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APPENDIX (FINANCE) A-1
Math Explorations follows several fundamental principles. It is important to carefully state these at the beginning, and describe how these are a perfect fit not only in educating the general student population, but also in teaching students whose native language is not English. These guiding principles will help the curriculum come alive for all students.

First, learning math is not a spectator sport. The activities that fill the text and accompanying workbooks encourage students to develop the major concepts through exploration and investigation rather than by given rules to follow. A crucial element is to understand the importance of small-group work, and to appreciate the extent to which everyone can benefit from working together. In fact, often the process of explaining how to work a problem helps the explainer as much or more than the person who asks the question. As every teacher knows, explaining an idea to someone else is one of the best ways to learn it for oneself. Some basic rules for discussion within a group include

1. Encourage everyone to participate, and value each person’s opinions. Listening carefully to what someone else says can help clarify a question. The process helps the explainer often as much as the questioner.

2. If one person has a question, remember that the chances are someone else will have the same question. Be sure everyone understands new ideas completely, and never be afraid to ask questions.

3. Don’t be afraid to make a mistake. In the words of Albert Einstein, “A person who never made a mistake never discovered anything new.” Group discussion is a time of exploration without criticism. In fact, many times mistakes help to discover difficulties in solving a problem. So rather than considering a mistake a problem, think of a mistake as an opportunity to learn more about the process of problem-solving.
4. Finally, always share your ideas with one another, and make sure that everyone is able to report the group reasoning and conclusions to the class. Everyone needs to know why things work and not just the answer. If you don’t understand an idea, be sure to ask “why” it works. You need to be able to justify your answers. The best way to be sure you understand why something works is to describe your solution to the group and class. You will learn more by sharing your ideas with one another.

If an idea isn’t clear, there are several things to try.

a. First, look for simpler cases. Looking deeply at simple cases can help you see a general pattern.

b. Second, if an idea is unclear, ask your peers and teacher for help. Go beyond “Is this the right answer?”

c. Third, understand the question being asked. Understanding the question leads to mathematical progress.

d. Focus on the process of obtaining an answer, not just the answer itself; in short become problem centered, not answer centered. One of the major goals of this book is to develop an understanding of ideas that can solve more difficult problems as well.

e. After getting help, work the problem yourself, and make sure you really understand. Make sure you can work a similar problem by yourself.

Some hints to help in responding to oral questions in group and class discussion: As you work through the Explorations in the book, working both individually and in groups can make understanding the material easier. Sometimes it is better to explore problems together, and other times try exploring first by yourself and then discuss your ideas with others.
When you discuss the problems as a group, it is more productive if you try to remember these simple rules:

1. Try not to interrupt when someone else is talking.

2. In class, be recognized if you want to contribute or ask a question.

3. Be polite and listen when others in your group or class are talking. This is one of the best ways to learn.

4. Finally, don’t be shy. If you have a question, raise your hand and ask. Remember, there is almost always someone else with a similar or identical question.

**Last, some general advice about taking notes in math:**

1. Reading math is a specific skill. Unlike other types of reading, when you read math, you need to read each word carefully. The first step is to know the mathematical meaning of all words.

2. It is often necessary to write definitions of new words and to include mathematical examples. Try to write definitions in your own words without changing the meaning or omitting any important point. When you write down a definition, look for an example that illustrates what you are learning. This will help you relate what you are learning to real world situations.

3. Explaining new ideas and definitions that you read to your peers and teachers is very helpful. This will provide practice with any new definitions, and make sure that you are using the words correctly. Explaining a concept can help to correct any misconceptions and also reinforces learning.

4. If possible, try to draw a visual representation to make a difficult or new concept clear. It is really true that “a picture is worth a thousand words.” Visual cues help you understand and remember definitions of new terms.
Throughout this book, students learn algebraic thinking and the precise use of mathematical language to model problems and communicate ideas. The communication that makes this possible can be in small groups, in class discussion, and in student notes.

It is important to note that the use of variables and algebra is not an afterthought, but is woven throughout all of our books. By using language purposefully in small groups, class discussions, and in written work, students develop the ability to solve progressively more challenging problems.

The authors are aware that one important member of their audience is the parent. To this end, they have made every effort to create explanations that are as transparent as possible. Parents are encouraged to read both the book and the accompanying materials.

Possibly the most unique aspect of this book is the breadth and span of its appeal. The authors wrote this text for both a willing 4th or 5th grader and any 6th grader. Students may particularly enjoy the ingenuity and investigation problems at the end of each set of exercises which are designed to lead students to explore new concepts more deeply.

