MATH EXPLORATIONS PART 1

5th Edition

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Preface and Introduction

Math Explorations follows several fundamental principles. It is important to carefully state these at the beginning, and describe how these are a perfect fit not only in educating the general student population, but also in teaching students whose native language is not English. These guiding principles will help the curriculum come alive for all students.

Learning math is not a spectator sport. The activities that fill the text and accompanying workbooks encourage students to develop the major concepts through exploration and investigation rather than by given rules to follow. **A crucial element is to understand the importance of small-group work, and to appreciate the extent to which everyone can benefit from working together.** In fact, often the process of explaining how to work a problem helps the explainer as much or more than the person who asks the question. As every teacher knows, explaining an idea to someone else is one of the best ways to learn it for oneself.

Some basic rules for discussion within a group include:

1. Encourage everyone to participate and value each person's opinions. Listening carefully to what someone else says can help clarify a question.
2. If one person has a question, remember that the chances are someone else will have the same question. Be sure everyone understands new ideas completely and never be afraid to ask questions.
3. Don't be afraid to make a mistake. In the words of Albert Einstein, "A person who never made a mistake never discovered anything new." Group discussion is a time of exploration without criticism. In fact, many times mistakes help to identify difficulties in solving a problem. Rather than considering a mistake a problem, think of a mistake as an opportunity to learn more about the process of problem-solving.
4. Always share your ideas with one another, and make sure that everyone is able to report the group reasoning and conclusions to the class. Everyone needs to know why things work and not just the answer. If you don't understand an idea, be sure

to ask "why" it works. You need to be able to justify your answers. The best way to be sure you understand why something works is to describe your solution to the group and class. You will learn more by sharing your ideas with one another.

If an idea isn't clear there are several things to try:

1. Look for simpler cases. Looking deeply at simple cases can help you see a general pattern.
2. Ask your peers and teacher for help. Go beyond "Is this the right answer?"
3. Understand the question being asked. Understanding the question leads to mathematical progress.
4. Focus on the process of obtaining an answer, not just the answer itself; in short become problem-centered, not answer-centered. One major goal of this book is to develop an understanding of ideas that can solve more difficult problems as well.
5. After getting help, work the problem yourself, and make sure you really understand. Make sure you can work a similar problem by yourself.

Suggestions for responding to oral questions in group and class discussion:

As you work through the Explorations in the book, working both individually and in groups can make understanding the material easier. Sometimes it is better to explore problems together, and other times you may want to explore first by yourself and then with others by discussing your ideas. When you discuss the problems as a group, it is more productive if you try to remember these simple rules:

1. Try not to interrupt when someone else is talking.
2. In class, be recognized if you want to contribute or ask a question.
3. Be polite and listen when others in your group or class are talking. This is one of the best ways to learn.

4. Finally, if you have a question, raise your hand and ask. Remember, there is almost always someone else with a similar or identical question.

Advice about reading and taking notes in math.

1. Reading math is a specific skill. When you read math, you need to read each word carefully. The first step is to learn the mathematical meaning of all words. Some words may be used differently in math than in everyday speech.
2. It is often necessary to write definitions of new words and to include mathematical examples. Try to write definitions in your own words without changing the meaning or omitting any important point. When you write down a definition, look for an example that illustrates what you are learning. This will help you relate what you are learning to real world situations.
3. Explaining new ideas and definitions that you read to your peers and teachers is very helpful. This will provide practice with any new definitions, and make sure that you are using the words correctly. Explaining a concept can help to correct any misconceptions and also reinforces learning.
4. If possible, try to draw a visual representation to make a difficult or new concept clear. It is really true that "a picture is worth a thousand words." Visual cues can help you understand and remember definitions of new terms.

Throughout this book, students learn algebraic thinking and the precise use of mathematical language to model problems and communicate ideas. The communication that makes this possible can be in small groups, in class discussion, and in student notes.

It is important to note that the use of variables and algebra is not an afterthought, but is woven throughout all of our books. By using language purposefully in small groups, class discussions, and in written work, students develop the ability to solve progressively more challenging problems.

The authors are aware that one important member of their audience is the parent. Parents are encouraged to read both the book and the accompanying materials and talk to their students about what they are learning.

