Type of <br> Crystal <br> System | Number <br> of <br> Faces | Description of <br> Faces | Name of <br> Crystal | Surface <br> Area (in <br> $\left.\mathbf{c m}^{2}\right)$ |  |
| :---: | :---: | :---: | :---: | :---: | :--- |
| Rhombohedral | 6 | Rhombuses on all <br> sides, no right <br> angles |  |  |  |
| Monoclinic | 6 | 4 Rectangles, 2 <br> Parallelograms, <br> 16 right angles, 8 <br> other angles |  |  |  |
| Hexagonal | 6 | 6 parallelograms, <br> no right angles |  |  |  |

1. a. HGA, HGB, AGE, BGE, CEG, DEG
b. AHG, HAG, GHF, HGF, GFB, FBG, FGB, BGD, BDG, EGD, GDE, ACE
c. GBD, GAC, HFG
d. (HAG, GAC), (HFG, BFG), (CEG, GED), (HGA, AGE), (HGB, BGE)
e. (HGF, FGB), (BGD, DGE)

## REVIEW PROBLEMS

1. In the following figure, lines $\overleftrightarrow{A B}$ and $\overleftrightarrow{C D}$ are parallel.


Identify each of the following:
a. Right angles
b. Acute angles
c. Obtuse angles
d. Supplementary angle pairs
e. Complementary angle pairs
2. a. KJL: $180^{\circ}-\left(75^{\circ}+75^{\circ}\right)=30^{\circ}$
b. Note that students have not been taught that alternate interior angles are congruent, but this can still be worked if they extend side CA and use corresponding and supplementary angles to solve.

ADC: $180^{\circ}-\left(70^{\circ}+58^{\circ}\right)=52^{\circ}$
DAB $=A D C=52^{\circ}$
ADB: $180^{\circ}-\left(52^{\circ}+40^{\circ}\right)=88^{\circ}$
c. EFD: $180^{\circ}-\left(40^{\circ}+65^{\circ}\right)=75^{\circ}$

GDH $=E D F=65^{\circ}$
DHG: $180^{\circ}-\left(75^{\circ}+65^{\circ}\right)=40^{\circ}$
EDG $=\mathrm{FDH}=115^{\circ}$
3. BAE: $90^{\circ}$

ABC: $90^{\circ}-67^{\circ}=23^{\circ}$
AEB: $180^{\circ}-\left(90^{\circ}+23^{\circ}\right)=67^{\circ}$
BEF: $180^{\circ}-\left(67^{\circ}+72^{\circ}\right)=41^{\circ}$
EGB: $180^{\circ}-\left(41^{\circ}+67^{\circ}\right)=72^{\circ}$
FGC: $180^{\circ}-72^{\circ}=108^{\circ}$
CFG: $180^{\circ}-\left(108^{\circ}+27^{\circ}\right)=45^{\circ}$
DCF: $90^{\circ}-27^{\circ}=63^{\circ}$
CDF: $180^{\circ}-\left(63^{\circ}+63^{\circ}\right)=54^{\circ}$
EDF: $90^{\circ}-54^{\circ}=36^{\circ}$
EFD: $180^{\circ}-\left(72^{\circ}+36^{\circ}\right)=72^{\circ}$
2. In each of the figures below, fill in the missing angle measures with the information you are given.
a.

b. Lines $\overleftrightarrow{A B}$ and $\overleftrightarrow{C D}$ are parallel.

c. Lines $\overleftrightarrow{E F}$ and $\overleftrightarrow{G H}$ are parallel

3. Figure $A B C D$ is a rectangle. Fill in the missing angle measures with the information you are given.

4. $5^{2}+12^{2}=r^{2} . r^{2}=169 . r=13$.
a. Area $=\pi r^{2}=\pi(13)^{2}=169 \pi$.
c. $\quad \mathrm{A}=\pi(3.5)^{2}=12.25 \pi$
Circumference $=2 \pi r=2(13) \pi=26 \pi$.

$$
C=2 \pi(3.5)=7 \pi
$$

b. $\quad \mathrm{A}=\pi(1.6)^{2}=2.56 \pi$

$$
C=2 \pi(1.6)=3.2 \pi
$$

5. Area $=5$. You can calculate this either by noting that the triangle splits the parallelogram in half or by looking at the area formulas and noting that there is an extra $\frac{1}{2}$ factor in triangle area.
6. 


c.

b.

4. What is the area and circumference of the circle pictured?
a.

c.

b.

5. Parallelogram $A B C D$ has area 10. Triangle $A B C$ has the same base and same height as the parallelogram. What is the area of triangle $A B C$ ? Can you figure this out in multiple ways?

6. Find the sum of the angles in each polygon.
a.

b.

c.

7.
a. area: $A=\frac{1}{2}(6+9) 4=30$ square units
b. $\quad$ area: $A=b h=8(6)=48$
8. Angle DFE: $180^{\circ}-\left(61^{\circ}+83^{\circ}\right)=36^{\circ}$. Angle ABC: $180^{\circ}-\left(61^{\circ}+36^{\circ}\right)=83^{\circ}$. The two triangles are similar triangles with a scaling factor of 2 . So the lengths of the sides of DEF are $16,23.8$, and 27.2 , and the height is 14 . The area of $D E F=\frac{1}{2} \cdot 27.2 \cdot 14=190.4$. The perimeter of $D E F=16+27.2+23.8=67$.
9. The symmetry lines are vertical lines through every point and bisecting the angles of the equilateral triangles.
7. Find the area of each of the following figures:
a.

b.

8. What is the area and perimeter of triangle DEF?

9. Describe the lines of symmetry in this figure.

10.
a. $6^{2}+8^{2}=c^{2}$
b. $4^{2}+5^{2}=c^{2}$
$36+64=100=c^{2}$
$16+25=41=c^{2}$
c. $\begin{aligned} & a^{2}+5^{2}=8^{2} \\ & a^{2}=64-25=43 \\ & \\ & a=\sqrt{43} \approx 6.5\end{aligned}$
$c=10$
$c=\sqrt{41} \approx 6.4$
11. $V=\pi r^{2} h=\pi\left(6^{2}\right)(8)=288 \pi$ cubic feet, or about 904.32 cubic feet of water
12. $V=B \cdot h=(2 \cdot 7)(12)=14 \cdot 12=168$ cubic inches $\mathrm{SA}=2 \mathrm{~B}+\mathrm{ph}=2(7 \cdot 2)+(18 \cdot 12)=28+216=244$ square inches
13. $\quad V=s^{3}=3.5^{3}=42.875$ cubic cm

$$
S A=s^{2} \cdot 6=(3 \cdot 5)^{2} \cdot 6=73.5 \text { square } \mathrm{cm}
$$

14. 98.125 cubic yards
15. Find the length of the missing sides of each of the following right triangles:
a.

b.

c.

16. The fire department has a cylindrical holding tank of water that measures 8 feet deep and has a diameter of 12 feet. How much water can the tank hold?
17. The measurements of a cereal box are 12 inches by 2 inches by 7 inches. Determine the volume and the surface area of the cereal box.
18. Determine the volume and surface area of the given figure.

19. Determine the volume of the following cylinder.

20. 

a. $\quad 2(8)(3)+2(7)(8)+2(7)(3)=202 \mathrm{~m}^{2}=S A$
b. $\quad$ Volume $=(3)(8)(7) \mathrm{m}^{3}=168 \mathrm{~m}^{3}$
c. newSA: $2(2 \bullet 8)(2 \bullet 3)+2(2 \bullet 7)(2 \bullet 8)+2(2 \bullet 7)(2 \bullet 3)=4[2(8)(3)+2(7)(8)+2(7)(3)]=4(202)=808 \mathrm{~m}^{2}$ new $V=2 l(2 w)(2 h)=8(l w h)=8(168)=1344 m^{3}$

16.
a. $a^{2}+b^{2}=c^{2}-a=3, b=4$ so $c^{2}=3^{2}+4^{2}$ so $c^{2}=25$. Therefore, $c=5$.
b. $\quad S A=2 B+p h=2 \cdot \frac{1}{2} \cdot 3 \cdot 4+(3+4+5) \cdot(6)=12+72=84$ square units $V=B \cdot h=\frac{1}{2} \cdot 4 \cdot 3 \cdot 6=36$ cubic units
15. Draw a net of the rectangular prism below.
a. What is the prism's surface area?
b. What is its volume?
c. What would be the volume and surface area if a new rectangular prism was formed by doubling the dimensions of the old rectangular prism?

16. We have the a right triangular prism as pictured.
a. What is the length of the third side of the triangular base?
b. What is its volume and its surface area?


## Section 11.1: 1

Solution: Since $x^{2}$ must be less than $90^{\circ}$, the only solution is $x=9$.

## Section 11.2: 4

Solution: . A rectangle has 4 right angles. The angles of an $n$-gon add up to $180 \mathrm{n}-360$ degrees. If we have 5 angles equal to 90 degrees, then that leaves 180n-360-450 $=180 \mathrm{n}-810$ degrees for the other $\mathrm{n}-5$ angles. At least one of those angles must be at least as large as the average $\frac{180 n-810}{n-5}=180+\frac{90}{n-5}$, but a convex polygon can't have an angle larger than 180 degrees.

Section 11.3: 3 sq. meters
Solution: We do not want two fence posts on the same side, since then our triangle would have a base of 1 and a maximum of height of 3 (area 1.5). Without loss of generality we can choose any one wooden post to be a vertex where the side across from it does not have a vertex. Then to maximize the area, the other two vertices should be as far away as possible, i.e. the far fence posts on the remaining two sides. Thus we have a triangle with a base of 3 and a height of 2 , giving an area of 3 .

Section 11.4: $\quad 9 \pi-18=10.27 \mathrm{sq}$. in.

## CHALLENGE PROBLEMS

## Section 11.1:

For how many values of x are the angles $\mathrm{x}^{0}$ and $\left(\mathrm{x}^{2}\right)^{\circ}$ complementary?

## Section 11.2:

A polygon is called convex if every line segment joining any two vertices lies inside the polygon or forms one of its sides. At most how many right angles can a convex polygon have?

## Section 11.3:

A square garden with sides of length 3 m is surrounded by a fence. There is a metal fence post in each corner and a wooden fence post every meter along the sides. What is the largest area of a triangle whose vertices are all at wooden fence posts?

## Section 11.4:

Points A and B lie on a circle with center O and a radius of 6 in. If OA and OB are perpendicular and chord $A B$ is drawn, what is the area enclosed by chord $A B$ and the shorter arc AB?

## Section 12.1 - Measures of Central Tendency

## Big Idea:

Recognizing and using measures of central tendency

## Key Objectives:

- Review the vocabulary of data sets.
- Recognize and use the mean.
- Recognize and use the median.
- Recognize and use the mode.
- Select which measure of central tendency is best to use in a given situation.


## Materials:

Calculators (optional)

## Pedagogical/Orchestration:

This section is a relatively traditional presentation of the measures of central tendency. Arguably the most important practical lesson is which measure is preferable, given a particular situation. Because many of your students will be reviewing the definitions and the skills to identify the measures, you might concentrate on the more analytical question of relative utility.

## Activity:

"Wake Up Early"; "Eating Lunch"; "Drawing to Win"; and "Our Class Data" at the end of the section and on CD.

## Vocabulary:

data analysis, data, data point, measure of central tendency, range, mean, arithmetic mean, average, median, frequency, mode, skewed, sample

## TEKS:

6.10(B);
7.1(A); 7.9(A); 7.11(A,B); 7.13(A,B,C);
7.14(A); 7.15(A); 7.12(A,B);
8.12(A);
8.13(A,B);
8.14(A); 8.15(A); 8.16(A)

## WARM-UPS for Section 12.1 (Measures of Central Tendency)

1. A football team ran the ball 8 straight plays. The results are in the following list: gained 8 yards, lost 3 yards, gained 7 yards, lost 5 yards, gained 11 yards, gained 8 yards, lost 4 yards, and gained 16 yards.
Part 1: Which of the following is the average yardage gained per play?
a. 4.5 yards
b. 4.75 yards
c. 5.25 yards
d. 5.5 yards

Ans: (b) Net yardage was 38 yards, so $38 \div 8=4.75$
Part 2: How many first downs did the team make during these 8 plays? Ans: 3
2. The science club wants to have shirts made for their trip to the planetarium. Cheapshirts.com charges a $\$ 20$ set up fee and $\$ 5$ per $t$-shirt printed. Use this information to fill in the table below.

| Shirts | Price |
| :---: | :---: |
| 0 |  |
| 1 |  |
| 2 |  |
| 4 |  |
| 10 |  |
| 15 |  |

3. a. In triangle $D O G, m<D=48^{\circ}$ and $m<0=48^{\circ}$. What is the $m<G$ ? Ans: $84^{\circ}$
b. If the side opposite angle $D$ is 14 cm in length, what is the length of the opposite angle 0 ? Explain.

## Ans: 14 cm because triangle DOG is an isoceles triangle.

## Launch for Section 12.1:

For this lesson, discuss with your students the following questions: "What would be a good way to summarize a whole set of data like the heights of every student in this class? What if you wanted to compare the heights of the students in this class with the heights of students in other math classes?" Let students give their ideas, and if no one mentions a measure of central tendency, lead them by asking, "How could we use just one number to represent your heights? Would the highest number be a good value to represent the set of heights?" Allow for responses, listening for mentions of mean, median or mode. Let your students know that there could be more than one number that could represent the data, and that different measures offer different information about the data. The mean, median and mode all describe the center of the data in different ways, and are called measures of central tendency. Have your students stand in front of the room in a line and ask how they could arrange themselves so that they could figure out the median and mode for their heights. The students can stand from shortest to tallest, and then measure and record their heights in inches. A physical demonstration of having the two students on each end take a step forward and working their way to the center is an excellent visual for finding the median. Something similar can be done for the mode having students of identical heights step forward and recording the data. Keep track of all measures on the board or on chart paper. Have students sit down and figure the mean using the heights of the

## LAUNCH CONT.

students that were written on the board or chart paper. Tell your students that their data will be compared with students in other math classes, and that today they will be exploring the use of measures of central tendency and will learn to choose the measure that is preferable for a given situation.

## EXPLORATION 1

This is an excellent opportunity to discuss measurement and the conversion from mixed feet/inches to inches.


## SECTION 12.1 MEASURES OF CENTRAL TENDENCY

Sets are useful for grouping interesting and related numbers. One such set is the heights of all of the people in your class. In order to use these sets, we need to analyze the numbers, or data, in context. The first step in data analysis, the process of making sense of a set, is collecting data. In data analysis, the idea of a data set is slightly different from that of a set. Unlike regular sets, data sets can have repetition of elements, and the order or arrangement matters.

## EXPLORATION 1

Measure each person in your class in inches and record their name, age in months, and height in inches in a table like the one below. The numbers below were taken from another class, so your own class will have different results. Try to find ways to summarize the information in the table so that you can share your results with a friend without showing her the whole table. Would your strategy still work if there were 100 people in the survey? 1000 people?

Notice that the range is a single number. For example, in Exploration 1, students will naturally want to say that the range of heights is from 49 to 62 , but the range $=62-49=13$.

Note: dot plots are often seen using an x in place of the dot.

| Name | Height (in) | Age (months) |
| :--- | :---: | :---: |
| Sophia | 52 | 113 |
| Rhonda | 51 | 112 |
| Edna | 57 | 112 |
| Danette | 61 | 115 |
| Hesam | 55 | 117 |
| Eloi | 62 | 110 |
| Vanessa | 58 | 113 |
| Michelle | 60 | 108 |
| Mari | 58 | 125 |
| Calvin | 56 | 129 |
| Moises | 57 | 124 |
| Amanda | 57 | 120 |
| Hannah | 55 | 131 |
| Tricia | 55 | 129 |


| Name | Height (in) | Age (months) |
| :--- | :---: | :---: |
| Kristen | 57 | 130 |
| Max | 52 | 135 |
| Jim | 50 | 142 |
| Karen | 57 | 136 |
| Diane | 49 | 138 |
| Tiankai | 58 | 138 |
| Oscar | 51 | 137 |
| Jenny | 60 | 138 |
| Bence | 59 | 142 |
| Pat | 53 | 134 |
| Teri | 59 | 135 |
| Sally | 57 | 139 |
| Will | 57 | 140 |

The entire collection of numbers is called the data and each individual piece of information is called a data point. Data is plural for datum.

A major goal of data analysis is to find a simple measure of the data, called a measure of central tendency, that summarizes or represents, in a general way, the majority of the data. There are three common measures of central tendency: the mean, median, and mode. The mean, median and mode are different ways to identify the center of the data. We are also interested in how spread out our data is. The range, the difference between the largest and smallest values of the data, provides a simple measure of how much the data varies.

A dot (line) plot orders data and displays frequency. Each data point is represented by a dot on the line. For example, given the following data set $\{2,8,4$, $8,8,6,5,7,9,3,7,5\}$, the corresponding dot plot is shown below.


## EXAMPLE 1

Use the dot plot above the answer the following questions.
a. What percent of the data has a value of 5 or less?
b. What percent of the data has a value of 7 or more?
c. What is the proportion of the data with a value of 7 or less to data with a value of 8 or more?
d. What is the proportion of data that is 5 or less to data that is 7 or more?

The mean, also called the arithmetic mean or average, is the sum of all the data values divided by the number of data points. For a visual example, suppose we have five containers, each containing a certain number of blocks:


These data can be grouped into a data set: $\{7,3,5,7,3\}$. There are 25 blocks total. The mean number of blocks in a container is the number of blocks each container has if these 25 blocks are distributed evenly among the 5 containers: $\frac{25}{5}=5$.


While the mean and median only make sense for quantitative data, the mode can be computed for quantitative and categorical data. However, the mode is not a very useful measure when quantitative data can take many distinct values.

The median is the value of the middle data point when the values are arranged in increasing order. If the data set has an even number of data points, the median is the average of the two middle values. To find the median value for the container example, order the data, with the smallest number of blocks first and the largest number last:


The median is the number of blocks in the middle, or third container with respect to the sorted ordering. The median is a helpful measure of central tendency because half of the values are less than or equal to the median and the other half of the values are greater than or equal to it.

Frequency is the number of times a data point appears in a data set. For example, if there are 4 people in the class who are 56 inches tall, then the frequency of the height 56 inches in the class is 4 . The mode is the value or element that occurs the most often or with the highest frequency in the data set. In Exploration 1 the mode in our data is 57 because it appears 7 times in the data. What is the mode in your class' data? A set of data can have more than one mode. For the containers of blocks example, the modes are 3 and 7 because both appear twice.

## PROBLEM 1

Use the data table from Exploration 1 for the following questions.
a. Construct a dot (line) plot of students' heights.
b. What qualitative characteristics do you notice about the dot plot, such as the shape, the centers, or the spread? What other statements can you infer about the class from the data?
c. Determine the mean, median, mode and range for the data set.

Age and height measurements for another class in the same school were collected and are shown in the following table.

| Name | Height <br> (in) | Age (months) | Name | Height <br> (in) | Age (months) |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Alexandra | 63 | 140 | Brittany | 64 | 145 |
| Ian | 66 | 138 | Shazil | 69 | 148 |
| Jongwook | 62 | 136 | Stanley | 68 | 138 |
| Keith | 58 | 158 | Eric | 65 | 139 |
| Alison | 57 | 145 | Gregory | 58 | 141 |
| Ming | 54 | 126 | Maurice | 59 | 150 |
| Izzy | 64 | 169 | Jazmine | 57 | 137 |
| Aishu | 59 | 126 | Maria | 61 | 145 |
| Nannette | 60 | 154 | Jeffrey | 56 | 129 |
| Patty | 61 | 127 | Amy | 56 | 138 |
| Michael | 63 | 136 | Carlos | 57 | 137 |
|  |  |  | Lisa | 68 | 141 |

d. Create a dot plot of the student's heights for this class. What statements can be made about the data from this class based on the dot plot?
e. Compare the dot plots for the two classes. Compare their means, medians and range. Do you notice any similarities or differences between the two classes?
f. What percent of the students are five feet tall or shorter?
g. What percent of the students are at least 5 feet 3 inches tall?
h. What is the proportion of data that is 7 or less to data points that is 8 or more?
i. What is the proportion of data that is 5 or less to data that is 7 or more?

## EXAMPLE 2

Find the mean, median, mode, and range of the following data set.
$\{2,8,4,8,8,6\}$

PROBLEM 2
The mean is 8.5 . The median is 8 . The mode is 5 . The range is 11 .

PROBLEM 3
The mean is given by $(95+30+98+93+100) / 5=83.2$.
The median is the middle value when the data is ordered $(30,93,95,95,100)$, so it is 95 .

## SOLUTION

The mean is found by adding the values together and dividing by the number of values. The sum of the values is $2+8+4+8+8+6=36$. The number of values in the set is 6 . The mean is $\frac{36}{6}=6$.

Putting the data set into order from smallest to greatest value results in $\{2,4,6,8,8,8\}$. Because there are an even number of values in the set, the median is the average of the two middle values. The median is the average of 6 and $8, \frac{(6+8)}{2}=\frac{14}{2}=7$.

The most commonly occurring value in the data set is 8 , so 8 is the mode.
The range is the difference between the highest and lowest value. The range is $8-2=6$.

## PROBLEM 2

In the following data set, what is the mean? the median? the mode? the range?

$$
\{4,9,12,5,9,14,11,15,5,6,7,5\}
$$

The mean depends on all the numbers in the data, but the median only depends on the value of the data point in the middle position. That does not, however, suggest that the mean is a better measure of central tendency than the median.

## PROBLEM 3

Find the mean and median of the following six weeks test grades:

$$
\{95,30,98,93,100\} .
$$

Compare the value of each as a measure of the data.

## PROBLEM 4

Two high school soccer teams have a scheduled game. The rosters from each school are reflected in the chart

## EXERCISES

## EXPLORATION 2

Original mean $=56.04$ inches; new mean $=68.32$ inches.
Original median =57 inches; new median $=57$ inches.
The mean has increased, but the median has not changed.
4.

The range is 37 . The modes are 86 and 91 . The median is 83 . The mean is 82.30 .
5. The mean is 55 inches. It is impossible to tell what the median is. You might discuss this with your class. If the data is not easily ordered, the mean is an easier measure to find than the median. Otherwise, the median is easier.
6. Original total $=(58)(17)=986$. After Rhonda, the new mean is $\frac{986+65}{18}=\frac{1051}{18}=58.39 \mathrm{in}$.

| School | Freshmen | Sophmore | Juniors | Seniors |
| :---: | :---: | :---: | :---: | :---: |
| A | 1 | 3 | 6 | 8 |
| B | 5 | 7 | 4 | 2 |

a. Construct dot plots for both of these teams.
b. Compare these dot plots to determine if one team is more experienced than the other team.

## EXPLORATION 2

Using the data from Exploration 1, compute the mean and the median of the heights of the class. Then, imagine that a giant who is 400 inches tall joins the class. Compute the new mean and find the new median. How has each changed?

If the data is skewed, or uneven, a median value is a more accurate picture of the representative value than the mean is. Exploration 2 had a very tall giant join the class. The mean was affected by this outlier, a term used to refer to a value that is drastically different from most of the data values. The median, however, was not affected. The mean is usually more influenced by extreme values than the median.

Let us review the ways in which we summarized data in this section.
If we have a set of $n$ values, then we can find the following measures:

- Find the mean by adding the values and dividing by $n$.
- Find the median by ordering the values and finding the value that is in the middle, if $n$ is odd, or taking the average of the middle values, if $n$ is even.
- The mode is the most frequent value that occurs. There could be two or more such values.
- The range is the difference between the largest and the smallest values in the set.

1. a. The mean age in years is $\left(\frac{1}{12}\right)\left(\frac{3442}{27}\right) \sim 10.6$.
b., c., d. Answers will vary. The mean height of the example class is 4 feet 8 inches.

You might mention that in the 18th and 19th centuries, descendants of Europeans living in North America were far taller than those in Europe. In fact, they were the tallest in the world. Now, according to a new survey, Americans are shorter than Europeans, on average.
2. Answers will vary.
4. a. Mean: 8.3 , median: 8 , mode: 8 , range: 16
b. Mean: 67.85 , median: 66 , mode: 66 , range: 14

## EXERCISES

1. Using the example class from Exploration 1,
a Find the mean age, in years, to the nearest tenth.
b. Now look at the ages of students in your own class. Find the mean, median and mode of the ages, in years, to the nearest tenth.
c. How are the two means different? What are the factors that cause the difference?
d. To the nearest inch, is the median height of students in your class different from the mean height? If so, why do you think they are different?
2. Separate the data from your class into categories by age and find the mean height for each age. Does the mean height increase with age? Explain the results of your analysis.
3. An English teacher assigned the same writing assignment to two different classes. In class $x$, the teacher only gave written instructions about the assignment. In class y , the teacher had the class discuss the assignment in small groups as well handing out written instructions. The assignment was graded on a scale of A, B, C, D or F. The results are in the chart below:

| Class | A | B | C | D | F |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $X$ | 2 | 4 | 9 | 5 | 3 |
| $Y$ | 7 | 8 | 7 | 3 | 0 |

a. Construct dot plots for both of these classes.
b. Compare these dot plots to determine which class performed better than the other. Why did class y have better results on the assignment?
4. Find the mean, median, mode and range of the following data sets:
a. $\{8,6,10,14,10,9,8,3,8,16,0,8,6,2,15,2,12,8,16,5\}$
b. $\{74,66,66,66,64,66,71,66,71,66,74,64,66,73,71,60,71,65,63,74\}$
c. Contruct a dot plot for the data in part b.
d. Use the dot plot to find the percent of the data points between 60 and 69 .
e. What is the ratio of data between 60 and 69 to those between 70 and 79 ?
9. Total height of the class $=(6)(55)=330$. Let Hannah's score $=x$. New total $=330+x$ and $(7)(56)=392$. So $330+x=392$, and $x=62$ inches.
10. The 14 -person total $=(54)(14)=756$. The 12 -person total $=(50)(12)=600$. The new average $=\frac{756+600}{26}$ $=52.15 \mathrm{in}$.
5. Use the following results of a math test as data to create a dot plot. Determine the range, mean, mode, and median for the math test.
$\{100,92,79,65,86,80,78,63,91,83,91,87,79,86,85,92,75,76,95$, $78,68,67,73,76,71,86,89,85,91,96,83,77,93$
a. Use the dot plot to find the percent of test scores 90 or above?
b. What is the ratio of test scores above 90 to those below 90 ?
6. Two brothers live 300 miles from each other. Tom lives near the coast and his brother Mike lives inland. They each keep a record of the number of inches of rainfall each month for a year. The results are:

Tom: $\{3,4,6,2,4,4,5,3,4,6,5,4\}$
Mike: $\{1,2,0,1,0,1,8,9,1,3,2,4\}$
a. Make a dot plot for each of these sets of data.
b. Determine the mean, median, mode, and range for each set of data. Compare the measures and make some generalizations about Tom's weather versus Mike's.
c. Compare the shape of each dot plot. What does each shape indicate about the annual rainfall in the two areas?
7. A class has 17 students and the total height of all the students is 935 inches. What is the mean height of the class? What is the median height?
8. Rhonda joins a class that has 17 students. The class mean height was 58 inches. Rhonda is 65 inches tall. What is the new mean height for the class with Rhonda as an additional student? Give your answer to the nearest hundredth of an inch.
9. A class has six students with a mean height of 55 inches. The class mean height changes to 56 inches after Hannah, a new student, joins the class. How tall is Hannah?
10. A 14-person class with an average height of 54 inches merges with a 12 -person class with an average height of 50 inches. What is the average height of the combined class to the nearest hundredth of an inch?
12. Median, because the outlier of 51 skews the mean. There is no mode and the range does not provide any appropriate information.
13. Students will need to solve all measures. mean $=169.3$; median $=186$; mode $=186$; range $=148$
14. Yes, pack different clothes. The range of temperatures in Castolon means you would need a wider variety of clothes for that location. Galveston is right next to the Gulf of Mexico while Castolon is far inland. The large body of water is why Galveton has a smaller temperature range; the water moderates the daily swing in temperature.
11. Which measure of central tendency is most helpful in representing the following?
a. Your grade in math. mean
b. The winner of the race for mayor. mode
12. Choose the appropriate measure of central tendency or range to describe the data in the table. Justify your reasoning.

| School | Number of Teachers |
| :--- | :---: |
| Jones Middle School | 32 |
| Lampasas Middle School | 36 |
| Falls Middle School | 28 |
| Miller Middle School | 37 |
| Fossum Middle School | 51 |

13. The heights of various buildings in the city are listed. Which measure of central tendency or range would make the heights appear tallest? 168 ft ., 186 ft ., $221 \mathrm{ft} ., 73 \mathrm{ft} ., 152 \mathrm{ft} ., 186 \mathrm{ft}$., 199 ft .
14. The January mean daily temperatures for Castolon, TX and Galveston, TX are approximately the same. However, their ranges are quite different. The temperature data, in ${ }^{\circ}$, from NOAA are

| City | Maximum | Minimum | Mean | Range |
| :---: | :---: | :---: | :---: | :---: |
| Galveston | 61.9 | 49.7 | 55.8 | 12.2 |
| Castolon | 67.7 | 33.6 | 50.7 | 34.1 |

Even though Galveston and Castolon have about the same daily mean temperature for January, would you consider packing different clothes for the two places? Which measure of central tendency influenced your decision? Why?
15. The height seems to be going up about 3 inches per year. So the median height for 15 -year-olds might be 68 inches. However, at some point, growth stops. So the mean height for 24- and 25-year olds is probably the same.

## Ingenuity

15. Start counting the digits in the one-digit numbers. Then look at the two-digit numbers, taking the total number of two-digit numbers times the number of digits in each two-digit number. Follow that pattern, subtracting from 1263. Remind the students who need it that digits are the ten symbols used to produce all numbers in the decimal system.
1-9 use 9 digits. Pages 10-99 use 2(90) = 180 digits. Pages 100-999 use 3 digits each. The pages of the book have another $(1263-189)=1074$ digits. This requires $\frac{1074}{3}=358$ more pages. So that total number of pages is $9+90+358=457$.

## Investigation

17. Answers will vary. One possibility is that the median is a better measure than mean for something like the median home price or median family income because the prices and the number of people in each category vary so much.
Geography majors from University of North Carolina have a higher average salary than geography majors from other colleges. Provide a reason to account for this occurrence. Michael Jordan majored in geography while as a student at UNC so his salary is what we call an outlier, a number that skews the data. Median is a more appropriate measure of salary.
18. On the right are estimated national median heights in inches for 9- through 14 -year-olds in 2000, according to the National Center for Health Statistics (NCHS).

Based on this data, what is your estimate for the median height for 15 -year-olds? Do you think the median heights for 24-year-olds and 25 -year-olds are that much different? Explain.

| Age group | Height (in) |
| :--- | :---: |
| 9-year-olds | 52.5 |
| 10-year-olds | 54.5 |
| 11-year-olds | 56.5 |
| 12 -year-olds | 59.0 |
| 13 -year-olds | 62.0 |
| 14 -year-olds | 65.0 |

16. Ingenuity:

It takes 1263 digits to number all the pages of a book. How many pages are there in the book?

## 17 Investigation:

Supply an example that applies to your hometown where the median is a more appropriate measure than mean

## Our Class Data



Objective: The students will collect a variety of data from their classmates, and use their knowledge of displaying data to make a bar graph and a pie chart.

## Materials:

Paper and pencil to record data collection
Rulers
Protractors
Markers, crayons or colored pencils
White paper

## Activity Instructions:

1) Divide your class up into groups of two. Explain to each group that they are to collect data from each of their classmates about what type of pets live in their house.
2) After collecting data, groups should brainstorm about how they are going to organize their data into both a bar graph and a pie graph.
3) After brainstorming, students will pick up two pieces of white paper and their rulers and protractors and start creating their graphs. Groups can either work together, or they can divide the work and make one graph each.
4) After each group has created two graphs, students can use their markers, crayons or colored pencils to make their graphs more creative.

## Drawing to Win!

Objective: Students will discover that the ratio of the number of favorable outcomes to the number of possible outcomes is equal to the theoretical probability of the outcome. Students will understand the relationship between experimental probability and theoretical probability.

## Materials:

25 Colored blocks ( 10 red, 12 yellow, 3 blue, note that block should be the same size and shape) Bucket or container to hold the blocks (bucket should not be clear)

## Activity Instructions:

Teacher fills the container with the blocks and leads a game called "Drawing to Win!" One at a time, students will choose a block from the container. Before each block is drawn, students will try to guess the color that they will draw. *Note that the students should not know the number of each of color of block in the container, students should only know the different colors which are in the container. After they draw, students will return their block to the container. Students will keep a tally of the results. Students should record what color was chosen, the color of the guess, and whether the guess was correct or incorrect. Each student keeps a tally of the results of each draw. At the end, the class will compare and discuss their results to complete (Part I), Experimental Probabilities, as shown below.

After all of the Experimental Probabilities are found, the teacher should reveal all of the blocks that were in the container. The class will use this new knowledge to find the Theoretical Probabilities (Part II), as shown below.

```
Part I
Experimental Probability:
P(red) = total red blocks drawn : total number of trials
P(blue) = total blue blocks drawn : total number of trials
P(yellow) = total yellow blocks drawn : total number of trials
Part II
Theoretical Probability:
P(red) = total red blocks : total number of blocks
P(blue) = total blue blocks : total number of blocks
P(yellow) = total yellow blocks : total number of blocks
```


## Drawing to Win Worksheet

## First find the Experimental Probability:

$\mathrm{P}($ red $)=$ total red blocks drawn : total number of trials $=$
$\mathrm{P}(\mathrm{blue})=$ total blue blocks drawn : total number of trials $=$
P(yellow) = total yellow blocks drawn : total number of trials =

## Second find the Theoretical Probability:

$\mathrm{P}($ red $)=$ total red blocks : total number of blocks =
$\mathrm{P}($ blue $)=$ total blue blocks : total number of blocks $=$ P (yellow) = total yellow blocks : total number of blocks =

## Third answer the following questions:

1. What is the sum of the theoretical probabilities?
2. How do the experimental and theoretical probabilities compare from parts I \& II?
3. Does each block have an equally likely chance to be drawn? Explain.
4. Does each color have an equally likely chance to be drawn? Explain.
5. Which person is most likely to pick a yellow block- the first person to pick from the bucket or the last person? Explain.

## Eating Lunch

Objective: Students will flip a coin to collect data and find experimental probability of an event.

## Materials:

## Coin

Chart or table for data collected

## Activity Instructions:

Teacher will give students the following situation:
"Fernando always eats lunch at school with his girlfriend Terry. His favorite food is pizza, but his girlfriend is encouraging him to eat healthier. So every day, he flips a coin. If the coin falls on heads, he eats pizza and if it falls on tails, he eats a healthy salad".

Students will conduct an experiment to predict how many days, in April, Fernando is going to eat pizza. They will record their data in a table for each day in April, one toss per day. Below is a sample of how their table should look.