The text has its origins in the Texas State Honors Summer Math Camp (HSMC), a six-week residential program in mathematics for talented high school students. The HSMC began in 1990 modeled after the Ross program at Ohio State, teaching students to “think deeply of simple things” (A. E. Ross). Students learned mathematics by exploring problems, computing examples, making conjectures, and then justifying or proving why things worked. The HSMC has had remarkable success over the years, with over 115 students being named semi-finalists, regional finalists, and national finalists in the prestigious Siemens Competition in Math, Science, and Engineering. Initially supported by grants from the National Science Foundation and RGK Foundation, the HSMC has also received significant contributions from Siemens Foundation, Intel, SBC Foundation, Coca-Cola, the American Math Society Epsilon Fund, and an active, supportive Advisory Board.
In 1996, two San Marcos teachers, Judy Brown and Ann Perkins, suggested that we develop a pipeline to the HSMC that would introduce all young students to algebra and higher-level mathematics. Following their suggestion, we began the Junior Summer Math Camp (JSMC) as a two-week program for students in grades 4–8. We carefully developed the JSMC curriculum by meeting regularly with Judy and Ann, who gave us invaluable feedback and suggestions.

With support from the Fund for the Improvement of Postsecondary Education (FIPSE), Eisenhower Grants Program, Teacher Quality Grants, and the Texas Education Agency, we developed the JSMC into a replicable model that school districts throughout the state could implement. The JSMC curriculum was designed to prepare all students for higher-level mathematics. In some districts the JSMC targeted gifted students; in other districts the program was delivered to mixed group of students. In every setting, the program had remarkable results in preparing students for algebra as measured by the Orleans-Hanna algebra prognosis pre- and post-tests.

Over the years, we trained hundreds of teachers and thousands of students. Although we cannot thank each personally, we should mention that it has been through their suggestions and input that we have been able to continually modify, refine, and improve the curriculum.

A concern with the JSMC curriculum was that it was only supplementary material for teachers, and many of the state-required mathematics topics were not included. The Math Explorations texts that we have written have taken the JSMC curriculum and extended it to cover all of the TEKS (Texas Essential Knowledge and Skills), for grades 6-8 while weaving in algebra throughout. The third volume for 8th graders allows all students to complete Algebra I. This is an integrated approach to algebra developed especially for younger students. By learning the language of mathematics and algebra, young students can develop careful, precise mathematical models that will enable them to work multi-step problems that have been a difficult area for U.S. students on international tests.
An accompanying Teacher Edition (TE) has been written to make the textbook and its mathematical content as clear and intuitive for teachers as possible. The guide is in a three-ring binder so teachers can add or rearrange whatever they need. Every left-hand page is filled with suggestions and hints for augmenting the student text. Answers to the exercises and additional activities are available in the TE and companion CD.

This project had wonderful supporters in the Meadows Foundation, RGK Foundation, and Kodosky Foundation. A special thanks to our Advisory Board, especially Bob Rutishauser and Jeff Kodosky, who have provided constant encouragement and support for our curriculum project. The person who motivated this project more than any other was Jeff Kodosky, who immediately realized the potential it had to dramatically change mathematics teaching. Jeff is truly a visionary with a sense for the important problems that we face and ideas about how to solve them. His kind words, encouragement, and support for our JSMC and this project have kept me going whenever I got discouraged.

Our writing team has been exceptional. The primary basis for the book was our junior summer math camp curriculum, coauthored with my wife Hiroko, and friend, colleague and coauthor Terry McCabe. The three of us discuss every part of the book, no matter how small or insignificant it might seem. Each of us has his or her own ideas, which together I hope have made for an interesting book that will excite all young students with the joy of mathematical exploration and discovery. Over the summers of 2005-2012, we have been assisted by an outstanding group of former Honors Summer Math Camp students, undergraduate and graduate students from Texas State, as well as an absolutely incredible group of pilot teachers. While it would take a volume to list everyone, we would be remiss not to acknowledge the help and support from this past summer.
Terry McCabe led the team in developing this book, Math Explorations (ME) Part 2, assisted by Hiroko Warshauer and Max Warshauer. Alex White from Texas State provided valuable suggestions for each level of the curriculum, and took over the leadership of the effort for the third volume, Algebra 1.