Possibly the most unique aspect of this book is the breadth and span of its appeal. The authors wrote this text for both a willing 4th or 5th grader and any 6th grader. Students may particularly enjoy the ingenuity and investigation problems at the end of each set of exercises which are designed to lead students to explore new concepts more deeply.

The text has its origins in the Texas State Honors Summer Math Camp (HSMC), a six-week residential program in mathematics for talented high school students. The HSMC began in 1990 and is modeled after the Ross program at Ohio State, teaching students to "think deeply of simple things" (A. E. Ross). Students learned mathematics by exploring problems, computing examples, making conjectures, and then justifying or proving why things worked. The HSMC has had remarkable success over the years, with over 150 students being named semi-finalists, regional finalists, and national finalists in the prestigious Siemens Competition in Math, Science, and Engineering. Initially supported by grants from the National Science Foundation and RGK Foundation, the HSMC has also received significant contributions from Siemens Foundation, Intel, SBC Foundation, Coca-Cola, the American Math Society Epsilon Fund, and an active, supportive Mathworks Advisory Board.

In 1996, two San Marcos teachers, Judy Brown and Ann Perkins, suggested that we develop a pipeline to the HSMC that would introduce all young students to algebra and higher-level mathematics. Following their suggestion, we began the Junior Summer Math Camp (JSMC) as a two-week program for students in grades 4–8. We carefully developed the JSMC curriculum by meeting regularly with Judy and Ann, who gave us invaluable feedback and suggestions.

With support from the Fund for the Improvement of Postsecondary Education (FIPSE), Eisenhower Grants Program, Teacher Quality Grants, and the Texas Education Agency, we developed the JSMC into a replicable model that school districts throughout the state could implement. The JSMC curriculum was designed to prepare all students for higher-level mathematics. In some districts, the JSMC targeted gifted students; in other districts the program was delivered to mixed groups of students. In every setting, the program had remarkable results in preparing students for algebra as measured by the Orleans-Hanna algebra prognosis pre- and post-tests.

Over the years, we trained hundreds of teachers and thousands of students. Although we cannot thank each personally, we should mention that it has been through their suggestions and input that we have been able to continually modify, refine, and improve the curriculum.

A concern with the JSMC curriculum was that it was only supplementary material for teachers, and many of the state-required mathematics topics were not included. The Math Explorations texts that we have written have taken the JSMC curriculum and extended it to cover all of the TEKS (Texas Essential Knowledge and Skills), for grades 6-8 while weaving in algebra throughout. The third volume for 8th graders allows all students to complete Algebra I. This is an integrated approach to algebra developed especially for middle school students. By learning the language of mathematics and algebra, young students can develop careful, precise mathematical models that will enable them to work multi-step problems that have been a difficult area for U. S. students on international tests.

An accompanying Teacher Edition (TE) has been written to make the textbook and its mathematical content as clear and intuitive for teachers as possible. The guide is in a three-ring binder so teachers can add or rearrange whatever they need. Every left-hand page is filled with suggestions and hints for augmenting the student text. Answers to the exercises and additional activities are also provided in the TE.

This project had wonderful supporters in the Meadows Foundation, RGK Foundation, and Kodosky Foundation. A special thanks to the Mathworks Steering Committee, especially

Bob Rutishauser and Jeff Kodosky, who have provided constant encouragement and support for our curriculum project. The person who motivated this project more than any other was Jeff Kodosky, who immediately realized the potential it had to dramatically change mathematics teaching. Jeff is truly a visionary with a sense for the important problems that we face and ideas about how to solve them. His kind words, encouragement, and support for our JSMC and this project have kept me going whenever I got discouraged.

Our writing team has been exceptional. The primary basis for the book was our JSMC curriculum, coauthored by my wife, Hiroko Warshauer, and friend and colleague, Terry McCabe. The three of us discuss every part of the book, no matter how small or insignificant it might seem. Each of us has his or her own ideas, which together I hope have made for an interesting book that will excite all young students with the joy of mathematical exploration and discovery. During the summers of 2005-2013, we have been assisted by an outstanding group of Honors Summer Math Camp alumni, undergraduate and graduate students from Texas State, as well as an absolutely incredible group of pilot teachers. While it would take a volume to list everyone, we would be remiss not to acknowledge the help and support from these past summers.

