MATRICES

A matrix is a rectangular array of numbers. The numbers in the array are called the elements of the matrix. The array is enclosed with brackets.

- If \( m \) and \( n \) are positive integers, then an \( m \times n \) matrix (read "m by n") is a matrix that contains \( m \) rows (horizontal lines) and \( n \) columns (vertical lines).

- An array composed of a single row of numbers is called a row matrix.

- An array composed of a single column of numbers is called a column matrix.

- A matrix with the same number of rows and columns is said to be a square matrix.

- Two matrices are equal matrices if they have the same order \( (m \times n) \) and their corresponding components/elements are equal.

- An \( n \times n \) matrix that consists of 1’s (ones) on its main diagonal and 0’s (zeros) elsewhere is called the identity matrix of order \( n \) and is denoted by \( I \) or \( I_n \).

\[
I_{3 \times 3} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

- Note: An identity matrix must be square. If \( A \) is an \( n \times n \) matrix, then the identity matrix has the property that \( AI = A \) and \( IA = A \).

- If \( A \) is a square matrix and if there exists a matrix \( A^{-1} \) such that \( AA^{-1} = I = A^{-1}A \), then \( A^{-1} \) is called the inverse of \( A \).

- Matrix operations include:
  - addition/subtraction of matrices of the same order.
  - scalar multiplication—multiplication of a matrix by a number (usually called "scalar").
  - matrix multiplication—multiplication of a matrix by another matrix.

**Examples of Matrices:**

<table>
<thead>
<tr>
<th>2x3 matrix</th>
<th>Row matrix</th>
<th>Column matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E = \begin{bmatrix} 4 &amp; -4 &amp; 3 \ 7 &amp; -2 &amp; -5 \end{bmatrix} )</td>
<td>( E = \begin{bmatrix} 1 &amp; 3 &amp; -1 \ 7 &amp; -1 &amp; 2 \end{bmatrix} )</td>
<td>( T = \begin{bmatrix} .5 \ .75 \ .92 \end{bmatrix} )</td>
</tr>
<tr>
<td>Square matrix</td>
<td>Identity matrices</td>
<td></td>
</tr>
<tr>
<td>( U = \begin{bmatrix} 3 &amp; 2 \ 1 &amp; 7 \end{bmatrix} )</td>
<td>( I = \begin{bmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{bmatrix} ), ( I = \begin{bmatrix} 1 \end{bmatrix} )</td>
<td></td>
</tr>
</tbody>
</table>
MATRIX OPERATIONS

1. Matrix Addition and Subtraction

Two matrices can only be added and a matrix can only be subtracted from another if the matrices have the same number of rows and columns.

To add two matrices simply add the elements in corresponding positions.

\[
\begin{bmatrix}
5 & 3 \\
2 & 6 \\
9 & 4 \\
\end{bmatrix} + \begin{bmatrix}
-1 & 3 \\
-5 & -6 \\
\end{bmatrix} = \begin{bmatrix}
5+5 & 3+6 \\
2+(-1) & 6+3 \\
9+(-5) & 4+(-6) \\
\end{bmatrix} = \begin{bmatrix}
10 & 9 \\
1 & 9 \\
4 & -2 \\
\end{bmatrix}
\]

Note that the sum \( \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \end{bmatrix} \) is undefined.

To subtract a matrix from another proceed similarly:

\[
\begin{bmatrix}
5 & 3 \\
2 & 6 \\
9 & 4 \\
\end{bmatrix} - \begin{bmatrix}
-1 & 3 \\
-5 & -6 \\
\end{bmatrix} = \begin{bmatrix}
5-5 & 3-6 \\
2-(-1) & 6-3 \\
9-(-5) & 4-(-6) \\
\end{bmatrix} = \begin{bmatrix}
0 & -3 \\
3 & 3 \\
14 & 10 \\
\end{bmatrix}
\]

2. Matrix Multiplication

The product of two matrices is defined only if the number of columns of the first matrix equals the number of rows of the second matrix. That is,

\[
A_{m \times n} B_{n \times p} = AB_{m \times p}
\]

**Example 1:** Find the product \( AB \) where

\[
A = \begin{bmatrix}
-1 & 3 \\
4 & -2 \\
5 & 0 \\
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
-3 & 2 \\
-4 & 1 \\
\end{bmatrix}
\]

**Solution:**

First check if the product \( AB \) is defined. Since the order of \( A \) is \( 3 \times 2 \) and the order of \( B \) is \( 2 \times 2 \) and the number of columns of \( A \) is equal to the number of rows of \( B \) we know that the product \( AB \) is defined and is of order \( 3 \times 2 \).
To find $c_{11}$ we multiply corresponding entries in the first row of $A$ and the first column of $B$ and add these products. 

$$
C_{11} = (-1)(-3) + (3)(-4) = -9
$$

To find $c_{12}$ we multiply corresponding entries in the first row of $A$ and the second column of $B$ and add these products. 

$$
C_{12} = (-1)(2) + (3)(1) = 1
$$

Continuing this pattern, we find the values 

$$
C_{21} = (4)(-3) + (-2)(-4) = -4 \\
C_{22} = (4)(2) + (-2)(1) = 6 \\
C_{31} = (5)(-3) + (0)(-4) = -15 \\
C_{32} = (5)(2) + (0)(1) = 10
$$

This product is 

$$
AB = \begin{bmatrix}
-1 & 3 \\
4 & -2 \\
5 & 0
\end{bmatrix} \begin{bmatrix}
-3 & 2 \\
-4 & 1
\end{bmatrix} = \begin{bmatrix}
-9 & 1 \\
-4 & 6 \\
-15 & 10
\end{bmatrix}
$$

Example 2: Find the product $CE$ where 

$$
C = \begin{bmatrix}
18 & 3 \\
25 & 0
\end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix}
12 & 15 \\
45 & 38 \\
15 & 17
\end{bmatrix}
$$

Solution:

The product $CE$ is defined since the number of columns of $C$ is equal to the number of rows of $E$. 

$$
CE = \begin{bmatrix}
18 & 3 & 25
\end{bmatrix} \begin{bmatrix}
12 & 15 \\
45 & 38 \\
15 & 17
\end{bmatrix} = [b_{11} \quad b_{12}] = [726 \quad 809]
$$

$$
b_{11} = (18)(12) + (3)(45) + (25)(15) = 726 \\
b_{12} = (18)(15) + (3)(38) + (25)(17) = 809
$$
Example 3: Find the product \( AB \) where

\[
A = \begin{bmatrix}
5 & 2 \\
4 & 7 \\
9 & 3
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
6 & 1 \\
9 & 4 \\
3 & 2
\end{bmatrix}
\]

Solution:
The product \( AB \) is not defined.
The order of \( A \) is \( 3 \times 2 \) and the order of \( B \) is \( 3 \times 2 \).
Since the number of columns of \( A \) is not equal to the number of rows of \( B \), we cannot multiply \( A \) by \( B \).