We made numerous refinements to the curriculum in summer 2012 incorporating the 2012 revised TEKS, additional exercises, new warm-ups, and an accompanying collection of workbook handouts that provides a guide for how to teach each section. Many of these changes were inspired by and suggested by our pilot site teachers. Amy Warshauer and Alexandra Eusebi from Kealing Middle School in Austin, who piloted Part 2 of the curriculum with their 6th grade students, helped design and put together a fabulous collection of workbook handouts that accompany the revised text. Melissa Freese from Midland provided valuable assistance on both the workbooks and in making new warm-up problems. Additional edits and proofreading was done by Michael Kellerman, who did a wonderful job making sure that the language of each section was at the appropriate grade level. Robert Perez from Brownsville developed special resources for English Language Learners, including a translation of key vocabulary into Spanish.

ME Part 1 should work for any 6th grade student, while ME Part 2 is suitable for either an advanced 6th grade student or any 7th grade student. Finally, Math Explorations Algebra I completes algebra for all 8th grade students. The complete set of 3 books covers all of the Texas Essential Knowledge and Skills (TEKS) for grades 6-8, while also covering Algebra I.

The production team was led by Claudia Hernandez, an undergraduate student in mathematics at Texas State. Claudia did an outstanding job of laying out the book, editing, correcting problems, and in general making the book more user friendly for students and parents. She has a great feel for what will excite young students in mathematics, and worked tirelessly to ensure that each part of the project was done as well as we possibly could. As we prepared our books for state adoption, Bonnie Leitch came on board to help guide and support the entire project. Bonnie
worked tirelessly to find where each of the Texas Essential Knowledge and Skills (TEKS) and English Language Proficiency Skills (ELPS) was covered in both the text and exercises. We added additional exercises and text to cover any TEKS that were not sufficiently addressed. Bonnie also edited these revisions and gave a final proofreading for each of the books, working with the authors to proofread every edit. However, in the end the authors take total responsibility for any errors or omissions. We do, however, welcome any suggestions that the reader might have to help make future editions better. In short, we had an incredible, hard-working team that did the work of an entire textbook company in a few short weeks! Without their help the project would not have reached its present state.

Any curriculum will only be as effective as the teachers who use it, and without the support and encouragement of the administration and parents, this can never happen. In this, we have been very fortunate to be able to work with dedicated teachers and administrators from San Marcos, McAllen, New Braunfels, Midland, and Austin. The Mathworks staff gave invaluable help. Andrew Hsiau, and Patty Amende have provided support whenever needed. I hope you will join our team by giving us feedback about what works, what doesn’t and how we can improve the book. By working together, I believe that we can develop a mathematics curriculum that will reach out to all students and that will engage students at a higher level than we have previously been able to achieve.

Max Warshauer

Texas Mathworks,
Director

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SECTION 1.1 BUILDING NUMBER LINES

Let’s begin by thinking carefully about numbers. Numbers are part of the mathematical alphabet, just like letters are used in English to form words. We use numbers for counting and representing quantities. When we think of the number one, we have in mind a picture:

Similarly, the number two describes a different quantity:

We could use a picture with dots to describe the number two. For instance, we could draw:

We call this way of thinking of numbers the “set model.” There are, however, other ways of representing numbers.
Another way to represent numbers is to describe locations with the **number line model**, which is visually similar to a thermometer. To construct a number line, begin by drawing a straight line and picking some point on the line. We call this point the **origin**. Label the origin with the number 0. We can think of 0 as the address of a certain location on the number line. Notice that the line continues in both directions without ending. We show this with arrows at the ends of the line.

Next, mark off some distance to the right of the origin and label the second point with the number 1.

Continue marking off points the same distance apart as above and label these points with the numbers 2, 3, 4, and so on.

The points you have constructed on your number line lead us to our first definitions.

**DEFINITION 1.1: COUNTING NUMBERS (POSITIVE INTEGERS)**

The **counting numbers** are the numbers in the following never-ending sequence:

\[ 1, 2, 3, 4, 5, 6, 7, \ldots \]

We can also write this as

\[ +1, +2, +3, +4, +5, +6, +7, \ldots \]

These numbers are also called the **positive integers** or **natural numbers**.
One interesting property of the natural numbers is that there are “infinitely many” of them; that is, if we write down a list of natural numbers, there is always some natural number that is not on the list.

When we include the number 0, we have a different collection of numbers that we call the whole numbers.

**DEFINITION 1.2: WHOLE NUMBERS (NON-NEGATIVE INTEGERS)**

The whole numbers are the numbers in the following never-ending sequence:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, …

These numbers are also called the non-negative integers.

In order to label points to the left of the origin, we use negative integers: -1, -2, -3, -4, …  The sign in front of the number tells us on which side of zero the number is located. Positive numbers are to the right of zero; negative numbers are to the left of zero. Zero is not considered to be either positive or negative.

