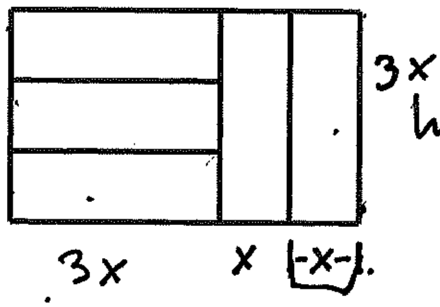
1. The figure below consists of five congruent rectangles that together form one large rectangle. If the area of the large rectangle is 60 square centimeters, what is the perimeter of the large rectangle?

$60cm^2$



Welcome!
We invite you to be on camera if you can.

$$(3x)(5x) = 60$$
$$15x^2 = 60$$
$$x^2 = 4$$
$$x = 2$$

Perimeter $= 2(5x + 3x) = 32\text{ cm}$

2. In the following list, each positive integer appears exactly seven times, and all integers occur in nondecreasing order. There are seven consecutive numbers in the list that add to 365. What is the middle number in this set of seven consecutive numbers?

1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 5, 5, . . .

$$\begin{array}{r} 52 \\ 7\,\overline{)365} \\ \underline{35} \\ 15 \\ \underline{14} \\ 1 \end{array}$$

$$(7)(52) = 364$$

$$51, 52, \underbrace{52, 52, 52, \boxed{52}, 52, 52, 53}_{365}$$

3. The integers a, b and c are all positive.

Given that a + b + c = 14 and 156a + 13 b + c = 873, find 100a + 10b + c.

$a = 6$? $156a > 900$ no good

$a = 4$? $156a = 624$ → too small.

$a = 5$ $156a = 780$

$13b + c = 93$

$b + c = 9$

$12b = 84$

$b = 7$ $c = 2$

$100a + 10b + c = 572$

4. Suppose each cell of the 3 × 3 grid shown below is filled with one of the digits 2, 3, 5, 6, or 7 so that the following conditions are met:

1. No two adjacent cells (horizontally or vertically) contain the same digit.
2. The three-digit number in the bottom row is equal to the sum of the three-digit number in the top row and the three-digit number in the middle row,
3. The three-digit number in the bottom row is as large as possible.

What is the sum of all 9 digits entered in the cells?

5	2	7
2	3	6
7	6	3

2, 3, 5, 6, 7

$5 + 2 + 7 + 2 + 3 + 6 + 7 + 6 + 3$

$= 41$

5. Positive integers x and y satisfy the equation $x^3 - y^3 = 218$. Find x + y.

$$X^3 - Y^3 = 218$$

$$(X-Y)(X^2+XY+Y^2) = 2(109)$$

$\boxed{\text{Both odd}}$ or both even.

X, Y odd. $X - Y = 2$

3, 1 ? $9^3 - 1^3 \neq 218$

5, 3 ? $125 - 27 \neq 218$

7, 5 ? $343 - 125 = 218$ ☺

$$X + Y = 7 + 5 = 12$$

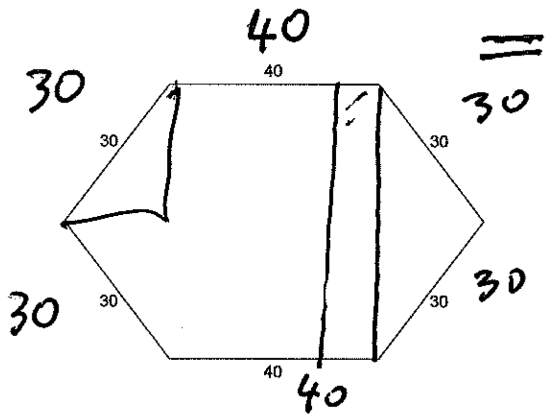
6. A six-digit positive integer is *sixish* if it has six different **non-zero** digits and the product of each pair of adjacent digits is divisible by 6. What is the greatest sixish six-digit positive integer?

9	8	6	4	3	2

7?

5?

9. The hexagon shown below has sides of length 40, 30, 30, 40, 30, and 30, and six interior angles each measuring 120°. If a *triangle unit* is the area of an equilateral triangle with sides of length 1, how many triangle units are in the area of the hexagon shown?