Coin Toss Data

| Day | Result of Toss <br> (H or T) | Number of Heads <br> So Far | Fraction of <br> Heads So Far | Percent of <br> Heads So Far |
| :---: | :---: | :---: | :---: | :---: |
| 1 |  |  |  |  |
| 2 |  |  |  |  |

EXTENSION: Teacher will ask students to explain what happens to the percent of tosses that are heads as we add more data. To do this, the class will gather data from all the groups in class and answer the following questions:

What percent of the total number of tosses for your class is heads?
As you add more data, what happens to the percent of tosses that are heads?
Based on your results for April, how many times do you predict Fernando will eat pizza in May?
Students will explain their reasoning.

## Waking Up Early

Objective: Students will collect data by using a spinner to determine the probability of an event.

## Materials:

Probability spinner
Paper clips

## Activity Instructions:

Teacher will ask students: " During the summer, how early would you be willing to wake up if your parents paid you for getting up early? Your parents will decide "how early" using the spinner that is attached.

Students are given a copy of the attached spinner divided into equally spaced sections labeled 9:00, 10:00, and 11:00. The teacher should ask the students which time of the morning they think will occur most often in this experiment. Students will vote for the most popular time (probably 11:00 am).

Each student uses a paper clip and spins as many time as decided by the teacher (or as time permits) to figure out the probability that the paper clip will land on 9:00, 10:00, or 11:00. Then, they will answer the following questions once they have gathered their data.

1) What is the experimental probability that the clip falls on 9:00, 10:00, and 11:00?
2) What do you think will happen if you spin the clip 100 times?

Students determine the fraction of each section of each time allotted in this spinner to answer the following questions:
3) What is the theoretical probability that the clip falls on 9:00, 10:00 and 11:00 (assume all sections are equally spaced). Explain your answer.
4) Based on the theoretical probability, which time(s) is/are most favorable? Does your answer change if you spin the spinner 100 times?
5) Compare the experimental and theoretical probabilities?

## Waking Up Early Spinner



## Section 12.2-Graphing Data

## Big Idea:

Graphing data

## Key Objectives:

- Read and interpret a bar graph.
- Read and interpret a pie graph.
- Analyze data using graphs.


## Materials:

Grid paper, Protractor, Compass or handouts with pre-made circles, Large area for making human graphs, String or yarn

## Pedagogical/Orchestration:

This section gives an explanation of bar and pie graphs and the relative use of each type. The exercises are rich with applications involving graphing data.

## Activity:

"What's Your Shoe Size" and "Comparing Median and Mean" on CD and at end of section

## Vocabulary:

bar graph, circle (pie) graph, dot (line) plot, histogram(See CD: stem-and-leaf plot)

## TEKS:

$6.10(A)(D) ; \quad 7.1(B) ; \quad 7.2(A)(D) ; \quad 7.3(A) ; \quad 7.11(A)(B) ; \quad 7.13(A) ; \quad 7.14(A) ; \quad 8.5(A) ; \quad 8.12(A, C) ; \quad 8.13(A, B) ;$
8.14(A,B,C); 8.15(A)

## WARM-UPS for Section 12.2

1. A cardboard box (rectangular prism) has a square base and has no top. If the length of the base is 12 inches and the height of the box is 8 inches, what is the area of the cardboard used to make this box?
a. $4\left(12^{2}\right)+8^{2}$
b. $12^{2}+4(8)(12)$
c. $4\left(12^{2}\right)+4(8)(12)$
d. $12^{2}+2(8)(12)$

## Ans: (b) because there is no top.

2. Rachel goes shopping so she has a great outfit to wear when she goes dancing. Her total cost was $\$ 132.74$. She bought a skirt for $\$ 31.25$ and 3 tops for the same price each. How much did one top cost?

## Ans: $\$ 33.83$ per top

3. Melanie wants to save $\$ 130.20$ to buy a nice birthday gift for her mother whose birthday is 12 weeks away. How much does Melanie need to save each week to reach her goal?

## Ans: $\$ 10.85$ each week.

## Launch for Section 12.2:

Tell your students, "Today we will experiment with two other representations called bar graphs and circle graphs. Look at what you are wearing and decide what is the most dominant color of your clothes." Then the class will need to decide which four colors are represented the most. The fifth category will be labeled "other." Once the decision is made on the four dominant colors, then have the students line up against a wall and tell them they are to become a human bar graph, grouping by color of clothing. There should be five bars, four for the main colors, and one for the "other" category. Ask students, "Where are the x and y axes? What are the important characteristics of a bar graph we need to remember?" Some things they should consider: 1) The bars should be equally spaced apart on the $x$-axis. 2) The intervals or scaling on the $y$-axis should be equally spaced. In other words, people should be the same distance apart. 3) The graph on paper would need labels for each axis and a title for the whole graph. Ask students what the labels would be for each axis. This human bar graph is extremely visual being as the students are wearing the colors that each bar represents. This is a great photo opportunity if you are blessed with a 2nd story window, bleachers or even a ladder to get a good shot of the graph from above.

This paragraph revisits the vocabulary of intervals and spacing from Chapter 1. You can also discuss whether the labeling on the axes matters.

## EXAMPLE 1

Emphasize that 12 is somewhat arbitrary. Any integer slightly larger than the highest frequency, 10, will do. Have students discuss their answers. While we can make a bar graph with any order of colors, alphabetical or in order of frequency are the most common.

## SECTION 12.2 GRAPHING DATA

When collecting data, it is often useful to draw a picture or graph to represent the data that has been collected. A graph of the data gives a quick, easy way to see what the data represents.

## EXAMPLE 1

Ms. Garcia's class has twenty students. Each student was asked which color they like best. The survey shows that 10 students prefer red, 5 students prefer green, and 5 students prefer blue. What are the best ways to represent or display this information?

## SOLUTION

One way to display the data is to make a special kind of graph called a bar graph. To construct a bar graph, draw an $x$ - and $y$-axis, subdivide the horizontal or $x$-axis into three equally-spaced intervals and label the intervals with the categories Red, Green, and Blue. Then label the vertical or $y$-axis with points from 0 to 12 . For each color, draw a vertical bar equally separated from the other bars. The height of each bar represents the number of people who liked a particular color best. The bar graph should look like this:


## Favorite Color

Explain why the vertical axis has a number scale from 0 to 12 . Why is there no number scale on the horizontal axis? Explain whether the order of the colors is important.

PROBLEM 1


Favorite Color

Another way to represent this data is by using percents. Because there are 20 students in all, $\frac{10}{20}$ of the class likes the color red. Convert $\frac{10}{20}$ to the decimal 0.50 and then to the percent $50 \%$. Similarly, $\frac{5}{25}=0.25=25 \%$ of the class likes green.

## PROBLEM 1

Calculate the percents of the other two colors. Use these percents as data on the vertical axis to build another bar graph. Label the axes and draw the graph.

a. Calculate the percents of green and blue. What do they represent in the situation?
b. What proportion of the students like red?
c. What is the proportion of students who like green to those who like red?

Although the shape of the bar graph is the same, the percentage graph gives a picture of the relative quantity of the class' preference for each color, not the number of students directly. Instead, the bar graph shows the relative number or percentage of students immediately.

Another way to represent the percentage data is to use a circle or a pie graph of the data. Use your protractor to draw a circular outline for the circle graph. You have already computed the percentage of each color. The proportion of the circle graph with a given color corresponds to the percentage of students who prefer that color. The larger the sector of the circle graph, the greater the percentage of people who liked the color.

Because $50 \%$ of the students prefer the color red, the question is what angle corresponds to $50 \%$ of the circle? To calculate $50 \%$ of $360^{\circ}$, multiply ( 0.50 ) $(360)=180^{\circ}$. Construct an angle that measures $180^{\circ}$ and color that part of the circle red and label it. You can also calculate $50 \%$ of $360^{\circ}$ by using a proportion as follows:

$$
\begin{gathered}
50 \%=\frac{50}{100}=\frac{x}{360} \\
\frac{1}{2}=\frac{x}{360} \\
(360) \frac{1}{2}=\frac{x}{360}(360) \\
\frac{360}{2}=\frac{x}{1} \\
180=x
\end{gathered}
$$

Next, calculate $25 \%$ of 360 and construct a section of the circle with this angle and color it green. Label this region as well. Finish making the circle graph by computing the angles and then coloring and labeling the last region. The completed circle graph on the next page clearly represents which color students like the best and makes the result of the survey visually obvious. Why does a circle graph make it easy to catch computation errors in converting from percents to degrees?


## PROBLEM 2

Since the total number of students is different in the two classes, the bar graph with counts is deceiving. It is preferable to use percentages. This is one of the reasons we would use percentages instead of counts.



Green: $180^{\circ}$, Red: $120^{\circ}$, Blue: $30^{\circ}$, Purple: $30^{\circ}$
c) Circle Graphs display data as a part of the whole while bar graphs retain the initial data.

EXPLORATION 1


## PROBLEM 2

Mr. Ruiz asked his 12 students which color they liked best as well. In his class, he found that 6 students prefer green, 4 students prefer red, 1 student prefers blue, and 1 student prefers purple.
a. Make a bar graph to represent the data from Mr. Ruiz's class.
b. Make a circle graph to represent the same data.
c. What differences do you notice between the bar graph and circle graph?

## EXPLORATION 1

The rainfall record for a region over an 8-year period from 1990 to 1997 is listed to the right.
a. Plot the data on a coordinate plane. Label the horizontal axis to represent time in years, and the vertical axis to represent inches of rainfall. To convert the set of points to a line graph, connect the points sequentially with straight lines. Line graphs are typically used when the first data deal with change in time.

| Year | Rainfall |
| :---: | :---: |
| 1990 | 30 inches |
| 1991 | 32 inches |
| 1992 | 24 inches |
| 1993 | 18 inches |
| 1994 | 28 inches |
| 1995 | 36 inches |
| 1996 | 42 inches |
| 1997 | 31 inches |

## EXPLORATION 2

Another way to represent the data is in a histogram. A histogram is a special kind of bar graph that shows the frequency of data between set intervals. The chart below shows the results of Mr. Red's students' final exam grades.

| Student | Grade |
| :---: | :---: |
| Joe | 78 |
| Bryan | 97 |
| Cathy | 98 |
| Kendall | 85 |
| Amy | 68 |

B. We chose this range because all intervals must have the same range.

| Michelle | 71 |
| :---: | :---: |
| Melissa | 94 |
| Max | 100 |
| Terry | 66 |
| Alex | 76 |
| Victoria | 73 |
| Penelope | 65 |
| Nathan | 75 |
| Jeremy | 77 |

A. Complete the following table with the grades organized into the proper range.

| Range | Number of Grades (Frequency) |
| :---: | :---: |
| $91-100$ | 4 |
| $81-90$ | 1 |
| $71-80$ | 6 |
| $61-70$ | 3 |

The histogram organizes the data in the following chart.

B. Why did we choose the range $91-100$ instead of $90-100$ in the table?
C. When you look at the histogram, what do you notice about the grade distribution?
D. From the histogram, what score(s) do most of the students make?

## EXPLORATION 2

Go over the construction of the Venn diagram very carefully. The wording used to describe the items/elements in a set region in the Venn diagram is very important. The students will often make the common mistake of accounting for 71 students having a brother to mean "only a brother" and not realize that they could have a sister, as well as a pet but some of them have been counted in the 59 students that have a sister and others in the 35 students that have a pet. The numbers in the Venn diagram should be those that are in the region and only in that region. The information should therefore be analyzed very carefully. In this case, the intersection of the three sets would be better starting point than any one of the three large sets.


Organizing data is an important part in the process of analyzing data. Venn Diagrams are a way to represent data that have different or similar characteristics.

Venn Diagrams commonly use circles to represent sets of items from the data. The circles are usually within a large rectangle that represents all the data gathered.


It may have two or more circles that may or may not overlap, depending on the relationship among the sets. The sets should have a description and the Venn Diagram may contain numbers in the regions of the diagram that tell us how many elements are in those specific regions of the circles.

## EXPLORATION 3

If the three sets in the example below represent students who have a brother(s), who have a sister(s), who have a pet(s), then explore what some of the overlapped parts of the circle can represent. Be very careful with how you word your descriptions. Within the regions in the circle, we can write the description as well as place numbers to indicate how many elements are in that region of the circle.

Remember that a person can have a brother and also have a sister.


In a survey of 100 students at Fossum Middle School, the following results were obtained. Use the Venn Diagram above and write the description and the numbers in the appropriate regions.
a. 15 students have a brother, sister and a pet
b. 45 students have a brother and a sister
c. 25 students have a brother and a pet
d. 17 students have a sister and a pet
e. 59 students have a sister
f. 35 students have a pet
g. 71 students have a brother
h. Are there any students in your survey that have no brother, sister nor pet? Explain your reasoning.

## EXAMPLE 2

The Venn Diagram below shows the number of students at Miller Junior High School who have a favorite sports team, a favorite movie, and a favorite singer. The total number of students surveyed was 200.


Using the information from the above Venn Diagram, answer the following questions and explain your reasoning.

a. How many students have only a favorite singer?
b. How many students have a favorite singer?
c. How many students have a favorite sports team?
d. How many students have all three favorites?
e. How many students have no favorites?
f. What percent of students have no favorites?

## SOLUTION

The three circles represent the sets of students that have a favorite singer, S ; favorite sports team, T ; and favorite movie, M . The numbers represent students that have the respective favorites.

The number of students that have only a favorite singer is 50 . However, if we want the number of students that have a favorite singer, then we must add up all the numbers that are in the various regions that make up the circle for S. For example, a person may have a favorite singer and also have a favorite movie. Some of them may have a favorite singer and have a favorite movie and a favorite sports team. The total number of students that have a favorite singer is then $50+15+10+4=79$.

Similar to the argument about the students who have a favorite singer, the number of students with a favorite sports team is equal to 70 . This is the result of adding $36+20+10+4$.

The number of students that have all three favorites is 10 . Notice this is the intersection of the three circles.

The number of students that have no favorites is 8 . These are the students not situated in any of the circular regions.

## PROBLEM 3

The results of a random survey of 2000 people about their preference for the color of a car are shown in the circle graph below. Answer the following questions:

a. How many people surveyed preferred a blue car?
b. How many people surveyed preferred a black car?
c. What is the ratio of the number of people who preferred a blue car to the number of people who preferred a black car?
d. What is the ratio of the number of people who preferred a red car to the number of people who preferred a green car?
e. What is the ratio of the number of people who preferred a black car to the number of people who preferred a green car?
f. If you were an automotive executive, what would you decide to do, based on the data above?

## EXAMPLE 3

Twelve students received the following scores on a math test:

$$
97,75,73,75,83,85,54,98,97,65,75 \text {, and } 83 \text {. }
$$

a. Graph this data with a box and whisker plot.
b. What percentage of the class has scores that appear in the box?
c. What is the interquartile range of the data?
d. What is the range of the data?

## SOLUTION

A box and whisker plot divides the data into 4 parts, each containing $\frac{1}{4}$ of the data points. First, order the data to find the points that divide the data into fourths. We will need to graph five numbers:

1. The minimum - the left-hand end of the left whisker.
2. The maximum - the right-hand end of the right whisker.
3. The median, which is the same as the second quartile, is Q2, the middle of the box..
4. The left side of the box is the first quartile Q1
5. The right side of the box is the third quartile Q3.

The first step is to arrange the data in order, from smallest to largest:

$$
54,65,73,75,75,75,83,83,85,97,97,98
$$

1,2 : The minimum is 54 , and the maximum is 98 .
3. The median is the middle number if there are an odd number of data points, and the average of the two numbers in the middle, if there are an even number of data points. Because there are an even number of data points, the average of the two numbers in the middle is $\frac{(75+83)}{2}=79$. This is the median and the second quartile Q2.
4. The first quartile Q1 is the median of the numbers to the left of the median: 54, $65,73,75,75,75$. Again there are an even number of numbers so the median is $\frac{(73+75)}{2}=74$. The first quartile Q1.
5. The third quartile Q 3 is the median of the numbers to the right of the median: $83,83,85,97,97,98$. So the third quartile Q3 is $\frac{(85+97)}{2}=91$.
Using these 5 numbers, we now make our box and whiskers plot:

b. $50 \%$ of the data is in the box.
c. The Interquartile Range (IQR) is the difference between the third quartile and the first quartile, $\mathrm{IQR}=\mathrm{Q} 3-\mathrm{Q} 1=91-74=17$. Notice that $50 \%$ of the data is in the IQR. The IQR gives the spread of the middle half of the data.
d. The range of the data is the difference between the maximum value and the minimum value. For our problem, the range $=98-54=44$.

## EXAMPLE 4

a. Now graph the same data with a stem and leaf plot.
b. What is the advantage of the stem and leaf plot over the box and whisker in demonstrating how the numbers are distributed?
c. Make a histogram that shows the number of students with scores in the 50 's, $60^{\prime}$ s, $70^{\prime}$ 's, $80^{\prime}$ s, and $90^{\prime}$ s.
d. Compare the stem and leaf plot to the histogram.

## SOLUTION

a. When we make a stem and leaf plot, the stems are the tens digit

| Stem |  |
| :---: | :--- |
| 5 |  |
| 6 |  |
| 7 |  |
| 8 |  |
| 9 |  |
|  |  |

The leaves are the one's digits:

| Stem | Leaf |  |  |  |  |
| :---: | :--- | :--- | :--- | :--- | :--- |
| 5 | 5 | 4 |  |  |  |
| 6 | 6 | 5 |  |  |  |
| 7 | 7 | 3 | 5 | 5 | 5 |
| 8 | 8 | 3 | 3 | 5 |  |
| 9 | 9 | 7 | 7 | 8 |  |

b. The score that appears most often, the mode, is easy to see.
c. A histogram, shows the same data and shape in a vertical direction.

The horizontal axis labels are the scores $50,60,70,8090$ that show the range of the data.
The vertical axis is the number of scores in each range

50's 1
60's 1
70's 4
80's 3
90's 3

d. The number of scores in the histogram is the number of leaves on each stem.

Both show the distribution of scores.

## EXAMPLE 5

The range of salaries in San Marcos compared with Austin is summarized in the box and whiskers plot below:

San Marcos:


Austin:

a. From the box and whisker plot, which city has the higher median salary?
b. Compare the distribution of salaries in San Marcos and Austin.
c. Which city has the highest salaries?

## SOLUTION

a. Austin has a higher median salary-- $\$ 35,000$ compared to $\$ 28,000$ in San Marcos.
b. San Marcos salaries are a little lower. Also, there is a wider range of salaries in Austin.
c. Austin also has the highest salaries.

## PROBLEM 3

1. A Venn Diagram is given below with the three sets representing:
$E=\{2,4,6,8,10,12,14,16,18,20,22,24\}$
$F=\{3,6,9,12,15,18,21,24\}$
$G=\{2,3,5,7,11,13,17,19,23\}$

2 The number of elements in set E is 12 , the number of elements in set F is 8 , the number of elements in set G is 9 . The number of elements that are prime, multiples of 3 and even is 0 . The number of elements that are prime and even is 1 , namely the number 2 . The number of elements that are even and multiples of 3 is 4 , namely 6,12 18 and 24 . The number of elements that are prime and multiples of 3 is 1 , namely 3 .

## PROBLEM 4

a. Bar Graph
Data compares different colors
b. Circle Graph
Compares part to whole
c. Venn Diagram
Compares when multiple data points can be chosen
d. Line Graph
Shows change over time.

## PROBLEM 4

Consider the numbers from $1-25$. Let E equal the set of all the even numbers from $1-25$ and list them. Let F equal the set of all multiples of 3 from $1-25$ and list them. Let G equal the set of all prime numbers from $1-25$ and list them.

1) Construct a Venn Diagram with the appropriate elements listed in the regions of the Venn Diagram. Describe, in words, each region.
2) Construct a different Venn Diagram and write in each appropriate region the number of elements in the region.
When might a Venn Diagram be a good choice for representing data? How is this better than a bar graph? A circle graph?

The following table summarizes the type of graph chosen depending on the information provided.

| Graph Type | When To Use |
| :--- | :--- |
| Bar Graph | To compare groups of data |
| Circle (pie) Graph | To compare parts to the whole |
| Histogram | To show frequency within intervals |
| Venn Diagram | To show different or similar characteristics |
| Line Graph | To show change over time |
| Dot (line) Plot | Orders data and displays frequency |

## PROBLEM 5

Choose the appropriate type of graph for each situation.
a. Favorite colors of students.
b. The number of students who prefer pepperoni pizza to all types of pizza.
c. Students were polled according to their favorite subjects in school. They were given the choice of math, science, English, and social studies. How many chose math and science?
d. Jose's height over the past year.

1. Summer is the favorite season and fall is the least favorite.

## EXAMPLE 6

The scores on a test are given in the Stem and Leaf graph below:

| Stem | Leaf |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 2 | 5 |  |  |  |  |  |  |  |
| 6 | 3 | 4 | 4 | 4 | 5 |  |  |  |  |
| 7 | 1 | 3 | 3 | 5 | 5 | 6 | 6 | 9 |  |
| 8 | 0 | 0 | 2 | 4 | 8 |  |  |  |  |
| 9 | 9 | 1 | 2 | 2 | 8 |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |

a. How many students took the test?
b. What score occurred the most often?
c. What is the median score?
d. What is the highest score?
e. What is the range of the scores ?

## SOLUTION

a. 24 , the total number of leaves
b. 64 occurred most often.
c. The median is the average of 75 and $76,75.5$.
d. The highest score is 98 .
e. The range of scores is from 52 to 98 , so the range is $98-52=46$.

## EXERCISES

1. a. Describe what the data in the circle graph shows about the favorite time of the year of students in the class.
b. What information do you need about the class to give more specific information about the class' preference for time of year?
c. The approximate percent of students who favor winter.
d. The approximate percent of students who favor
 summer.
e. The approximate percent of students who favor fall.
2. Answers will vary. Discuss the different strengths of pie and bar graphs.
f. The ratio of students who favor summer to those who favor fall.
g. The ratio of students who favor fall to those who favor winter.
h. The ratio of students who favor summer to those who favor spring.
3. Poll your class. Ask all the students what their favorite month is. Make a bar graph and a circle graph for data gathered from a survey of favorite months.
a. The percent that favor the summer months.
b. The percent that favor November or December.
c. The ratio between those who favor summer months to those who favor non-summer months.
d. The ratio between those who favor June to those who favor September.
4. Poll your class for size-of-family data. Make a bar graph and a circle graph using this data.
5. A poll is taken as to what is each person's favorite season. The results are

| Season | Number of People |
| :---: | :---: |
| Winter | 10 |
| Spring | 35 |
| Summer | 10 |
| Fall | 40 |

a. Make a bar graph using the numbers in each group.
b. Make an equivalent bar graph showing the percent of those polled favoring each season.
c. Make a circle graph using the numbers in each group
d. Make an equivalent circle graph showing the percent of those polled favoring each season.
5. Biologists found the estimated population of rabbits in the Palo Duro Canyon during the 8-year period from 1990 to 1997. Make a bar graph

| Year | Population |
| :---: | ---: |
| 1990 | 12,500 |
| 1991 | 13,000 |
| 1992 | 11,000 |
| 1993 | 7,500 | | Year | Population |
| ---: | ---: |
| 1994 | 8,000 |
| 1995 | 11,500 |
| 1997 | 15,000 | to represent the population estimates to the right.

7. a. 17 cars were sold in August.
b. Total sales amount in thousands $=(12)(1)+(13)(1)+(14)(1)+(15)(3)+(17)(5)+(19)(1)+(20)(3)+(22)$ $(1)+(23)(1)=293$. Therefore, $\$ 293,000$ total.
c. Connie's $1 \%$ commission is $\$ 2,930$. When a person earns pay on commission, it means that one earns a percentage of the sales. In this case, multiply the commission rate of $1 \%=0.01$ times the total amount of sales to determine the dollar amount earned on commission.
8. Note: This is very difficult to see without color. Indicate $0 \%$ with a labeled line or don't include it.

9. Make a line graph to represent the data from Exercise 5.
10. Connie was working on important data when her computer crashed and she lost all the numbers. Luckily, she had printed a bar graph of her data earlier.


Based on the data from the bar graph:
a. What percent of cars sold cost less than $\$ 20,000$ ?
b. What percent of cars sold cost more than $\$ 17,000$ ?
c. What is the proportion of cars sold that cost less than $\$ 19,000$ to the number that cost more than $\$ 20,000$ ?
d. How many cars were sold?
e. What is the total sales amount for the month of August?
f. If Connie gets a $1 \%$ commission on the sales amount, how much did Connie earn from her commission in the month of August?
8. Make a circle graph for the birthday data to the right. How do you represent months with no birthdays?

| Month | Birthdays |
| :--- | :---: |
| January | 12 |
| February | 13 |
| March | 0 |
| April | 13 |


| Month | Birthdays |
| :--- | :---: |
| May | 15 |
| June | 22 |
| July | 24 |
| August | 25 |


| Month | Birthdays |
| :--- | :---: |
| September | 12 |
| October | 13 |
| November | 0 |
| December | 23 |

9. The second quarter has the most birthdays.
10. Make a histogram based on the birthday data to the right. Group the months by quarters of the year. Which quarter of the year has the most birthdays?
11. The box and whisker plot below shows the distribution of ages of workers at Max's Department Store.

a. From this box and whisker plot, what is the median age?
b. What is the youngest employee's age?
c. What is the oldest employee's age?
d. Do you think this department store caters to young or older customers? Explain.
e. What percentage of the employees is younger than 35 ?
f. What percentage of the employees is between 21 and 35 years old?
12. Rachel and Liz are preparing to bake cookies, but they want to bake the most popular kind. They polled their friends to find out what kind of cookie is the most preferred. The results are shown in the circle graph below.

Cookie Choices


1481 (571)
11. a. 3
b. $136.8^{\circ}$
c. 150 students
a. If 25 friends are polled, how many prefer Snickerdoodle?
b. What angle measure corresponds to sugar cookies?
c. They decide to poll the entire seventh grade. 57 said they like sugar cookies. How many students are in the seventh grade?
d. What percent of students preferred Sugar or Chocolate Chip cookies?
12. A survey was conducted that asked for the salaries of ten people:
\$25,700, \$32,000, \$14,000, \$115,000, \$39,000, \$36,000, \$74,000, \$18,000, $\$ 12,000$, and \$38,000
a. Graph this data with a box and whisker plot.
b. What percentage of the respondents has salaries that appear in the box between Q2 and Q3?
c. Graph this data with a stem and leaf plot.
d. From the stem and leaf plot, which salaries occur most often?
e. Make a histogram that shows the salaries in $\$ 10,000$ increments.
f. Compare the stem and leaf plot to the histogram
13. The coaches from junior high schools compared the total basket scored by their teams during the preceding season. The results are shown in the graph below. The combined total baskets scored by all teams was 1,340.

Total Baskets Scored

13. a. $26.1 \%$
b. Abell and San Jacinto
14. a. $\frac{45}{360}=0.125=12.5 \%$ have red hair.
b. $\quad(0.125)(24)=3$ students with red hair.
14. Dot Plot. This display shows data distribution.
15. Line Graph. This display shows change over time.
15. A circle chart is inappropriate for this data because the percentages are from different classes, so the "whole" is different for each class.


Ingenuity
16. For 1 person, 3 possible results: $A, B$, or $C$.

For 2 people, 6 possible results: $A A, A B, A C, B B, B C$, or $C C$.
For 3 people, 10 possible results: $A A A, B B B, C C C, A B B, A C C, A A B, B B C, C C B$, or $A B C$.
a. Approximately what percent of the combined total was scored by Goddard?
b. Which two teams were the closest in total baskets scored?
14. In a circle graph for hair color from another class, the angle for red hair is $45^{\circ}$.
a. What percent of the students have red hair?
b. How many of the 24 class members have red hair?
15. A circle chart cannot be made in all cases where data is presented in percentages. All four seventh grade teachers in a school asked their students which color they liked best. The table shows the results for what percentage in each class liked green best. Make a bar graph. Explain why you couldn't make a circle chart for the data in the table.

| Class | Percent that prefer green |
| :--- | :---: |
| Ms. Garcia | $32 \%$ |
| Mr. Ruiz | $50 \%$ |
| Ms. Serviere | $25 \%$ |
| Ms. Voigt | $20 \%$ |

16. Which type of graph would be most appropriate to display the grade distribution in Dr. Warshauer's Algebra class? Why?
17. Which type of graph would be most appropriate to display the change in temperature over the month of February? Why?
18. All the students in a class measured their heights. The results of this survey are given in the stem and leaf graph below, with the heights in inches. The stem is the tens digit, and the leaf is the units digit.

| Stem | Leaf |
| :---: | :---: |
| 5 | 34445669 |
| 6 | 1112222345889 |
| 7 | 000125 |

Investigation
17. Answers will vary. A line graph does not make sense because the order of the data does not matter.
a. How many students were surveyed?
b. What height occurred the most often?
c. What is the median height of the class?
d. What is the highest?
e. What is the range of heights?

## 19. Ingenuity:

Each time a question is given with three possible answers, it is possible to get a different result. For example, if the possible responses to the survey question are $A, B$ and $C$, one possibility with 5 people is $2 A^{\prime} s, 1 B$ and 2 C's. How many different results are possible if there is 1 person? 2 people? 3 people?

## 20. Investigation:

Conduct a survey of 20 people on a question of your choice that has three possible answers. Collect your data into a table, and display the results with a bar graph, a line graph, if possible, and a circle graph. Which representation seems most appropriate for your survey? Why

## Comparing Median and Mean

Objective: Students will compare the median and mean of sets of data. They will determine how the distribution of the grades affects whether the mean will be greater than, less than or equal to the median.

## Materials:

Comparing Median and Mean Handout, Pencil, Colored Pencils, Calculator (optional)

## Activity Instructions:

1) Find the median and mean of each set of test data, and plot the (median, mean) as ordered pairs for each set. There should be 6 ordered pairs when done.
2) Explain to students what the graph of $y=x$ would look like and help them graph it on the grid. Have a discussion on what it means for y to equal x in this context.
3) Students will answer questions \#1-4, and create their own set of data for question 5.

## Answers to Activity:

The (median, mean) ordered pairs for the tests are as follows:
Test 1(90,78); Test 2(92,92); Test 3(90,87.8); Test 4(22, 37.4); Test 5(75, 75.4); Test 6(95,83.2)

1) The line $y=x$ will be a diagonal line that includes points $(1,1),(2,2)$, etc. If data point is above the line, this means that the mean is greater than the median $(y>x)$. If data point is below the line, then the mean is below the median $(y<x)$. If the data point falls on the line then the mean equals the median $(y=x)$.
2) The median and mean for Tests 2,3 and 5 are close to or equal to each other. The grades for these tests are evenly distributed about the median. For instance, on Test 5 one test is 5 below the median and the other test is 20 below the median for a total of 25 below the median. The two tests above the median add up to 27 points above the median. This caused the mean to be just slightly above the median.
3) Tests 1 and 6 have means that are much less than the median. The grades higher than the median on Test 1 add up to 15 points higher than the median, whereas the points lower add up to 75 points. This is due to the outlier, 20 , which had a great affect on the mean, and caused the mean to be much lower than the median.
4) Test 4 has a mean much greater than the median. The grades below the median range only 7 points whereas the grades above the median range 73 points. This unequal distribution caused the mean to be greater than the median.
5) Answers will vary, but the point when plotted should be at least 10 units above the line $y=x$. The data should be chosen so that the test grades above the mean have a much greater difference with the median than the test grades below the median.

## Comparing Median and Mean

|  | Test 1 | Test 2 | Test 3 | Test 4 | Test 5 | Test 6 | Test 7 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | 95 | 90 | 90 | 35 | 70 | 60 |  |
|  | 100 | 91 | 75 | 95 | 55 | 62 |  |
|  | 85 | 92 | 82 | 22 | 95 | 95 |  |
|  | 90 | 93 | 100 | 20 | 82 | 99 |  |
| (median, mean) | 20 | 94 | 92 | 15 | 75 | 100 |  |

These are the grades of five students in your class for the last 6 tests that have been given. Calculate the median and mean for each of the 6 tests, and then plot the data as points on a coordinate grid with median on the $x$-axis and mean on the $y$-axis.


1. Graph the line $y=x$ and compare the points you plotted to this line. What does it mean if your data point is above this line? What does it mean if it is below the line? What does it mean if the data point is on the line?
2. For which tests are the median and mean close to or equal to each other? What do you notice about the grades for these tests that caused the mean to be close to the median?
3. For which tests are the mean much less than the median? What do you notice about the grades for these tests that caused the mean to be less than the median?
4. For which test is the mean much greater than the median? What do you notice about the grades for this test that caused the mean to be greater than the median?
5. Create a set of test grades for Test 7 so that the mean is at least 10 points greater than the median. Plot this point on the grid. Where is the point in comparison to the line $y=x$ ? Describe how you chose the data in order to make the mean greater than the median.

## What's Your Shoe Size?

Objective: Students will collect and record data about their class shoe sizes to find mean, median, mode and range of a set of data.

## Materials:

Scratch paper for tracing
Measuring tape/ruler (one per group)
Large chart paper (optional)

## Activity Instructions:

1) Each student will trace his/her foot on a sheet of paper, leaving their socks ON! Students will use measuring tape or rulers to measure their foot length-wise (toe to heel) and record it in cm . Students will record this data and their actual shoe size (in either whole or half sizes), and then give this data to their teacher as well.
2) The teacher will collect the data from each student and display this data in a place in the room where all students can access the information. Each student will record the data for the entire class, both their foot lengths and their shoe sizes.
3) Students will figure out the mean, median, mode, and range of the foot lengths and shoe sizes and record this data in the attached table.
4) Have students compare lengths to shoe sizes, checking to see if the measurements are consistent.
5) Have a Winning Prize for the student with the lengthiest foot or largest shoe size, or the smallest (outliers) etc. . .for fun!!

## WHAT'S YOUR SHOE SIZE? WORKSHEET



## Section 12.3 - Probability

## Big Idea:

Exploring probability of events using sample spaces, and simple and compound experiments

## Key Objectives:

- Understand events and sample spaces in an experimental setting.
- Understand the difference between a simple and a compound experiment.
- Know how to compute the probability of an event from the data.
- Understand the difference between experimental and theoretical probability.
- Learn how to display data in a compound experiment.
- Use probability in daily situations.


## Materials:

Two different colored number cubes, Coins for flipping

## Pedagogical/Orchestration:

This section lays the groundwork for later compound events. It defines a simple experiment with its events making a sample space. Probability is defined mathematically. There is also a discussion of theoretical versus empirical probability. Finally, the discussion shows one of the most efficient ways to display a compound event, like rolling two number cubes.