So we have seen that numbers can be used in different ways. They can help us to describe the quantity of objects using the set model or to denote a location using the number line model. Notice that the number representing a location also can tell us the distance the number is from the origin if we ignore the sign.

**DEFINITION 1.3: INTEGERS**

The collection of integers is composed of the negative integers, zero, and the positive integers:

..., -4, -3, -2, -1, 0, 1, 2, 3, 4, …
Definition 1.3 leads to the **trichotomy** property, which states that there are exactly three possibilities for an integer: positive, negative, or zero.

Such collections of numbers are often called sets of numbers. For example, assume $S$ is the set \{1, 2, 3, 4, 5\} and $T$ is the set \{2, 3, 4\}. Notice that every number in $T$ is also an element of the set $S$. That means that $T$ is a subset of $S$. This relationship is represented by the following diagram:

![Diagram of sets S and T]

EXEMPLARY 1

Call the set of positive integers $P$, the set of whole numbers $W$, and the set of integers $Z$. Draw a diagram of the relationship of these three sets.

![Diagram of sets P, W, and Z]
EXPLORATION: CONSTRUCTING A NUMBER LINE

1. Draw a straight line.
2. Pick a point on the line and call this point the origin. Label the origin with the number 0.
3. Locate the numbers 1, 2, 3, …, 10, and −1, −2, −3, …, −10.
4. Where would 20, 30, 50 be located? 100? 1000?
5. Find the negative numbers corresponding to the numbers in question 4.

EXERCISES

1. The post office is located at the origin of Main Street. We label its address as 0. The laboratory has address 6 and the zoo has address 9. Going in the other direction from the origin, we find a candy shop with address −4 and a space observatory with address −7. Draw a number line representing Main Street. Label each of the above locations on the number line. Watch your spacing.

2. a. Copy the line below to mark off and label the integers from 0 to 10 and from 0 to −10. Use a pencil to experiment because you might need to erase. Watch your spacing.

   ![Number Line Diagram](image)

b. Make a number line from −20 to 20.

3. Draw a section of the number line containing the number 77. Mark the number 77 on your line. Now label your number line with the first few integers after 77 and the last few before 77, at least three each way.

4. Do the same thing you did in the previous exercise, but this time start with the number −77 instead of 77.
5. Use a line like the one below to mark off the numbers with equal distances by tens from 0 to 100 and from 0 to –100. Use a pencil to experiment.

![Number Line Diagram]

a. What is the distance from 0 to 50 and from 50 to 100? Are they the same?
b. What is the distance from 10 to 20, 30 to 40, and 70 to 80? Are they the same?
c. Explain whether you need to rework your markings on the number line.
d. **Estimate** the location of the following numbers and label each on your number line:
   

6. Draw a number line so that the number –1000 is at the left end and 1000 is at the right end. **Estimate** the locations of the following integers:

   500, -500, 250, -100, -800, 10, -990, 342, -781, 203, -407

7. Draw a number line. Find all the integers on your number line that are greater than 15 and less than 20. Color each of these locations blue.

8. If you find two numbers on the number line, how do you decide which number is greater?

   Notice that we can move the number line from the horizontal position to a vertical position. We would then have a number line that looks like a thermometer. Draw a thermometer (vertical number line) on the side of your paper to help you answer questions 9 through 12.

9. The chart below shows the monthly average temperatures for the city of Oslo, Norway. Based on the data, put the twelve months in order from coldest to warmest.

<table>
<thead>
<tr>
<th>Month</th>
<th>Temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>-9.2</td>
</tr>
<tr>
<td>February</td>
<td>-9.9</td>
</tr>
<tr>
<td>March</td>
<td>-3.6</td>
</tr>
<tr>
<td>April</td>
<td>2.3</td>
</tr>
<tr>
<td>May</td>
<td>3.4</td>
</tr>
<tr>
<td>June</td>
<td>7.6</td>
</tr>
<tr>
<td>July</td>
<td>12.2</td>
</tr>
<tr>
<td>August</td>
<td>13.8</td>
</tr>
<tr>
<td>September</td>
<td>9.4</td>
</tr>
<tr>
<td>October</td>
<td>3.2</td>
</tr>
<tr>
<td>November</td>
<td>-2.3</td>
</tr>
<tr>
<td>December</td>
<td>-8.7</td>
</tr>
</tbody>
</table>

Based on the data, put the twelve months in order from coldest to warmest.
10. Chris visits Edmonton, Canada where it is \(-7^\circ C\). Carmen visits Winnipeg, Canada where it is \(9^\circ C\). Which temperature is closer to the freezing point? Explain. Remember, when we measure temperature in degrees Celsius (°C), \(0^\circ C\) is the freezing point of water.

