OPTIMIZATION

Optimization problems are word problems dealing with finding the maximum or minimum solutions to a problem. Examples of optimization problems are as follows:

1. Given 20sq. ft. of cardboard, what are the dimensions of the biggest box that can be made?
2. If you wanted to construct a cylindrical tin can that would hold 10 fluid ounces of a liquid, what are the dimensions that will use the least amount of material?

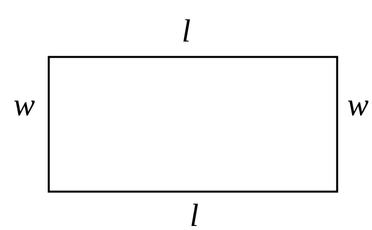
Before working optimization problems, a student must be aware of a couple of concepts from calculus.

1. A student must know how to take the first derivative of a function of one variable.
2. A student must be aware of the fact that in order to find the maximum or minimum points on a graph, one needs to take the first derivative of the function, set it equal to zero and solve for the unknown variable.

Let us look at an optimization problem. Be aware of the steps involved.

Example: A farmer wants to build a rectangular fence that will enclose 120 square feet for his dog Miff. The two long sides of the fence are to be made of Styrofoam at a cost of $5 per foot. The two shorter sides are to be made of wire at a cost of $6 per foot. What are the dimensions of the fence that will minimize cost?

First, let us draw a picture of the problem.



Next, let us write out our **Objective Equation**. The object of the problem is to minimize cost, so our Objective Equation must be an equation that represents the total cost of the fence (*Cost Equation)*.

$$\text{Cost} = 5l + 6w + 5l + 6w$$

$$= 5l + 5l + 6w + 6w$$

$$= 10l + 12w$$

Our next step is to see if we are constrained in some way. If so, we must write an equation representing this constraint (**Constraint Equation**). We were told that we had to have an area of 120 square feet, so our constraint equation would be the equation for area.

$$A = wl = 120$$

Since the object of this problem is to minimize cost, we know that we will have to take the first derivative of the cost equation, set it equal to zero and solve for the unknown variable. Right now, however, we cannot take the first derivative of the cost equation because it is not in terms of one variable; therefore, our next step is to get the **Objective Equation** (*Cost Equation*) in terms of one variable. It is usually easiest to do this by using the **Constraint Equation** to solve for one of the variables in the objective equation

Since $wl = 120$, then $w = 120/l$

and so substituting $120/l$ for w in the *Cost Equation*, we get:

$$\text{Cost} = 10l + 12(120/l)$$

$$= 10l + 1440/l$$

$$= 10l + 1440l^{-1}$$

Now we are ready to take the first derivative of the cost equation

$$C = 10l + 1440l^{-1}$$

$$C' = 10 + (-1440)l^{-2}$$

$$C' = 10 - 1440l^{-2}$$

Now we set the first derivative of the cost equation equal to zero and solve for l

$0 = 10 - 1440l^{-2}$ Adding $1440l^{-2}$ to both sides we get

$1440l^{-2} = 10$

$1440/l^2 = 10$ Multiplying by l^2 on both sides we get

$1440 = 10l^2$ Dividing by 10 on both sides we get

$144 = l^2$

$l = \pm 12$ Since we cannot have a negative length, our only remaining answer is

$l = 12$

Our final step is to <u>make sure we have solved the problem</u>. The problem was to find the dimensions that will give us a minimum cost. We have l but we still need to find w.

Since $w = 120/l$ (from the constant equation)

$w = 120/12 = 10$ $l = 12$ $w = 10$

The problem is solved.

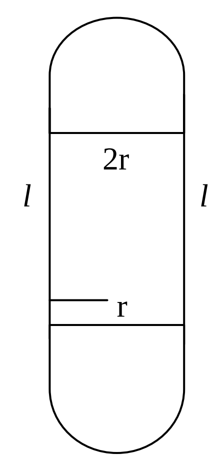
General Steps to Use When Solving Optimization Problems

1. Draw a picture.
2. Determine your **Objective Equation**. The Objective Equation is the equation that illustrates the object of the problem. If asked to maximize area, an equation representing the total area is your objective equation. If asked to minimize cost, an equation representing the total cost is your objective equation.
3. Determine your **Constraint Equation**. The Constraint Equation is an equation representing any constraints that you are given in the problem. **Note**: There may not always be a constraint in the problem**. This may imply that the objective equation is already in one variable**.
4. Make sure that the objective equation is in terms of one variable. You will probably be able to use the constraint equation in some way to complete this step.
5. Take the first derivative of the objective equation, set it equal to zero, and solve for your variable.
6. Go back and make sure you have solved for all variables in the problem.

Example: What is the maximum area of a rectangular window with semicircle caps (top and bottom) if the perimeter of the window is 30 feet.

1. Draw a picture

2. Determine your **Objective Equation**



Area of 2 semi-circles | Area of rectangle

$A = \pi r^2 + l \bullet 2r$

$\pi r^2 + 2lr$

3. Determine **Constraint Equation**

$P = 2l + 2\pi r = 30$

4. Express the **Objective Equation** in terms of one variable.

$A = \pi r^2 + 2\boldsymbol{l}r$

(where $\boldsymbol{l}$ is $\underline{15 - \pi r}$ as determined next)

Since $2l + 2\pi r = 30$

$2l = 30 - 2\pi r$

$l = 1/2\,(30 - 2\pi r)$

$l = \boxed{15 - \pi r}$

Then $A = \pi r^2 + 2(15 - \pi r)r$

$= \pi r^2 + 30r - 2\pi r^2$

$A = 30r - \pi r^2$

5. Take the first derivative of the **Objective Equation**, set it equal to zero, and solve for your variable.

$A' = 30 - 2\pi r$

$30 - 2\pi r = 0$

$30 = 2\pi r$

$30/2\pi = r$

$r = 15/\pi$

6. Go back and make sure that you have solved the problem.

$A = 30r - \pi r^2$

$= 30(15/\pi) - \pi(15/\pi)^2$

$= 30(15/\pi) - \pi(225/\pi^2)$

$= 450/\pi - 225/\pi$

$= 225/\pi$

The maximum area of the window is $225/\pi$ square feet.

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Source: Learning Skills Center
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