## Internet Resource:

Battleship Game: Probability- http://www.quia.com/ba/51817.html

## Activities:

"Flower Arrangement" on CD and at the end of the section

## Vocabulary:

experiment, outcomes, sample space, simple experiment, compound experiment, probability, theoretical probability, empirical (experimental) probability

## TEKS:

6.3(C);
6.9(A,B);
6.11(D);
6.12(A);
6.13(A,B);
7.1(A); 7.2(A, B, C,D,F,G);
7.13(A,B); 7.10(A,B); 8.11(B,C); 8.14(A)

## WARM-UPS for Section 12.3 (Probability)

1. A class conducted a survey of its members about whether they liked or disliked the flavors vanilla, chocolate and strawberry. The class had 29 students. If you are given only the following information, how many students liked vanilla, how many students liked chocolate and how many students liked strawberry?
5 students liked all three
2 students liked only chocolate and strawberry
4 students liked only chocolate and vanilla
1 students liked only strawberry and vanilla
8 students liked only chocolate
6 students liked only vanilla
2. A recipe for 21 muffins calls for $1 \frac{3}{4}$ cups of blueberries. If you have 10 cups of blueberries, how many muffins can you make using this recipe?
a. 110 muffins
b. 115 muffins
c. 120 muffins
d. 125 muffins

Ans: (c) because $10 \div\left(\frac{7}{4}\right)=10\left(\frac{4}{7}\right)=\frac{40}{7}$. So ( $\frac{40}{7}$ recipes) $(21$ muffins $/$ recipe $)=120$ muffins
3. The minute hand of a clock rotates or turns $360^{\circ}$ in an hour.
a. How many degrees does it turn each minute?

Ans: $6^{\circ}$
b. How many degrees will it turn in moving from point at 12 to pointing at 3 ?

Ans: $90^{\circ}$
c. How many degrees will it turn in moving from 12 to 6 ?

Ans: $180^{\circ}$
d. How many degrees in turning from 6 to 8 ? Ans: $60^{\circ}$

## Launch for Section 12.3:

Teachers, ask your students "What is the chance of it snowing tomorrow in Texas?" and "What is the chance that tomorrow is Sunday?" Lead the class in a discussion as to the probability of any events. Ask a variety of questions that are unlikely to happen. Facilitate a discussion that probability can be between 0 and 1. An unlikely event is closer to 0 and a more likely event is closer to 1 . Lesson 12.3 is an introduction to probability. Tell your students, "Today we will be playing a probability game that involves flipping two coins." The students need to get into groups of three, and each group will be handed two coins. Player A gets a point if the two coins land on Heads when flipped, Player B gets a point if the two coins land on Tails when flipped, and Player C gets a point if one coin lands on Heads and the other on Tails. They are to flip the coins 20 times each. Each person in the group has a role. Two can be assigned to flipping the coins and the third person can record the results of the flips, also known as the outcomes. Once the game is concluded, have groups share their results. It is likely that Player C won in all groups, so ask your student why they think Player C dominated. Once students have given their opinions, tell them they will be examining the outcomes from each group. Ask each group the following information and record it on the board or chart paper: "How many outcomes were two heads? How many outcomes were two tails? How many outcomes were one of each?" Add up the totals of each outcome for the class, and tell them these are the results of the experiment that they have just conducted and they will be comparing these "empirical" (experimental) outcomes to all the possible outcomes known as the "theoretical" outcomes. Tell them this list of all the possible theoretical outcomes is also known as the sample space. Ask them to make the sample space for the data. It helps if they are to specify the coins as Coin A and Coin B in a table such as this:

| Coin A | Coin B |
| :---: | :---: |
| $H$ | $H$ |
| $T$ | $T$ |
| $H$ | $T$ |
| $T$ | $H$ |

Students should recognize that the sample space signifies why Player C dominated the game, and that the game as set up was not a fair game. Look at the empirical outcomes again and see if they are similar to the sample space. Tell your students that today they will be learning more about probability and how it allows us to make educated guesses about what might happen in the future.

Transition from the launch- experimental- and compare it to theoretical.

Contrary to popular belief, $80 \%$ chance of rain does not mean that $80 \%$ of the area will get rain. It means that in the past, when the area had similar climatic situations, it rained four out of five times, or $80 \%$ of the time.

## SECTION 12.3 PROBABILITY

The study of probability allows us to make educated guesses about what might happen in the future based on past experience, to determine how likely different outcomes are. This knowledge can help us make the informed choices.

You have heard people say, "There is a 50-50 chance of getting heads on a coin flip," or "There is a $80 \%$ chance of rain today." What do statements like these mean? How can we determine the chance that some event will happen?

To study this question, we first need to define or review a few important terms. An experiment is a repeatable action with a set of outcomes. For example, the experiment of flipping a coin has two possible outcomes, either heads or tails. The set of all possible outcomes of an experiment is called the sample space. There are different ways to display the probable outcomes, some of which are creating a tree diagram, a list, or a table.

In studying an experiment, the question is, "What are all the possible outcomes?" If you flip a coin once, you will observe only one of the two possible outcomes, heads or tails. The key to finding any probability is to determine the likelihood of each possible outcome.

It is often helpful to use abbreviations or draw diagrams to represent the outcomes of the experiment. For example, when flipping a coin, we often write H for the outcome of getting a head and $T$ for the outcome of getting a tail. The sample space is the set $\{\mathrm{H}, \mathrm{T}\}$; there are only two possible outcomes.

The sample space $\{\mathrm{H}, \mathrm{T}\}$ represents a simple coin-flipping experiment. Each event arises from a single toss. Other examples of simple experiments are listed below:
a. Roll a number cube or die once. The possible outcomes are the sample space $\{1,2,3,4,5,6\}$.
b. Randomly pick a marble from a bag containing a blue, a red, a green, and a purple marble. The sample space for this simple experiment is $\{B, R, G, P\}$.

Some student might pick the sample space \{all heads, one of each, all tails\}. This is a possible sample space but each outcome is not equally likely because \{one of each\} can occur two different ways. We prefer that the sample space consist of all possible simple and equally likely outcomes.

Another way to represent a sample space for these simple experiments is to draw a tree diagram. For the sample of flipping a coin, the tree diagram might be drawn like this:


## EXAMPLE 1

Now consider a more complicated experiment, flipping a coin twice. What is the set of all possible outcomes? How many outcomes show no tails?

## SOLUTION

The order of outcomes is important. The outcome of getting a head and then a tail, denoted by HT , is a different outcome from getting a tail and then a head, denoted by TH. This experiment has the sample space $\{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{T} T\}$. Notice that there are four possible outcomes. When the experiment involves flipping a coin twice, $\{H\}$ is an impossible outcome. The simple event $\{\mathrm{HH}\}$ is the subset containing the outcome that both flips show heads and is the only outcome that shows no tails. The following chart lists all the possible outcomes of this experiment.

| 1st flip | 2nd flip | Outcome |
| :---: | :---: | :---: |
| $H$ | $H$ | $H H$ |
| $H$ | T | HT |
| T | T | TT |
| T | H | TH |

We can also summarize all the possible outcomes with a Tree Diagram, as shown below.


In both of the experiments so far, each outcome in the sample spaces has the same chance of occurring as any other outcome. Each outcome is said to be equally likely.

## DEFINITION 12.1: EVENT, SIMPLE EVENT AND COMPOUND EVENT

An event is any subset of the sample space. A simple event is a subset of the sample space containing only one possible outcome of an experiment. A compound event is a subset of the sample space containing two or more outcomes.

The words simple and compound are used to describe both events and experiments. The main thing to remember is that a simple event has just one outcome while a compound event has more than one outcome listed. A simple experiment has just one action, such as pick a card, roll a number cube, or flip a coin. A compound experiment may roll two number cubes or flip a coin more than once.

Call $E$ the event of "getting at least one head" in the two-flip experiment. The outcomes $\mathrm{HH}, \mathrm{HT}$, and TH satisfy the criteria for being in $E$, so $E=\{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}\}$. Because $E$ contains three possible outcomes, it is a compound event.

In order to study a complicated event $E$, a technique that is often useful is to find the possible outcomes that are not in $E$. We call this set the complement of $E$, or $\mathbf{E}^{\mathbf{C}}$. $\{T \mathrm{~T}\}$, the event that there are no heads, is the complement of the event of at least one head, $\{T H, H T, H H\}$.

Therefore, the probability of showing no heads is $\frac{1}{4}$, because 1 of the 4 equally likely outcomes show no heads. The probability of showing at least one head is $\frac{3}{4}$.

## DEFINITION 12.2: PROBABILITY

In an experiment in which each outcome is equally likely, the prob-
ability $P(A)$ of an event $A$ is $\frac{m}{n}$ where $m$ is the number of outcomes in the subset $A$ and $n$ is the total number of outcomes in the sample space $S$.

$$
P(A)=\frac{\# \text { of outcomes }}{\text { total number of possible outcomes }}
$$

One way to show your class the distinction between $P(A)=0$ or $P(A)=1$ is to ask the following two questions. If the day of this lesson is Monday, ask, "What is the probability that tomorrow is Tuesday?" It is one, of course. It has to be Tuesday. Then ask, "What is the probability that tomorrow is Saturday?" Unfortunately, that is impossible, so $P(A)=0$.

Empirical is a big word that means what we experience. Empirical knowledge is knowledge we gain through our experience.

Sometimes in middle school texts and material, empirical probability is called experimental probability.

Notice that the probability of an event from an experiment is always a number between 0 and 1 . Explain why this is true, using the coin-flipping example if necessary. Explain why $P(S)=1$, where $S$ is the sample space for an experiment.

Explain why $P\left(E^{C}\right)=1-P(E)$.

## EXAMPLE 2

Consider the experiment of rolling one number cube.
a. What is the probability of rolling a four?
b. What is the probability of not rolling a four?

## SOLUTION

Identify the sample space, $S=\{1,2,3,4,5,6\}$, for the experiment of rolling one number cube. Recall, a sample space is the set of all the possible outcomes of an experiment.
a. Let $\mathrm{A}=$ the event of rolling a 4 in the experiment. Written as a subset of the sample space we have $A=\{$ rolling a 4$\}$ or just $\{4\}$. $A$ is a simple event. The probability of $A$ is the fraction with the numerator equal to 1 , the number of outcomes in A and the denominator equal to 6 , the number of total outcomes in the same space $S$. Therefore, $P(A)=\frac{1}{6}$.
b Let $B=$ the event of not rolling a 4 in this experiment. Written as a subset, we have $B=\{1,2,3,5,6\}$. Notice that $B$ is the complement of event A because the outcomes in B are "not 4." B is a compound event. The probability of \{not getting a 4$\}=P(B)=\frac{5}{6}$. Another way to think about the probability of "not an event" is 1 minus probability of the event. A notation for this is:

$$
P(B)=P\left(A^{C}\right)=1-\frac{1}{6}=\frac{5}{6}
$$

When we consider an experiment of rolling one number cube, we do not actually roll a number cube. Instead, we think about what could possibly happen if we rolled a number cube. This is called theoretical probability. If we actually rolled the number cube, that would be experimental (empirical) probability.

To simulate 6 experiments that involve flipping a fair coin 50 times, run the FLIP 50 program, from the CD, on a TI-83.

## PROBLEM 1

a. Since 5 out of 10 of the marbles are green, the probability of drawing a green marble is $\mathrm{P}($ green $)=\frac{5}{10}=\frac{1}{2}$
b. There are 2 red marbles out of the 10 marbles in the bag. The probability of drawing a red marble is $\mathrm{P}($ red $)=$ $\frac{2}{10}=\frac{1}{5}$
c. If you were to conduct this experiment 100 times, you would expect approximately 50 of the draws to result in a green marble, 30 blue, and 20 red. This estimate is based on the theoretical probability. The actual results of your experiment for each color, however, may be slightly higher or lower. In fact, if your classmates were to conduct the same experiment, they would find similar values that may not be exactly identical to the theoretical. Based on the theoretical probability, it is anticipated that they get results close to 50 for green, 30 for blue, and 20 for red. You would expect very few to be far away from these values.

To see the difference, go back to the simple experiment of tossing a coin. The theoretical probability of getting heads is $\frac{1}{2}$. Now, gather some empirical data by performing the experiment many times: Toss a coin 50 times and record the heads and tails. What was the result of your empirical experiment? Was the ratio of heads to total tosses $\frac{1}{2}$ ? Was it close to $\frac{1}{2}$ ? On average, as the number of tosses, or repetitions of the experiment, increases, the experimental probability gets closer to the theoretical probability.

## PROBLEM 1

a. Draw a tree diagram for the simple experiments of rolling the number cube from part a.
b. Draw a tree diagram for the simple experiment of picking a marble out of the bag from part $b$.

## EXPLORATION 1

Many middle school students prefer red black tennis shoes, but some student prefer neither. How might the percent of middle school students who prefer red tennis shoes be estimated? Explain.

## EXPLORATION 2

Bilateral Junior School has 1000 students, and each student prefers either the color red or blue. Each student in this unusual school also prefers either chocolate or strawberry ice cream. Use a survey to calculate the percent of your class that favors colors red or blue. Repeat for the percent favoring chocolate or strawberry ice cream. Then find the following percentages:
a. the number of students who favor red and chocolate
b. the number who favor red and strawberry
c. the number who favor blue and chocolate
d. the number who favor blue and strawberry

Use the work above to predict
e. the number of students in the school who favor red and chocolate.
f. the number in the school who favor red and strawberry.
g. the number in the school who favor blue and chocolate.
h. the number in the school who favor blue and strawberry.

## PROBLEM 2

Possible outcomes are: HHH, HHT, HTH, THH, TTH, THT, HTT, TTT. This can be shown in the following tree diagram. Each outcome is a possible branch.

i. P(\{favors red and chocolate\})
j. P(\{favors red and strawberry\})
k. P(\{favors blue and chocolate\})
I. $\quad P(\{f a v o r s$ blue and strawberry $\})$

## PROBLEM 2

Suppose there is a bag containing 10 marbles: 5 green, 3 blue and 2 red.
a. What is the theoretical probability of randomly drawing a green marble from the bag?
b. What is the probability of drawing a red marble from your original bag of 10 marbles?
c. Imagine you conducted an experiment 100 times. Each time you would draw a marble from the bag, record the color, and then replace it in the bag. How many times do you anticipate you would draw each color?

## PROBLEM 3

Consider an experiment in which we flip a coin.
a. If we flip the coin 3 times, what are the possible outcomes? Draw a tree diagram and make a list for the sample space.
b. Conduct an experiment of flipping a coin twenty times. How many outcomes are heads? How does this compare to the probability of getting heads from part a?
c. Conduct the same experiment flipping the coin fifty times. How do the experimental and theoretical probabilities compare?

The above example demonstrates the law of large numbers, which states as more experiments are conducted, experimental probability will approach theoretical probability.

## EXPLORATION 3

There are three bags containing red and blue marbles, as shown below.

## EXPLORATION 1

Possible misconceptions may occur, such as in part a picking bag 3 because it contains the most blue marbles.
a. Bag 1: $\frac{25}{100}=\frac{1}{4}=25 \%$
Bag $2: \frac{20}{60}=\frac{1}{3}=33 \%$
Bag $3: \frac{25}{125}=\frac{1}{5}=20 \%$

Bag 2 will give you the best chance.
b. Bag $1: \frac{75}{100}=\frac{3}{4}=75 \% \quad$ Bag $2: \frac{40}{60}=\frac{2}{3}=66 \% \quad$ Bag $3: \frac{100}{125}=\frac{4}{5}=80 \%$

Bag 3 will give you the best chance.
c. Add 5 red, remove 5 blue or add 20 red marbles to the bag.
d. Add 5 red marbles from the bag and add 5 blue.
e. Answers will vary.

See full explanation of this exploration at the end of this section.
PROPORTIONAL REASONING II: BAGS OF MARBLES


Total 100 marbles


Total 60 marbles


Total 125 marbles
a. Each bag is shaken. If you were to close your eyes, reach into a bag, and remove one marble, which bag would give you the best chance of picking a blue marble? Explain your answer.
b. Which bag gives you the best chance of picking a red marble?
c. How can you change Bag 2 to have the same chance of getting a blue marble as Bag 1? Explain how you reached this conclusion.
d. How can you change Bag 2 to have the same chance of getting a blue marble as Bag 1 if Bag 2 must contain 60 total marbles?
e. Consider only Bags 1 and 2. Make a new bag of marbles so that this bag has a greater chance of getting a blue marble than Bag 1, but less of a chance of getting a blue marble than Bag 2. Explain how you arrived at the number of blue and red marbles for your new bag.

## EXAMPLE 3

As an experiment, a class bought five bags of the same brand of candy from five different stores, opened them and randomly selected 1000 pieces. There were 198 red, 305 green, and 497 yellow candy. Next they opened another bag and randomly selected 150 pieces of candy. Approximately how many pieces of each color might the class expect to be in this sample?

## SOLUTION

The percent of each candy in this sample might be as follows:

$$
\begin{aligned}
& \frac{198}{1000} \text { is approximately } \frac{200}{1000}=.20=20 \% \\
& \frac{305}{1000} \text { is approximately } \frac{300}{1000}=.30=30 \% \\
& \frac{497}{1000} \text { is approximately } \frac{500}{1000}=.50=50 \%
\end{aligned}
$$

So the class expected about $20 \%$ of the last sample to be red, or $(0.20)(150)=30$ pieces, about $30 \%$ of the last sample to be green, or $(.30)(150)=45$ pieces and about $50 \%$ of the last sample to be yellow, or $(.50)(150)=75$ pieces

## PROBLEM 4

In a survey of middle school students, 180 have only a dog, 60 have neither a dog nor cat, 160 have both a cat and a dog, and 200 have only a cat.
a. What is the probability that a student selected at random will have a cat?
b. What is the probability that a student selected at random will have a dog?
c. What is the probability that a student selected at random will have both a cat and a dog?
d. What is the probability that a student selected at random will have neither a cat nor a dog?
e. What is the probability that a student selected at random will have only a cat?
f. In a random sampling of 30 students, about how many might have a cat?

There are many applications of probability in society. Every day the newspaper and other news outlets use surveys and polls to predict the outcome of an upcoming election or to discover public opinion about issues like whether to build a new highway or to allow drivers to text while driving. Professional statisticians use a random sample of between several hundred to a few thousand people and ask their opinion. For example, the size of the random sample is 300 people. From the sample, 180 favor candidate A, 90 favor candidate B, and 30 are undecided. From this data pollsters conclude that on election day approximately $\frac{180}{300}=60 \%$ of the voters to vote for candidate A , or more if she attracts some undecided voters. So, if about 10,000 people vote in the election, unless something happens, about $60 \%$ of the voters will cast their ballot for candidate A.

This is a likely outcome if the random sample accurately reflects the people who actually vote on the day of the election, if the undecided vote is not significant or something happens to change the predicted outcome. For instance, the voters who want vote for candidate A might be affected by bad weather, a flu epidemic or bad press.

## EXAMPLE 4

Watch the ways students organize their data. Discuss with the students that making a tree diagram gets "messy" and over-whelming and that making a list can get confusing. Discuss that making a table can sometimes be the best option for keeping this data organized.

Recall that this is a question from Section 12.1 without the notation.

PROBLEM 5
$A=\{(1,6),(2,6),(3,6),(4,6),(5,6),(6,6),(6,5),(6,4),(6,3),(6,2),(6,1)\}$, so $P(A)=\frac{11}{36}$. B has 6 outcomes and $C$ has 6 outcomes, so $P(B)=\frac{6}{36}=\frac{1}{6} ; P(C)=\frac{6}{36}=\frac{1}{6}$.

## EXAMPLE 4

If we roll two number cubes, determine the sample space using a list (table).

## SOLUTION

What is a possible outcome? You might roll a 3 and a 4. As with tossing a coin twice, the order matters. Rolling a 3 and a 4 is different from rolling a 4 and a 3 . To see this more easily, think about rolling one red number cube and one green number cube. Rolling a red 4 and a green 3 is a different outcome from rolling a red 3 and a green 4 .

To list all the outcomes of this experiment, abbreviate the outcome red 3 and green 4 as $(3,4)$. The sample space for the two-number cube experiment can be listed in several ways. One convenient method is to make a table like the one below

| $(R, G)$ | 1 | 2 | 3 | 4 | 5 | 6 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | $\{(1,1)$ | $(2,1)$ | $(3,1)$ | $(4,1)$ | $(5,1)$ | $(6,1)$ |
| 2 | $(1,2)$ | $(2,2)$ | $(3,2)$ | $(4,2)$ | $(5,2)$ | $(6,2)$ |
| 3 | $(1,3)$ | $(2,3)$ | $(3,3)$ | $(4,3)$ | $(5,3)$ | $(6,3)$ |
| 4 | $(1,4)$ | $(2,4)$ | $(3,4)$ | $(4,4)$ | $(5,4)$ | $(6,4)$ |
| 5 | $(1,5)$ | $(2,5)$ | $(3,5)$ | $(4,5)$ | $(5,5)$ | $(6,5)$ |
| 6 | $(1,6)$ | $(2,6)$ | $(3,6)$ | $(4,6)$ | $(5,6)$ | $(6,6)\}$ |

How do we represent the above solution as a tree diagram?

## PROBLEM 5

Consider again the experiment of rolling two number cubes. What is the probability of each of the following events?
a. $A=\{$ getting at least one 6$\}$
b. $B=\{$ getting a double (both number cubes have the same number) $\}$
c. $C=\{$ the sum of the two number cubes is 7$\}$

Flipping a coin twice or flipping two coins have equivalent sample spaces. So do rolling a number cube twice and rolling two number cubes. However, the experiments are different, just as flipping a coin is different from running a computer simulation of flipping a coin, even if the probabilities are the same. Extend the idea a bit further: 'it's a lot easier to keep track of an experiment in which you flip a single coin one thousand times than one in which you flip one thousand coins once each; but on the other hand, if you've flipped one thousand coins, you don't have to record the results until the end, since at the end, you can see exactly how each of the coins flipped.

What is the difference between flipping a coin two times and flipping two coins simultaneously? What is the difference between rolling a number cube two times and rolling two number cubes? Run the thought experiments for both the coins and the number cubes to see the difference or similarities.

Two groups are surveyed about their preference for sodas or diet sodas. A random sample of 500 people between 45 and 55 years of age was selected. Another group of 400 people, between the ages of 18 and 28 , was randomly selected. The results of the survey are in the table below.

| Group | Regular Soda | Diet Soda |
| :---: | :---: | :---: |
| A: $18-28$ age | 284 | 116 |
| B: $45-55$ age | 188 | 312 |

The percent of the group A sample that favor diet soda is $29 \%$. The percent of the group B sample that favor diet soda is $63.4 \%$. Explain how the percentages were reached. From these random samples, why can you infer that someone in the age range of 45 to 55 is twice as likely to prefer diet soda?

## SOLUTION

The percent of the group A sample who favor diet soda is $116 / 400=29 \%$. The percent of group B sample who favor diet soda is $312 / 500=63.4 \%$. From these random samples, you can infer that someone 45 to 55 years old is twice as likely to prefer diet soda. Because the data was derived from a random sample of a relatively large group, it is safe to say that the probability that someone between the ages of 18 and 28 prefers diet soda is approximately $29 \%$, and the probability that someone between the ages of 45 and 55 prefers diet soda is approximately 63.4\%.

## EXERCISES

1. A citrus farmer has 2,000 orange trees. He suspects that he might have a fruit fly problem. Devise a plan that would enable the farmer to decide whether or not his orchard has fruit flies and what percent of his orange trees might be infested.
2. For each of the following simple experiments, write a sample space and draw a tree diagram to represent the sample space:
a. Roll a 10 -sided number cube.
b. Pick a coin from a hat that contains a penny, a nickel, a dime, a quarter and a fifty-cent coin.
c. Pick a person from a group containing Bob, Betty, Brenda, Brian, and Bentley.
3. Write a sample space $S$ for the experiment of rolling a number cube.
a. Let $B=\{4\}$ be the simple event of getting a 4 . What is $\mathrm{P}(B)$ ?
b. Write the subset of the complement of $B, B^{c}$. Is $B^{c}$ a simple event or a compound event? Explain how you found the complement of $B$.
c. What is $\mathrm{P}\left(B^{\mathrm{C}}\right)$ ? $\mathrm{TE}: \mathrm{P}\left(B^{c}\right)=5 / 6=1-1 / 6=1-\mathrm{P}(B)$.
d. Call $A$ the event of rolling a number that is at least 5 . Write $A$ as a subset of $S$. Is event $A$ simple or compound? What is $P(A)$ ? Does your answer make sense?
e. Call $D$ the event of getting an even number. Write $D$ as a subset of $S$. What is $P(D)$ ?
f. What is the complement of $D, D^{c}$ ? Explain how to find $D^{c}$. What is $P\left(D^{c}\right)$ ?
4. A random group of sixty students from a Fastfood Junior School were asked the following questions: Do you prefer your French fries with or without catsup? Do you prefer regular of diet soda? The follow tree diagram shows the outcome in this sample:

A. Estimate the following percentages:
a. students who favor catsup and regular soda
b. those who favor catsup and diet soda
c. those who favor no catsup and regular soda
5. a. $A=\{5,6\}$, so $P(A)=\frac{2}{6}=\frac{1}{3}$.
b. $\{2,4,6\}$, so $P(\{$ rolling an even number $\})=\frac{3}{6}=\frac{1}{2}$.
d. those who favor no catsup and diet soda
B. Fastfood Junior School has 900 students. Predict the number of students who:
a. favor catsup and regular soda.
b. favor catsup and diet soda.
c. who favor no catsup and regular soda.
d. favor no catsup and diet soda.
C. Compute the experimental probabilities for each of the following events:
a. P(\{favors catsup and regular soda\})
b. P(\{favors catsup and diet soda\})
c. P(\{favors no catsup and regular soda\})
d. P(\{favors no catsup and diet soda\})
6. In a large lake, a scientist randomly catches forty small mouth bass using a net, tags each fish, and returns her catch. A week later, she returns to the lake and randomly catches 100 small mouth bass. She finds that ten of these fish have her tags. How many small mouth bass might be in this lake?
7. Marta has a large bag of marbles that are red, blue or green. She draws out 20 marbles and gets 7 red, 9 blue and 4 green marbles. She knows there are exactly 500 marbles in the bag, predict how many of each color marble there might be in the bag.
8. Two large containers are filled with red and white balls. In container $A$, the proportion of red to white balls is 3:2. In container $B$, the proportion of red to white balls is $5: 4$. If a ball is randomly selected from each container, which ball is more likely to be white, the ones for container $A$ or $B$ ?
9. Consider the experiment of rolling 1 number cube.
a. What is the probability of $A=$ \{rolling a number that is at least 5\}?
b. List the outcomes that involve rolling an even number. What is the probability of this event?
c. Perform an experiment by rolling the number cube 50 times and record the outcomes in a table. Using these results, compute the experimental probability of \{getting a 1$\}$ or $\mathrm{P}(1)$.
10. $S=\{(H, 1),(H, 2),(H, 3),(H, 4),(H, 5),(H, 6),(T, 1),(T, 2),(T, 3),(T, 4),(T, 5),(T, 6)\}$. There are 12 outcomes.

Ask students to show how they counted them. They might discover the tree method or just use the listing method.
10. Look for a tree, list, or table $\mathrm{E}=\{$ getting an A or a 1\}

The set of outcomes is $\{\mathrm{A} 1, \mathrm{~A} 2, \mathrm{~A} 3, \mathrm{~A} 4, \mathrm{~A} 5, \mathrm{~A} 6, \mathrm{~B} 1, \mathrm{C} 1\}$
So $P(E)=\frac{8}{18}$
11. The sample space is $\{H H H, H H T, H T T, H T H, T H H, T H T, T T T, T T H\} . P(\{$ getting 3 heads $\})=P(\{H H H\})=\frac{1}{8}$.
d. Compute the following experimental probabilities: $P(2), P(3), P(4), P(5)$ and $\mathrm{P}(6)$.
e. Compare these experimental probabilities to the theoretical probabilities. Now average the individual experimental probabilities from everyone in class. What do you notice?
9. Perform the experiment of first flipping a coin and then rolling a number cube. What is the sample space of this experiment? How many outcomes does it have?
10. Suppose we spin the spinner and roll the number cube, as shown below.


What is the sample space? How did you organize your sample space? How many outcomes are there? What is P (getting an A and a 1 )? What is P (getting an A or a 1)?
11. Consider the experiment of flipping four coins simultaneously. What is the sample space? What is the probability of getting 4 heads?
12. Krystal bought a bag of 24 jellybeans at the store. The bag has 8 red jellybeans, 6 yellow jellybeans, 6 green jellybeans, and 4 white jellybeans.
a. What is $P($ Green $)$ ?
b. What is $P($ Red $)$ ?
c. What is $P$ (Yellow)?
d. Compute the sum $P($ Green $)+P($ Red $)+P($ Yellow $)+P($ White $)$. Explain why the answer to this sum makes sense.
13. a. $\{T T T T\}=1$
b. $\{$ HTTT, THTT, TTHT, TTTH $\}=4$
c. $\{$ HHTT, HTHT, HTTH, THHT, THTH, TTHH $\}=6$
d. $\{H H H T$, HTHH, HHTH, THHH $\}=4$
e. $\{H H H H\}=1$
f. $\{H T T T$, THTT, TTHT, TTTH, HHTT, HTHT, HTTH, THHT, THTH, TTHH, HHHT, HTHH, HHTH, THHH, HHHH\} $=15$
7. a. Use the chart in Example 3 and count, or calculate $1-\frac{25}{36}=\frac{11}{36}$.
b. $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)=\frac{1}{4}$
c. Use the chart to find $\frac{6}{36}=\frac{1}{6}$.
17. For this problem, think of two letter code words, where the first letter is $\mathrm{G} 1, \mathrm{G} 2, \mathrm{G} 3, \mathrm{G} 4, \mathrm{G} 5$, or G 6 . The second letter is R1, R2, R3, R4, R5, or R6. or set up a 5 -by- 5 red-green matrix with 25 possibilities.
a. 25
b. The pip is the individual colored indention on the face of a die that makes up the numbered display. The outcomes where the sum of the pips are less than 7 are:
(G1, R1), (G1, R2), (G1, R3), (G1, R4), (G1, R5),
(G2, R1), (G2, R2), (G2, R3), (G2, R4),
(G3, R1), (G3, R2), (G3, R3),
(G4, R1), (G4, R2),
(G5, R1)
c. A double means that both dice show the same number:
(G1, R1), (G2, R2), (G3, R3), (G4, R4), (G5, R5)
d. (G1, R1), (G2, R2), (G3, R3)
e. The product of two odd numbers is also odd:
(G1, R1), (G1, R3), (G1, R5), (G3, R1), (G3, R3), (G3, R5), (G5, R1), (G5, R3), (G5, R5)
13. Consider the experiment of flipping a coin four times.
a. How many outcomes get no heads?
b. How many outcomes get exactly 1 head?
c. How many outcomes get exactly 2 heads?
d. How many outcomes get exactly 3 heads?
e. How many outcomes get exactly 4 heads?
f. How many outcomes get at least 1 head?
14. Roll two number cubes, one red and one green.
a. What is $P(E)$ for the event $E$ that at least one number cube rolled a 3?
b. What is $P(F)$ for the event $F$ that neither of the number cubes rolled a number greater than 3 ?
c. What is $P(G)$ for the event $G$ that the numbers on the number cubes add to 7 ?
15. The city of Carthage has a population of about 500,000 . It has about 240,000 registered voters. In an upcoming bond election, voters will be asked whether they support a new water treatment plant. A poll is taken using 500 voters chosen at random asking whether they support the new plant. The poll results are 220 in favor, 200 against, and 80 undecided. What can you say about the chances of the bond issue passing?
16. A factory has 850 employees. Of these, 550 work the day shift and 300 work the night shift. The day shift has 240 males and the night shift has 120 females. If an employee is selected at random, what is the probability that the employee is
a. a female who works the night shift.
b. a male who works the night shift.
c. a male who works the day shift.
d. a female who works the day shift.
17. Think about rolling two standard five-sided number cubes: a green number cube and a red number cube.
19. Since order doesn't matter, the possible servings are $\{C C, C S, C V, S S, S V, V V\}$. If a cone is used as opposed to a bowl, order may matter depending on your perspective. You may want to discuss this in class and discuss that the choices available to Tommy changes.
20. Because order matters, both have an equally likely probability: $\frac{1}{16}$
a. What is the set of all possible outcomes from rolling the two number cubes?
b. What is the set of all possible outcomes where the sum of the pips is less than 7 ?
c. What is the set of all possible outcomes where the outcome is a double?
d. What is the intersection of the sets from parts $\mathbf{b}$ and $\mathbf{c}$ ?
e. What is the set of all possible outcomes where the product of the pips is odd?
18. Two groups were surveyed about their preference, milk chocolate or dark chocolate. A random sample of 250 people between the ages of 15 and 20 was selected from group A. Another group of 280 people between the ages of 35 and 40 was randomly selected. The results were

| Group | Milk Chocolate | Dark Chocolate |
| :---: | :---: | :---: |
| A: $15-20$ age | 194 | 56 |
| B: $35-40$ age | 92 | 188 |

a. What percent prefer dark chocolate in group A?
b. What percent prefer dark chocolate in group B ?
c. What can you infer from these surveys about the differences in the two populations?
d. What two criteria in the survey allow anyone to infer results from the data?
e. Estimate the probability of that someone between the age of 15-20 might prefer milk chocolate to dark chocolate.
f. Estimate the probability of that someone between the age of 35-40 might prefer milk chocolate to dark chocolate.
19. Tommy bought a bowl of ice cream with two scoops. For each of the scoops, he can choose chocolate, strawberry or vanilla. List the different two-scoop choices Tommy can make.
20. Consider flipping a coin four times. Which outcome is more likely, getting HHHH or HTHT?
21. a. List the elements in the sample space. This set should consist of 20 elements: $\{(n 1, d 1),(n 1, d 2),(n 2, d 1)$, ( $n 2, d 2$ ), ( $n 3, d 1$ ), ( $n 3, d 2$ ), ( $n 1, n 2$ ), ( $n 1, n 3$ ), ( $n 2, n 3$ ), (d1, d2), (d1, n1), (d2, n1), (d1, n2), (d2, n2), (d1, $n 3),(d 2, n 3),(n 2, n 1),(n 3, n 1),(n 3, n 2),(d 2, d 1)\}$.
b. In order to get 15 cents, Harry needs one dime and one nickel. Looking at the sample space, we see that there are 12 ways this can be done. $P(15)=\frac{12}{20}=\frac{3}{5}$.
c. The sample space now has (5)(5) or 25 elements. This set looks like \{(n1, d1), ( $n 1, d 2$ ), ( $n 2, d 1$ ), ( $n 2$, $d 2$ ), (n3, d1), (n3, d2), (n1, n2), (n1, n3), (n2, n3), (d1, d2), (d1, n1), (d2, n1), (d1, n2), (d2, n2), (d1, n3), (d2, $n 3),(n 2, n 1),(n 3, n 1),(n 3, n 2),(d 2, d 1),(n 1, n 1),(n 2, n 2),(n 3, n 3),(d 1, d 1),(d 2, d 2)\}$.
d. $P(15)=\frac{12}{25}$.