11. The temperature in Toronto, Canada, one cold day, is \(-10^\circ C\). The next day the temperature is \(4^\circ C\). Which temperature is closer to the freezing point?

12. The temperature in Fargo, North Dakota is \(-15^\circ F\) while it is \(-20^\circ F\) in St. Paul, Minnesota. Which temperature is colder? How much colder?

13. **Ingenuity:**

On a cold winter day in Iowa, the temperature at 6:00 P.M. is \(10^\circ F\). If the temperature decreases an average of \(4^\circ F\) for each of the next five hours, what will the temperature be at 11:00 P.M.?

14. **Investigation:**

Use the number line below as a thermometer with the Celsius scale above the line and the Fahrenheit scale below the line to discover how the two scales are related.

\[\begin{array}{c}
\text{°C} \\
\text{°F} \\
\text{freezing point} \\
\text{boiling point}
\end{array}\]
a. At what temperature does water boil on each scale?
b. At what temperature does water freeze on each scale?
c. What is the Fahrenheit reading for 50 °C?
d. Is the Celsius reading for 25 °F a positive or negative number?
e. A nice day is 77 °F. What is this temperature in Celsius?
f. A hot day is 100 °F. Estimate this temperature in Celsius.
SECTION 1.2 LESS THAN AND GREATER THAN

We say that 2 is less than 5 because 2 is to the left of 5 on the number line. “Less than” means “to the left of” when comparing numbers on the number line. We use the symbol “<” to mean “less than.” We write “2 is less than 5” as “2 < 5.” Some people like the “less than” symbol because it keeps the numbers in the same order as they appear on the number line.

We also say that 5 is greater than 2 because 5 is to the right of 2 on the number line. “Greater than” means “to the right of” when comparing numbers on the number line. We use the symbol “>” to mean greater than, so we write “5 is greater than 2” as “5 > 2.”

DEFINITION 1.4: LESS THAN AND GREATER THAN

Suppose that \( x \) and \( y \) are integers. We say that \( x \) is less than \( y \), \( x < y \), if \( x \) is to the left of \( y \) on the number line. We say that \( x \) is greater than \( y \), \( x > y \), if \( x \) is to the right of \( y \) on the number line.

Here \( x \) and \( y \) are called variables, which we will formally introduce in Chapter 3. A variable is a letter or symbol that represents an unknown quantity. Variables give us a convenient way to describe properties and ideas because variables can represent many values. Variables give us a simple way to describe math objects and concepts. In this case, \( x \) and \( y \) represent two integers, and the way that we tell which is greater is to compare their positions on the number line. The two number lines below demonstrate the cases \( x < y \) and \( x > y \). Can you tell which is which?

\[
\begin{array}{c}
\text{x} \\
\text{y}
\end{array}
\]

\[
\begin{array}{c}
\text{y} \\
\text{x}
\end{array}
\]
Mathematicians use the symbols “≤” and “≥” to mean “less than or equal to” and “greater than or equal to.” We can write “2 ≤ 5” because 2 is less than 5, but we can also write “5 ≤ 5” because 5 is equal to 5.

EXAMPLE 1
For each pair of integers below, determine which one is greater and which one is less. Express your answer as an inequality of the form \( x < y \) or \( x > y \), where \( x \) and \( y \) are the given integers.

\[
\begin{align*}
a. & \quad 3 \quad 7 \\
b. & \quad -2 \quad 9 \\
c. & \quad -1 \quad -5 \\
d. & \quad 4 \quad -4
\end{align*}
\]

SOLUTION

a. Begin by drawing a number line from \(-10\) to 10. Using this number line, we see that 3 is to the left of 7, so \( 3 < 7 \).

b. We observe that \(-2\) is to the left of 9 on the number line, so \( -2 < 9 \). We can also see this in a different way: We know that \(-2\) is to the left of 0, because \(-2\) is negative, and 0 is to the left of 9, because 9 is positive. Thus \(-2\) must be to the left of 9, and we have \( -2 < 9 \).

c. We notice that \(-5\) is to the left of \(-1\), so \( -5 < -1 \) or \( -1 > -5 \).

d. Because \(-4\) is to the left of 0, and 0 is to the left of 4, we have \( -4 < 4 \) or alternatively \( 4 > -4 \).