Hiroko Warshauer led the team in developing this book, Math Explorations Part 1, assisted by Terry McCabe and Max Warshauer. Terry McCabe led the team in developing Math Explorations Part 2. Alex White from Texas State University provided valuable suggestions for each level of the curriculum, and took over the leadership of the effort for Math Explorations Part 3: Algebra 1.

We made numerous refinements to the curriculum in the school year 2012-2013, incorporating the 2012 revised TEKS, additional exercises, new warm-ups, and an accompanying collection of workbook handouts that provides a guide for how to teach each section. Many of these changes were inspired by and suggested by our pilot site teachers. Amy Warshauer, Alex Eusebi, and Denise Girardeau from Austin's Kealing

Middle School did a fabulous job working with our team on edits, new exercises, and the accompanying student workbook. Additional edits and proofreading was done by Michael Kellerman, who did a wonderful job making sure that the language of each section was at the appropriate grade level. Robert Perez from Brownsville developed special resources for English Language Learners, including a translation of key vocabulary into Spanish. Finally, Sam Baethge helped with putting together the glossary, proofreading, and checking and correcting the answer keys for all of the problems.

The production team was led by Namakshi P. Kant, a graduate student in mathematics education at Texas State. She did an outstanding job of laying out the book, editing, correcting problems, and in general making the book more user friendly for students and parents. Nama has a great feel for what will excite young students in mathematics, and worked tirelessly to ensure that each part of the project was done as well as we possibly could. As we prepared our books for state adoption, Bonnie Leitch came on board to help guide and support the entire project. Bonnie worked tirelessly to find where each of the Texas Essential Knowledge and Skills (TEKS) and English Language Proficiency Skills (ELPS) was covered in both the text and exercises. We added additional exercises and text to cover any TEKS that were not sufficiently addressed. Bonnie also edited these revisions and gave a final proofreading for each of the books, working with the authors to proofread every edit. However, in the end the authors take total responsibility for any errors or omissions. We do, however, welcome any suggestions that the reader might have to help make future editions better. In short, we had an incredible, hard-working team that did the work of an entire textbook company in a few short weeks! Without their help the project would not have reached its present state.

Finally, in this newest 5th edition (2016), we have continued to make edits and improvements. Genesis Dibrell, an undergraduate at Texas State, did a fabulous job of adding in these corrections, formatting the text to be consistent, and correcting any typos. Everyone at Mathworks contributed to the final product, especially Michelle Pruett, our new curriculum coordinator, and Patty Amende, who helped oversee the entire project. We could never have completed the project without the incredible help and dedication of our whole team.

Math Explorations Part 1 should work for any 6th grade student, while Math Explorations Part 2 is suitable for either an advanced 6th grade student or any 7th grade student. Finally, Math Explorations Part 3: Algebra 1 is a complete algebra course for any 8th grade student. The complete set of 3 books covers all of the Texas Essential Knowledge and Skills (TEKS) for grades 6-8 while also covering Algebra 1.

Math Explorations Part 1 was piloted by teachers and students in San Marcos, Austin, New Braunfels, and Midland. The results of these pilots have been extremely encouraging. We are seeing young 6th and 7th grade students reach (on average) 8th grade level as measured by the Orleans-Hanna test by the end of 7th grade.

Any curriculum will only be as effective as the teachers who use it, and without the support and encouragement of the administration and parents, this can never happen. In this, we have been very fortunate to be able to work with dedicated teachers and administrators from San Marcos, McAllen, New Braunfels, Midland, and Austin. The Mathworks staff gave invaluable help. Michelle Pruett, Andrew Hsiau, and Patty Amende provided support whenever needed. I hope you will join our team by giving us feedback about what works, what doesn't and how we can improve the book. By working together, I believe that we can develop a mathematics curriculum that will reach out to all students and that will engage students at a higher level than we have previously been able to achieve.