## Ingenuity

22. a. 6
b. 6 for the first and 5 for the second $=(6)(5)=30$ possible outcomes. Discuss why in the first step you have 6 choices and at the second step you only have 5 choices left in the hat to choose from. If your class makes an array, $A B$ is different from $B A$, and doubles like $A A$ don't occur. So the array is $6 \times 6$ with the diagonal of doubles removed.
c. From part b, the 30 outcomes came from AB being different from BA. Without order, they and every other pair are the same, so there are half the possibilities: (6)(5) $\frac{1}{2}=15$ possible outcomes. However, caution the pattern-seekers against assuming that when order doesn't matter, the possibilities will always be halved.

## Investigation

23. a. There are 13 hearts out of 52 cards, so $P(H)=\frac{13}{52}=\frac{1}{4}$. There are also 13 clubs out of 52 cards, so $P(C)$ $=\frac{1}{4}$. Or consider that hearts is 1 suit out of 4 so $P(H)=\frac{1}{4}$.
b. Hearts and clubs combine to make up 26 of the 52 cards (or half of the cards), so $P(H$ or $C)=\frac{1}{2}$. Discuss with students the importance of distinguishing between the conjunctions "or" and "and."
c. In 1 draw, only one suit shows, so $P(H$ and $C)=0$. Discuss with students the importance of distinguishing between the conjunctions "or" and "and."
d. There are 12 face cards out of 52 cards in the deck, so $P($ face card $)=\frac{12}{52}=\frac{3}{13}$.
e. $\left(\frac{1}{4}\right)\left(\frac{3}{13}\right)=\frac{3}{52}$.
24. Harry has 5 coins in his pocket: 3 nickels and 2 dimes.
a. He pulls a coin from his pocket. Then, without replacing the first coin, he pulls a second coin from his pocket. What is the sample space for this experiment? (Hint: Consider labeling the nickels n1, n2, and n3, and the dimes d 1 and d 2 .)
b. What is the probability he pulled exactly 15 cents from his pocket?
c. Now, he pulls a coin from his pocket, replaces it and then pulls a second coin from his pocket. What is the sample space for this experiment?
d. What is the probability he pulled exactly 15 cents from his pocket in the second experiment?
25. Ingenuity:

A group of 6 students place their names in a hat.
a. Consider the experiment of drawing a name from the hat. How many outcomes are there for this experiment?
b. Consider another experiment of drawing two names from the hat, where the order in which the names are drawn matters. How many possible outcomes are there?
c. Now consider the same experiment of drawing two names from the hat, but this time simultaneously, so that order does not matter. How many possible outcomes are there for this experiment?

## 23. Investigation:

Consider the experiment of drawing one card randomly from a standard 52-card deck.
a. What is the probability of getting a heart? a club?
b. What is the probability of getting a heart or a club?
c. What is the probability of getting a heart and a club?
d. What is the probability of getting a face card: a jack, queen or king?
e. What is the probability of getting a heart and a face card?

# Flower Arrangement 



Objective: The students will use their knowledge of probability to design a flower arrangement that meets a certain criteria.

## Materials:

Paper and Pencil
White paper
Markers, crayons or colored pencils

## Activity Instructions:

1) Tell your students that they are to design a flower arrangement in which the probability of choosing a red flower out of the arrangement is $\frac{1}{10}$, the probability of choosing a pink flower is $\frac{2}{5}$, and the probability of choosing a white flower is $\frac{1}{2}$. These are the only three colors of flowers in the arrangement.
2) Your students will then decide how many total flowers might be in the arrangement, and then how many roses of each color there are.
3) After they have answered item 2, they are to draw a flower arrangement with the appropriate number of each color of flower on the plain white paper. Encourage creativity and beauty.

## Proportional Reasoning II: Bags of Marbles

Orchestrating Discussions: Five practices constitute a model for effectively using student responses in whole-class discussion that can potentially make teaching with high-level tasks more manageable for teachers by Smith, Hughes, Engle, and Stein in May 2009 Mathematics Teaching in the Middle School
(Materials: Graph paper with 1-cm grids; pencil; calculator (optional))

## Objective:

Students will be able to solve problems involving proportional relationships in probability.

## Prior Knowledge:

Suppose a baseball team is made up of 6 boys and 3 girls. Each person writes his or her name on a piece of paper and puts it in a hat. The coach draws one piece of paper from the hat. Which name is more likely to be drawn from the hat, a boy's name or a girl's name? Why do you think that? What is the chance that the name will be a boy? What is the chance that the name will be a girl? Explain.
(Do the above of a similar background check as a launch to make sure the ideas of probability from section 12.3 are secure.)

## Motivating Problem:

There are three bags containing red and blue marbles. The three bags are labeled as shown below.

Bag 1: $\quad 75$ red and 25 blue for a total of 100 marbles
Bag 2: $\quad 40$ red and 20 blue for a total of 60 marbles
Bag 3: $\quad 100$ red and 25 blue for a total of 125 marbles
Each bag is shaken. If you were to close your eyes, reach into a bag, and remove one marble, which bag would give you the best chance of picking a blue marble?
Justify your answer.
(Have students work individually on the problem and observe different approaches.)
(Once the students have all had an opportunity to make sense of and come up with tentative ideas for the solution, have them work in small groups to share their ideas.)

Discuss your answer and explanation with your group.
(Students' discussions may include discussion that bag 1 is $1 / 4$ blue, bag 2 is $1 / 3$ blue and bag 3 is $1 / 5$ blue. Others may use percents: bag 1 is $25 \%$ blue, bag2 is $331 / 3 \%$ blue, bag 3 is $20 \%$ blue. Others may argue incorrectly that bag 1 and bag 3 have more blue marbles than bag 2 so NOT bag 2 . Some areas of confusion may be in looking at ratios of Blue to Red rather than Blue to the total Blue + Red, though they do provide some information.)

## Reflection:

A. Each group reports their answer and explanation.
B. Explain why this bag gives you the best chance of picking a blue marble? You may use the diagram above in your explanation.
C. What math tools did you use to help understand the problem? (e.g. Table, picture, fractions?)
D. Did you use a table? Did you use fractions to explain your answer? How?

## Further Exploration:

A. Which bag gives you the best chance of picking a red marble? Explain why.
B. How can you change Bag 2 to have the same chance of getting a blue marble as Bag 1? Explain how you got your answer.
(The students may wish to add marbles of either color, for example if 20 red balls are added to Bag 2 then the chance of getting a blue is 20/80 $=1 / 4$.
C. How can you change Bag 2 to have the chance of getting a blue as Bag 1 if Bag 2 must contain 60 total marbles? ( 15 blue and 45 red makes a chance of getting a blue to be $15 / 60=1 / 4$ )

## Extension:

Consider only Bags 1 and 2. Make a new bag of marbles so that this bag has a greater chance of getting a blue than Bag 1 but less of a chance of getting a blue than Bag 2. Explain how you arrived at the number of blue and red marbles for your new bag.
(Students may try adding or subtracting quantities in either or both bags. You may monitor their efforts and ask what effect their changes have. Have them explain or show what is changing. Some ways that this new bag can be obtained include:
Taking 25/100 = $1 / 4$ in Bag 1 and 20/60 = $1 / 3$ in Bag 2 and determining a number between the two. Students may find a common denominator such as 12,24 , etc. and find the equivalent fractions for $1 / 4=3 / 12$ and $1 / 3=4 / 12$. While these two fractions make it difficult to see what lies between them, the equivalent fractions $1 / 4=6 / 24$ and $1 / 3$ $=8 / 24$ would lead to $7 / 24$ as a candidate for the new bag. Namely, a bag with 24 marbles of which 7 are blue and 17 are red. Another way is to look at the decimal representation for $1 / 4=.25$ and $1 / 3=.3333 \ldots$. A decimal such as $.3=3 / 10$ is between .25 and $.3333 \ldots$ so a bag with 10 marbles, of which 3 are blue and 7 red would also work. The students may come up with other interesting compositions. In fact, with $1 / 4$ and $1 / 3$, the fraction $(1+1) /(3+4)=$ $2 / 7$ is a fraction between $1 / 3$ and $1 / 4$ !)

## Section 12.4 - Independent Events

## Big Idea:

Understanding probability of independent events

## Key Objectives:

- Understand what makes an event independent.
- Use tree diagrams to study compound events.
- Learn the Rule of Product and when it applies.
- Learn the Rule of Sum and when it applies.
- Solve probability problems involving compound events.


## Materials:

Decks of cards, Dice, Coins, Paper bags, Index cards for Launch

## Pedagogical/Orchestration:

This section combines the elements of probability to understand independent events. It explains when to sum and when to multiply and why the two operations are appropriate at different times. It is a very active, experimental section.

## Activity:

"Crabs" at the end of the section and on CD

## Vocabulary:

independent, root, mutually exclusive, (See CD: combination, permutation)

## TEKS:

$6.9(A)(B) ; \quad 7.2(A)(B)(C)(F)(G) ; \quad 7.11(A)(B) ; \quad 7.13(A)(B)) ; \quad 7.10(B) ; \quad 8.11(A, B, C) ; \quad 8.14(A)$

## WARM-UPS for Section 12.4 (Independent Events)

1. A soccer league orders 30 jerseys for $\$ 195$. If the company gives them the same rate, how much would 18 jerseys cost?
a. $\$ 117$
b. $\$ 118$
c. $\$ 119$
d. $\$ 120$

Ans: (a) because $(\$ 195 / 30)=(x / 18)$ gives $x=\$ 117$.
2. List the following decimals in order from least to greatest. Use a number line if needed.
$0.3,0.03,-0.3,-0.03,0.303,-0.033,-0.003$
Ans: $-0.3,-0.033,-0.03,-0.003,0.03,0.3,0.303$
3. Marian works 35.6 hours for the week at Pepperoni's Pizza Pub
a. If she makes $\$ 9.50$ per hour, how much will she make for the week? Ans: $\$ 338.20$
b. Marian gets a raise of $\$ 0.25$ the following week. If she works the same number of hours, how much will she earn? Ans: \$347.10

## Launch for Section 12.4:

This launch will incorporate Example 1 from Lesson 12.4 with the students actually conducting the experiment. Give each group two paper bags: Bag A includes cards 1 through 4 and Bag B includes cards A, B, and C. Show your students that you are selecting a number and a letter from the bag and recording them on a piece of paper. Do not show them which letter and number you picked, and return your selections to the bags. Make a show of hiding the paper in a place inaccessible to the students, and tell students there will be a prize for the group that ended up with that outcome the most times. Tell your students they will be conducting an experiment and one student from each group will need to record the outcomes. Have them blindly draw a card from each bag, record the outcomes and replace the cards in the bags. They are to repeat this process 20 times. Once they have completed the experiment, ask students if they thought all of the outcomes were equally likely? At this time, take the paper from its hiding place and give the prize to the group that matched your outcome the most times. Tell them that today they will be figuring out the sample space for this experiment later on in the lesson and will compare it to their experimental (empirical) outcomes

## EXAMPLE 1

Have your class conduct a thought experiment like this, either as a class or in groups. This experiment is a combination of two independent experiments: (1) choose the number and (2) choose the letter.

## SECTION 12.4 INDEPENDENT EVENTS

One of the major goals of mathematics is to find simple underlying ideas to explain how and why things work. To do this, mathematicians analyze problems by breaking them into simpler steps.

## EXAMPLE 1

Suppose you have a hat and a box. The hat contains the numbers 1, 2, 3, and 4, and the box contains the letters $\mathrm{A}, \mathrm{B}$ and C . Imagine the following experiment: Without looking, reach into the hat and pull out one number, and then reach into the box and without looking, pull out one letter. What is a possible outcome? How can you represent all the possible outcomes? How many possible outcomes are there?

## SOLUTION

This experiment can be broken down into two simpler experiments:
Step1: Draw a number from the hat. This has sample space $S_{1}=\{1,2,3,4\}$. There are 4 possible outcomes.

Step 2: Draw a letter from the box. This has sample space $S_{2}=\{A, B, C\}$ with 3 possible outcomes.

For the combined experiment, list the outcomes as ordered pairs, like $(2, B)$ and $(3, A)$, or shorten the notation to $2 B$ and 3 A . Always try to write the sample space with some sort of order, if possible, so you do not miss a possible outcome. In this case, the sample space is

$$
S=\{1 A, 1 B, 1 C, 2 A, 2 B, 2 C, 3 A, 3 B, 3 C, 4 A, 4 B, 4 C\} .
$$

The outcomes can also be listed in a table:

|  | $A$ | $B$ | $C$ |
| :---: | :---: | :---: | :---: |
| 1 | $1 A$ | $1 B$ | $1 C$ |
| 2 | $2 A$ | $2 B$ | $2 C$ |
| 3 | $3 A$ | $3 B$ | $3 C$ |
| 4 | $4 A$ | $4 B$ | $4 C$ |

Because the table is a 4 by 3 table, the number of outcomes is $4 \cdot 3=12$. We will come back to this in a bit.

Teachers, again, have the class go through this if there are any questions. Ask if the tree could start from the top and go down or start on the bottom and go up.

What is the size of the sample space? How does the arrangement in the table help to count the number of outcomes?

A tree diagram is another method to visually represent the sample space:


The 4 branches that come from the left node or root of the tree represent the first experiment, choosing from $\{1,2,3,4\}$. Each branch splits into three smaller branches, representing the second experiment, choosing from $\{A, B, C\}$. There are 12 branches on the right. To obtain the 12 outcomes in the combined experiment, follow the branches from left to right, reading the labels to obtain the 12 outcomes in the combined experiment.

The action of drawing a number from the hat and drawing a letter from the box are said to be independent, meaning that the probabilities of one of these actions does not depend on the other. Because they are independent, the two actions can occur either in succession or simultaneously, it doesn't matter which. Using the table model, the 4 rows represent the 4 choices in the hat. The choices correspond to the first 4 branches in the tree model. Each row has 3 columns that represent the 3 choices in the box. The rows correspond to the second level of branching. For each of the first 4 choices, there are 3 second choices. So to count the number of members in the array, or the number in the sample space, compute the sum of 4 rows with 3 outcomes in each row: $3+3+3+3$. This is the same as the area model for multiplication. The total number of outcomes is $4 \cdot 3$.

This process is a formal rule in counting:

## THEOREM 12.1: THE RULE OF PRODUCT

If one action can be performed in $m$ ways and a second independent action can be performed in $n$ ways, then there are $m \cdot n$ possible ways to perform both actions.

Next, let's use our counting principles to compute the probabilities. For the experiment above, consider the following two events:

Let event $E$ be the event that the number drawn from the hat is a 2 .
Let event $F$ be the event that the letter drawn from the box is an $A$.
These events are independent since the probability of either experiment does not depend on the other. The probability of $E$ is $\frac{1}{4}$. You can compute this in two ways. If you think of $E$ as the experiment of drawing a number from a hat, then there are 4 equally likely outcomes. So, $P(E)=\frac{1}{4}$. However, if you think of the combined experiment, then the sample space $S$ has 12 possible outcomes, and event $E=$ $\{2 A, 2 B, 2 C\}$. So, $P(E)=\frac{3}{12}=\frac{1}{4}$.
Similarly, the probability of $F$ is $\frac{1}{3}$, since the sample space contains 3 equally likely outcomes. Alternatively, $F=\{1 \mathrm{~A}, 2 \mathrm{~A}, 3 \mathrm{~A}, 4 \mathrm{~A}\}$. So, $P(F)=\frac{4}{12}=\frac{1}{3}$.

We can now extend this problem and ask:

## EXAMPLE 2

What is the probability that both $E$ and $F$ will occur?

## SOLUTION

Looking at our tree diagram, the only way that both $E$ and $F$ will occur is if the number drawn is a 2 and the letter drawn is an A . So, the probability of $E$ and $F$ is $\frac{1}{12}$.

Another way to think about this is entirely with probabilities. If $E$ occurs $\frac{1}{4}$ of the time, and an independent event $F$ occurs $\frac{1}{3}$ of the time, then both $E$ and $F$ will occur $\frac{1}{3} \cdot \frac{1}{4}=\frac{1}{12}$ of the time.

We can summarize this rule, which extends the rule of product, as:

## THEOREM 12.2: THE RULE OF PRODUCT OF PROBABILITIES

If two events, $E$ and $F$, are independent, then the probability that both will occur is the product of their probabilities, namely

$$
P(E \text { and } F)=P(E) \cdot P(F)
$$

This leads to the idea of "conditional probability." The question is what do we mean when we talk about the "probability of event $E$, if event $F$ occurs?" To explain this, consider the following example:

## EXAMPLE 3

Consider the experiment of rolling a number cube. Let $E$ be the event that the outcome is a 1 or 2 . Let $F$ be the event that the outcome is odd.

1. What is the probability of event $E$ ?
2. What is the probability of event $F$ ?
3. What is the probability of event $E$, if event $F$ occurs, written $P(E \mid F)=P(E$ given that F occurred)?

## SOLUTION

1. The sample space of this experiment is $S=\{1,2,3,4,5,6\}$. Event $E=\{1,2\}$. So, $P(E)=\frac{2}{6}=\frac{1}{3}$.
2. Event $F=\{1,3,5\}$. The probability of $F$ is $\frac{3}{6}=\frac{1}{2}$.
3. If you are given that the event $F$ occurs, then the new sample space is $F$. The question is, how likely is it that $E$ occurs if we know that the outcome is odd? The only way that $E$ could occur would be if the outcome is a 1 . So,

You can assume this is the same as putting all seven elements together in one container. This makes choosing each element equally likely.

The idea of equally likely outcomes is very important. When we roll a die, we are assuming that the outcome of 1 is the same as $2,3,4,5$, or 6 . We say that the probability of rolling a 1 is $\frac{1}{6}$. Similarly, the probability of rolling a 2 or a 3 or a 4 or a 5 or a 6 is each $\frac{1}{6}$. In flipping a coin, heads is as likely to come up as tails so we say the probability of a $H$, or $P(H)=\frac{1}{2}$ and $P(T)=\frac{1}{2}$.

When we say that two actions are mutually exclusive, we mean that the two events cannot occur at the same time. For example, in a roll of a die, the sample space is $\{1,2,3,4,5,6\}$. Consider the event of getting an even roll, $\{2,4$, $6\}$, and the event of rolling an odd, $\{1,3,5\}$. The two events are mutually exclusive because you cannot get a roll that is both even and odd at the same time.

$$
P(E \mid F)=\frac{\text { \# of outcomes in both E andF }}{\text { \# of outcomes in } F}
$$

i.e. $P(E \mid F)=\frac{1}{3}$. Since this is the same as the probability of event $E, E$ and $F$ are independent events. By this, we mean that $P(E)$ is the same, whether or not event $F$ occurs. We can write the set of objects in both $E$ and $F$ as $E \cap F$. So,

$$
P(E \mid F)=\frac{\# \text { of elements in } E \cap F}{\# \text { of elements in } F}=\frac{P(E \text { and } F)}{P(F)} .
$$

## EXAMPLE 4

Suppose, with the same hat of numbers and box of letters, we change the experiment to the following: Assume cards with numbers 1-4 and cards with letters A-C are same shape and size. Put all of the cards in some box. Select one card. How many possible outcomes are there for this experiment?

## SOLUTION

This is the same as choosing one item from the combined sample space $\{1,2,3,4, A, B, C\}$, which is the set $S_{1}$ or $S_{2}$. The combined sample space has seven elements, or possibilities.

The main difference between the earlier situation and this one lies in the change of one word. In the first example we chose from the hat and the box, while in the second we chose from the hat or the box. This illustrates the importance in mathematics of reading words carefully, especially words like "and" and "or." The following rule captures the number of ways to perform one action "or" another:

## THEOREM 12.3: THE RULE OF SUM

If one action can be performed in $m$ ways and a second action can be performed in $n$ ways, then there are $(m+n)$ ways to perform one action or the other, but not both. This assumes that the two actions have no elements in common.

The idea of equally likely outcomes is very important. When we roll a die, we are assuming that the outcome of 1 is the same as $2,3,4,5$, or 6 . We say that the probability of rolling a 1 is $\frac{1}{6}$. Similarly, the probability of rolling a 2 or a 3 or a 4 or a 5 or a 6 is each $\frac{1}{6}$. In flipping a coin, heads is as likely to come up as tails so we say the probability of a $H$, or $P(H)=\frac{1}{2}$ and $P(T)=\frac{1}{2}$.

When we say that two actions are mutually exclusive, we mean that the two events cannot occur at the same time. For example, in a roll of a die, the sample space is $\{1,2,3,4,5,6\}$. Consider the event of getting an even roll, $\{2,4$, $6\}$, and the event of rolling an odd, $\{1,3,5\}$. The two events are mutually exclusive because you cannot get a roll that is both even and odd at the same time.

## PROBLEM 1

a. $\frac{2}{4} \cdot \frac{1}{3}=\frac{4}{12}=\frac{1}{6}$.
b. $\frac{3}{4} \cdot \frac{2}{3}=\frac{6}{12}=\frac{1}{2}$.
c. This is the complement of the event in part b. $1-\frac{1}{2}=\frac{1}{2}$.
d. This is the complement of the event of drawing neither an odd number nor a B. $1-\frac{2}{4} \cdot \frac{2}{3}=1-\frac{4}{12}=\frac{8}{12}$ $=\frac{2}{3}$.
e. Example 1 events are not mutually exclusive. (See example 1 sample space) The event 1 A exists and would be double counted.

The rule of sum and the rule of product provide a powerful way to examine experiments made up of several actions. By carefully using the rule of sum and the rule of product, it can often be far easier to compute the number of possible outcomes in such a compound experiment.

Note: Two events that have no elements in common are said to be "mutually exclusive." Using Example 4, where the sample space is $\{1,2,3,4, A, B, C\}$, let E be the event of getting a number and $F$ be the sound of getting a letter. Then $P(E)=$ $\frac{4}{7}$ and $P(F)=\frac{3}{7} \cdot P(E \cup F)=P(E)+P(F)=\frac{4}{7}+\frac{3}{7}=\frac{7}{7}=1$.

## THEOREM 12.4: THE RULE OF MUTUALLY EXCLUSIVE EVENTS

For mutually exclusive events, $E$ and $F$, the probability of $E$ or $F$, written $P(E$ or $F)$, occurring is given by the rule of sum

$$
P(E \text { or } F)=P(E)+P(F) \text {. }
$$

This follows since the number of ways that $E$ or $F$ can occur is the number of outcomes for event $E$ plus the number of outcomes for event $F$, by the rule of sum.

## PROBLEM 1

For the experiment in Example 1, compute the probability of each of the following events:
a. The event of drawing an even number and an A.
b. The event of drawing neither a 1 nor an A.

Use the definition of complement to answer c and d .
c. The event of drawing a 1 or an $A$.
d. The event of drawing an odd number or a B.
e. Why can't we use the rule of mutually exclusive events to answer c and d?

## PROBLEM 2

a. The probability of selecting a green pair of socks in the first draw is $\frac{4}{10}=\frac{2}{5}$. Because the socks from the first draw are replaced, the probability of selecting green socks in the second draw is also $\frac{2}{5}$. The probability of selecting green socks for both draws is $\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)=\frac{4}{25}$.
b. Because the socks from the first draw were replaced, the probability of selecting green socks in the second draw is unaffected by the color selected in the first draw, so the conditional probability is $\frac{4}{10}$.
c. The probability of selecting a yellow pair of socks in the first draw is $\frac{4}{10}=\frac{2}{5}$. However, in the second draw, the socks selected from the first draw are no longer available to be selected. If Rhonda selected yellow in the first draw, the drawer contains 3 pairs of yellow, 3 pairs of blue, 2 pairs of pink, and 1 pair of white. The probability of selecting yellow socks in the second draw is $\frac{3}{9}=\frac{1}{3}$. The probability of selecting yellow both times is $\left(\frac{2}{5}\right)\left(\frac{1}{3}\right)=\frac{2}{15}$.
d. As discovered in part c , the conditional probability of selecting yellow socks in the second draw, given that yellow was selected in the first draw, is $\frac{1}{3}$, which is different from the probability of selecting yellow socks in the first draw.
e. In Rhonda's case, the outcome of the first selection does affect the outcome in the second draw because she does not replace the socks from the first draw and is facing a different set of socks when she selects her second pair. Thus Experiment 1 is not independent. But in Sally's case, the socks are replaced after the first selection, so the outcome in the second draw is not affected by the first draw. Thus Experiment 2 is independent.

## PROBLEM 2

We describe two experiments that each involve a two-step process. Determine whether each experiment is independent or not. Why?

Experiment 1: Sally opens a drawer with pairs of socks. There are 4 pairs of green, 3 pairs of purple, 2 pairs of red, and 1 pair of black socks. Without looking, she pulls out one pair of socks. If she doesn't like the color, she puts it back and pulls out another pair of socks. But Sally is not careful, so each time she puts the socks back, they get all mixed up.
a. What is the probability that the first two pairs of socks that Sally selects are both green?
b. Suppose the first pair of socks was green; what is the probability that the second pair will be green also?

Experiment 2: Rhonda wants to pack two pairs of socks for a trip. In her drawer, there are 4 pairs of yellow, 3 pairs of blue, 2 pairs of pink, and 1 pair of white socks. Without looking, she pulls out one pair of socks, puts it in her suitcase, and then selects another pair.
c. What is the probability of Rhonda pulling out two pairs of socks that are both yellow?
d. Suppose the first pair of socks was yellow; what is the probability that the second pair will be yellow also?
e. Compare Sally's and Rhonda's two-step processes. Does the outcome of the second step (the color of the second pair of socks selected) depend on the outcome of the first step? Explain.

## EXERCISES

1. a. 9 possible outcomes: \{(Tim walk, Diana bike), (Tim walk, Diana swim), (Tim walk, Diana surf), (Tim volleyball, Diana bike), (Tim volleyball, Diana swim), (Tim volleyball, Diana surf), (Tim shells, Diana bike), (Tim shells, Diana swim), (Tim shells, Diana surf)\}
b. 6 possible outcomes: \{walk, volleyball, collect shells, bike, swim, surf\}
2. $(4)(3)(2)=24$ possible outfits.

Label the shirts S1, S2, S3, and S4. Label the pants P1, P2, P3, and P4. Label the hats H1 and H2.
$\{(S 1, P 1, H 1),(S 1, P 1, H 2),(S 1, P 2, H 1),(S 1, P 2, H 2),(S 1, P 3, H 1),(S 1, P 3, H 2),(S 2, P 1, H 1),(S 2, P 1, H 2),(S 2$, P2, H1), (S2, P2, H2), (S2, P3, H1), (S2, P3, H2), (S3, P1, H1), (S3, P1, H2), (S3, P2, H1), (S3, P2, H2), (S3, P3, H1), (S3, P3, H2), (S4, P1, H1), (S4, P1, H2), (S4, P2, H1), (S4, P2, H2), (S4, P3, H1), (S4, P3, H2)\}
Emphasize to students that it is important to be systematic so they don't skip an event. You can also use a tree.
3. Make sure students leave enough room horizontally to expand the tree diagram. They might want to work with graph paper sideways. This would make a great class exercise, using the board or butcher paper.
a. $\quad(2)(2)=4 .\{(H, H),(H, T),(T, T),(T, H)\}$
b. $\quad(2)(2)(2)=8 .\{(H, H, H),(H, H, T),(H, T, H),(T, H, H),(T, T, H),(T, H, T),(H, T, T),(T, T, T)\}$
c. Four flips: $(2)(2)(2)(2)=16$. Five flips: $(2)(2)(2)(2)(2)=32$.
d. Each time the number of coins increases, the number of events doubles. Answers will vary. The pattern is $2^{n}$, where $n$ is the number of flips.
4. a. $(5)(5)=25$. Notice if students make a tree or a list or a square array. Which of these forms helps see the number of outcomes in the sample space?
b. The number of two-letter code words that starts with the letter $\mathrm{A}=(1)(5)=5$. Then $\mathrm{P}($ starts with A$)=\frac{5}{25}$ $=\frac{1}{5}$.
c. The number of two-letter code words that has neither a B nor a $\mathrm{C}=$ the number of two-letter code words based on $A, D$, and $E=(3)(3)=9$. The number of code words that has a $B$ or $C$ is the complement: $25-9$ $=16$. The probability that the code word has a B or $\mathrm{C}=\frac{16}{25}$.

## EXERCISES

1. Tim and Diana plan to spend a day at the beach. Tim wants to take a walk, play volley ball or collect shells. Diana wants to bike, swim or surf.
a. If they each choose one of their own activities, what are the possible outcomes for the day? Draw and label a tree diagram representing all the possible outcomes.
b. If Tim and Diana decide to spend the day together doing the same activity, what are the possible outcomes?
2. Buddy has 4 shirts, 3 pairs of pants and 2 hats, all color coordinated. How many different outfits can he wear? An outfit is different from another outfit if just one of the articles of clothing is different. Draw a tree for this sample space. Also list the sample space using symbols.
3. Imagine an experiment that involves flipping a coin several times.
a. Draw a tree for the experiment of flipping a coin two times. How many branches are there?
b. Use a different color pen or pencil to extend the tree to represent flipping the coin three times.
c. Extend it again to represent a four-flip experiment and then a five-flip experiment.
d. Describe the process of extending the tree to include one more flip. What is the pattern for the total number of branches at each stage?
4. Use the letters $A, B, C, D$ and $E$, to make code words.
a. How many of two-letter code words can you make?
b. If you pick one of these words at random, what is the probability that it starts with the letter A?
c. If you pick one of these words at random, what is the probability that it has a B or C?
5. The sample space contains $(2)(2)(2)(2)=16$ possible outcomes.
a. $\frac{1}{16}$
b. One head can occur in 4 ways. $\frac{4}{16}=\frac{1}{4}$.
c. $\frac{1}{16}$ This can only happen one way, with heads on the first flip. In part $b$, there are three other ways to get one head.
d. Set up the sample space to find 6 ways to get 2 heads. $\frac{6}{16}=\frac{3}{8}$.
e. $P($ at least $2 H)=1-P($ exactly $1 H)-P($ exactly $0 H)=1-\frac{4}{16}-\frac{1}{16}=\frac{11}{16}$ or $\frac{1}{16}+\frac{4}{16}+\frac{6}{16}=\frac{11}{16}$
6. 

a. $\quad P($ getting a red $)=P($ red $)=\frac{4}{10}$. Simple Event
b. $\quad P($ blue $)=\frac{6}{10}$. Simple Event
c. $P($ red and red $)=P($ red $) P($ red $)$ (Independent)
d. $P($ getting two reds without replacement $)=P($ red $) P($ red given first was red $)=\frac{4}{10}\left(\frac{3}{9}\right)=\frac{12}{90}=\frac{2}{15}$ (Dependent)
e. $P($ getting two blue without replacement $)=P($ blue $) P($ blue given the first was a blue $)=\frac{6}{10} \cdot\left(\frac{5}{9}\right)=\frac{30}{90}$ $=\frac{1}{3}$
(Dependent)
f. $\quad P($ getting a red then a blue without replacement $)=P($ red $) P($ blue given that the first was red $)=\frac{4}{10}\left(\frac{6}{9}\right)$ $=\frac{24}{90}=\frac{4}{15}$. (Dependent)
g. $\quad P($ getting one of each color without replacement $)=P($ red then blue or blue then red $)=\frac{4}{10}\left(\frac{6}{9}\right)+\frac{6}{10}($ $\left.\frac{4}{9}\right)=\frac{8}{15}$. (Dependent)
7. a. $\frac{13}{52}=\frac{1}{4}$
b. $\frac{13}{52}\left(\frac{13}{52}\right)=\frac{1}{16}$
c. $\frac{13}{52}\left(\frac{12}{51}\right)=\frac{1}{17}$
8. a. (4)(4)(4) $=64$. Some of these words are also English words: e.g. mat, ham, hat. Other words are not English: e.g. thm, hma, aht. Discuss the merits of the following methods: make a tree, a table (which can be tricky with more than two categories), or just use the rule of product.
b. The number of words that start with $\mathrm{m}=(1)(4)(4)=16$. The probability $=\frac{16}{64}=\frac{1}{4}$.
c. The number of words that do not have an $\mathrm{m}=(3)(3)(3)=27$. The probability $=\frac{27}{64}$.
d. This is the complement of the probability in part c. $1-\frac{27}{64}=\frac{37}{64}$.
5. In the experiment of flipping a coin four times, what is the probability of
a. getting no heads?
b. getting exactly one head?
c. getting \{HTTT\}? Why is this different from part b ?
d. getting exactly 2 heads?
e. getting at least 2 heads?
6. Suppose you have a bag with 4 red and 6 blue marbles. Determine if the compound events are independent or dependent.
a. If you randomly draw one marble, what is the probability of getting a red?
b. If you randomly draw one marble, what is the probability of getting a blue?
c. If you draw a marble, replace it and draw another marble, what is the probability of getting two reds?
d. If you draw two marbles, what is the probability of getting two red marbles?
e. If you draw two marbles, what is the probability of getting two blue marbles?
f. If you draw two marbles, what is the probability of getting a marble of each color?
7. John has a deck of cards.
a. If he draws a card, what is the probability that it is a heart?
b. If he draws a card, replaces it, then draws a card again, what is the probability that both cards are hearts?
c. If he draws 2 cards, what is the probability that both are hearts?
8. You have only the following words to work with: $\{m, a, t, h\}$ ?
a. How many 3 -letter code words can be made from the four letters?
b. What is the probability that the word starts with m ?
c. What is the probability that the word does not contain $m$ ?
d. What is the probability that the word contains at least one $m$ ?
9. a. 4
b. $(4)(4)=16$
c. $(4)(4)(4)=64$
d. $(4)(4)(4)(4)=256$
e. $(4)(4)(4)(4)(4)=1024$
f. The pattern is $4^{n}$, where $n$ is the length of the code word, so $4^{10}=1,048,576$
10. $6+10+26=42$.
11. (6)(6)(6) $=216$
12. Only 1 is neither even nor prime, so the probability is $\frac{5}{6}$. The probability of each event is $\frac{3}{6}$, but the events are not mutually exclusive because a 2 is both even and prime.
13. $(52)(26)=1352$
14. $\frac{4}{52}+\frac{12}{52}=\frac{16}{52}=\frac{4}{13}$

Ingenuity
15. a. The number of ways are there to answer the first question correctly but the other four incorrectly is (1)(3) (3)(3)(3). To answer only the second question correctly is (3)(1)(3)(3)(3)... and so on. So to answer exactly one question correctly is $(5)\left(3^{4}\right)=405$.
b. The number of ways to answer 0 correctly is $3^{5}=243$.
c. At least 2 correctly is the complement of exactly 1 or exactly 0 . We find the total possible test answers, $4^{5}$ $=1024$, and subtract from it $(405+243)=648$, the number of ways the answers are not correct at all or only one correct. 1024 - 648 is the number of ways that the test will have at least two correct answers. The probability of this is $\frac{(1024-648)}{1024}=\frac{376}{1024}=\frac{47}{128}$.
9. Given a four-letter alphabet, find the following:
a. How many one-letter code words are possible?
b. How many two-letter code words are possible?
c. How many three-letter code words are possible?
d. How many four-letter code words are possible?
e. How many five-letter code words are possible?
f. Use a pattern that you observe above to determine how many ten-letter code words are possible.
10. Think about the experiment of rolling a number cube or choosing a digit or choosing a letter from the alphabet. How many outcomes are possible?
11. Consider the experiment of rolling three number cubes: 1 green, 1 blue and 1 red. How many outcomes are there in the sample space?
12. Think about the experiment of rolling a number cube. What is the probability of getting an even number or a prime number when you roll the number cube? Is this answer different than you might expect? Explain your reasoning.
13. Kristen draws a card from a standard 52 -card deck and then selects a letter from the alphabet. How many outcomes are possible?
14. Bill draws one card from a standard 52 -card deck. How many ways can he draw a 10 or a face card? What is the probability of getting a ten or a face card?