EXAMPLE 2
Put the following integers in order from least to greatest:

\[
2, -2, 7, -1, -4, -5, 4, 6, 3
\]

SOLUTION

Again, we can use the number line to help us put the integers in order:

\[
\begin{array}{c}
-10 & -5 & 0 & 5 & 10 \\
\end{array}
\]
We can locate our nine given integers on the number line. You might try doing this by copying the diagram above and labeling the given numbers on your diagram. After comparing the nine numbers given, we get the following order:

\[-5, -4, -2, -1, 2, 3, 4, 6, 7\]

We can write this list using the < symbol as:

\[-5 < -4 < -2 < -1 < 2 < 3 < 4 < 6 < 7\]

**EXERCISES**

1. Rewrite each of the following as a statement using <, >, ≤, or ≥. Compare your statements to the relative locations of the two numbers on the number line.  
   Example: \(-3\) is less than \(8\). \(-3 < 8\).
   
   a. 9 is greater than 6.  
   b. 4 is less than 7.  
   c. \(-3\) is greater than \(-5\).  
   d. \(-4\) is less than 1.  
   e. \(8\) is greater than or equal to 0.  
   f. \(-6\) is less than or equal to 2.

2. Compare the numbers below and decide which symbol, < or >, to use between the numbers. Make a number line to show the relationship.

   a. \[\begin{array}{c}
   3 \\
   4
   \end{array} \quad \begin{array}{c}
   4 \\
   3
   \end{array}\]

   b. \[\begin{array}{c}
   0 \\
   1
   \end{array} \quad \begin{array}{c}
   1 \\
   0
   \end{array}\]

   c. \[\begin{array}{c}
   -2 \\
   0
   \end{array} \quad \begin{array}{c}
   0 \\
   -2
   \end{array}\]

   d. \[\begin{array}{c}
   -2 \\
   -3
   \end{array} \quad \begin{array}{c}
   -3 \\
   -2
   \end{array}\]

   e. \[\begin{array}{c}
   -8 \\
   9
   \end{array} \quad \begin{array}{c}
   9 \\
   -8
   \end{array}\]

   f. \[\begin{array}{c}
   6 \\
   -5
   \end{array} \quad \begin{array}{c}
   -5 \\
   6
   \end{array}\]

3. Compare the numbers below and decide which symbol, < or >, to use. Use a number line to show the relationship of these numbers.

   a. \[\begin{array}{c}
   5 \\
   -5
   \end{array} \quad \begin{array}{c}
   3 \\
   -3
   \end{array}\]

   b. \[\begin{array}{c}
   -10 \\
   10
   \end{array} \quad \begin{array}{c}
   -4 \\
   4
   \end{array}\]

   c. \[\begin{array}{c}
   -12 \\
   12
   \end{array} \quad \begin{array}{c}
   0 \\
   0
   \end{array}\]

   d. \[\begin{array}{c}
   -15 \\
   -5
   \end{array} \quad \begin{array}{c}
   -13 \\
   -3
   \end{array}\]

   e. \[\begin{array}{c}
   -9 \\
   -9
   \end{array} \quad \begin{array}{c}
   -7 \\
   -7
   \end{array}\]

   f. \[\begin{array}{c}
   -21 \\
   -21
   \end{array} \quad \begin{array}{c}
   21 \\
   21
   \end{array}\]

4. Describe any patterns you see in Exercises 2 and 3.
5. Compare the numbers below and decide which symbol, < or >, to use. Use your rules from Exercise 4 to help you.

a. 6 2
   2 6
d. 0 14
   -14 0
b. 13 -11
e. 12 28
   -13 11
   -12 -28
c. -9 3
   -3 9
   -12 -28

6. List the following integers in order using a number line.

-62, -75, 26, 83, -59

7. What are the possible values for an integer that is greater than 3 and less than 7? Mark these possible values on a number line.

8. Determine whether each of the following statements is true or false. Explain your answers.
   a. If an integer is greater than 5, then it is greater than -5.
   b. If an integer is less than 5, then it is less than -5.

9. In Madison, Wisconsin, the morning temperature is -2 °C. In the evening the temperature reads -6 °C. Did the temperature rise or fall? How much did it change?

10. Albert is on a flight of stairs 87 steps above the ground floor. Elaine has gone into the sub-basement 78 steps down from ground level (let’s call it the -78th step). Who is farther from ground level? Why?

11. What are the possible values for an integer that is closer to 5 than it is to -2? Mark these possible values on the number line.

12. Earlier we introduced “greater than or equal to” and “less than or equal to.” Write a formal definition, using definition 1.4 as a model, for these two concepts.