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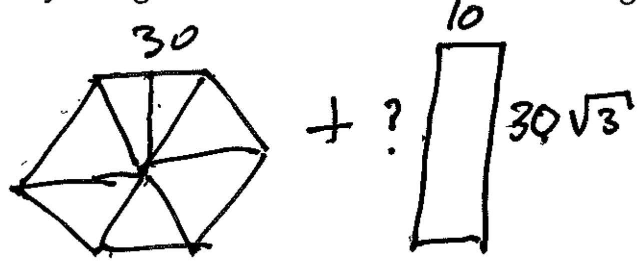
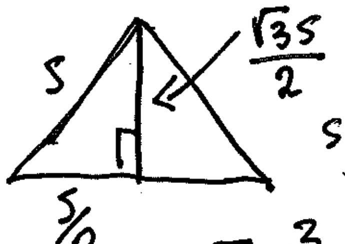


$$6(30)^2\frac{\sqrt{3}}{4} + (300)4\frac{\sqrt{3}}{4}$$



$s=30 \rightarrow \frac{\sqrt{3}s}{2} = 15\sqrt{3}$

$$= (5400+1200)\frac{\sqrt{3}}{4}$$
$$= 6600\frac{\sqrt{3}}{4}$$
$$= 6600\ \Delta\text{-units}$$

$$\text{Area} = \frac{\sqrt{3}\,s^2}{4}$$

Δ-unit $= \frac{\sqrt{3}}{4}$ sq units

10. In a sequence of 1000 positive integers, the first number is 42, and each subsequent number is double the previous one. How many of the 1000 numbers in the sequence have odd tens digits?

42, 84, 168, * 36, 72, 144, 88, 76, . . .

no carry $\rightarrow$ next 10s digit is even.

carry $\rightarrow$ next 10s digit is odd

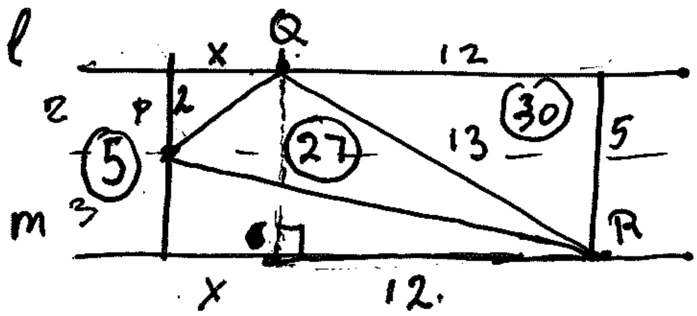
One's 2, 4, 8, 6, 2, 4, 8, 6, 2, 4, 8, 6, . . .

ten's E, E, E, O, O, E, E, O, O,

1000 of them:

500 − 1 = 499

11. A point P lies between two parallel lines l and m. The perpendicular distances from P to l and m are 2 units and 3 units, respectively. Points Q and R lie on lines l and m, respectively, so that QR = 13, the area of ΔPQR is 27 square units, and the distance from P to R is greater than the distance from P to Q. What is the distance PR?



Big $\square$ has area $5(x+12)$

$= 30 + 27 + x + \frac{3}{2}(x+12)$

$5x + 60 = 57 + \frac{5}{2}x + 18$

$\frac{5}{2}x = 15 \qquad x = 6$

~~$PQ = \sqrt{4+36}$~~

$PR = \sqrt{3^2 + 18^2} = 3\sqrt{37}$

12. An integer is *repetitive* if it uses the same digit at least twice in a row in base ten. Thus 100, 110, 998 and 999 are the first two and last two repetitive three-digit integers. Find the sum of all the three digit repetitive integers.

All same: xxx $111 + 222 + \cdots + 999$ (555 average!)

$= 9(555)$

xxy (no zero) $\rightarrow 72(555)$

xyy (no zero) $\rightarrow 72(555)$

xx0 $\rightarrow 9(550)$

x00 $\rightarrow 9(500)$

Ans $= 555(9 + 72 + 72) + 9(550 + 500)$

$= 94{,}365$

$1 + 2 + 3 + \cdots + 9 = 45$

13. The numbers 1, 2, ..., 8 are reordered into a sequence $x_1, x_2, \ldots, x_8$. How many possibilities are there for this sequence if the first two terms differ by at least 2, and the last two terms also differ by at least 2?

NO CONDITIONS: $N = \#\text{ways} = 8! = 8(7)(6)\ldots 2(1)$

X_1, X_2 adjacent 1 2 3 4 5 6 7 8   3,4 — — —

$|x_1 - x_2| = 1$

$\#\text{ways} = (7)\ (2)\ 6!$

1st two?

X_7, X_8 diff by 1: $\#\text{ways} = 7(2)6!$

X_1, X_2 and $X_7 X_8$ both ~~equal~~ differ by 1

#choices for 2 pairs

12 → 5
23 → 4
34 → 3
45 → 2
56 → 1

$(15)2(2)(2)4!$

$\text{ANS} = 8! - 4(7!) + 120(4!)$

$= 23040$

14. Trey and Quatro play a game where they repeatedly flip a fair coin until it either comes up heads three times in a row or tails four times in a row. Trey wins if the coin comes up heads three times in a row, and Quatro wins if it comes up tails four times in a row. What is the probability Trey wins the game?