## 15. Ingenuity:

A short quiz has five multiple-choice questions. Each question has four choices with one correct and the other three incorrect.
a. How many ways are there to answer the five questions?
b. How many ways are there to miss every question?
c. If you guess randomly for each question, what is the probability you will answer at least two correctly?

Investigation
16. Choosing students is like counting the number of outcomes from flipping a coin. You can think of H as meaning that the student was selected and T as meaning that the student was not selected.
a. 1
b. 4
c. 6
d. 4
e. 1
f. 0 heads: 1 way; 1 head: 4 ways; 2 heads: 6 ways; 3 heads: 4 ways; 4 heads: 1 way.

## 16. Investigation:

There are 4 students who volunteered to help their teacher on a project.
a. How many ways can the teacher choose 0 students to help?
b. How many ways can the teacher choose 1 student to help?
c. How many ways can the teacher choose 2 students to help?
d. How many ways can the teacher choose 3 students to help?
e. How many ways can the teacher choose 4 students to help?
f. How is the pattern in the answer above related to the sample space of flipping a coin 4 times?


Objective: This activity is a twist on the traditional gambling game of Craps. Students will practice betting techniques when rolling two die, and then, after several trials, strategize using their knowledge of probability and independent dependent events to try to make better bets.

## Materials:

Two standard 6-sided die
Crabs Board (preferably blown up into a poster or recreated on the floor)
Some sort of coin or chip to be used to place bets (integer chips work just fine)

## Directions:

Hand each of your students several coins or chips to be used to place bets. Place the Crabs board in some central location in your room. Read the following directions below aloud to the class. Then, ask your students to place their bets.
> "Today we are going to be playing a game called CRABS. The object of this game is to be the best better so that you can win a lot of chips. The bets will all be placed on the spaces on the CRABS board. Everyone can place up to 3 bets for each round. You can either place all 3 chips on one number, or you can place all three chips on different numbers. Once you have made your bets, I'm going to roll the die. After I've rolled, if the number on either of the die matches the number you bet - YOU WIN. Also, if the sum of the numbers on both die matches the number you bet - YOU WIN AGAIN. There may be more than one winner for each round that we play. If there is more than one winner, all winners will split the winning chips. Good luck. Here we go!"

Once all bets have been made, roll the dice to see who wins. Winner will take all of the chips. If there is more than one winner, all winners will split the chips equally. Continue playing this way for at least 10 rounds.

After 10 rounds, it's time to stop playing and start strategizing. Ask your students:

1) How were they choosing their bets? Were they just choosing randomly, or was there a reason they were betting a certain way?
2) Which number(s) do they think have the best chance of showing up on the die? Which have the least chance?
3) How could we figure out mathematically which number(s) have the best chance?

It is now time to lead your students into organizing their thoughts. Draw a diagram replicating the Crabs Board in a place for the whole class to see. Have the students help you fill in the diagram with the exact number of ways you could possibly get each of the numbers on the board. When the diagram is complete (as shown below), your students should be able to see exactly which number(s) have the best and least chance. Play a few more rounds of the game and you should see how their bets all change. It would be interesting to ask the students if they find the game more or less fun now with their new found knowledge.

| \# or Sum | Possible Outcomes | \# or Sum | Possible Outcomes |
| :---: | :---: | :---: | :---: |
| 1 | 11 | 7 | 6 |
| 2 | 12 | 8 | 5 |
| 3 | 13 | 9 | 4 |
| 4 | 14 | 10 | 3 |
| 5 | 15 | 11 | 2 |
| 6 | 16 | 12 | 1 |



| Number <br> Or <br> Sum | Place <br> Your <br> Bets | Number <br> Or <br> Sum | Place <br> Your <br> Bets |
| :---: | :---: | :---: | :---: |
| 1 |  | 7 |  |
| 2 |  | 8 |  |
| 3 |  | 9 |  |
| 4 |  | 10 |  |
| 5 |  | 11 |  |
| 6 |  | 12 |  |

1. Mean: 2; Median: 2; Mode: 1
a. $\quad 15$
b. $\quad 0,0,1,1,1,1,1,2,2,2,2,3,4,4,4$
c. 2
d. mode is 1
e. $\frac{28}{15}=1.86 \overline{66} \approx 1.9$


## REVIEW PROBLEMS

1. Mr. Johnson's class recorded the number of siblings each member of the class has in the bar graph below. Calculate the mean, median, and mode of the data.

a. How many students were in the class?
b. Make an ordered list of the data parts.
c. What is the median ?
d. What is the mode?
e. What is the mean?
f. Make a circle graph of this data
2. Kate went to a family reunion. During her time there, she recorded which of her family members had blue eyes, which had brown hair, and which had green eyes. She recorded the data as a Venn diagram.

3. 10 Brown Hair only, 10 Nothing, 23 Green Eyes only, 5 Green Eyes and Brown Hair, 15 Blue Eyes and Brown Hair

2.f. the number of family members that don't have green eyes, blue eyes, or brown hair. Total number at family reunion $=70$
4. 



Favorite Family Member
$P$ (Parent): $\frac{11}{30} ; P$ (Female): $\frac{17}{30}$
a. How many of Kate's family members have green eyes?
b. How many of her family members have brown hair? 30
c. How many have green eyes AND brown hair? 5
d. how many have neither green eyes nor brown hair? 7
e. How many of Kate's family members that have green eyes DON'T have brown hair? 23
f. What does the 10 on the outside of the circles represent? How many family members attended the reunion?
3. Mr. Greenstein asks his class who their favorite member of their immediate family is and records it in a pie chart. Graph this data as a bar graph. If you select a member of the class at random, what is the probability that a student's favorite family member is a parent? What is the probability that the favorite family member is a female? Is it easier to figure this out from the pie chart or the bar graph?

4. Mean: 195.75; Mode: \{156, 250\}; Median: 156
a. $\quad P($ mean $)=0$
b. $\quad P($ median $)=\frac{2}{24}=\frac{1}{12}$
c. $\quad P($ mode $)=\frac{4}{24}=\frac{1}{6}$
d. $P($ mean or median $)=0+\frac{1}{12}=\frac{1}{12}$
e. $\quad P($ median and mode $)=P(156)=\frac{2}{24}=\frac{1}{12}$
f. $\quad P($ mean, median, or mode $)=\frac{4}{24}=\frac{1}{6}$
5. Possible outcomes $=6 \cdot 6=36$

Ways to get sum of $5:(1,4),(2,3),(3,2),(4,1)=4$ so $P($ sum of 5$)=\frac{4}{36}=\frac{1}{9}$
Ways to get sum of $3:(1,3),(3,1)=2$ so $P($ sum of 3$)=\frac{2}{36}=\frac{1}{18}$
$P($ sum greater than 7$)=\frac{15}{36}$
6. Possible outcomes $=6 \cdot 6=36$

Ways to get product of $5:(1,5),(5,1)=2$ so $P($ product of 5$)=\frac{2}{36}=\frac{1}{18}$
Ways to get product of $12:(2,6),(3,4),(6,2),(4,3)=4$ so $P($ product of 12$)=\frac{4}{36}=\frac{1}{9}$
7. \{TTT, HTT, THT, TTH, HHT, HTH, THH, HHH\}

The number of possible outcomes $=2 \cdot 2 \cdot 2=8$
$P($ exactly one head $)=\frac{3}{8}$

| Day of the Year Born |  |  |  |
| :--- | :--- | :--- | :--- |
| Wilby | 335 | Jose | 301 |
| Bence | 12 | Jeffrey | 123 |
| Danette | 107 | R.J. | 260 |
| Terry | 156 | David | 74 |
| Trisha | 92 | Jacob | 103 |
| Kristen | 237 | Karen | 250 |
| Vanessa | 155 | Maja | 13 |
| Teri | 352 | Jennifer | 156 |
| Wesley | 274 | Tiffany | 293 |
| Donavion | 50 | Kaitlyn | 45 |
| Ben | 250 | Rhonda | 216 |
| Andrew | 43 | Michelle | 218 |

4. The table shown lists 24 people and the day of the year they were born. Determine the mean, median, mode and range of the data to determine the probability of the following:
a. mean of the data set
b. median of the data set
c. mode of the data set
d. the mean or the median of the data set
e. the median and the mode of the data set
f. the mean, the median, or the mode of the data set
5. If you roll two number cubes, what is the probability that the sum of the two numbers rolled is 5 ? What is the probability the sum is 3 ? What is the probability that the sum will be greater than 7 ?
6. If I roll two six-sided number cubes, what is the probability that the product of the two numbers rolled is 5 ? What is the probability the product is 12 ?
7. Write out the set of all the possible outcomes when you flip a coin three times. How do you know that you've listed them all? What is the probability of getting exactly one head?
8. 

$\begin{array}{ll}\text { a. } & \frac{4}{10}=\frac{2}{5} \\ \text { b. } & \frac{60}{10}\left(\frac{6}{10}\right)=\frac{36}{100}=\frac{9}{25} \\ \text { c. } & \frac{60}{10}\left(\frac{5}{9}\right)=\frac{1}{3}\end{array}$
9. Possible code words $=4 \cdot 4 \cdot 4=64$

Possible code words that start with $P=1 \cdot 4 \cdot 4=16$
$P($ starts with $P)=\frac{16}{64}=\frac{1}{4}$
8. Suppose you have a bag with 4 red marbles and 6 blue marbles.
a. If you pick a marble, what is the probability of getting a red marble?
b. If you pick a marble, replace it in the bag and then pick another marble, what is the probability of getting a blue marble on each pick?
c. If you pick 2 marbles (without replacement), what is the probability of getting 2 marbles?
9. Suppose we form 3 -letter code words from the alphabet $\{a, o, p, t\}$. How many of these code words are there? How many of these code words start with $p$ ? What is the probability that one of these code words starts with $p$ ?

Section 12.1: Area 22 with radius 7
Solution: The area of 22 in a circle of radius 7 represents $\frac{22}{49 \pi}=0.1429146 \ldots$, or $14.29146 \%$. The area of 2263 in a circle of radius 71 represents $\frac{2263}{5041 \pi}=0.1428953 \ldots$, or $14.28953 \%$, just $0.00193 \%$ less.

## Section 12.2: 5

Solution: Ideally six students will pick 1 and the other five will pick 10 (or vice versa), making the median 1 and the mean $\frac{56}{11}=5 \frac{1}{11}$. The difference is $4 \frac{1}{11}$, and the least integer greater than or equal to that is 5 .

Section 12.3: $\frac{7}{8}$
Solution: Player A always has a $\frac{1}{8}$ chance of winning on the first 3 flips. If player B chooses THH , then player A cannot win after the third flip, so player B wins with probability $\frac{7}{8}$.

Section 12.4: $\frac{219}{256}$
Solution: We can compute the probability of getting 6,7 , or 8 heads, then subtract from 1 . There is only 1 way to get 8 heads, namely HHHHHHHH. There are 8 ways to get 7 heads since there are 8 possible places for the T. There are $\frac{8 \cdot 7}{2}=28$ ways to get 6 heads since there are 28 possible places for two T's. There are $2^{8}=256$ possible outcomes, so the probability of getting more than 5 heads is $\frac{1+8+28}{256}=\frac{37}{256}$, thus the probability of getting at most 5 heads is $1-\frac{37}{256}=\frac{219}{256}$.

## CHALLENGE PROBLEMS

## Section 12.1:

Which represents a larger percentage: a slice of area 22 in a pie chart of radius 7 or a slice of area 2263 in a pie chart of radius 71 ?

## Section 12.2:

A class of 11 students was given the following extra credit question on a test:
Pick a positive integer between 1 and 10 , inclusive: $\qquad$
(The least integer greater than or equal to the nonnegative difference between the mean and the median of the answers given will be added to everyone's score.)

What was the maximum bonus that the class could have earned?

## Section 12.3:

Two bored statisticians play a coin-flipping game as follows: Each player chooses a sequence of 3 heads or tails, then a fair coin is flipped until one of the sequences arises in consecutive flips, making that player the winner. If player A chooses HHH and player B chooses optimally (he gets to see player A's choice first, and hence cannot choose HHH ), then what is the probability that player B wins?

## Section 12.4:

A fair coin is tossed 8 times. What's the probability of getting at most 5 heads?

## Section 13.1 - Measures of Central Tendency

Big Idea:

Key Objectives:
-
-
-
-
-

Materials:

Pedagogical/Orchestration:

Activity:

Vocabulary:

TEKS:

WARM-UPS for Section 12.1 (Measures of Central Tendency)

Launch for Section 12.1:

## Section 13.1 - Measures of Central Tendency

Big Idea:

Key Objectives:
-
-
-
-
-

Materials:

Pedagogical/Orchestration:

Activity:

Vocabulary:

TEKS:

WARM-UPS for Section 12.1 (Measures of Central Tendency)

Launch for Section 12.1:

# PERSONAL FINANCE 

13

## SECTION 13.1 SIMPLE AND COMPOUND INTEREST

In earlier chapters, you learned about percentages and percent discounts and explored different kinds of taxes. In this chapter, after a review of these basic ideas, you will begin to put all of the pieces together. The goal is to see how mathematics provides an essential tool for managing money. Using mathematics, you will learn to set up a system to organize your finances and make good financial decisions.

Begin by reviewing simple interest, then learn how banks extend the idea. First, what does it mean to invest a principal amount of money $P$ in a bank at an interest rate $r$ ? Look at the following example:

## EXAMPLE 1

Bobby invests $\$ 100$ at a simple interest rate of $6 \%$. How much money will he have after two months?

## SOLUTION

The amount of money in Bobby's account after two months is the original principal $P=\$ 100$, plus the interest earned, I . The $6 \%$ interest of any principal is paid after one year. In two months, Bobby will earn $\frac{2}{12}$ of the yearly interest. The interest he earns is

$$
I=\$ 100 \cdot(0.06) \cdot\left(\frac{2}{12}\right)=100 \cdot \frac{0.12}{12}=100 \cdot 0.01=1
$$

The amount in Bobby's account after two months is the original principal plus the interest earned, or $100+1=\$ 101$.

The simple interest earned is $I=P \bullet r \bullet t$ or $I=$ Prt. Explain why.

## SIMPLE INTEREST FORMULA

When a principal amount of money $P$ is invested at an interest rate in decimal form $r$ for $t$ years, the simple interest earned will be $I=$ Prt.

If $t$ is less than one year, the fractional part of the year represents $t$, as above.
To compute the amount in an interest-bearing account at the end of $t$ years, called $A$, combine the original principal amount $P$ with the interest made to get the formula: $\quad A=P+I=P+\operatorname{Prt}=P(1+r t)$

In the shortened form, to compute the amount $A$ in an interest-bearing account after $t$ years, using the inverse of the distributive property, multiply the original amount by ( $1+r t$ ).

## EXAMPLE 2

Maggie invests $\$ 500$ in an account that earns $10 \%$ interest. How much will be in her account after 6 months?

## SOLUTION

There are two ways to solve the problem:
A. First find the interest earned in six months. To do this, convert the time to years: $t=6$ months $=6$ months $\bullet\left(\frac{1 \text { year }}{12 \text { months }}\right)=\frac{6}{12}$ year $=\frac{1}{2}$ year. The interest earned is $I=\operatorname{Prt}=(500)(0.10)\left(\frac{1}{2}\right)=25$. So the amount after six months will be $\$ 500+\$ 50=\$ 550$.
B. Alternatively, to compute the amount:
$A=P(1+r t)=500\left(1+0.10 \cdot \frac{1}{2}\right)=\$ 500(1.05)=\$ 550$.
Which way do you prefer?
Do banks usually use simple interest? The answer is no. Bank customers use them to build their savings or to get a loan. In both cases, banks use compound interest. Generally, they add or charge interest to an account at least four times a year, or quarterly, sometimes more often. Banks compute interest using a

## compound interest formula.

To do this, the banks divide the year into a number of periods $m$ per year. If they are computing interest quarterly, $m=4$. If they are computing interest daily, $m=365$. If they are computing interest monthly, $m=12$. Each period, they adjust accounts or loans by adding the simple interest earned during the period. When the next period begins, the account or loan has more money than it did at the beginning of the given period. So in the next period, the account or loan will earn or be charged interest on both the original principal as well as any interest earned or charged the previous period.

This sounds complicated, but actually the idea is simple.
Step 1: At the beginning of a period, $P$ is the amount of money in an account. How much money will be in a bank account at the end of the period?

The amount at the end of the period will be $P+I$, where I is the interest earned during the period. So what is I? In order to compute $I$, the bank uses two numbers:

1. The interest rate r .
2. The time period $t$. If there are $m$ periods per year, each period will be $\left(\frac{1}{\mathrm{~m}}\right)$ of a year. First, compute the interest earned during one period:

$$
I=\operatorname{Pr}\left(\frac{1}{\mathrm{~m}}\right)=P \bullet\left(\frac{\mathrm{r}}{\mathrm{~m}}\right)
$$

For short, use $I$ for the amount of interest and let $i=\frac{r}{m}$. Then $I=P i$, so the amount at the end of the period is $P+I=P+P i=P(1+i)$.

To find the amount at the end of a period, multiply the amount at the beginning of the period by $(1+i)$.

At the end of each period, multiply the amount at the beginning of the period by the factor $(1+i)$.

| \# of Period | Amount in account at the beginning of period | Amount in account at end of period |
| :---: | :---: | :---: |
| 1 | $P$ | $P(1+i)$ |
| 2 | $P(1+i)$ | $P(1+i)(1+i)$ |
| 3 | $P(1+i)(1+i)$ | $P(1+i)(1+i)(1+i)$ |
| .......... | ............. | .............. |
| $n$ | $P(1+i)^{n-1}$ | $P(1+i)^{n}$ |

The amount at the end of period 1 is equal to the amount at the beginning of period 2. Each time a period passes, the amount at the beginning of the period is multiplied by the factor $(1+i)$. Using exponential notation yields the compound interest formula:

## COMPOUND INTEREST FORMULA

If $P$ is the principal amount, $r$ is the interest rate, and $m$ is the number of times the interest is compounded per year, then the amount $A$ after $n$ periods is
$A=P(1+i)^{n}$

## EXAMPLE 3

A $\$ 500$ amount is invested at $10 \%$ rate of interest compounded monthly. How much will be in the account after two years?

## SOLUTION

Use the compound interest formula, and find

$$
P=500 \quad r=10 \%=0.10 \quad m=12 \quad \text { so } \quad i=\frac{r}{m}=\frac{0.10}{12}
$$

The next question is how many periods are there in two years? Because each period is one month, there will be 12 periods in a year, or 24 periods in two years. So $n=24$.

Substitute the values into the compound interest formula.
$A=500\left(1+\frac{0.10}{12}\right)^{24}$. Computing this longhand is more than a nightmare. Using a calculator, the amount of money in the account after two years is $A=$ $610.195=\$ 610.20$.

## EXAMPLE 4

Suppose $\$ 500$ is invested in a bank at $10 \%$ simple interest rate. How much will be in the account after two years? Compare the earnings from a simple interest rate of $10 \%$ with a compound interest rate of $10 \%$ compounded monthly from Example 3.

## SOLUTION

Use the simple interest formula, $A=P(1+r t)$

$$
A=500(1+0.10 \bullet 2)=\$ 600 .
$$

By compounding monthly, a customer will earn $\$ 10.20$ more with compound interest than with simple interest. Practice using simple interest and compound interest formulas in the problems below.

## EXERCISES

1. Sam invests $\$ 100$ in the bank at a simple interest rate of $8 \%$. How much will be in his account after
a. 2 months?
b. 6 months?
c. 1 year?
d. $\quad 2$ years?
e. 5 years?
2. How much money must be invested at a simple interest rate of $12 \%$ to have $\$ 1000$ at the end of seven years?
3. Sue invests $\$ 500$ at a simple interest rate of $12 \%$. How much interest will she earn after
a. 2 months?
b. 6 months?
c. 1 year?
d. $\quad 2$ years?
4. A $\$ 100$ amount is invested at an interest rate of $8 \%$ compounded monthly. How much will be in the account after
a. 2 months?
b. 6 months?
c. 1 year?
d. 2 years?
e. 5 years?
5. How much money must be invested at an interest rate of $12 \%$ compounded monthly to have $\$ 10,00$ at the end of six years?
6. If interest is compounded monthly, how many periods are there in
a. 3 months?
b. 1 year?
c. 5 years?
7. Compare the difference in simple interest from money invested at $8 \%$ and interest compounded monthly at $8 \%$ after five years.
8. Explain to a fifth grader the difference between simple interest and compound interest.

## 9. Exploration:

How long will it take money to double at a compound interest rate of $12 \%$ compounded monthly? At 8 \%? Research and explain the Banker's Rule of 72.

## Section 13.2-Making Up a Personal Budget

Big Ideas:

Key Objectives:
-
Materials:
-
Pedagogical/Orchestration:
-

Activity:

Vocabulary:

TEKS:

WARM-UPS for Section 11.2 (Angles in a Triangle)

Launch for Section 11.2:

## SECTION 13.2 MAKING UP A PERSONAL BUDGET

In order to keep a good credit history, it's necessary to pay all bills on time. This can be a challenge because often people do not have enough money to buy everything that they want. And even if people have enough money at a given time, they might need money to pay future expenses, like a house or college. For example, on payday Frank receives a monthly check of $\$ 2500$. Does he have enough money to buy a $\$ 500$ I-Pad? To know this, Frank must look at his upcoming expenses and develop a plan for how he will use his income so that he doesn't run out of money he needs for essentials, like food.

To answer any financial question, it is necessary to examine the parts of a family budget. First, start with income, both from wages and savings. Family members might also have other sources of income, for example presents from a birthday or miscellaneous income from a part-time job. Ask your parents if they can think of other income sources.

Next, consider the expenses that a family needs to plan for. First, there are probably mortgage payments for a house or monthly rent. Second, budget for food. Next, might be car expenses, like monthly car payments, car and house insurance, emergency funds, savings for retirement, and local, state, and federal taxes. Your parents might also be saving money to send you to college, as well as for a vacation.

Some of these expenses are fixed expenses, they do not change each month. For example, rent is a fixed monthly expense. Other expenses are variable. A variable expense is the grocery bill. To figure variable expenses, just estimate the average amount figured from previous expenses. If the January grocery bill totaled $\$ 120$, $\$ 150$ in February, and $\$ 60$ in March, then a good estimate of average monthly food expenses is

$$
\frac{(120+150+60)}{3}=\$ 110
$$

## EXAMPLE 1

Sally made a list of her expenses and income. Her table is given below:

| Monthly Income: |  |
| :--- | ---: |
| Wages from job | 2500 |
| Interest on Savings account | 25 |
| Weekend typing - self-employment income | 200 |
| TOTAL | 2725 |


| Monthly Expenses |  |
| :--- | ---: |
| Rent | 750 |
| Food | 500 |
| Car insurance | 120 |
| Car payment | 300 |
| House insurance | 85 |
| Clothes allowance | 100 |
| Savings for college | 100 |
| Savings for emergencies | 50 |
| Savings for trips | 100 |
| Savings for taxes | 200 |
| Savings for retirement | 100 |
| TV Cable and Phone | 100 |
| Cell Phone | 50 |
| Savings for car repairs | 75 |
| Gas | 20 |
| Savings for presents for family and <br> friends |  |

## EXAMPLE 2

a. What are her total monthly expenses?
b. What percentage is each category of the total budget for expenses? Use a pie chart to get a visual understanding of her expenses.

## SOLUTION

a. To find her total expenses, add up all of the expenses above.
b. In order to determine the percentage of each category, divide the expenses in that category by the total expenses. For example, the percentage of expenses for clothes is (100/2680) $=0.0371=3.7 \%$
When a person makes up a budget, the expenses will depend on the local situation. For example, if the person lives in a small city, the rent might be less than in a larger city. The family expenses also depend on several variables, like the number and age of the children, whether they need to go to day care, educational expenses if they are in school, and extracurricular expenses if they play music or sports.

## PROBLEM 1

Sally family includes her father, mother, and two brothers aged 9 and 12 . She lives in San Marcos, a small town in central Texas.
a. Create a family budget, and estimate the minimum household budget to meet her family's basic needs.
b. Estimate the average hourly wage that Sally's parents need to make, working 40 hours per week, to meet basic needs.
Sally has a friend, Victoria, who lives in San Antonio, a larger town south of San Marcos. She also has five members in her family: her father, mother, and two sisters aged 5 and 15.
a. Create a family budget, and estimate the minimum household budget to meet her family's basic needs.
b. Estimate the average hourly wage that Victoria's parents need to make, working 40 hours per week, to meet basic needs.

Now that you have an idea what a budget might look like, it's time to look at the overall financial situation. To do this, look at all assets and liabilities. An asset is something owned, like a house. A liability is money owed, like a mortgage.

If a person's house is worth $\$ 150,000$ then she has an asset of $\$ 150,000$. However, if she still owes $\$ 90,000$ to the bank on the house, then her house is also a liability of $\$ 90,000$. The net worth of the house, $\$ 150,000-\$ 90,000$, is $\$ 60,000$. Although the house is a $\$ 150,000$ asset, the person really owns only $\$ 60,000$ of the value of the house. The remaining $\$ 90,000$ is considered a liability.

## EXAMPLE 3

Sally from Example 1 made the following list of all of her assets and liabilities.

| Assets: |  |
| :--- | ---: |
| Car | $\$ 18,500$ |
| Clothes | $\$ 800$ |
| Television | $\$ 150$ |
| Piano | $\$ 900$ |
| Savings account | $\$ 1,200$ |
| Savings for College | $\$ 4,000$ |
| Retirement account | $\$ 25,300$ |
| Total Assets: |  |


| Liabilities |  |
| :--- | :--- |
| Outstanding car loan | $\$ 14,300$ |
| Credit Card Balance | $\$ 2,540$ |
| Total Liabilities |  |

Organize the above items into a spreadsheet, like Excel or a graphing calculator. Use the spreadsheet's formulas to compute Sally's net worth.

$$
\text { Net Worth = Total Assets }- \text { Total Liabilities }
$$

## EXERCISES

1. Make a list of items that might be part of a personal budget. List at least three possible sources of income and ten different expenses that you need to be budgeted.
2. Total the income and expenses. Calculate the percentage of the total budget in each category.
3. List ten examples of assets a family might own, and choose a value for each.
4. Use these to create a table of assets. Then group appropriate assets into categories, like savings.
5. List ten examples of liabilities a family might owe, and estimate a reasonable value for each.
6. Use these to create a table of liabilities. Group appropriate liabilities into an overall category, like credit card debt if there are several types of credit cards.
7. Create a net worth statement by combining the two tables above. How much is the family worth in this example?
8. Estimate the average hourly wage, assuming a 40-hour work week, needed for the family to meet its basic needs, using exercises 1-7.
9. Suppose the family moves to another city. Explain how this might affect the budget
10. Develop your own sample budget for your city. Make another budget for a large nearby city. Compare the average hourly wage needed for your family to meet its basic needs in each city.

## Section 13.3 - Taxes

Big Ideas:

## Key Objectives:

- 

Materials:
-
Pedagogical/Orchestration:
-

Activity:

Vocabulary:

TEKS:

NOTE TO TEACHER: Make sure students have completed their 8-10 triangles in previous day's homework in order to have a successful launch.

WARM-UPS for Section 11.2 (Angles in a Triangle)

Launch for Section 11.2:

## SECTION 13.3 TAXES

Taxes are financial charges made by a governing body such as a city, state, or federal government on an individual or property. One example is the sales tax on items that you purchase. Another example is the income tax on money that you earn.

1. What are some reasons governments charge a tax?
2. How are the tax revenues used?
3. What are some differences between a sales tax and an income tax?
4. Determine what is the sales tax rate in your city. Are all sales tax rates the same in your state? Find two other tax rates. Are the sales tax rates the same in Illinois? Compare the tax rate in Chicago with your city. How much would a shirt costing $\$ 29.99$ cost with tax in Chicago. How much would the same shirt cost in your city?
5. Notice that the income tax is a tax on what you earn while a sales tax is a tax on what you buy. Investigate the income tax rates for a wage earner. You may need to consult the Internal Revenue Office Tax Table at : www.irs.gov/pub/irs-pdf/i1040tt.pdf
Here is the expected 2012 Federal income tax brackets according to the Forbes website:
http://www.forbes.com/sites/moneybuilder/2011/09/30/2012-federal-income-tax-brackets-irs-tax-rates/

| Tax Bracket | Married Filing Jointly | Single |
| :--- | :--- | :--- |
| $10 \%$ Bracket | $\$ 0-\$ 17,400$ | $\$ 0-\$ 8,700$ |
| $15 \%$ Bracket | $\$ 17,400-\$ 70,700$ | $\$ 8,700-\$ 35,350$ |
| $25 \%$ Bracket | $\$ 70,700-\$ 142,700$ | $\$ 35,350-\$ 85,650$ |
| $28 \%$ Bracket | $\$ 142,700-\$ 217,450$ | $\$ 85,650-\$ 178,650$ |
| $33 \%$ Bracket | $\$ 217,450-\$ 388,350$ | $\$ 178,650-\$ 388,350$ |
| $35 \%$ Bracket | Over $\$ 388,350$ | Over $\$ 388,350$ |

a. Suppose you are a single worker and earned $\$ 34,500$ in 2012. Accord--ing to the information in the above table, how much can you expect to pay in taxes?
b. Suppose you are a married worker and earned \$73,000 in 2012. According to the information in the above table, how much can you expect to pay in taxes, if you file jointly with your partner?
c. Your coworker is single and earned the same amount of $\$ 73,000$ in 2012. According to the information in the above table, how much can she expect to pay in taxes?

Find a website, for example, http://www.irs.gov/pub/irs-pdf/i1040tt.pdf, you will find two ways that explains how to calculate taxes. Notice that tax filers can either use a tax table or a tax rate schedule. For either method, the first step involves deciding which category the filer is in: single, married filing jointly, married filing separately, or head of a household. The other component is the filer's net income.

## EXAMPLE 1

Susan has a net income of $\$ 69,500$ and is single. Sarah and Bob are married and have a joint income of $\$ 69,500$. Compute the tax they owe with a tax table or with a tax rate schedule. Which method is easier?

## SOLUTION 1

Use a tax table. Find the row of the tax table that includes $\$ 69,500$. Then locate the correct column to find the taxes owed. The table looks like:

| If line 43 is- |  | And you are - |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- |
| At least | But less <br> than | Single | Married <br> filing <br> jointly | Married filing <br> separately | Head of <br> Household |
| $\$ 69,000$ |  |  |  |  |  |
| 69,000 | 69,050 | 13,286 | 9,484 | 13,286 | 11,901 |
| 69,050 | 69,100 | 13,299 | 9,491 | 13,299 | 11,914 |
| 69,100 | 69,150 | 13,311 | 9,499 | 13,311 | 11,926 |
| 69,150 | 69,200 | 13,324 | 9,506 | 13,324 | 11,939 |
| 69,200 | 69,250 | 13,336 | 9,514 | 13,336 | 11,951 |


| If line 43 is- |  | And you are - |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- |
| At least | But less <br> than | Single | Married <br> filing <br> jointly | Married filing <br> separately | Head of <br> Household |
| 69,250 | 69,300 | 13,349 | 9,521 | 13,349 | 11,964 |
| 69,300 | 69,350 | 13,361 | 9,529 | 13,361 | 11,976 |
| 69,350 | 69,400 | 13,374 | 9,536 | 13,374 | 11,989 |
| 69,400 | 69,450 | 13,386 | 9,544 | 13,386 | 12,001 |
| 69,450 | 69,500 | 13,399 | 9,551 | 13,399 | 12,014 |
| 69,500 | 69,550 | 13,411 | 9,559 | 13,411 | 12,026 |
| 69,550 | 69,600 | 13,424 | 9,566 | 13,424 | 12,039 |
| 69,600 | 69,650 | 13,436 | 9,574 | 13,436 | 12,051 |
| 69,650 | 69,700 | 13,449 | 9,581 | 13,449 | 12,064 |
| 69,700 | 69,750 | 13,461 | 9,589 | 13,461 | 12,076 |
| 69,750 | 69,800 | 13,474 | 9,596 | 13,474 | 12,089 |
| 69,800 | 69,850 | 13,486 | 9,604 | 13,486 | 12,101 |
| 69,850 | 69,900 | 13,499 | 9,611 | 13,499 | 12,114 |
| 69,900 | 69,950 | 13,511 | 9,619 | 13,511 | 12,126 |
| 69,950 | 70,000 | 13,524 | 9,626 | 13,524 | 12,139 |

From this table and using the first column, Single, Susan should pay $\$ 13,411$ in federal income taxes. Sarah and Bob use the married column to find that they owe \$9,559.

Why do you think that Sarah and Bob have less to pay in taxes? The main reason is that married people together generally have more expenses than a single person, so they need more after-tax income so they're charged a little less.

## EXAMPLE 2

Using Schedule $X$ and Schedule Y, explain how 1) Susan and 2) Sarah and Bob figure their taxes.