Max Warshauer

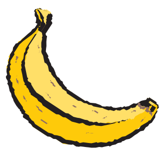
Max Warshauer
Director of Texas State Mathworks

EXPLORING INTEGERS

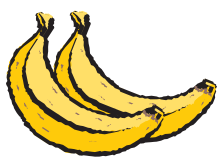
1

SECTION 1.1
BUILDING NUMBER LINES

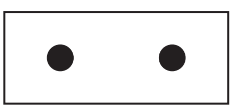
Let's begin by thinking carefully about numbers. Numbers are part of the mathematical alphabet, just like letters are used in English to form words. We use numbers for counting and representing quantities. When we think of the number one, we have in mind a picture:



Similarly, the number two describes a different quantity:

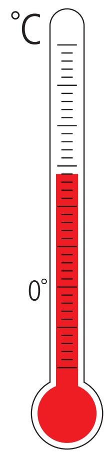


We could use a picture with dots to describe the number 2. For instance, we could draw:



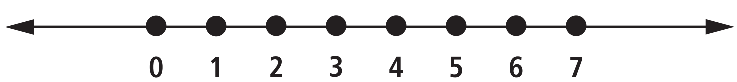
We call this way of thinking of numbers the "set model." There are, however, other ways of representing numbers.

Another way to represent numbers is to describe locations with the **number line model**, which is visually similar to a thermometer. To construct a number line, begin by drawing a straight line and picking some point on the line. We call this point the **origin**. Label the origin with the number 0. We can think of 0 as the address of a certain location on the number line. Notice that the line continues in both directions without ending. We show this with arrows at the ends of the line.



Next, mark off some distance to the right of the origin, and label the second point with the number 1.

Continue marking off points the same distance apart as above, and label these points with the numbers 2, 3, 4, and so on. Deciding how we label our marks is called **scaling**.



DEFINITION 1.1: COUNTING NUMBERS (POSITIVE INTEGERS)

The **counting numbers** are the numbers in the following never-ending sequence

$$1, 2, 3, 4, 5, 6, 7, \ldots$$

We can also write this as

$$^{+}1, ^{+}2, ^{+}3, ^{+}4, ^{+}5, ^{+}6, ^{+}7, \ldots$$

These numbers are also called the **positive integers,** or **natural numbers**.

One interesting property of the natural numbers is that there are "infinitely many" of them; that is, if we write down a list of natural numbers, there is always some natural number that is not on the list.

When we include the number 0, we have a different collection of numbers that we call the **whole numbers**.

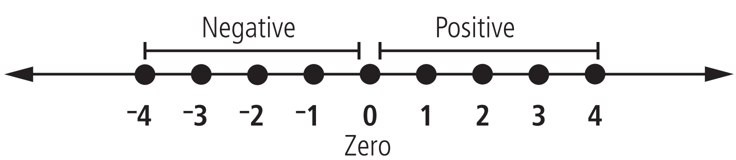
DEFINITION 1.2: WHOLE NUMBERS (NON-NEGATIVE INTEGERS)

The **whole numbers** are the numbers in the following never-ending sequence:

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots$$

These numbers are also called the **non-negative integers**.

In order to label points to the left of the origin, we use **negative integers**: $^{-}1$, $^{-}2$, $^{-}3$, $^{-}4$, … The sign in front of the number tells us on what side of zero the number is located. Positive numbers are to the right of zero; negative numbers are to the left of zero. Zero is neither positive nor negative.



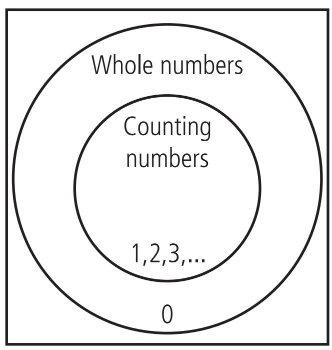
We have seen that numbers can be used in different ways. They can help us describe the quantity of objects using the set model or to denote a location using the number line model. Notice that the number representing a location can also tell us the distance the number is from the origin if we ignore the sign.