P = Prob Trey wins

P_H = prob Trey wins if 1st is heads

P_T = " " " " 1st is tails

$P = \frac{1}{2} P_H + \frac{1}{2} P_T$

$P_H = \frac{1}{4} + \frac{3}{4} P_T$

$P_T = \frac{7}{8} P_H$

$P_H = \frac{1}{4} + \frac{3}{4} \cdot \frac{7}{8} P_H$

$\frac{11}{32} P_H = \frac{1}{4}$ $P_H = \frac{8}{11}$ $P_T = \frac{7}{11}$

H H H   H

HT

HHT

T TTT   1/8

TH   7/8

TTH

TTTH

$P = \frac{1}{2}\left(\frac{8}{11} + \frac{7}{11}\right) = 15/22$

1, 2, 3, 4, 5, 6, 7, 8

Which pairs go at ends?

1,2 → 34, 45, 56, 67, 78   5

3,4 →   4

$5+4+3+2+1 = 15$

for which pairs.

3,4   6, 7

→ start or end?   2 choices

6	7					4	3

which ~~way~~ is first, 3 or 4 ?   2

6 or 7 ?   2

4 choices   3 . . .

$(15)(8)(4!)$

Geometric Series Method

HHH

get tail, don't lose at once win

$$\frac{1}{8} + \frac{7}{8}\cdot\underset{H}{\frac{7}{8}}\cdot\frac{1}{4} + \frac{7}{8}\cdot\frac{7}{8}\left(\frac{3}{4}\cdot\frac{7}{8}\right)\cdot\frac{1}{4}$$

get tail don't lose don't win

$$+ \frac{7}{8}\cdot\frac{7}{8}\cdot\frac{3}{4}\cdot\frac{7}{8}\cdot\frac{3}{4}\cdot\frac{7}{8}\cdot\frac{1}{4} + \cdots$$

$$= \frac{1}{8} + \left(\frac{7}{8}\right)^2\left(\frac{1}{4}\right)\left[1 + \frac{21}{32} + \left(\frac{21}{32}\right)^2 + \cdots\right]$$

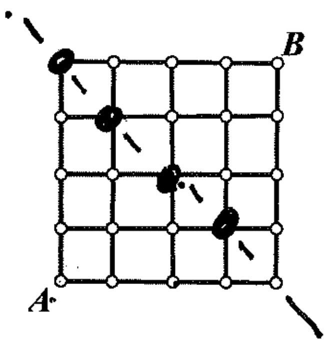
$$= \frac{1}{8} + \frac{7^2}{8^2}\cdot\frac{1}{4}\left(\frac{1}{1 - 21/32}\right)$$

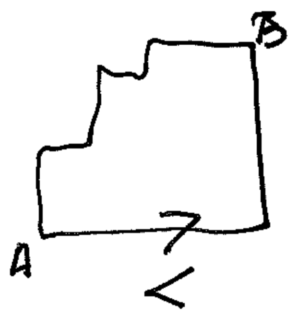
$$= \frac{1}{8} + \frac{7^2}{8^2}\cdot\frac{1}{4}\,\frac{32}{11}$$

$$= \frac{1}{8} + \frac{49}{88} = \frac{11+49}{88}$$

$$= \frac{60}{88} = \frac{15}{22}$$

15. A circuit is a path which has the same starting and ending point, but never visits any other point twice. How many circuits are there on the grid which start at A and follow a path to B along edges moving right or up only at each stage, and then returning to A moving along edges going left or down at each stage? [The direction in which a circuit is traversed matters, so that reversing a circuit gives a different circuit.]

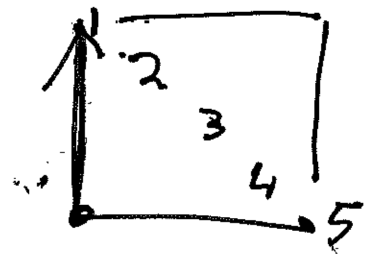




Anti-clockwise
clockwise.

Count clockwise routes, × 2

CROSS \ exactly twice



12	1^2
13	3^2
14	3^2
15	1^2
23	6^2
24	8^2
25	3^2
34	6^2
35	3^2
45	1^2

3 patterns below

$$\text{Ans} = 2(1^2+3^2+3^2+1^2+6^2+8^2+3^2+6^2+3^2+1^2)$$

$$= 350$$