13. **Ingenuity:**
    Suppose that the tens digit of a whole number between 80 and 90 is greater than the ones digit, but less than twice the ones digit. If the integer is even, what is its value?
14. **Investigation:**

Make a large timeline from the year 2000 BC to the year 2100 AD. Research the years that mark the following periods on this timeline.

a. The life of William Shakespeare
b. The U.S. Civil War
c. The Mayan civilization
d. The Roman Empire
e. People driving cars
f. The United States has been a country
g. Texas was a state
h. Texas was a country
SECTION 1.3 DISTANCE BETWEEN POINTS

We locate the numbers 10 and -10 on the number line.

![Number line with 10 units left of zero and 10 units right of zero]

Notice that 10 and -10 are each 10 units from 0. We have a special name for the distance of a number from 0: the absolute value of the number.

In mathematics, we have a special symbol to represent absolute value. For example, we write |10| and read it as “absolute value of 10.” We write |-10| and read it as “absolute value of -10.” Because 10 and -10 are both 10 units from 0 we have the following:

The absolute value of 10 equals 10 or |10| = 10.

The absolute value of -10 equals 10 or |-10| = 10.

The absolute value gives us a measure of the size or the magnitude of a number, or its distance from the origin. The positive or negative sign tells us the direction of the number relative to 0. Because 10 and -10 are the same distance from 0, they have the same absolute value. In other words, -10 < 10 but |-10| = |10|.

Note: The absolute value symbol ‘| |’ should not be confused with parentheses ‘()’.

EXPLORATION

Using the number line that you have constructed, find the distance between each pair of numbers:

a. 2 and 8  
d. 5 and -3  
g. 9 and 0
b. -4 and -1  
e. 0 and 6  
h. 0 and 9

c. -4 and 1  
f. 0 and -6  
i. 0 and -9

In addition to the previous examples, you may also see -|10| which is read as “the negative absolute value of 10” or |-10| which is read as “the negative absolute value of -10.” Since |10| = 10 and |-10| = 10, then we have

-|10| = -10 and -|-10| = -10.
EXERCISES

1. Find the absolute values of the following numbers.
   a. \(-7\)  d. \(10\)  g. \(-21\)
   b. \(8\)  e. \(-10\)  h. \(-42\)
   c. \(-8\)  f. \(19\)  i. \(33\)

2. Calculate the following:
   a. \(|-23|\)  b. \((15)\)  c. \(|0|\)  d. \(-|17|\)  e. \(-|-34|\)

3. What is the absolute value of the absolute value of \(-44\)?

4. Find the distance between each number and zero:
   a. \(9\)  b. \(0\)  c. \(-9\)

5. For each pair of numbers below, place the correct symbol <, >, or =.
   a. \(|-7|\)  \(|5|\)  f. \(|-6|\)  \(|-8|\)
   b. \(|7|\)  \(|-5|\)  g. \(|-6|\)  \(8\)
   c. \(|9|\)  \(9\)  h. \(6\)  \(|-8|\)
   d. \(|9|\)  \(|-9|\)  i. \(25\)  \(|28|\)
   e. \(|32|\)  \(|-47|\)  j. \(|-28|\)  \(25\)

6. Find the distance between 5 and 8. Did you use the number line? Can you use absolute values?

7. For each pair of integers given below, find the distance between the two integers on the number line.
   a. 2 and 3  b. 4 and 9  c. 25 and 17  d. \(-12\) and \(12\)
   e. 2 and \(-3\)  f. \(-4\) and 9  g. 17 and 25  h. 12 and 12
   i. \(-2\) and 3  j. \(-4\) and \(-9\)  k. \(-17\) and 25
   l. \(-2\) and \(-3\)  m. \(4\) and \(-9\)
8. For each pair of integers given below, find the distance between the two integers on the number line.
   a. 40 and 70  
   b. 30 and 20  
   c. -80 and -40  
   d. -126 and -64  
   3 and 8  
   -2 and -5  
   -7 and -9  
   43 and 78  
   28 and 15  
   -87 and -49

9. A number is a distance of 5 from 13. Is it possible to determine a single value for the number? Explain using a number line.

11. a. Find numbers that are a distance 5 from 2.
    b. Find numbers that are a distance 4 from -3.
    c. Find numbers that are a distance 3 from -8.
    d. If x represents some integer, what numbers are distance 5 from x?
    e. What numbers are distance x from 2?

12. If x is the same distance from 6 as it is from 2, what is x?

13. Create a number line that extends from -15 to 15.

14. **Ingenuity:**

   The distance between two cities on a highway is 118 miles. If all the exits between these two cities are at least 5 miles apart, what is the largest possible number of exits between these two cities?