## SOLUTION 2

Use a tax rate schedule. Susan will use Schedule $X$ for filing status Singe, and Sarah and Bob will use Schedule Y-1.

| Schedule $\mathbf{X}$ |  |  |  |
| :--- | ---: | :--- | :--- |
| If your taxable <br> income is: |  | The tax is: |  |
| Over - | But not over - |  | of the amount <br> over - |
| $\$ 0$ | $\$ 8,700$ | $\ldots \ldots \ldots . . . . . . . . .10 \%$ | $\$ 0$ |
| 8,700 | 35,350 | $\$ 870.00+15 \%$ | 8,700 |
| 35,350 | 85,650 | $4,867.50+50 \%$ | 35,350 |
| 85,650 | 178,650 | $17,442.50+28 \%$ | 85,650 |
| 178,650 | 388,350 | $43,482.50+33 \%$ | 178,650 |
| 388,350 | $\ldots \ldots . . . . . . . . . . . . ~$ | $112,683.50+50 \%$ | 388,350 |


| Schedule Y-1 |  |  |  |
| :---: | :---: | :---: | :---: |
| If your taxable income is: |  | The tax is: |  |
| Over - | But not over - |  | of the amount <br> over - |
| \$0 | 17,400 | ............... 10\% | \$0 |
| 17,400 | 70,700 | \$1,740.00 + 15\% | 17,400 |
| 70,700 | 142,700 | 9,735.00 + 50\% | 70,700 |
| 142,700 | 217,450 | $27,735.00+28 \%$ | 142,700 |
| 217,450 | 388,350 | $48,665.00+33 \%$ | 217,450 |
| 388,350 | ................... | 105,062.00 + 35\% | 388,350 |

There are different income brackets in each schedule.
First compute Susan's taxes with Schedule $Y$.

In the 10\% bracket, for an income up to 8,700, 10\% of net income.
In the $15 \%$ bracket, pay $10 \%$ of the first $\$ 8,700$ of net income, then $15 \%$ of the amount above $\$ 8,700$, but less than $\$ 35,350$.
In the $25 \%$ bracket, pay the same taxes as the lower bracket on net income to $\$ 35,350$, and then $25 \%$ of the income between $\$ 35,350$ and $\$ 85,650$.

Using Schedule Y, Susan will owe $\$ 4,867.50+25 \%$ of the amount between $\$ 35,350$ and $\$ 69,500$. The amount between $\$ 69,500$ and $\$ 35,350$ is
\$69,500 - \$35,350 = \$34,150.

Compute 25\% of the amount to obtain

$$
(0.25)(\$ 34,150)=\$ 6,537.50 .
$$

Then add the original $\$ 4,867.50$ to obtain $\$ 13,405$, almost exactly the same amount as the tax table amount of $\$ 13,411$.

Now compute Sarah and Bob's joint taxes using Schedule Y-1.
In the 10\% bracket, for an income up to 17,400, 10\% of net earned income.
In the $15 \%$ bracket, pay $10 \%$ of the first $\$ 17,400$, then $15 \%$ of the amount above $\$ 17,400$, but less than $\$ 70,400$. The amount that Sarah and Bob will pay $15 \%$ on is $\$ 69,500-\$ 17,400=\$ 52,100$. Together the tax bill is $\$ 1,740+(0.15)$ $(\$ 52,100)=\$ 9,555$. Again, this is almost exactly the same as the tax table result of $\$ 9,559$.

## EXERCISES

1. John earned $\$ 69,900$ in 2012 . How much did he owe, using the tax table, if he was single?
2. Compute the amount of tax John owed using Schedule Y. Show your work.
3. Sally and Bob filed a joint return and earned a net income of \$69,900 in 2012. How much did they owe using the tax table?
4. How much did they owe using Schedule Y-1? Document your work.
5. If you earned $\$ 100,000$ in 2012, what tax bracket will you be in? Using Schedule Y-1, how much did you owe in federal income taxes?
6. If you earned $\$ 100,000$, would you pay $25 \%$ taxes on all of you net income? Explain.
7. Make a list of items that might be part of a personal budget. List at least three possible sources of income and ten different expenses that you need to be budgeted.
8. Total the income and expenses. Calculate the percentage of the total budget in each category.
9. List ten examples of assets a family might own, and choose a value for each.
10. Use these to create a table of assets. Then group appropriate assets into categories, like savings.
11. List ten examples of liabilities a family might owe, and estimate a reasonable value for each.
12. Use these to create a table of liabilities. Group appropriate liabilities into an overall category, like credit card debt if there are several types of credit cards.
13. Create a net worth statement by combining the two tables above. How much is the family worth in this example?
14. Estimate the average hourly wage, assuming a 40-hour work week, needed for the family to meet its basic needs, using exercises 1-7.
15. Suppose the family moves to another city. Explain how this might affect the budget

# MATH EXPLORATIONS Glossary of Terms 

Absolute value - 1 . The absolute value of a number is its distance from zero.

$$
-2 \text {. For any } x,|x| \text { is defined as follows: }|x|=x \text {, if } x \geq 0
$$

$|x|=-x$, if $x<0$

Acute angle - An angle whose measure is greater than 0 degrees and less than 90 degrees.

Acute triangle - A triangle in which all three angles are acute angles.

Altitude of a triangle - A segment drawn from a vertex of the triangle perpendicular to the opposite side of the triangle, called the base, (or perpendicular to an extension of the base).

Angle - An angle is formed when two rays share a common vertex.

Area model - A mathematical model based on the area of a rectangle, used to represent multiplication or to represent fractional parts of a whole.

Arithmetic sequence - A sequence $a_{1}, a_{2}, a_{3}, a_{4}, \ldots$ is an arithmetic sequence if there is a number $c$ such that for each $n, a_{n+1}=a_{n}+c$, that is $a_{n+1}-a_{n}=c$.

Attribute - A distinguishing characteristic of an object such as angles or sides of a triangle.

Axis - A number line in a plane. Plural form is axes. Also see: Coordinate Plane

# MATH EXPLORATIONS Glossary of Terms 

Bar graph - A graph in which rectangular bars, either vertical or horizontal, are used to display data.

Base -1 . Any number $x$ is raised to the $n$th power, written as $x^{n}, x$ is called the base of the expression; 2. Any side of a triangle ; 3. Either of the parallel sides of a trapezoid. 4. Either of the parallel sides of a parallelogram.

Box and Whisker Plot - For data ordered smallest to largest the median, lower quartile and upper quartile are found and displayed in a box along a number line. Whiskers are added to the right and left and extended to the least and greatest values of the data.

## Cartesian coordinate system - See: Coordinate Plane

Center of a circle - A point in the interior of the circle that is equidistant from all points of the circle.

Chord - A segment whose endpoints are points of a circle.

Circle - The set of points in a plane equidistant from a point in the plane..

Circumference - The distance around a circle. Its length is the product of the diameter of the circle and pi.

# MATH EXPLORATIONS Glossary of Terms 

Coefficient - In the product of a constant and a variable the constant is the numerical coefficient of the variable and is frequently referred to simply as the coefficient.

Common Denominator - A common multiple of the denominators of two or more fractions. Also see: Least Common Denominator

Common Factor - A factor that two or more integers have in common. Also see: Greatest Common Factor.

## Common Multiple - See: Least Common Multiple.

Complement - The complement of a set is a set of all the elements of the universal set that are not in the given set.

Complementary Angles - Two angles are complementary if the sum of their measures totals $90^{\circ}$.

Composite number - A prime number is an integer $p$ greater than 1 with exactly two positive factors: 1 and $p$. A composite number is an integer greater than 1 that has more than two positive factors. The number 1 is neight a prime nor a composite number.

Concentric circles - Circles with the same center and in the same plane that have different radii.

# MATH EXPLORATIONS Glossary of Terms 

Cone - A three-dimensional figure with a circular base joined to a point called the apex.

Congruent - Used to refer to angles or sides having the same measure and to polygons that have the same shape and size.

Conjecture - An assumption that is thought to be true based on observations.

Constant - a fixed value.

Constant of Proportionality- If a function has a rule in the form $y=K x$, then for any input $x \neq 0$, the quotient of $\frac{y}{x}$ will always have the value $K$. The number $K$ is called constant of proportionality.

Coordinate(s) - A number assigned to each point on the number line which shows its position or location on the line. In a coordinate plane that ordered pair, $(x, y)$, assigned to each point of the plane showing its position in relation to the $x$-axis and $y$-axis.

Coordinate Plane - A plane that consists of a horizontal and vertical number line, intersecting at right angles at their origins. The number lines, called axes, divide the plane into four quadrants. The quadrants are numbered I, II, III, and IV beginning in the upper right quadrant and moving counterclockwise.

# MATH EXPLORATIONS <br> Glossary of Terms 

Corresponding Angles - 1. If two lines are cut by a transversal the angles on the same side of the transversal and on the same side of the two lines are corresponding angles. If the lines are parallel the pairs of angles will have equal measure. 2. If two polygons are similar the angles that are int he same relative position in the figures are corresponding angles and have equal measures.

Corresponding Sides - If two polygons are similar the sides of the polygons in the same relative positions are corresponding sides and the ratio of the lengths of each pair is the same.

Counterclockwise - A circular movement opposite to the direction of the movement of the hands of a clock.

Counting numbers - The counting numbers are the numbers in the following never-ending sequence: $1,2,3,4,5,6,7 \ldots$ We can also write this as $+1,+2$, $+3,+4,+5,+6,+7, \ldots$ These numbers are also called the positive integers or natural numbers.

Cube - 1. A three-dimensional shape having six congruent square faces. 2. The third power of a number.

Cylinder - A three-dimensional figure with parallel circular bases of equal size joined by a lateral surface whose net is a rectangle.

Data - A collection of information, frequently in the form of numbers.

# MATH EXPLORATIONS <br> Glossary of Terms 

Data analysis - The process of making sense of collected data.

Degree - 1. The circumference of a circle is divided into 360 equal parts or arcs. Radii drawn to both ends of the arc form an angle of 1 degree. 2. The degree of a term is the sum of the exponents of the variables. The degree of a polynomial is the highest degree of any of its terms.

Denominator - The denominator of a fraction indicates into hoiw many equal parts the whole is divided. The denominator appears beneath the fraction bar.

Diameter -A segment with endpoints on the circle that passes through its center.

Distance - For any two numbers $x$ and $y$, the distance between $x$ and $y$ is the absolute value of their difference; that is, Distance $=|x-y|$.

Dividend - The quantity that is to be divided.

Divisibility - Suppose that $n$ and $d$ are integers, and that $d$ is not 0 . The number $n$ is divisible by $d$ if there is an integer $q$ such that $n=d q$. Equivalently, $d$ is a factor of $n$ or $n$ is a multiple of $d$.

Division Algorithm - Given two positive integers $a$ and $b$, we can always find unique integers $q$ and $r$ such that $a=b q+r$ and $0 \leq r<b$. We call $a$ the dividend, $b$ the divisor, $q$ the quotient, and $r$ the remainder.

# MATH EXPLORATIONS Glossary of Terms 

Divisor - The quantity by which the dividend is divided.

Domain - The set of input values in a function.

Edge - A segment that joins consecutive vertices of a polygon or a polyhedron.

Elements - Members of a set.

Empirical probability - Probability determined by real data collected from real experiments.

Empty set - Also called a Null Set. A set that has no elements.

Equation - A math sentence using the equal sign to state that two expressions represent the same number.

Equilateral triangle - An equilateral triangle is a triangle with three congruent sides. An equilateral triangle also has three congruent angles.

Equivalent -1. A term used to describe equations or inequalities that have the same solution. 2. A term used to describe fractions or ratios that are equal. 3. A term used to describe fractions, decimals and percents that are equal.

# MATH EXPLORATIONS Glossary of Terms 

Event - An event is any subset of the sample space. A simple event is a subset of the sample space containing only one possible outcome of an experiment. A compound event is a subset of the sample space containing two or more outcomes.

Experiment - A repeatable action with a set of outcomes.

Exponent - Suppose that n is a whole number. Then, for any number x , the $\mathrm{n}^{\text {th }}$ power of $x$, or $x$ to the $n^{\text {th }}$ power, is the product of $n$ factors of the number $x$. This number is usually written $x^{n}$. The number x is usually called the base of the expression $\mathrm{x}^{\mathrm{n}}$, and n is called the exponent.

Exponential Notation - A notation that expresses a number in terms of a base and an exponent.

Face - Each of the surface polygons that form a polyhedron.

Factor - An integer that divides evenly into a dividend. Use interchangeably with divisor except in the Division Algorithm.

Factorial - The factorial of a non-negative number $n$ is written $n!$ and is the product of all positive integers less than or equal to $n$. By definition $0!=1!=1$. .

Frequency - The number of times a data point appears in a data set.

# MATH EXPLORATIONS Glossary of Terms 

Function - A function is a rule which assignes to each member of a set of inputs, called the domain, a member of a set of outputs, called the range.

Graph of a function - The pictorial representation of a function.

Greater than, Less Than - Suppose that x and y are integers. We say that x is less than $\mathrm{y}, \mathrm{x}<\mathrm{y}$, if x is to the left of y on the number line. We say that x is greater than $\mathrm{y}, \mathrm{x}>y$, if x is to the right of y on the number line.

Greatest common factor, GCF - Suppose $m$ and $n$ are positive integers. An integer $d$ is a common factor of $m$ and $n$ if $d$ is a factor of both $m$ and $n$. The greatest common factor, or GCF, of $m$ and $n$ is the greatest positive integer that is a factor of both $m$ and $n$. We write the GCF of $m$ and $n$ as GCF ( $m, n$ )..

Height - The length of the perpendicular between the bases of a parallelogram or trapezoid; also the altitude of a triangle.

## Horizontal axis - See Coordinate Plane.

Hypotenuse - The side opposite the right angle in a right triangle.

Improper fraction - A fraction in which the numerator is greater than or equal to the denominator.

# MATH EXPLORATIONS Glossary of Terms 

Independent events - If the outcome of the first event does not affect the outcome of the second event.

Input values - The values of the domain of a function.

Integers - The collection of integers is composed of the counting numbers, their negatives, and zero; ... $-4,-3,-2,-1,0,1,2,3,4 \ldots$

Intersection of sets - A set whose elements are all the elements that the given sets have in common, written $A \cap B$.

Irregular polygon - A polygon that is not a regular polygon.

Isosceles triangle - A triangle with at least two sides of equal length is called an isoceles triangle.

Lateral Area - The surface area of any three-dimensional figure excluding the area of any surface designated as a base of the figure.

Lattice point - A point of the coordinate plane, $(x, y)$, in which both $x$ and $y$ are integers.

Least Common Denominator - The least common denominator of the fractions $\frac{p}{n}$ and $\frac{k}{m}$ is $\operatorname{LCM}(n, m)$.

# MATH EXPLORATIONS Glossary of Terms 

Least common multiple, LCM - The integers $a$ and $b$ are positive. An integer $m$ is a common multiple of $a$ and $b$ if $m$ is a multiple of both $a$ and $b$. The least common multiple, or LCM, of $a$ and $b$ is the smallest integer that is a common multiple of $a$ and $b$. We write the LCM of $a$ and $b$ as LCM $(a, b)$.

Legs -1 . The two sides of a right triangle that form the right angle. 2. The equal sides of an isoceles triangle or the non-parallel sides of a trapezoid.

## Less than - See: Greater Than.

Line graph - A graph used to display data that occurs in a sequence. Consecutive points are connected by segments.

Line Plot - A graph that shows frequency of data along a number line.

Line of symmetry - Line $L$ is a line of symmetry for a figure if for every point $P$ of the figure there is a point Q of the figure so that L is the perpendicular bisector of segment PQ.

Linear Model for Multiplication - Skip counting on a number line

Magnitude - The absolute value of a number; its distance from zero.

Mean - The average of a set of data; sum of the data divided by the number of items.

# MATH EXPLORATIONS Glossary of Terms 

Measures of central tendency - Generally measured by the mean, median or mode of the data set

Median - The middle value of a set of data arranged in increasing or decreasing order. If the set has an even number of items the median is the average of the middle two items.

Missing Factor Model - A model for division in which the quotient of an indicated division is viewed as a missing factor of a related multiplication.

Mixed fraction - The sum of an integer and a proper fraction.

Mode - The value of the element that appears most frequently in a data set.

Multiplicity - The number of times a factor appears in a factorization.

## Natural numbers - See: Counting Numbers

Negative integers - Integers less than zero.

Nets - One way to see the surface area of a three dimensional figure by cutting along its edges to produce a two dimensional figure.

# MATH EXPLORATIONS Glossary of Terms 

Notation - A technical system of symbols used to convey mathematical information.

## Null set - See: Empty Set.

Numerator - The expression written above the fraction bar in a common fraction to indicate the number of parts counted.

Obtuse Angle - An angle whose measure is greater than 90 degrees and less than 180 degrees.

Obtuse Triangle - A triangle that has one obtuse angle.

Ordered pair - A pair of numbers that represent the coordinates of a point in the coordinate plane with the first number measured along the horizontal scale and the second along the vertical scale.

Origin - The point with coordinate 0 on a number line; the point with coordinates $(0,0)$ in the coordinate plane.

Outcomes - The set of possible results of an experiment.

Outlier - A term referring to a value that is drastically different from most of the other data values.

# MATH EXPLORATIONS Glossary of Terms 

Output Values - The set of results obtained by applying a function rule to a set of input values.

Parallel lines - Two lines in a plane that never intersect.

Parallelogram - A parallelogram is a four-sided figure with opposite sides parallel.

Percent - A way of expressing a number as parts out of 100 ; the numerator of a ratio with a denominator of 100 .

Perfect Cube - An integer $n$ that can be written in the form $n=k^{3}$, where $k$ is an integer.

Perfect Square - An integer $n$ that can be written in the form $n=k^{2}$, where $k$ is an integer.

Perimeter - The perimeter of a polygon is the sum of the lengths of its sides.

Perpendicular - Two lines or segments are perpendicular if they intersect to form a right angle.
$\mathbf{P i}$ - The ratio of the circumference to the diameter of any circle, represented either by the symbol $\pi$, or the approximation $\frac{22}{7}$ or $3.1415926 \ldots$

# MATH EXPLORATIONS Glossary of Terms 

Pie graph - A graph using sectors of a circle that are proportional to the percent of the data represented.

Polygon -A polygon is a simple, closed, plain figure formed by three or more line segments..

Polyhedron - A three-dimensional figure with four of more faces all of which are polygons.

Positive integers - See: Counting Numbers

Power - See: Exponent

Prime Number - See: Composite Number

Prime Factorization - The process of finding the prime factors of an integer. The term is also used to refer to the result of the process.

Prism - A type of polyhedron that has two bases that are both congruent and parallel, and lateral faces which are parallelograms.

Probability - In an experiment in which each outcome is equally likely, the probability $P(A)$ of an event $A$ is $\frac{m}{n}$ where $m$ is the number of outcomes in the subset $A$ and $n$ is the total number of outcomes in the sample space $S$.

# MATH EXPLORATIONS Glossary of Terms 

Proper fraction - A fraction whose value is greater than 0 and less than 1 .

Proportion - An equation of ratios in the form $\frac{a}{b}=\frac{c}{d}$, where $b$ and $d$ are not equal to zero.

Protractor - An instrument used to measure angles in degrees.

Pyramid - A type of polyhedron that has one face, called a base, and triangular lateral faces that meet at a point called the apex.

Pythagorean Theorem - The formula that states that if $a$ and $b$ are the lengths of the legs of a right triangle and $c$ is the length of the hypotenuse, then $a^{2}+b^{2}=c^{2}$.

Quadrant - See Coordinate Plane.

Quotient - The result obtained by doing division. See the Division Algorithm for a different use of quotient.

Radius - The distance from the center of a circle a point of the circle. Plural form is radii.

Range - The difference between the largest and smallest values of a data set. See: Function for another meaning of range.

# MATH EXPLORATIONS Glossary of Terms 

Rate - A rate is a division comparison between two quantities with different units. Also see Unit Rate.

Ratio - A division comparison of two quantities with or without the same units. If the units are different they must be expressed to make the ratio meaningful.

Rational Number - A number that can be written as $\frac{a}{b}$ where $a$ is an integer and $b$ is a natural number.

Ray - Part of a line that has a starting point and continues forever in only one direction.

## Reciprocal - See: Multiplicative Inverse.

Reflection - The transformation that moves points or shapes by "flipping" them across a line or axis; a mirror image of the original set of points. If $B$ is the reflection of $A$ in line $L$, then $L$ is the perpendicular bisector of segment $A B$.

Regular Polygon -A polygon with equal length sides and equal angle measures.

Relatively Prime - Two integers $m$ and $n$ are relatively prime if the GCF of $m$ and n is 1 .

## Remainder - See: Division Algorithm.

# MATH EXPLORATIONS Glossary of Terms 

Repeating decimal - A decimal in which a cycle of one or more digits is repeated infinitely.

Right Angle - An angle formed by the intersection perpendicular lines; an angle with a measure of $90^{\circ}$.

Right Triangle -A triangle that has a right angle.

Sample Space - The set of all possible outcomes of an experiment.

Scaffolding - A method of division in which partial quotients are computed, stacked and then combined.

Scale Factor - If polygons $A$ and $B$ are similar and $s$ is a positive number so that for each side of A with length k there is a corresponding side of B with length sk , then $s$ is the scale factor of $A$ to $B$.

Scalene Triangle - A triangle with all three sides of different lengths is called a scalene triangle.

Scaling - 1. A process by which a shape is reduced or expanded proportionally. 2. Choosing the unit of measure to be used on a number line.

Sector - A region of a circle bounded by two radii and an arc of the circle which joins their endpoints.

# MATH EXPLORATIONS Glossary of Terms 

Sequence - A list of terms ordered by the natural numbers. The outputs of a function whose domain is the natural numbers or whole numbers.

Set - A collection of objects or elements.

Similar polygons - Two polygons whose corresponding angles have equal measures and whose corresponding side lengths form equal ratios.

Simple event - See: Event.

Simplest Form of a Fraction - A form in which the greatest common factor of the numerator and denominator is 1 .

Simplifying - The process of finding equivalent fractions to obtain its simplest form.

Skewed - An uneven representation of a set of data.

Slant Height - An altitude of a face of a pyramid or a cone.

Square Root - For non-negative numbers $x$ and $y, y=\sqrt{x}$, read " $y$ is equal to the square root of $x$ ", means $y^{2}=x$.

# MATH EXPLORATIONS Glossary of Terms 

Stem and Leaf Plot - A method of showing the frequency of a certain data by sorting and ordering the values.

Straight Angle - An angle with a measure of 180 degrees formed by opposite rays.

Subset - Set B is a subset of set $A$ if every element of set $B$ is also an element of set A.

Supplementary Angles- Two angles are supplementary if the sum of their measures totals $180^{\circ}$.

Surface Area - The total area of all the faces of a polyhedron. The total of the lateral area and base area of a cone. The total of the lateral area and the two bases of a cylinder.

Term - 1. Each member of a sequence. 2. Each expression in a polynomial separated by addition and subtraction signs.

Terminating Decimal - If $a$ and $b$ are natural numbers with $b \neq 0$ and $a \div b$ yields a finite quotient, the decimal formed is a terminating decimal.

Theoretical probability - Probability based on mathematical law rather than a collection of data.

# MATH EXPLORATIONS Glossary of Terms 

Translation - A transformation that moves a figure along a line in a plane but does not alter its size or shape.

Transversal - Any line that intersects two or more lines at different points.

Trapezoid - A four sided plane figure with exactly one set of parallel sides.

Tree diagram - 1. A process used to find the prime factors on an integer. 2. A method to organize the sample space of compound events.

Trichotomy - A property stating that exactly one of these statements is true for each real number: it is positive, negative or zero.

Union of Two Sets - A set that contains all of the elements that appear in either of the given sets, written $A \cup B$.

Unit Rate - A ratio of two unlike quantities that has a denominator of 1 unit.

Universal Set - A set containing all of the elements under consideration.

Variable - A letter or symbol that represents an unknown quantity.

Venn diagram - A diagram involving two or more overlapping circles that aids in organizing data.

# MATH EXPLORATIONS Glossary of Terms 

Vertex -1 . The common endpoint of two rays forming an angle. 2. A point of a polygon or polyhedron where edges meet.

Vertical angles - If two straight lines intersect at a point, then each line is divided into two rays. The angles formed by using opposite rays from each line are called vertical angles.

Vertical angle theorem - If two lines intersect at a point $P$, then the vertical angles formed will always have the same measure.

## Vertical Axis - See: Coordinate Plane.

Volume - A measure of space; the number of unit cubes needed to fill a threedimensional shape.

Whole numbers - The whole numbers are the numbers in the following neverending sequence: $0,1,2,3,4,5,6,7,8,9,10,11, \ldots$.
x-axis - The horizontal axis of a coordinate plane.
y -axis - The vertical axis of a coordinate plane.
zero pair - For any natural number $\mathrm{n}, \mathrm{n}+(-\mathrm{n})$ is called a zero pair because their sum is zero.

## MATH EXPLORATIONS Glossary of Terms

# MATH EXPLORATIONS <br> Summary of Ideas 

## Summary of Important Ideas

Additive Property of Equality - If $A=B$, then $A+C=B+C$.

Additive Identity - For any number $\mathrm{x}, \mathrm{x}+0=\mathrm{x}$.

Additive Inverses - For any number $x$, there exists a number - $x$, called the additive inverse of $x$, such that $x+(-x)=0$.

Area of a Circle - The area of a circle with radius $r$ is $A=r^{2}$ square units.

Area of a Parallelogram - The area of a parallelogram with base $b$ and height $h$ is given by $A=b h$.

Area of a Triangle - The area of a triangle with base $b$ and height $h$ is given by $A=\frac{1}{2}$ bh or $A=\frac{b h}{2}$.

Associative Property of Addition - For any numbers $x, y$, and $z,(x+y)+z=$ $x+(y+z)$.

Associative Property of Multiplication - For any numbers $x, y$, and $z,(x y) z$ $=x(y z)$.

Commutative Property of Addition - For any numbers $x$ and $y, x+y=y+x$.

# MATH EXPLORATIONS <br> Summary of Ideas 

Commutative Property of Multiplication - For any numbers A and $\mathrm{B}, \mathrm{AB}=$ BA.

Corresponding Angle Postulate - If two parallel lines are cut by a transversal, then the corresponding angles have the same measure, and if two lines are cut by a transversal so that the corresponding angles have the same measure, then the two lines are parallel.

Distributive Property of Multiplication Over Addition - Given two positive integers $n$ and $d$, we can always find unique integers $q$ and $r$ such that $n=d q$ $+r$ and $0 \leq r<b$. We call $n$ the dividend, $d$ the divisor, $q$ the quotient and $r$ the remainder.

Division Rules - 1) If dividend and divisor have different signs, (one positive, other negative), the quotient is negative. 2) If the dividend and the divisor have the same sign, (both positive or both negative), the quotient is positive.

Double Opposites Theorem - For any number $\mathrm{x},-(-\mathrm{x})=\mathrm{x}$.

Division Algorithm - Given two positive integers $a$ and $b$, we can always find unique integers $q$ and $r$ such that $a=b q+r$ and $0 \leq r \leq b$. We call $a$ the dividend, $b$ the divisor, $q$ the quotient, and $r$ the remainder.

Equivalent Fraction Property - For any number a and nonzero numbers $k$ and
b, $\frac{a}{b}=\frac{k a}{k b}=\frac{a k}{b k}=\frac{a k}{b k}$

# MATH EXPLORATIONS <br> Summary of Ideas 

Fractions and Division - For any number $m$ and nonzero $n$ the fraction $\frac{m}{n}$ is equivalent to the quotient $n \longdiv { m }$

Fundamental Theorem of Arithmetic- If $n$ is a positive integer, $n>1$, then $n$ is either prime or can be written as a product of primes $n=p_{1} \cdot p_{2} \cdot \cdots \cdot p_{k}$, for some prime numbersp1,p2, $\ldots$, pksuch that $p_{1} \leq p_{2} \leq \cdots \leq p_{k^{\prime}}$ where is a natural number. In fact, there is only one way to write $n$ in this form

Multiplication of Powers - Suppose that x is a number and a and b are whole numbers, then $x^{a} x^{b}=x^{(a+b)}$

Multiplicative Identity - The number 1 is the multiplicative identity; that is, for any number $n, n \cdot 1=n$.

Multiplicative Inverse - For every non-zero x there exists a number $\frac{1}{x}$, called the multiplicative inverse or reciprocal of $x$, such that $x \cdot \frac{1}{x}=1$.

Multiplying Fractions - The product of two fractions $\frac{a}{b}$ and $\frac{c}{d}$, where $b$ and $d$ are nonzero, is $\frac{a}{b} \cdot \frac{c}{d}=\frac{a c}{b d}$

Subtraction Property of Equality - If $A=B$, then $A-C=B-C$

Sums With Like Denominators - The sum of two fractions with like denominators, $\frac{a}{n}$ and $\frac{b}{n}$, is given by $\frac{a}{n}+\frac{b}{n}=\frac{a+b}{n}$.

# MATH EXPLORATIONS <br> Summary of Ideas 

Surface Area of a Cube - The surface area, SA, of a cube is given by the formula, $S A=6 s^{2}$ where $s$ is the length of a side.

Surface Area of a Cylinder - The total surface, SA, of a cylinder is the sum of the areas of the bases and the middle section, (lateral area), given by: $\mathrm{SA}=2 \mathrm{~B}$ $+\mathrm{Ph}=2 \pi r^{2}+2 \pi r h$, where $B$ is the area of the base, $P$ is the perimeter, $h$ is the height, and $r$ is the radius.

Surface Area of a Rectangular Prism - The surface area, SA, of a rectangular prism is given by the formula, $S A=2 B+P h$ where $B$ is the area of a base, $P$ is the perimeter of the base rectangle and $h$ is the height of the prism.

The Rule of Products - If one action can be performed in $m$ ways and a second independent action can be performed in $n$ ways, then there are $m \cdot n$ possible ways to perform both actions.

The Rule of Sums - If one action can be performed in $m$ ways and a second action can be performed in $n$ ways, then there are $(m+n)$ ways to perform one action or the other, but not both. This assumes that each action is equally likely and mutually exclusive.

Triangle Similarity Theorem - If two triangles have the same angle measures they are similar and the ratios of their corresponding sides are equal. Conversely, if two triangles have sides with the same ratio they are similar and their corresponding angles have equal measure.

# MATH EXPLORATIONS <br> Summary of Ideas 

Triangle Sum Theorem - The sum of the measures of the angles of a triangle equals $180^{\circ}$.

Unit Fraction - For any positive integer $n$, the multiplicative inverse or reciprocal of $n$ is the unit fraction $\frac{1}{n}$.

Vertical Angle Theorem - If two lines intersect at a point P, then the vertical angles will have the same measure.

Volume of a Cube - The volume of a cube with each side of length $s$ units is $s^{3}$ cubic units. $V=s^{3}$ or $V=B h$ where $B$ is the base area of the three-dimensional figure and h is the height.

Volume of a Cylinder - The volume of a cylinder with radius $r$ and height $h$ is given by $\mathrm{V}=\mathrm{Bh}=\pi r^{2 h}$.

Volume of a Prism - The volume of a prism is the area of the base of the three-dimensional figure multiplied by the height of the prism. $\mathrm{V}=\mathrm{B}$ h where B is the base area.

# MATH EXPLORATIONS 

Glosario

Absolute Value/ Valor Absoluto : La distancia de un numero a cero. Por<br>cualquier valor x , es definido como:<br>\[ \begin{aligned} \& |x|=x, si x \geq 0<br>\& |x|=-x, si x \geq 0 \end{aligned} \]

Acute Angle/ Angulo Agudo: Un angulo de medida mayor de 0 grados y menor de 90 grados.

Acute Triangle/ Triangulo Agudo: Un triangulo en el cual los tres angulos son agudos.

Altitude of a Triangle/ Altura de un Triangulo: un segmento dibujado de la vertice de el triangulo perpendicular al lado opuesto, llamado la base, (o perpendicular a una extension de la base).

Angle/ Angulo: Un angulo es formado cuando dos semirectas tienen el mismo extremo llamado vertice.

Area Model/ Modelo Basado en la Area: Un Modelo matematico basado en la area de un rectangulo, usado para representar dos numeros multiplicados ya sean enteros o quebrados (fracciónes.)

Arithmetic Sequence/ Suceción Aritmetica: Una suceción $\mathrm{a}_{1}, \mathrm{a}_{2^{\prime}} \mathrm{a}_{3^{\prime}} \mathrm{a}_{4^{\prime}}$. . es aritmetica si existe un numero $c$ de tal manera que por cada $n, a_{n}+1=a_{n}+c, y$ $a_{n+1}-a_{n}=c$.

# MATH EXPLORATIONS <br> Glosarios 

Attribute/ Atributo: Una caracteristica de un objeto tal como angulos o lados de un triangulo.

Axis/ Eje: Una recta numerica en un plano. Ver Coordinate Plane.

Bar Graph: Grafica de Barras: Una grafica en la cual barras rectangulares, ya sean horizontales o verticales, se usan para mostrar los datos.

Base/ Base: 1.Por cualquier valor $x$, elevado a la potencia de $n$, escrito como xn, $x$ es llamado la base de la expresión; 2. Cualquier lado de un triangulo; 3. Cualquiera de los lados paralelos de un trapezoide; 4. Cualquiera de los lados paralelos de un paralelogramo.

Box and Whisker Plot/ Digrama de Cajas y Brazos: Para datos ordenados de mas chico a mas grande, la mediana, y el cuartile bajo y cuartile alto son mostrados en cajas sobre una recta numerica. Los brazos muestran los extremos de los datos.

Cartesian Plane/ Plano Cartesiano: ver Coordinate Plane

Center of a Circle/ Centro del Circulo: Un punto en el interior del circulo que queda a la misma distancia de cualquier punto el la periferia del circulo.

Chord/ Cuerda: Un segmento que une dos puntos el la periferia del circulo.

# MATH EXPLORATIONS <br> Glosario 

Circle/ Circulo: Un conjunto de puntos en un plano de igual distancia de un punto en el plano.

Circumference/ Circumferencia: La distancia de la periferia del circulo. La medida es el producto de el diametro del circulo y pi.

Coefficient/ Coeficiente: En el producto de un constante y un variable, el constante es el factor numerico de el termino, y es referido como el coeficiente.

Common Denominator/ Denominador Comun: Un multiple comun de los denominadores de dos o mas fracciónes. Ver Least Common Denominator.

Common Factor/ Factor Comun; Un factor que dos o mas enteros tienen en comun. Ver Greatest Common Factor.

Common Multiple/ Multiple Comun: Ver Least Common Multiple.

Complement/ Complemento: El complemento de un conjunto es un conjunto de todos los elementos del conjunto universal que no estan en el conjunto dado.

Complementary Angles/ Angulos Complementarios: Dos angulos son complementarios si la suma de sus medidas es $90^{\circ}$.

# MATH EXPLORATIONS <br> Glosarios 

Composite Number/ Numero Compuesto: Un numero primo es un entero p mayor que 1 con exactamente dos factores positivos; 1 y p. un numero compuesto es un entero mayor que 1 que tiene mas de dos factores positivos. El numero 1 no es primo ni compuesto.

Compound Interest/ Interes Compuesto: Interes pagado sobre interes previamente ganado mas el principal.

Concentric Circles/ Circulos Concentricos: Circulos con el mismo centro en el mismo plano pero con diferentes radios.

Cone/ Cono: Una figura de tres dimensiónes con una base circular juntada a un punto llamado el apíce o cima.

Congruent/ Congruente: Usado para referir a angulos o lados teniendo la misma medida y para poligonos del mismo tamaño y figura.