DEFINITION 1.3: INTEGERS

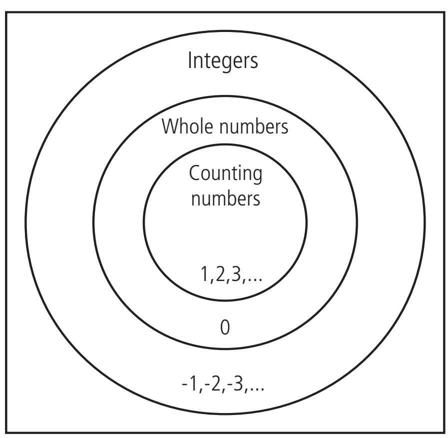
The collection of **integers** is composed of the negative integers, zero, and the positive integers:

$$\ldots, {}^-4, {}^-3, {}^-2, {}^-1, 0, 1, 2, 3, 4, \ldots$$

Notice that every counting number is a whole number, so the set of counting numbers is a subset of the set of whole numbers. One way to represent the relationship between these sets is to use a Venn Diagram. A **Venn Diagram** is simply a diagram that shows relationships between different sets in a visual way.



Notice that all of the whole numbers except 0 are inside the counting numbers. Similarly, every whole number is an integer. This is the Venn Diagram that relates the sets:



The integers that are not whole numbers are called the **negative integers**, so the integers are divided into three parts:

- The positive integers.
- Zero, which is a whole number but not a positive or negative integer.
- The negative integers.

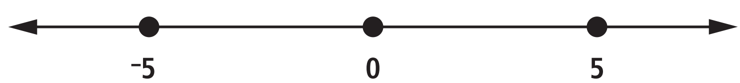
The fact that every integer is in exactly one of these three sets is called **trichotomy**, since it cuts the integers into these three parts. The prefix "tri" means three.

EXPLORATION 1: CONSTRUCTING A NUMBER LINE

1. Draw a straight line.
2. Pick a point on the line and call this point the origin. Label the origin with the number 0.
3. Locate and label the numbers $^-10$, $^-9$,...$^-2$,$^-1$, and 1, 2, 3, …, 10.
4. Where would 20, 30, 50 be located? 100? 1000?
5. Find the negative numbers corresponding to the numbers in question 4.

EXERCISES

1. At the zoo, the gift shop is located at the origin of Oak Street. Its address will be labeled as 0. Going right from the origin, the seal pool is located at address 8, and the monkey habitat is at address 4. Going in the other direction from the origin, the elephant habitat is located at address $^-3$, and the lion den is at address $^-7$. Draw a number line representing Oak Street. Label each of the locations on the number line. Watch your spacing.
2. a. Copy the line below to mark off and label the integers from 0 to 5 and from 0 to $^-5$. Use a pencil to experiment with the spacing because you might need to erase.



b. Make a new number line from ⁻10 to 10. Use what you learned about spacing to make it accurate.

3. Write an integer for each situation below. Find the point on the number line that corresponds to the integer. (create a number line from ⁻15 to 15, counting by fives but leaving a mark for each integer.)

a. Score 10 points

b. 8° below zero

c. A deposit of $12

d. A gain of 7 pounds

e. 4 ft. below sea level

f. A debt of $11

g. Neither positive nor negative

h. A withdrawal of $3

i. The opposite of ⁻4

j. Put in 9 gallons

4. For each of the following integers, create a section of a number line to illustrate which integer is immediately to the left of and which integer is immediately to the right of the given integer. Be sure to label which is which.

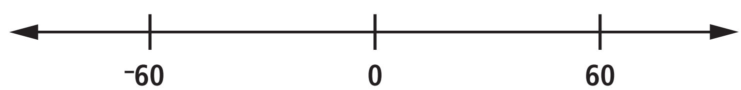
a. 5

b. ⁻3

c. 0

d. ⁻600

5. Draw a number line like the one below to mark off the numbers with equal distances by tens from 0 to 60 and from 0 to - 60. Use a pencil to experiment with scale.



a. Measure the distances from 0 to 30 and from 30 to 60. Are they the same?

b. Measure the distances from 10 to 20, 30 to 40, and 40 to 50. Are they the same?

c. Explain whether you need to rework your markings on the number line.

d. Estimate the location of the following numbers and label each on your number line: 25, 15,⁻35, ⁻7, ⁻54, 43, 18, 35, ⁻11, 48

6. Draw a number line so that the number ⁻600 is at the left end, and 600 is on the right end.

   a. What is the best amount to count by to ensure proper scaling?

   b. Sketch the locations of the following integers:

   25, 15, ⁻35, ⁻7, ⁻54, 43, 18, 35,⁻11, 48

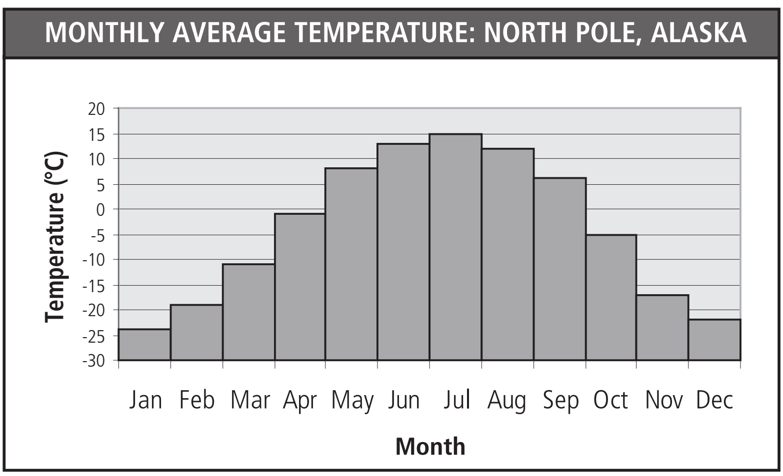
   0, 350, ⁻225, 250, ⁻450, ⁻25, ⁻599, ⁻50

7. Draw a number line. Find all the integers on your number line that are greater than 10 and less than 18. Circle the numbers you found.

8. How do you decide which number on a number line is greater? Draw 2 examples to explain your answer.

Notice that we can move the number line from the horizontal position to a vertical position. We would then have a number line that looks like a thermometer. Draw a thermometer (vertical number line) on the side of your paper to help you answer questions 9 through 12.

9. The chart below shows the monthly average temperatures for the city of North Pole, Alaska (not the actual North Pole, which is farther north). Based on the data, put the twelve months in order from coldest to warmest.



10. Nicholas visited his cousin, Marissa, in Anchorage, Alaska, where the temperature was $^{-}5$ °C. Michael visited his friend in Portland, Oregon, where the temperature was 8 °C. Which temperature is closer to the freezing point? Draw a thermometer to prove your answer. Remember, when we measure temperature in degrees Celsius (°C), 0 °C is the freezing point of water.

11. The temperature in Toronto, Canada, one cold day is $^{-}7$ °C. The next day, the temperature is 5 °C. Which temperature is closer to the freezing point? Draw a thermometer to prove your answer.

12. One cold day, the temperature in Oslo, Norway, is $^{-}9$ °C and the temperature in Stockholm, Sweden, is $^{-}13$ °C. Which temperature is colder? How much colder?

Spiral Review:

13. Name three fractions less than $\frac{2}{3}$.

14. Write $2\frac{3}{4}$ as an improper fraction.

15. **Ingenuity:**

    a. Draw a number line, then mark and label all of the integers from 3 to 11. How many integers have you marked?

    b. Draw a number line, then mark and label all of the integers from $^{-}7$ to 5. How many integers have you marked?

    c. Suppose we drew a number line and marked all of the integers from 12 to 37. How many integers would we mark if we did this?

    d. Suppose we drew a number line and marked all of the integers from 210 to 270. How many integers would we mark if we did this?

    e. Suppose we drew a number line and marked all of the integers from $^{-}120$ to 150. How many integers would we mark if we did this?

16. **Investigation:**

    For each of the following pairs of integers, decide which integer is further to the right on the number line.

    a. 5 and 12

b. 141 and 78

c. 7 and $^{-}1$

d. $^{-}4$ and $^{-}2$

e. $^{-}7$ and $^{-}10$

f. 9 and $^{-}55$

g. $^{-}8$ and $^{-}21$

h. $^{-}355$ and $^{-}317$