Conjecture/ Conjetura: Una afirmación basada en observaciónes.

Constant/ Constante: Un valor fijo.

Coordinate(s)/ Coordenado(s): Un numero asignando a un punto en la recta numerica. En el plano coordenado, el par ordenado ( $\mathrm{x}, \mathrm{y}$ ) situa cada punto y lo situa en respecto a los ejes de $x$, $y$ y .

# MATH EXPLORATIONS <br> Glosario 

Coordinate Plane/ Plano Coordenado: Un plano consistiendo de una recta numerica horizontal y una recta numerica vertical que se intersectan en el origen formando angulos rectos. Las rectas, llamadas ejes, dividen el plano en quatro quadrantes. Estos son numerados I, II, III, IV comensando con el quadrante en la esquina derecha de ariba y siguiendo contra-reloj.

Corresponding Angles/ Angulos correspondientes: 1. Si dos rectas son cortadas por una recta tranversal, los angulos del mismo lado del transversal y del mismo lado de las rectas son angulos correspondientes. Si las rectas son paralelas, las medidas de los angulos seran iguales. 2. Si dos poligonos son similares, los angulos en la misma posición relativamente seran angulos correspondientes y tendran medidas iguales.

Corresponding Sides/ Lados Correspondientes: Si dos poligonos son similares, los lados en la misma posición relativamente son lados correspondientes y la proporción de las medidas de cada par de lados es igual.

Counterclockwise/ Contra-reloj: un movimiento circular opuesto a la dirección de movimiento de las manecillas del reloj.

Counting Numbers/ Numeros de Conteo: Numeros usados para contar. Llamados numeros naturales o enteros, son en secuencia, 1, 2, 3, 4, 5, 6, 7...

Cube/ 1. Cubo 2. Elevado al Cubo: 1. Una figura de tres dimensiónes con seis caras cuadradas congruentes. 2. Elevar un numero a la tercera potentia.

# MATH EXPLORATIONS <br> Glosarios 

Cylinder/ Cilindro: Una figura de tres dimensiónes con bases circulares que son paralelas y del mismo tamaño unidas por una superficie lateral que tiene una plantilla rectangular.

Data (Data Set)/ Datos (Conjunto de Datos): Una colección de información frequentemente el forma de numeros.

Data Analysis/ Analize de Datos: El proceso de entender los datos colecciónados.

Degree/ Grado: 1. La circumferencia de un circulo es dividida en 360 partes iguales or arcos. Radios dibujados a los terminos de uno de estos arcos forman un angulo de 1 grado. 2. El grado del termino es la suma de las potencias de los variables. El grado de un polinomio es la potencia mas grande de sus integrantes.

Denominator/ Denominador: El denominador de una fracción indica a que tantas partes iguales se ha dividido el entero. El denominador aparese en la parte de abajo de la fracción.

Diameter/ Diametro: El segmento que une dos puntos el la periferia del circulo y pasa por el centro del circulo.

Distance/ Distancia: Por cualquier dos numeros x y y, la distancia entre x y y es el valor absolute de sus diferencias; es decir, Distancia $=$.

# MATH EXPLORATIONS <br> Glosario 

Dividend/ Dividendo: La cantidad que se divide.

Divisibility:/ Divisibilidad: Suponiendo que $n$ y $d$ son enteros, y que $d \neq 0$. El numero n es divisible por d si hay un entero q de tal manera que $\mathrm{n}=\mathrm{dq}$. Por igual, d es un factor de n o n es un multiple de d .

Division Algorithm/ Algoritmo de División: Dados dos enteros positivos a y $b$, siempre podemos encontrar enteros unicos $q$ y $r$ de tal manera que $a=b q+r$ y $0 \leq r<b$. Llamamos a el dividendo, $b$ el divisor, $q$ el cuociente, $y r$ el restante.

Divisor/ Divisor: La cantidad por cual el dividendo se divide.

Domain/ Dominio: El conjunto de primeros numeros de una función.

Edge/ Lado: Un segmento que une vertices consecutivas de un poligono o un poliedro.

Elements/ Elementos: miembros de un conjunto.

Empirical Probability/ Probabilidad Empírica: Probabilidad determinada por datos collecciónados de experimentos reales.

Empty Set/ Conjunto Vacio: Llamado conjunto nulo, es un conjunto que no tiene elementos.

# MATH EXPLORATIONS <br> Glosarios 

Equation/ Ecuación: Una proposición matematica con igualdad señalando que dos expresiónes son iguales.

Equilateral Triangle/ Triangulo Equilatero: Un triangulo con tres lados congruentes. Un triangulo equilatero tanbien tiene tres angulos congruentes.

Equivalent/ Equivalente: 1. Dos ecuaciónes o desigualdades son iguales si tienen la misma solución o conjunto de soluciónes. 2. Un termino para describir fracciónes o proporciónes que son igual. 3. Un termino para describir fracciónes, decimales, y porcientos que son iguales

Event/ Evento: Un evento es un subconjunto del espacio muestral. Un evento simple es un subconjunto del espacio muestral conteniendo solamente un resultado del experimento. Un evento compuesto es un subconjunto del espacio muestral conteniendo dos o mas resultados.

Experiment/ Experimento: Una actión que puede ser repetida con un conjunto de resultados.

Exponent/ Exponente (Potencia): Por cualquier valor x, elevado a la potencia de $n$, escrito como xn , x es llamado la base de la expresión, y n es el exponente o potencia.

## Exponential Notation/ Notación Exponencial (Notación Potencial):

Notación para expresar un numero en terminos de base y potencia.

# MATH EXPLORATIONS <br> Glosario 

Face:/Cara: La superficie de cada poligono que forma un poliedro.

Factor/ Factor: Un entero que divide un dividendo exactamente. Intercambiable con el divisor excepto en el Algoritmo de División.

Factorial/ Factorial: El factorial de un numero que no es negativo n es escrito como n ! y es el producto de todos los numeros positivos menor o igual a n . Por definición $\quad 0!=1!=1$.

Frequency/ Frecuencia: El numero de veces que un dato aparece en un conjunto.

Function/ Función: Una función es una regla que asigna a cada primer numero de un conjunto, llamado dominio, un numero de salida, Ilamado el rango. La regla no permite que los primeros numeros se repitan.

Graph of a Function/ Grafica de una Función: Una representación pictorica de una función graficando pares ordenados en el sistema coordinado.

Greater Than/ Mayor Que: ver Less Than.

Greatest Common Factor, GCF/ Máximo Común Factor, MCF: Si m y n son enteros positivos. Un entero $d$ es un factor común de $m \mathrm{y} n$ sid es un factor de m y de n . El máximo común factor o MCF de $m$ y n es el entero positive mas grande que es un factor de $m$ y $n$. Escrito como MCF de my n o como MCF(m,n).

# MATH EXPLORATIONS <br> Glosarios 

Height/ Altura: La longitud de el segmento perpendicular que une las bases de un paralelogramo o un trapezoide. Lo alto de un triangulo.

Horizontal Axis/ Eje Horizontal: ver Coordinate Plane.

Hypotenuse/ Hipotenusa: El lado opuesto al angulo recto en un triangulo recto.

Improper Fraction/ Fracción Impropia: Una fracción en la cual el numerador es mayor que o igual a el denominador.

Independent Event/ Evento Independiente: El resultado del primer evento no afecta el resultado del segundo evento.

Input Values/ Primeros Valores: Los valores del dominio de una función.

Integers/ Enteros: La colección de enteros es compuesta de numeros negativos, cero y los numeros positivos:..., $-4,-3,-2,-1,0,1,2,3,4, \ldots$

Intersection of Sets/ Traslape de Conjuntos: Un conjunto con elementos cuales son los elementos comunes de los conjuntos dados.

Irregular Polygon/ Poligono Irregular: Un poligono que no es regular.

# MATH EXPLORATIONS 

Glosario

Isosceles Triangle/ Triangulo Isósceles: Un triangulo con por lo menos dos lados de la misma medida es llamado un triangulo isósceles.

Lateral Area/ Area Lateral: La area de la superficie de cualquier figura de tres dimenciónes no incluyendo la area de la superficie designada como la base.

Lattice Points/ Puntos de Latice: Un punto en el plano coordinado ( $x, y$ ) en el cual x y y son enteros.

## Least Common Denominator, LCD/ Mínimo Común Denominador, MCDn:

El minimo comun denominador de las fracciónes y es MCM(n,m).

Least Common Multiple, LCM/ Mínimo Común Múltiple, MCM: Si a y $b$ son enteros positivos. Un entero $m$ es un múltiple común de a y $b$ si $m$ es un múltiple de $a$ y de $b$. El mínimo común múltiple o MCM de $a$ y $b$ es el entero positive mas chico que es un múltiple de $a$ y $b$. Escrito como MCM de a y b o como MCM(a,b).

Legs/ Catetos: Los lados que forman el angulo recto de un triangulo recto.

Less Than, Greater Than/ Menor Que, Mayor Que: El enunciado que un numero a es menor que un numero $b$, escrito $a<b$ significa que hay un numero positivo $x$ de tal manera que $b=a+x$. El numero $x$ tiene que ser $b-a$. Si $a$ es menor que $b, b$ es mayor que $a$, escrito $b>a$.

# MATH EXPLORATIONS <br> Glosarios 

Line Graph/ Grafica de Rectas: Una grafica usada para enseñar datos que ocuren en secuencia. Puntos consecutivos van conectados por segmentos.

Line Plot/ Grafica de Rectas: Una grafica usada para mostrar la frequencia de datos en una recta numerica.

Line of Symmetry/ Eje de Simetria: Recta L es un eje de simetria para la figura si por cada punto $P$ en la figura hay un punto $Q$ en la figura de tal manera que $L$ es el mediatriz del segmento PQ.

Linear Model for Multiplication/ Modelo Lineal para Multiplicar: Contar en grupos usando la recta numerica.

Magnitude/ Magnitud: El valor absoluto de un numero; su distancia de cero.

Mean/ Media Aritmetica: El promedio de un conjunto de datos; la suma de los datos dividida por el numero de datos.

Measures of Central Tendency/ Medidas de Tendencia Central: Medidas generalmente por la media, la mediana y la moda de el conjunto de datos.

Median/ Mediana: El valor medio en un conjunto de datos ordenados. Si el conjunto tiene un numero par de datos, la mediana es el promedio de los dos numeros de enmedio.

# MATH EXPLORATIONS 

Glosario

Missing Factor Model/ Modelo de Factor Faltante: Un modelo de división en el cual el cuociente de la división indicada es visto como un factor faltante de una multiplicación relaciónada.

Mixed Fraction(Mixed Number)/ Numero Mixto: La suma de un entero y una fracción propia.

Mode/ Moda: El valor del elemento que aparece mas veces en un conjunto de datos.

Multiplicity/ Multiplicidad: El numero de veces que un factor aparece en una factorización.

Natural Numbers/ Numeros Naturales: ver Counting Numbers

Negative Integers/ Enteros Negativos: Enteros con valor menor que cero.

Nets/ Plantillas: Una manera de ver la area de la superficie de una figura de tres dimensiónes cortando sobre los lados para producer una figura de dos dimensiónes.

Notation/ Anotación: Un sistema technico de simbolos usado para mostrar información matematica.

# MATH EXPLORATIONS <br> Glosarios 

Null Set/ Conjunto Nulo: ver Empty Set.

Numerator/ Numerador: La expressión en la parte de ariba de una fracción que indica el numero de partes contadas.

Obtuse Angle/ Angulo Obtuso: Un angulo con medida mayor que 90 grados y menor que 180 grados.

Obtuse Triangle/ Triangulo Obtuso: Un triangulo que tiene un angulo obtuso.

Ordered Pair/ Par Ordenado: Un par de numeros que representa los coordinados de un punto en el plano coordinado con el primer numero señalando la medida en el eje horizontal y el segundo señalando la medida en el eje vertical.

Origin/ Origen: El punto con el coordinado 0 el la recta numerica; el punto con los coordinados ( 0,0 ) en el plano coordinado.

Outcomes/ Numeros de Salida: El conjunto de resultados posibles de un experimento.

Outlier/ Valor Extremo: Un termino que se refiere a valor que es completamente diferente a los demas valores.

Output Values/ Valores de Salida: El conjunto de valores obtenidos al aplicar una función a un conjunto de primeros valores.

# MATH EXPLORATIONS <br> Glosario 

Parallel lines/ Lineas Paralelas: Dos lineas en un plano que nunca cruzan.

Parallelogram/ Paralelogramo: Es un cuádrilatero con los lados opuestos paralelos.

Percent/ Porciento: una manera de expresar un numero como partes de cien. El numerador de una proporción con 100 como denominador.

Perfect Cube/ Cubo Perfecto: Un entero $n$ que puede ser escrito en la forma $\mathrm{n}=$,
$y k$ es un entero.

Perfect Square/ Cuadrado Perfecto: Un entero $n$ que puede ser escrito en la forma $\mathrm{n}=, \mathrm{yk}$ es un entero.

Perimeter/ Perimetro: El perimetro de un poligono es la suma de la medida de los lados.

Perpendicular/ Perpendicular: Dos lineas o segmentos se dicen ser perpendiculares si cruzan a formar un angulo recto.

Pi/ Pi: La proporción de la circumferencia a el diametro de cualquier circulo, representado por el simbolo $\Phi$, o la aproximación o 3.1415926...

# MATH EXPLORATIONS <br> Glosarios 

Pie Graph/ Grafica de Sectores: Una grafica usando sectores de un circulo que son proporciónal al porciento de los datos representados.

Polygon/ Poligono: Un poligono es una figura simple plana formado por tres o mas segmentos.

Polyhedron/ Poliedro: Una figura de tres dimensiónes con cuatro o mas caras todas cuales son poligonos.

Positive Integers/ Enteros Positivos: ver Counting Numbers

Power/ Potencia: ver Exponents

Prime Number/ Numero Primo: ver Composite Number

Prime Factorization/ Factorización en Primos: El proceso de encontrar los factores primos de un entero. El termino tambien se refiere al resultado de este proceso.

Prism/ Prisma: Un tipo de poliedro que tiene dos bases que son paralelas y congruentes, y caras laterales que son paralelogramos.

# MATH EXPLORATIONS <br> Glosario 

Probability/ Probabilidad: En un experimento en el cual cada resultado tiene la misma oportunidad, la probabilidad $P(A)$ de un evento $A$ es donde $m$ es el numero de resultados en el subconjunto A y n es el numero total de resultados en el espacio muestral $S$.

Proper Fraction/ Fracción Propia: Una fracción con valor mayor que 0 y menor que 1.

Proportion/ Proporción: Una equación de razones en la forma , y b y d no son iguales a cero.

Protractor/ Transportador: Un instrumento para medir ángulos en grados.

Pyramid/ Pirámide: Un tipo de poliedro que tiene una cara llamada la base, y caras laterales triangulares que topan en un punto llamado el vertice.

Pythagorean Theorem/ Teorema de Pitagoras: Si a y b son las medidas de los catetos de un triangulo recto y c es la medida de la hipotenusa, entonces c2 $=\mathrm{a} 2+\mathrm{b} 2$.

Quadrant/ Cuadrante: ver Coordinate Plane

Quotient/ Cociente: El resultado obtenido al hacer división. Ver Division

# MATH EXPLORATIONS <br> Glosarios 

Algorithm para encontrar diferentes usos del cociente.

Radius/ Radio: La distancia del centro de un circulo a un punto en la periferia.

Range/ Rango: La diferencia entre el valor mas grande y el valor mas chico en un conjunto de datos. Ver Function por otras definiciónes de rango.

Rate/ Tasa de Variación: Una comparación por medio de un cociente, entre dos cantidades con diferentes unidades. Ver Unit Rates.

Ratio/ Razón: Una comparación por medio de un cociente. Si las unidades son diferentes, la razón tiene que tener sentido.

Ray/ Rayo: Parte de una recta con un extremo y se prolonga sin limite en una dirección.

Reciprocal/ Reciproco: ver Multiplicative Inverse

Reflection/ Reflexión: La transformación que mueve puntos o figuras al cruzar de una recta o eje; una imagen de espejo del conjunto original.

Regular Polygon/ Poligono Regular: Un poligono con lados de mismas medidas y angulos de mismas medidas.

# MATH EXPLORATIONS 

Glosario

Relatively Prime/ Numeros Primos Entre Si: Dos enteros m y n son numeros primos entre si, si el MCF de mynes 1 .

Remainder/ Restante: Ver Division Algorithm.

Repeating Decimal/ Decimal Periódico: Un decimal en el que se repiten uno o mas digitos sin terminación.

Right Angle/ Angulo Recto: un angulo formado cuando cruzan dos lineas perpendiculares; un angulo que mide 90 grados.

Right Triangle/ Triangulo Recto: un triangulo que tiene un angulo recto.

Sample Space/ Espacio Muestral: El conjunto de todos los resultados posibles de un experimento.

Scaffolding/ Andamiaje: Un metodo de división en el cual cocientes parciales son computados, apilados, y combinados.

Scale Factor/ Factor de Escala: Si poligonos A y B son similares y s es un numero positive a que para cada lado de poligono A con medida k hay un lado correspondiente en B con medida sk, entonces s es el factor de escala de A y B.

Scalene Triangle/ Triangulo Escaleno: Un triangulo con los tres lados de diferentes medidas es llamado un triangulo escaleno.

# MATH EXPLORATIONS <br> Glosarios 

Scaling/ Escalar: 1. El proceso en el cual una figura se reduce o aumenta proporciónalmente. Escojer la unidad de medida que sera usada en la recta numerica.

Sector/ Sector: Una región de un circulo rodeado por dos radios y un arco que une sus extremos.

Sequence/ Sucuencia: Un conjunto de terminos puestos en orden por los numeros naturales. Los numeros de salida de una función de dominio que incluye los numeros naturales o enteros.

Set/ Conjunto: una colección de objetos o elementos.

Similar Polygons/ Poligonos Similares: Dos poligonos cuales tienen angulos corespondientes de la misma medida y lados corespondientes de misma proporción.

Simple Event/ Evento Simple: Ver Event.

Simplest Form of a Fraction: Forma Mas Simple de una Fracción: Una forma en la cual el maximo comun factor del numerador y denominador es 1 .

Simplifying/ Simplificar: El proceso de encontrar fracciónes equivalentes para obtener la forma mas simple.

Skewed/ Sesgado: Una representación de un conjunto de datos desiguales.

# MATH EXPLORATIONS 

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Glosario
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Stem and Leaf Plot/ Diagrama de Tallo y Hojas: Un metodo para enseñar la frecuencia de ciertos datos al ordenarlos.

Straight Angle/ Angulo Llano: Un angulo que mide 180 grados formado por rayos opuestos.

Subset/ Subconjunto: Conjunto $B$ es un subconjunto de Conjunto $A$ si todos los elementos de conjunto B son parte de los elementos de Conjunto A.

Supplementary Angles/ Angulos Suplementarios: Dos angulos son suplementarios si la suma de sus medidas es $180^{\circ}$.

Surface Area/ Area de la Superficie: La area total de todas las caras en un poliedro. La area total de la lateral y la base en un cono. La area total de la lateral y las dos bases en un cilindro.

Term/ Termino: 1. un miembro de una suceción. 2. Cada expression en un polinomio separado por una suma o resta.

Terminating Decimal/ Decimo Finito: Si a y b son numeros naturales con $b \neq$ $0, y$ resulta en una cantidad finita, el numero decimal que resulta es un decimo finito.

Theoretical Probability/ Probabilidad Teórica: Probabilidad basada en leyes matematicas en ves de una colleccion de datos.

# MATH EXPLORATIONS <br> Glosarios 

Translation/ Traslacion: Una transformación que mueve una figura sobre el plano pero no altera el tamaño ni la forma.

Transversal/ Transversal: Cualquier recta que intersecta a dos o mas rectas en distintos puntos.

Trapezoid/ Trapezoide: Un cuadrilatero con exactamente un par de lados paralelos.

Tree Diagram/ Diagrama de Árbol: 1. Un proceso par encontrar los factores primos de un entero. 2. Un metodo para organizer el espacio muestral de eventos compuestos.

Trichotomy/ Tricotomia: Una propiedad declarando que exactamente una de estas declaraciónes es verdadera para cada numero real: es un numero positivo, negativo, o cero.

Union of Two Sets/ Unión de Dos Conjuntos: Un conjunto que contiene todos los elementos que aparecen en cualquiera de los conjuntos dados, escrito como A B.

Unit Rate/ Razon Unitaria: Una razon en la cual el denominador es una unidad.

Universal Set/ Conjunto Universal: Un conjunto conteniendo todos los elementos bajo consideración.

# MATH EXPLORATIONS 

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Variable:/ Variable: Una letra o simbolo que representa una cantidad desconocida.

Venn Diagram/ Diagram de Venn: Un diagrama con dos o mas circulos que intersectan para ayudar en la organización de datos.

Vertex/ Vertice: 1. El punto comun de los lados de un angulo. 2. Un punto extremo de un poligono o poliedro donde se juntan los lados.

Vertical Angles/ Angulos Verticales: Si dos rectas cruzan en un punto, cada recta se divide en dos formando dos rayos. Los angulos formados usando rayos opuestos de cada recta son llamados angulos verticales.

Vertical Angle Theorem/ Teorema de Angulos Verticales: Si dos rectas cruzan en un punto $P$, los angulos verticals formados siempre tendran la misma medida.

Vertical Axis/ Eje Vertical: ver Coordinate Plane.

Volume/ Volumen: La medida de un espacio; El numero de unidades cubicas necesarias para llenar una figura de tres dimensiónes.

Whole Numbers/ Numeros Enteros: Los numeros enteros son los numeros en la suceción: $0,1,2,3,4,5,6,7,8,9,10,11 \ldots$

# MATH EXPLORATIONS <br> Glosarios 

x-axis/ eje de x: El eje horizontal en el plano coordinado.
$y$-axis/ eje de y: El eje vertical en el plano coordinado

Zero Pair/ Par Formando Cero: Por cualquier numero natural $n, n+(-n)$ es llamado un par formando cero porque su suma es cero.

## MATH EXPLORATIONS Glosario

# MATH EXPLORATIONS Resumen de Ideas 

## MATH EXPLORATIONS

## Resumen de Ideas

Additive Property of Equality/ Propiedad Aditiva de la Igualdad: Si $\mathrm{A}=$ $B$, entonces $A+C=B+C$.

Additive Identity/ Identidad Aditiva: Por cualquier numero $\mathrm{x}, \mathrm{x}+0=\mathrm{x}$.

Additive Inverse/ Inverso Aditivo: Por cualquier numero $x$, existe un numero $-x$, llamado el inverso aditivo de $x$, de tal manera que $x+(-x)=0$.

Area of a Circle/ Area de un Circulo: La area de un circulo con radio res $\mathrm{A}=$ a r2 en unidades cuadradas.

Area of a Parallelogram/ Area de un Paralelogramo: La area de un paralelogramo con base by altura h es $\mathrm{A}=$ bh en unidades cuadradas.

Area of a Triangle/ Area de un Triangulo: La area de un triangulo con base by altura $h$ es $A=1 / 2 b h \circ A=$ en unidades cuadradas.

Associative Property of Addition/ Propiedad Asociativa de Adición: Por cualquier numeros $x, y$ and $z,(x+y)+z=x+(y+z)$.

# MATH EXPLORATIONS Resumen de Ideas 

Associative Property of Multiplication/ Propiedad Asociativa de Multiplicación: Por cualquier numeros $x, y$ and $z_{,}(x y) z=x(y z)$.

## Commutative Property of Addition/ Propiedad Conmutativa de Adición: Por cualquier numeros $x y y, x+y=y+x$.

## Commutative Property of Multiplication/ Propiedad Conmutativa de Multiplicación: Por cualquier numeros A y $\mathrm{B}, \mathrm{A} \cdot \mathrm{B}=\mathrm{B} \cdot \mathrm{A}$.

## Corresponding Angle Postulate/ Postulado de Angulos

Correspondientes: Si dos rectas paralelas son intersectadas por un transversal,los angulos formados correspondientes tienen la misma medida, y si dos rectas son intersectadas por un transversal de tal manera que los angulos correspondientes tienen la misma medida, entonces las rectas son paralelas.

Distributive Property of Multiplication over Addition/ Propiedad Distributiva de Multiplicación sobre Adición: Por Cualquier numeros $k$, $m$ $y n, n(k+m)=n k+m k$.

Division Rules/ Reglas de División: 1. Si el dividendo y el divisor tienen diferentes signos, ( uno positivo y uno negativo), el cociente es negativo. 2. Si el dividendo y el divisor tienen el mismo signo, ( los dos positivos o los dos negativos), el cociente es positivo.

Double Opposites Theorem/ Teorema de Opuestos Dobles: Por cualquier numero $x,-(-x)=x$.

# MATH EXPLORATIONS Resumen de Ideas 

Equivalent Fraction Property/ Propiedad de Fracciónes Equivalentes: por cualquier numero a y numeros no igual a cero $k$ y $b$.

Fractions and Division/ Fracciónes y División: Por cualquier numero my numero no igual a cero $n$, la fracción es equivalente a el cociente .

## Fundamental Theorem of Arithmetic/ Teorema Fundamental de la

Aritmetica: Si $n$ es un entero positivo, $n>1$, entonces $n$ es primo o puede ser escrito como el producto de primos $n=p 1 \cdot p 2 \cdot p 3 \cdot \ldots \cdot p k$, para unos numeros primos $\mathrm{p} 1, \mathrm{p} 2, \mathrm{p} 3, \ldots \mathrm{pk}$ de tal manera que $\mathrm{p} 1 \leq \mathrm{p} 2 \leq \mathrm{p} 3 \leq \ldots \leq \mathrm{pk}$. En realidad, nada mas hay una manera para escribir $n$ en esta forma.

Multiplication of Powers/ Multiplicación con Potencias: Si $x$ es un numero y a y b son numeros naturales, entonces $(x a)(x b)=x a+b$.

Multiplicative Identity/ Identidad Multiplicativa: El numero 1 es la identidad multiplicativa, es decir, por cualquier numero $n, n \cdot 1=n$.

Multiplicative Inverse/ Inverso Multiplicativo: Por cada x no igual a 0 , existe un numero , llamado el inverso multiplicativo o reciproco de $x$ de tal manera que

Multiplying Fractions/ Multiplicando Fracciónes: El producto de dos fracciónes y en donde by d son numeros no igual a cero, es .

# MATH EXPLORATIONS Resumen de Ideas 

Pythagorean Theorem/ Teorema de Pitagoras: Si a y b son las medidas de los catetos de un triangulo recto y c es la medida de la hipotenusa, entonces c2 $=a 2+b 2$.

Subtraction Property of Equality/ Propiedad Sustractiva de la Igualdad: Si $A=B$, entonces $A \quad C=B \quad C$.

Sums With Like Denominators/ Sumas de Fracciónes con
Denominadores Comunes: La suma de dos fracciónes con denominadores iguales, y es dado por .

Surface Area of a Cube/ Area de la Superficie de un Cubo: La area de la superficie, $S A$ de un cubo es dado por la formula, $S A=6 s 2$ donde $s$ es la medida de un lado.

Surface Area of a Cylinder/ Area de la Superficie de un Cilindro: La area de la superficie total SA de un cilindro es la suma de la area de las bases y la area lateral, dado por la equación: $S A=2 B+P h=2 \varpi r 2+2$ бrh, donde $B$ es la area de la base, P es el perimetro de el circulo (circumferencia), h es la altura y r es el radio del circulo.

Surface Area of a Rectangular Prism/ Area de la Superficie de una Prisma rectangular: La area de la superficie SA de una prisma rectangular es dado por la equación: $S A=2 B+P h$, donde $B$ es la area de la base, $P$ es el perimetro de la base rectangular, y h es la altura de la prisma.

# MATH EXPLORATIONS Resumen de Ideas 

The Rule of Products/ La Ley de Productos: Si una acción se puede hacer de $m$ numero de maneras y una segunda acción se puede hacer en n numero de maneras, entonces hay $m \cdot n$ numero de maneras para hacer las dos acciónes.

The Rule of Sums/ La Ley de Sumas: Si una acción se puede hacer de m numero de maneras y una segunda acción se puede hacer en $n$ numero de maneras, entonces hay $(m+n)$ numero de maneras para hacer una o la otra acción pero no las dos. Esto supone que las dos acciónes son exclusivas una de otra y tienen la misma oportunidad de acontecer.

Triangle Similarity Theorem/ Teorema de Semejanza de Triangulos: Si dos triangulos tienen angulos de mismas medidas, son semejantes o similares y las proporciónes de los lados correspondientes son iguales. Por igual, si dos triangulos tienen lados con la misma proporción, los triangulos son similares y sus angulos correspondientes tienen las mismas medidas.

Triangle Sum Theorem/ Teorema de la Suma de los Angulos de un Triangulo: La suma de las medidas de los angulos de un triangulo es igual a $180^{\circ}$.

Unit Fraction/ Fracción Unitaria: Por cualquier entero positivo n, el inverso multiplicativo o reciproco de $n$ es la fracción unitaria .

Vertical Angle Theorem/ Teorema de Angulos Verticales: Si dos rectas intersectan en un punto $P$, los angulos verticales formados tendran la misma medida.

# MATH EXPLORATIONS Resumen de Ideas 

Volume of a Cube/ Volumen de un Cubo: El volumen de un cubo con lados de medida s es $\mathrm{s} 3, \mathrm{~V}=\mathrm{s} 3 \circ \mathrm{~V}=\mathrm{Bh}$ donde B es la area de la base y h es la altura.

Volume of a Cylinder/ Volumen de un Cilindro: El volumen de un cilindro con radio ry altura h es $\mathrm{V}=\mathrm{Bh}$ donde B es la area del circulo $\mathrm{o} \mathrm{V}=\varpi \mathrm{r} 2 \mathrm{~h}$.

Volume of a Prism/ Volumen de una Prisma: El volumen de una prisma es la area de la base B multiplicada por la altura h de la prisma. Escrito $\mathrm{V}=\mathrm{Bh}$.

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## Finance

SECTION A. 1 INTEREST

Let's begin by looking at simple interest, and then see how banks extend this idea to everyday business. First, what does it mean to invest a principal amount of money $P$ in a bank at a simple interest rate $r$ ? The bank's customers are letting the bank use their money for a period of time, and in return the bank is willing to return their original amount at the end of the period plus extra money called Interest. The amount of extra money depends on the interest rate that the bank pays and the period of time.

So if Mrs. Hamlisch puts $\$ 200$ into a savings account at the beginning of the year at $12 \%$ interest rate, at the end of the year her account will have $\$ 200$ plus the interest that she earned. This interest is $12 \%$ of $\$ 200$, or, from the simple interest formula $I=P r t$,

$$
I=(0.12) 200=24
$$

So at the end of the year she will have $\$ 200+\$ 24=\$ 224$ in her bank account.

## EXAMPLE 1

Mr. Lablanc invests $\$ 500$ at a simple interest rate of $6 \%$. How much interest will he earn in one month? In six months? In one year?

Solution The yearly interest he will earn is $6 \%$ of his original principal if he put his money in the bank for one year. However, in one month, he will only earn $\frac{1}{12}$ of this amount. The interest earned in one month is $I=(500)(0.06)(1 / 12)=\$ 2.50$. Similarly, in 6 months, the interest he will earn is $I=(500)(0.06)\left(\frac{6}{12}\right)=\$ 15$.

## Chapter A Appendix

In one year the interest he will earn is $I=(500)(0.06)(1)=\$ 30$. Compare the three amounts of interest to see if they are reasonable.

Each case above uses the Simple Interest Formula:

## SIMPLE INTEREST FORMULA

If a principal amount $P$ is invested at an interest rate $r$ for $t$ years, then the simple interest earned will be $I=P r t$.

We can use the simple interest formula to find a formula for the amount of money $A$ that will be in a simple interest account after $t$ years. The amount $A$ is the original principal $P$ plus the interest $I$ earned over the period of time $t$. So:

$$
A=P+I=P+P r t .
$$

To simplify the formula factor $P$ from the right-hand expression to obtain

$$
A=P+P r t=P(1+r t) .
$$

## PROBLEM 1

Compute the total amount of money Mr. Lablanc will have in his account from Example 1. Do this in two ways:

1. Add the amount of interest earned to the initial amount. This is the total amount in the account after each time period.
2. Check your work by using the Simple Interest Formula for amount.

Most accounts, however, set more than one interest period a year. This is called compounding. Financial institutions divide the year into a certain number of periods and add simple interest to an
A-2
account after each period.

## EXAMPLE 2

If you invest a principal P at an interest rate $r$ compounded monthly, how much interest will $P$ earn in one period?

Solution Each period is one month long, so the length of time for one period is $\frac{1}{12}$ of a year. The interest earned for one period will be $I=P \times r \times\left(\frac{1}{12}\right)$. This is the same as we found in Example 1 .

In general, if there are $m$ periods in a year, the length of time for each period will be $\left(\frac{1}{m}\right)$ of a year. The interest earned in one period will be $I=P \frac{r}{m}$.

The periodic interest rate, then, is $\frac{r}{m}$. We can $I$ the periodic interest amount and $i=\frac{r}{m}$. So the interest earned in one period Is $I=P i$. That means the amount of money in an interest-earning account at the end of a period is $P+P i$. This looks just like the simple interest formula except the interest rate $r$ is replaced by the periodic interest rate $i=\frac{r}{m}$.

If an account earns interest compounded every six months, the periodic interest rate per each six-month period is $i=\frac{12 \%}{2}=6 \%$. If the account earns interest compounded quarterly, or four times a year, the periodic interest rate is $i=\frac{12 \%}{4}=3 \%$. Many accounts earn interest each month, so $i=\frac{r}{12}$.

## EXAMPLE 3

Suppose we deposit \$100 at 12\% per year compounded monthly. How much will be in the account after three months? Find a formula for the amount in the account at the end of $t$ months.

Solution Let's make a list of the amount of money in your account at the end of each of the first 3 months. Since one month is $\frac{1}{12}$ of a year, the interest rate for one month is $i=\frac{.12}{12}=.01$. The
amount in the account after one month would be $100+(.01)(100)=$ $100+1=101$.

Let $A(t)=$ amount in the account at the end of $t$ months. So,

$$
\begin{aligned}
A(0) & =100=P=\text { amt. in account at beginning of first month } \\
A(1) & =P+P r \\
& =P(1+i) \\
& =P(1.01)=\text { amt. in account at end of first month }
\end{aligned}
$$

Notice that we factor the $P$ out of the sum producing the factor $(1+i)$. Computing the next two months, we get

$$
\begin{aligned}
A(2)=A(1)+A(1) i & =A(1)(1+i) \\
& =[P(1+i)](1+i) \\
& =P(1+i)^{2} \\
& =P(1.01)^{2}
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
A(3)=A(2)+A(2) r & =A(2)(1+i) \\
& =\left[P(1+i)^{2}\right](1+i) \\
& =P(1+i)^{3} \\
& =P(1.01)^{3}
\end{aligned}
$$

Using this pattern, find the values for $A(4), A(5), A(12)$ and $A(t)$. Fill in the following table.
A-4

| month | $A(t)$ |
| :---: | :---: |
| 0 | $\$ 100$ |
| 1 | $\$ 101$ |
| 2 | $\$ 102.01$ |
| 3 | $\$ 103.03$ |
| 4 | $\$ 104.06$ |
| 5 |  |
| 12 |  |
| $t$ |  |

Of course this pattern works with different interest rates. The key idea is that for each period that passes, the amount at the end of the period is equal to the amount at the beginning of the same period multiplied by $(1+i)$, leading to the

## COMPOUND INTEREST FORMULA

If an initial principal $P$ is invested at an interest rate $r$ compounded $m$ times per year, then the amount in an account after $n$ periods is $A(n)=P(1+i)^{n}$, where $i=\frac{r}{m}$ is the interest earned each period.

## EXAMPLE 4

Sally invests $\$ 1000$ at $12 \%$ simple interest for three years. Ann invests $\$ 1000$ at a rate of $12 \%$ compounded monthly for three years.

1. How much money will Sally have after three years?
2. How much money will Ann have?
3. Are you surprised by the results in any way?
4. If the time invested were ten years, by how much would the amounts differ?

## Solution

1. The amount that Sally will have is $A=1000(1+0.12 * 3)=$
$\$ 1360$.
2. First, find the periodic interest rate: $i=\frac{0.12}{12}=0.01$. Second, find the number of periods in three years. Because $m=12$, there are 12 periods per year. In three years, there are 36 periods. Finally, use the compound interest formula to find Ann's amount, using a calculator:

$$
A=1000(1+0.12 / 12)^{36}=\$ 1430.77
$$

3. Ann will have $1430.77-1360=\$ 70.77$ more than Sally after 3 years.
4. Performing these calculations for ten years, Sally will have $A=$ $1000(1+.12 \times 10)=\$ 2200$. Ann will have $A=1000(1+$ $.01)^{120}=\$ 3300.39$.

So after ten years, Ann will have $\$ 1100.39$ more than Sally.

## PROBLEM 2

Suppose we deposit $\$ 1000$ at $12 \%$ per year compounded monthly. How much will be in the account after 6 months? Find a formula for the amount in the account at the end of $t$ months. Graph the function using your graphing calculator. Using the graph, estimate how many months it will take before you have $\$ 1100$.

## EXERCISES

1. Sue invests $\$ 500$ in the bank at a simple interest rate of $12 \%$. How much interest will she earn after
a. 2 months?
b. 6 months?
c. 1 year?
d. 2 years?
2. Chris invests $\$ 100$ in the Simple Bank of America at a simple interest rate of $8 \%$. How much will be in his account after
a. 2 months?
b. 6 months?
c. 1 year?
d. 2 years?
e. 5 years?
3. When interest is compounded monthly, how many periods are there in
a. 3 months?
b. 1 year?
c. 5 years?
4. Chris discovers he could invest Texas Compound Bank. In that bank he can invest $\$ 100$ in at an interest rate of $8 \%$ compounded monthly. How much will be in his account after
a. 2 months?
b. 6 months?
c. 1 year?
d. 2 years?
e. 5 years?
f. Compare with your answer from Exercise 2. Should he switch banks?
5. Jackie and Jennifer are sisters. On January 1, 1950, Jackie put $\$ 1000$ in her safe and forgot about it. Her sister Jennifer only had $\$ 100$ at that time. But she put her money into an account at the bank which paid $5 \%$ compounded yearly and forgot about it. On January 1,2000 , they decided to take out their money and throw a party. Who had more money to spend on the party? How much more?
6. A bank offers three types of accounts. In the Bronze account, you earn $12 \%$ annual interest compounded monthly. In the Silver account, you earn $12.2 \%$ compounded twice a year. Finally, in the Gold account you earn $12.4 \%$ compounded once a year. Which is the better deal? If you deposit $\$ 100$ how much will you have at the end of the year?
7. How much money will Mr. Garza need to deposit into an account earning $12 \%$ per year ( $1 \%$ per month) compounded monthly in order that he have $\$ 500$ at the end of 3 years?
8. John deposited $\$ 200$ in the bank at a interest rate of $9 \%$ compounded monthly. How much will be in the account after 6 months?
9. Victoria invests $\$ 100$ at a simple interest rate of $8 \%$. Penelope invests $\$ 100$ at a rate of $8 \%$ compounded monthly. How much will each girl have:
a. After 6 months?
b. After 1 year?
c. After 10 years?

Compare the amounts and summarize what you found.
10. Using the simple interest formula $A=P(1+r t)$,
a. Knowing $P, r$, and $t$, solve for $A$.
b. Given $A, r, t$, solve for the principal $P$.
c. Given $A, P, r$, solve for the time $t$. Using this information, find out how long will it take Sam to have $\$ 200$ in his account, after he invests $\$ 100$ at a simple interest rate of 8\%.
11. Solve the compound interest formula $A=P(1+i)^{n}$ for $P$. What does the new formula tell you? Make up a problem that you could use your new formula to solve.
12. Use your calculator to graph the following two functions. Then explain what information it gives.: $A=100(1+0.08 t)$. $A=$ $100(1+0.08 / 12)^{1} 2 t$. What difference do you notice in the two graphs? Which grows faster?
13. Compare the difference in simple interest from money invested at $8 \%$ and interest compounded monthly for money invested at $8 \%$ after five years.
14. Explain to a fifth grader the difference between simple interest and compound interest.
15. Investigation:

How long will it take money to double at a compound interest rate of $12 \%$ compounded monthly? At $8 \%$ ? Research and explain the Banker's Rule of 72.
16. Ingenuity:

Compute each of the products from a-d. Then speculate what the products are for parts e-f.
a. $\left(1+x+x^{2}\right)(1-x)$
b. $\left(1+x+x^{2}+x^{3}\right)(1-x)$
c. $\left(1+x+x^{2}+x^{3}+x^{4}\right)(1-x)$
d. $\left(1+x+x^{2}+x^{3}+\cdots+x^{8}\right)(1-x)$
e. $\left(1+x+x^{2}+\cdots+x^{19}+x^{20}\right)(1-x)$
f. $\left(1+x+x^{2}+\cdots+x^{(n-1)}+x^{n}\right)(1-x)$

## SECTION A. 2 COST OF CREDIT

What happens if instead of depositing money in the bank, we borrow money from the bank. For example, if we use a credit card. Do you think the interest rate will be the same, smaller, or larger? Now $A$ will be the amount we owe. How can we show that we owe the bank instead of the bank owing us?

Usually a traditional loan uses the compound interest formula. As with the simple interest formula, the principal $P$ is the amount of the loan, and the amount $A$ is the total amount to be repaid. Here is an example that uses the compound interest formula to understand loans.

## EXAMPLE 1

You use a credit card to purchase a $\$ 100 \mathrm{MP} 3$ player. The credit card company charges $24 \%$ per year compounded monthly. How much do you owe after $t$ months if you don't pay the credit card company any money? How much do you owe at the end of the year?

Solution You used the card to purchase a $\$ 100$ MP3 player so the initial value is $\$ 100$. To indicate that you owe the money, write $P=-100$. The annual rate is $24 \%$ or $i=2 \%$ per month. So the amount after $t$ months is

$$
A=-100(1+.02)^{t}=-100 \cdot 1.02^{t}
$$

At the end of the year, the amount is

$$
A=-100 \cdot 1.02^{12}=-126.8242
$$

Hence you owe $\$ 126.82$ to the credit card company.

## EXPLORATION 1

Many Americans have money in savings accounts that pay interest and at the same time owe money to credit card companies.

Suppose we put $\$ 1000$ dollars in the bank and the bank pays $12 \%$ compounded monthly. At the same time we use the credit card to purchase $\$ 1000$ of clothes. The credit card company charges $24 \%$ compounded monthly. How much do we have in the bank after $t$ months? How much do we owe to the credit card company after $t$ months? Does this make sense to you? What function could we graph to investigate what happens? Graph the function and see what happens at the end of the year.

When you borrow money to purchase a very expensive item like a car or a house, the lender (usually a bank) asks you to pay something each month. After each payment is made, that much less is owed the bank. In this way, you slowly pay off your loan. The formulas in this situation are complicated, but we can compute a few steps to explore how this works. Then we can use an online calculator to analyze a more realistic situation.

## EXPLORATION 2

Imagine you borrow $\$ 10000$ to purchase a car. The bank says it will charge $10 \%$ annual interest rate.

1. Suppose you pay the bank $\$ 500$ every 6 months.
a. How long do you think it will take to repay the loan?
b. Since you are paying each six months, the bank recomputes the interest and the money owed each six months. The money owed is called the principal of the loan. How much interest do you owe for the first 6 months? Adding the initial principal, \$10000, and the interest, how much do you owe right before your first payment? How much do you owe after your first payment? What do you notice?
c. Repeat the calculations above for the second 6 month period: How much interest do you owe for the second 6 months? Adding the amount owed after the first payment, and the interest, how much do you owe right before your second payment? How much do you owe after your second payment?
d. At this rate, how long will it take to pay off the loan?
2. Now suppose you pay the bank $\$ 1000$ every 6 months. You may use the table below to organize your calculations.
a. Now how long do you think it will take to repay the loan?
b. How much interest do you owe for the first 6 months? Adding the initial principal, $\$ 10000$, and the interest, how much do you owe right before your first payment? How much do you owe after your first payment?
c. Repeat the calculations above for the second 6 month period: How much interest do you owe for the second 6 months? Adding the amount owed after the first payment, and the interest, how much do you owe right before your second payment? How much do you owe after your second payment?
d. Compare the first payment and the second payment. How much did the principal go down after the first payment? How much did it go down after the second payment? Why are these different?

| Principal <br> Amt. at Beginning | Interest Owed <br> for Period | Amt. at End <br> of Period | Payment |
| :---: | :---: | :---: | :---: |
| 10000 |  |  | 1000 |
|  |  |  |  |

If you continue the calculations in 2 , you will see that it takes over 7 years to pay off the loan if you pay $\$ 1000$ every six months. In the middle of 8th year, you will have to make a final payment of \$210.71.

It is also possible to use an on-line calculator to make these computations. For example, use a search engine to find a loan calculator. What is the web address?

## EXPLORATION 3

Use an online calculator to investigate how car loans work. Typically, when you buy a car you negotiate with the car dealer or bank the terms of the loan. Two important features are the annual interest rate and total length of the loan. For a one year loan, you must pay
back the whole loan in one year ( 12 monthly payments). For a 6 year loan, you pay the loan back over a longer period ( 72 monthly payments). As you saw in Exploration 2, the amount you still owe on the loan and the interest on the loan are calculated after each payment.

Set the original amount of the loan to $P=\$ 10000$. Use the calculator to fill in the table.

1. Set the interest rate to $5 \%$. Fill in the following table:

| Length of Loan | Monthly Payment | Total Interest |
| :---: | :---: | :---: |
| 1 |  |  |
| 2 |  |  |
| 3 |  |  |
| 4 |  |  |
| 5 |  |  |
| 6 |  |  |

a. Using the data in the table, plot the Length of the Loan versus the Monthly Payment.
b. What happens to the Monthly Payment as the length of the loan increases? Why does this make sense?
c. Based on the graph, which type of loan would you prefer and why?
d. On a separate coordinate plane, now plot the Length of the Loan versus the Total Interest.
e. What happens to the Total Interest as the length of the loan increases? Why does this make sense?
f. Now which loan do you prefer?
2. Set the interest rate to $10 \%$. Fill in the following table:

| Length of Loan | Monthly Payment | Total Interest |
| :---: | :---: | :---: |
| 1 |  |  |
| 2 |  |  |
| 3 |  |  |
| 4 |  |  |
| 5 |  |  |
| 6 |  |  |

a. Add the data from this table to plot of the Length of the

## Chapter A Appendix

Loan versus the Monthly Payment above. If possible, use a different symbol to tell the two data sets apart.
b. Add the data from this table to plot of the Length of the Loan versus the Total Interest above. If possible, use a different symbol to tell the two data sets apart.
c. Now which loan would you prefer?
3. Interest rates on credit cards tend be very high. Predict what the graphs would look like if the interest rate was $20 \%$.

## PROBLEM 1

John has a yearly income of $\$ 45,000$. He is able to pay all of his bills on time each month, and has $\$ 425$ after regular expenses. John is considering buying a new car. Assuming that John qualifies for a loan at 3\% for 6 years, how expensive a car can he afford? John has three cars he is considering:

Car 1: Costs $\$ 18,000$ with other monthly expenses for insurance, gas and repairs of $\$ 120$ per month.
Car 2: Costs $\$ 24,000$ with other monthly expenses for insurance, gas and repairs of $\$ 150$ per month.
Car 3: Costs $\$ 30,000$ with other monthly expenses for insurance, gas and repairs of $\$ 180$ per month.

Which of these three options can John afford to buy? Would it be financially wise to spend his entire $\$ 425$ for the car? Explain.

Scientific and business calculators typically have options to compute monthly payments as well. In this example. we will show how one calculator can be used to compare loan options.

## EXAMPLE 2

Janice has a credit card bill of $\$ 2500$ that she needs to pay off. There are several options she is considering:

1. Repay the amount over a period of two years at an interest rate of $6 \%$ compounded monthly. If she chooses this method, what
will her monthly payment be, and what will the total cost of the loan be?
2. Repay the amount over a period of four years at an interest rate of $6 \%$ compounded monthly. What will her monthly payment be, and What will the total cost of the loan be?
3. Repay the amount over a period of six years, at an interest rate of $6 \%$ compounded monthly. What will her monthly payment be, and what will the total cost of the loan be?
In each of these options, Janice has the same interest rate, but a different length of time to make her payments. However, Janice needs to make sure that she makes her payments on time, or her interest rate will be increased to $18 \%$ compounded monthly.
4. How much is Janice's monthly payment if she repays the credit card bill of $\$ 2500$ over a two-year period at an interest rate of $18 \%$ compounded monthly. How much will the total cost of the loan be?
5. How much is Janice's monthly payment if she repays the credit card bill of $\$ 2500$ over a four-year period at an interest rate of $18 \%$ compounded monthly. How much will the total cost of the loan be?
6. How much is Janice's monthly payment if she repays the credit card bill of $\$ 2500$ over a six-year period at an interest rate of $18 \%$ compounded monthly. How much will the total cost of the loan be?

Solution The TI-83 Plus has special application to solve problems about loans and investments. To use the calculator

- Press APPS key. Choose 1: Finance.
- Choose 1: TVM Solver.
- A screen will appear where you enter the information. The following table explains each number you must enter.


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| Variable | Interpretation |
| :--- | :--- |
| $N$ | total number of payments |
| $I \%$ | annual interest rate (\% not decimal) |
| $P V$ | present value or principal |
| $P M T$ | payment |
| $F V$ | future value, worth at end |
| $P / Y$ | number of payments per year |
| $C / Y$ | number of times compounded each year |
| $P M T$ | payment at of end month or beginning |

When you are borrowing money, the present value is the amount you borrow. You get this money right away or at the present. You slowly pay off the loan until you owe no more money. So the future value is 0 . Since we don't know the monthly payment, we leave that blank. When you borrow money, your first payment is at the end of the first month, so we choose the END for the final option on the screen. To find the monthly payment for the Janice's first loan option, we enter

| Variable on TI-83/84 | Value |
| :--- | :--- |
| $N$ | 24 |
| $I \%$ | 6 |
| $P V$ | 2500 |
| $P M T$ |  |
| $F V$ | 0 |
| $P / Y$ | 12 |
| $C / Y$ | 12 |
| $P M T:$ | END |

To have the calculator compute the monthly payment, press $A L P H A$ and then press SOLVE. . The calculator should fill in the PMT cell. In this case, we get -110.80 . The number is negative because you pay that amount and the amount owed goes down.

The TVM solver does not report the total amount paid nor the interest paid. But these can be easily computed. To find the total amount paid multiply the monthly payment by the number of months. To find the interest paid, subtract the amount of the loan from the total amount paid.

The table below shows the difference between an interest rate of
$6 \%$ and of $18 \%$ compounded monthly.

| Amount | Months | Rate | Payment | Total | Interest Paid |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 2500 | 24 | 0.06 | $\$ 110.80$ | $\$ 2,659.20$ | $\$ 159.20$ |
| 2500 | 48 | 0.06 | $\$ 58.71$ | $\$ 2,818.08$ | $\$ 318.08$ |
| 2500 | 72 | 0.06 | $\$ 41.43$ | $\$ 2,982.96$ | $\$ 482.96$ |
| 2500 | 24 | 0.18 | $\$ 134.81$ | $\$ 3,235.44$ | $\$ 735.44$ |
| 2500 | 48 | 0.18 | $\$ 73.44$ | $\$ 3,525.12$ | $\$ 1,025.12$ |
| 2500 | 72 | 0.18 | $\$ 57.02$ | $\$ 4,105.44$ | $\$ 1,605.44$ |

Different Payment Methods Notice the significant difference that the higher interest rate makes both to monthly payments as well as the interest paid over the life of the loan. Also, the amount of interest increases with the time it takes to repay the loan. If you borrow at a high interest and pay the loan off over a long period of time, the amount of interest paid can be very large. In Example 2 , when the interest rate was $18 \%$ and the loan was over 6 years, the interest paid was $\$ 1605$. This is $64.2 \%$ the original price of the car. However, some consumers cannot afford the monthly payment necessary to repay quickly, so they choose a longer term for the loan.

## The benefits and costs of financial responsibility Credit

 Bureaus are organizations that collect information about individuals' borrowing and bill-paying habits. The credit bureau then uses this information to assign a credit score to everyone who wants to borrow money. Banks use this credit score to determine what interest rate they should charge a particular borrower. The lower the score, the higher the interest rate you will be charged. Hence, it is important to have a good credit history to qualify for a lower interest rate so borrowing is not so costly. In fact, with a good credit history, consumers may qualify for higher loan amounts than usual. Factors that influence a credit history include the timely payment of all bills are, a stable and adequate source of income, and the absence of bankruptcy history. Bankruptcy is a legal term used when a person cannot repay his debts.It is financially beneficial to pay all credit card bills on time. It is absolutely financially necessary to make the minimum required payment. Not meeting the required payment usually results in a very high interest rate. The table above demonstrated that a higher

## Chapter A Appendix

interest rate results in much higher monthly payments, and costs much more money over the life of the loan.

## Credit Cards: The Real Story

Credit cards are a very convenient method of payment. But it is important to recognize some key facts:

- Eventually always must pay the bill.
- Interest rates on credit cards tend to be higher than other types of credit or loans.
- Typically, if you pay the entire balance of the credit card bill at the end of each month, you are not charged any interest.
- If you don't pay the entire bill at the end of the month, the amount you owe can increase rapidly.

Let's explore a typical situation.

## EXPLORATION 4

A credit card account charges no interest on purchases made during a month, if you pay the entire balance at the end of the month. If you don't pay the entire bill at the end of the month, you are required to pay a minimum payment that is equal to $2 \%$ of the balance and you are charges $15 \%$ annual interest, compounded monthly on the unpaid charges. Make a table that identifies the advantages and disadvantages of paying the balance on a credit card monthly versus paying the minimum required. Use examples and calculations to support your claims.

One type of loan is called an easy-access loan. Many banks assess a penalty if a customer overdraws his account or charges more than his credit limit. To protect against these accidents, some people take out an "easy access" loan, or line of credit. In this case, when the bank account is overdrawn or the credit card is charge too much, the bank will not charge an overdraft penalty. Instead, the bank will loan the money through as an easy-access loan. This loan is like a regular loan and will need to be repaid.

Typically, easy-access loans are for much shorter duration than a regular loan. Usually these will be repaid almost immediately, but nonetheless the customer is charged interest. In Problem 2 below, you will calculate the cost of an easy access loan with different interest rates and for different time periods. Again, use a calculator to find the monthly payment.

## PROBLEM 2

Sarah took out an easy-access loan of $\$ 750$. The bank has two rates for easy-access loans. For customer with good credit, the rate is $6 \%$. However, if a customer's credit is not so good, the rate is $12 \%$. And if any payments are missed, the rate is $18 \%$. There are different options for the length of time to repay the easy-access loan. The options for payment periods are one month, three months, six months, or one year. Fill out the table below to determine the monthly cost of repaying an easy-access loan, and the total cost of the loan.

| Amount | Months | Rate | Payment | Total | Interest |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 750 | 1 | 0.06 |  |  |  |
| 750 | 3 | 0.06 |  |  |  |
| 750 | 6 | 0.06 |  |  |  |
| 750 | 12 | 0.06 |  |  |  |
| 750 | 1 | 0.12 |  |  |  |
| 750 | 3 | 0.12 |  |  |  |
| 750 | 6 | 0.12 |  |  |  |
| 750 | 12 | 0.12 |  |  |  |
| 750 | 1 | 0.18 |  |  |  |
| 750 | 3 | 0.18 |  |  |  |
| 750 | 6 | 0.18 |  |  |  |
| 750 | 12 | 0.18 |  |  |  |

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## EXERCISES

Use an online loan calculator when necessary.

1. Explain the difference between a credit card loan and an easyaccess loan.
2. Change the principal in Example 2 to $\$ 5000$ and compute the costs.
3. Lisa wants to buy a new car that costs $\$ 18,500$. There is an $8.25 \%$ sales tax, a charged based on a given percentage of the selling price, so that her total cost will be the cost of the car plus sales tax. She needs to save up for a $10 \%$ down payment of the total cost.
a. How much is her down payment?
b. How much will Lisa need to borrow?
4. Sue want to purchase a new car and that costs $\$ 20,000$, including taxes. She will make a down payment of $\$ 2000$ and will borrow the remaining $\$ 18,000$. The interest rate she qualifies for depends on her credit score. The possible interest rates are $2 \%$ compounded monthly, $6 \%$ compounded monthly, and $12 \%$ compounded monthly. How does the interest rate affect her monthly payment?
5. Sue also needs to decide how long she should take to repay the loan. The options are three years, five years, or six years. If Sue only qualifies for the $6 \%$ interest rate, how does the loan option she chooses affect her monthly payment? How does the length of time for repaying the loan affect the amount of interest she will pay over the life of the loan?
6. Sarah has a credit card bill of $\$ 3500$ that she cannot pay all at once. So she decides to repay it over time. The possible rates she will have to pay are $6 \%, 12 \%$, and $18 \%$ compounded monthly. Calculate the total cost of repaying her credit card bill with each of these rates over a three-year period.
7. Sarah also needs to decide how long she should take to repay the loan. The options are three years, five years, or six years. If Sarah qualifies for the $6 \%$ interest rate, how does the loan option she chooses affect her monthly payment? How does the length of time to repay the credit card bill affect the amount
of interest she will pay over the life of the loan?
8. Janet has overdraft protection on her checking account and accidentally overdrafted her account by $\$ 500$. She decided to take out an easy-access loan for $\$ 500$. There are several possible interest rates she might be charged: $5 \%, 10 \%$, or $15 \%$. Use a calculator to calculate the total cost of repaying the loan with each of the above rates over a two-year period. Over a four-year period. Which costs more in interest: 5\% over 3 years or $10 \%$ over 2 years? Explain.
9. Below are two payment options for buying a $\$ 250,000$ house.
a. Obtain a $6 \%$ loan for 30 years.
b. Obtain a $4 \%$ loan for 15 years.

Explain the advantages and disadvantages of each option.
10. Nancy is getting ready to buy a car but is not sure how expensive a car she can afford. With her credit score, she qualifies for a $6 \%$ loan compounded monthly for three years, or a $12 \%$ loan compounded monthly for six years. What are the advantages and disadvantages of the two different payment methods? Explain. Why might the type of payment method Nancy chooses determine how expensive a car she can afford?
11. Sandra and Bill have a combined yearly income of $\$ 115,000$. They are able to pay all of their bills on time each month, and have $\$ 1250$ after expenses. Their rent is $\$ 800$ per month. Sandra and Bill are considering buying a new house. Assuming they qualify for a loan of $3 \%$ for 30 years, how expensive a house could they afford? There are three houses they are considering-one costs $\$ 100,000$, one costs $\$ 150,000$, and one costs $\$ 200,000$. Which of these three options can they afford? Are there other house expenses they need to consider besides the monthly payment? Which of the three choices is financially responsible? What are some of the possible costs that could result from not being able to make their monthly payments?
12. Identify some of the benefits of financial responsibility. What does it mean to be financially responsible?
13. Identify some of the costs of financial irresponsibility. How can these costs rise and cause future problems?

## SECTION A. 3 PLANNING FOR THE FUTURE

Cost of College One of the most important decisions you will make is your career. Once you decide, it may take many years to prepare for your chosen career. Many times preparing for a career involves getting a college education. Perhaps one of the best reasons to go to a two-year or four-year college is that this will give you more career options. There have been many studies about whether college graduates earn more money than their counterparts. While there might be some debate about how much more money a they will earn, there is widespread agreement that college graduates have more options than their contemporaries.

Anyone interested in a career in Science, Technology, Engineering, or mathematics (STEM) will almost certainly need a college degree, and quite possibly even more advanced training like a masters or PhD. Many other higher-paying careers require a college education, or an associate two-year degree from a community or junior college. One advantage of going to a junior college is that this will let you try it out, and it provides a convenient stopping point after two years if you decide you don't want to continue. And if you do decide to continue, then you may transfer to a 4 -year college and receive credit for your first two years at the junior college. However, there can be problems making this transfer as well since some of your courses taken at the junior college might not match the requirements of the degree program at the 4 -year college.

As reported by cbsnews.com the highest-paying bachelor's degrees for new graduates in 2012 were

- Electrical engineering \$52,307
- Chemical engineering $\$ 51,823$
- Mechanical engineering $\$ 51,625$
- Computer engineering \$50,375
- Computer programming \$48,714
- Industrial engineering \$48,566
- Computer science $\$ 47,561$

Table A. 1 The average cost of college: 2012-13

|  | Public 2-year <br> (in-state) | Public 4-year <br> (in state) | Private 4-year |
| :--- | :--- | :--- | :--- |
| Tuition \& fees | $\$ 3,131$ | $\$ 8,655$ | $\$ 29,056$ |
| Room, board, books, etc. | $\$ 12,453$ | $\$ 13,606$ | $\$ 14,233$ |
| Total cost | $\$ 15,584$ | $\$ 22,261$ | $\$ 43,289$ |
| Net price <br> (after scholarships, grants, aid) | $\$ 4,350$ | $\$ 5,750$ | $\$ 15,680$ |

- Civil and environmental engineering $\$ 45,621$

Notice that almost all of these are in engineering, and they all require a strong mathematics background.

Table A. 1 shows the average cost of colleges from 2012. On average, for a public school, an education at a two-year college will cost $\$ 15,584$ and at a four-year state college will cost $\$ 22,261$ per year. The net price reflects the fact that most students obtain scholarships, grants, and financial aid to help pay for college. The net cost is what you and your family need to pay on average, assuming that you can obtain scholarships and other financial aid to cover the rest of the cost.

## EXPLORATION 1

Work in groups. Have each group member explore one two-year college and one public four-year college. Compare their costs, including the family contribution at each. Discuss in your group how the costs vary from college to college. Is there anything that surprises you?

Saving for College It's not too early to make a savings plan to save the money needed for at least the first year of college. Because the amount of scholarships and financial aid available is unknown, you should plan to save $\$ 22,261$, an amount necessary to attend the first year of some four-year public colleges. This assumes that the costs of college will not rise over the next few years, which is

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probably wishful thinking.
The savings plan will involve you (or your parents) making monthly payments into a savings account. It is hard to find a savings account that pays a high interest rate, but let's assume that you can find an account that pays an interest rate of $3 \%$ compounded monthly.

## EXAMPLE 1

How much must be invested each month into a savings account that pays $3 \%$ interest compounded monthly to have $\$ 22,261$ after four years? How much to have $\$ 22,261$ after eight years? After twelve years? What action do your answers suggest?

Solution To solve this problem we will use the TVM solver on the calculator. In this case, you want to invest enough money monthly in an savings account, so that at the end you have enough money for college. Since we start with no money, the present value is 0 , but the desired future value is $\$ 22,261$. The process doesn't start until we invest our first payment, so the payments occur at the beginning of each month. To see how much you must invest monthly over 4 years, enter the following into the calculator:

| Variable on TI-83/84 | Value |
| :--- | :--- |
| $N$ | 48 |
| $I \%$ | 2 |
| $P V$ | 0 |
| $P M T$ |  |
| $F V$ | 22261 |
| $P / Y$ | 12 |
| $C / Y$ | 12 |
| $P M T:$ | BEGIN |

To have the calculator compute the monthly payment, press ALPHA and then press SOLVE. The calculator should fill in the PMT cell. In this case, we get -435.99 . The number is negative because you pay that amount each month. If we change the number of years we have to invest and repeat the calculation, we can make a table:

| Years | Monthly Payment (\$) |
| :---: | :---: |
| 4 | 435.99 |
| 8 | 204.95 |
| 12 | 128.30 |

Clearly, the longer you wait to save for college the more money you will need to invest each month. So it is a good idea to start early.

## EXPLORATION 2

Unfortunately, the cost of going to college does not stay the same over time. Due to inflation, the costs increase each year. Suppose that the cost of college increases by $3 \%$ per year and it costs $\$ 22261$ this year.

1. How much will college cost in 4 years?
2. How much more will you have to save each month in order to pay for college in four years? How does this compare to Example 1.
3. What if you plan to attend college 8 years from now?

To offset the cost of saving, plan to apply for scholarships and financial aid. In this way, your family contribution on average could be reduced to the net price in the table above, less than one-third of the total cost.

Father Off in the Future Just as small amounts of money invested monthly amount to significant savings for college, the same is true of a retirement plan. Investing or paying into an account at regular intervals, establishes an annuity or tax-free savings account.

## EXAMPLE 2

The Ortizes are newly-weds and have figured that they need to save $\$ 50,000$ to make the down payment on their dream house. To do this, they plan to make monthly deposits into an account that pays an interest rate of $r=6 \%$ compounded monthly. They can afford to save $\$ 600$ dollars each month. How long must they wait until
they have money they need?

Solution We can use TVM solver to determine this. In this case we know the monthly payment, but do not know how many months are needed. The future value is $\$ 50000$ and the present value is $\$ 0$. The payment is -600 because the Ortizes will pay this each month. Enter the following into the calculator:

| Variable on TI-83/84 | Value |
| :--- | :--- |
| $N$ |  |
| $I \%$ | 6 |
| $P V$ | 0 |
| $P M T$ | -600 |
| $F V$ | 50000 |
| $P / Y$ | 12 |
| $C / Y$ | 12 |
| $P M T:$ | BEGIN |

Leave the $N=$ cell blank and press APLHA and then SOLVE. The calculator should fill in $N=69.54$. So the Ortizes need to save for 70 months or almost 6 years.

It is also important to start saving for retirement early as well. Advances in medicine has increased the number of years that Americans are living after retirement.

## EXPLORATION 3

Planning for retirement is a complicated issue. There are even retirement specialists who help others make decisions about their retirement. However, there is a lot of useful information available on the web. Use the internet to explore the following questions.

1. How much money do experts suggest you need to save before you retire?
2. How much money should you invest each year in order to achieve this goal?
3. What is the effect of waiting 10 years before you start saving
for retirement? What about waiting 20 years?
4. There are many ways to save or invest for retirement. Describe at least three of these. How do they compare?

## EXERCISES

1. Find the tuition and fees of two in-state junior colleges. Estimate the total cost of attending each. Find the commuting and residential cost for each.
2. Find the tuition and fees of one public in-state four-year college, one out-of-state public college, and one private fouryear college. Estimate the total cost of attending each. Include travel costs, and assume that you will be living away from home.
3. Investigate three colleges that have programs you might be interested in attending. How much does each cost to attend per year?
4. Devise a savings plan to make a college education possible. Explain how small amounts of money invested each month could enable a future student to save enough to help pay for college.
5. Discuss retirement plans with your parents. Are they able to save money each month for retirement?

## SECTION A. 4 CHAPTER REVIEW

## Key Terms

| congruent | rotation |
| :--- | :--- |
| dilation | scale factor |
| irrational number | similar |
| radical | translation |
| reflection |  |

## Formulas Simple Interest: Compound Interest: <br> $$
I=P r t \quad A(n)=P\left(1+\frac{r}{m}\right)^{n}
$$

## Practice Problems

1. Angelina invests $\$ 900$ in the bank at a simple interest rate of $3 \%$. How much interest will she earn after
a. 1 month?
b. 1 year?
c. 2 years?
2. Valerie deposited $\$ 1200$ into a bank account and left the account alone. The account collects 4\% interest, compounded quarterly.
a. How much will be in the account after 5 years? After 10 years? After 20 years?
b. Write an equation to model the amount of money in the account.
3. Suppose that you use a new credit card to help furnish your home. After all the purchases, the total balance on your card is $\$ 12,457$ (Wow!). The credit card charges $18.8 \%$ interest compounded monthly. If your were not to make a payment for 6 months, what would your balance be?
4. As a vehicle ages, its value decreases or depreciates. The function that models this is $V(t)=I(.082)^{t}$, where $I$ is the original cost of the vehicle when it was new, $t$ is the time in years from the date of purchase and $V(t)$ is the value after $t$ years. Patricia's SUV cost $\$ 32,000$ brand new.
a. Find the value of Patricia's vehicle 3 years after she purchased it.
b. Instead of buying the car, Patricia could have leased the car. She would have paid $\$ 400$ per month and then would have returned the car at the end of 3 years. Would this have been a better deal? Explain.
5. Susana Ormany is financial advisor on television. One day she says, its better to have compounding work for by investing, than against you by borrowing. Explain what she means using two examples.
6. Lalo and Lazaro are brothers. Lalo is very financially responsible and always pays his bills on time. Lazaro, on the other hand, often pays late. The brothers go to a bank separately to ask for a loan. Which brother is likely to have to pay the higher interest rate? What will the effect of this be on the monthly payments, each one has to pay?

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[^0]:    A. $(2,3) ;(-4,3)$ Line a
    $(-1,0) ;(-1,4)$ Line $b$
    You can recognize the pattern for line $a$ is that all $y$-values are the same.
    You can recognize the pattern for line $b$ is that all $x$-values are the same.
    B. Make sure your students' column headings are labeled properly: $x$ and $y$.
    C. Order is very important in finding patterns. Make sure your students have checked their work with the line.
    D. At this point all your students, we would hope, have noticed that for line $a$, all the $y$-values are the same and for line $b$, all the $x$-values are the same.
    E. $(20,3)$ Line a
    $(-1,-10)$ Line $b$
    F. This can be written as $(x, 3)$ which means $y=3$.
    G. This can be written as $(-1, y)$ which means $x=-1$.