15. **Investigation:**

   Write a process for finding the distance between two numbers. Remember to address all possible cases: two positive numbers, two negative numbers, one of each, and at least one number equal to zero.
REVIEW PROBLEMS

1. For each part below, draw a number line with the three given integers marked.
   a. 3, -4, 6
   b. 20, -45, 55
   c. -8, 0, -3
   d. -1214, -1589, -1370

2. At 7:00 A.M., Chicago’s O’Hare Airport has a temperature of -9 °C. At 11:00 A.M. that same day, the temperature reads 4 °C. Did the temperature rise or fall? Why? Can you determine by how many degrees?

3. At 4 P.M. in London, the temperature was 77 °F. At 6 A.M. the same day, the temperature was 57 °F. Did the temperature rise or fall? By how much?

4. Maggie has locked all of her money inside a safe and forgotten the combination. Luckily, Maggie left this note for herself: “The combination to open this safe is three positive integers. These positive integers are represented, in order, by the variables a, b, and c, where c is three, b < c, and a < b.” What is the combination to open the safe?

5. Homer was a Greek poet who produced several well-known epics during his lifetime. He wrote The Odyssey around 680 B.C.E. and The Iliad around 720 B.C.E. Which of these two works was written first? How many years passed between these two dates?

6. a. Which of the following numbers is farthest from 0: -2, 3, or -5?
   b. Which of the following numbers is closest to 14: 28, -1, or -5?

7. Compare the pairs of numbers below and place the appropriate symbols between them. Use < or >.
   a. 457 81
   b. -23 -32
   c. 191 -3
   d. |-9| 5
   e. |-17| |-22|
   f. |156| |204|
8. Find the distance between the following pairs of numbers.

Example: The distance between $-2$ and $9$ is 11

a. $0$ and $-1$

b. $-8$ and $0$

c. $0$ and $-14$

d. $14$ and $0$

e. $-12$ and $16$

f. $-4$ and $24$

g. $-20$ and $10$

h. $-20$ and $-40$

i. $20$ and $-20$

j. $25$ and $-15$

k. $-25$ and $25$

l. $17$ and $17$

9. Answer the following questions using the table of information for each planet in the solar system. Justify your answers.

<table>
<thead>
<tr>
<th>Studying the Planets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Planet</td>
</tr>
<tr>
<td>---------</td>
</tr>
<tr>
<td>Mercury</td>
</tr>
<tr>
<td>Venus</td>
</tr>
<tr>
<td>Earth</td>
</tr>
<tr>
<td>Mars</td>
</tr>
<tr>
<td>Jupiter</td>
</tr>
<tr>
<td>Saturn</td>
</tr>
<tr>
<td>Uranus</td>
</tr>
<tr>
<td>Neptune</td>
</tr>
</tbody>
</table>

a. Which planet is the hottest? Which is the coldest?

b. Which planet is closest to the sun? Which is the farthest from the sun?

c. Is there a relationship between a planet’s distance from the sun and its average temperature? Explain.
CHALLENGE PROBLEMS

Section 1.2:
If a, b, and c are positive integers such that a<b<c<10, how many possible values are there for the three numbers?

Section 1.3:
A collection of integers has the property that every integer from 1 to 10 inclusive is the distance between some two numbers in the collection. What is the least possible number of integers in the collection?
SECTION 2.1 ADDITION OF INTEGERS

Addition is a mathematical operation for combining integers. Pictorially, using the "set model," when we add two integers we are combining the sets. To add 4 and 3 we draw the picture below:

\[
\begin{array}{c}
\hline
\text{\includegraphics[width=2cm]{dice.png}} \\
\text{\includegraphics[width=2cm]{dice.png}} \\
\text{\includegraphics[width=2cm]{dice.png}} \\
\text{\includegraphics[width=2cm]{dice.png}} \\
\text{\includegraphics[width=2cm]{dice.png}} \\
\end{array}
\begin{array}{c}
\hline
\text{\includegraphics[width=2cm]{dice.png}} \\
\text{\includegraphics[width=2cm]{dice.png}} \\
\text{\includegraphics[width=2cm]{dice.png}} \\
\end{array}
\begin{array}{c}
\hline
\text{\includegraphics[width=2cm]{dice.png}} \\
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\text{\includegraphics[width=2cm]{dice.png}} \\
\text{\includegraphics[width=2cm]{dice.png}} \\
\end{array}
\begin{array}{c}
= \\
\end{array}
\begin{array}{c}
\hline
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\end{array}
\begin{array}{c}
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\text{\includegraphics[width=2cm]{dice.png}} \\
\end{array}
\begin{array}{c}
\hline
\text{\includegraphics[width=2cm]{dice.png}} \\
\text{\includegraphics[width=2cm]{dice.png}} \\
\end{array}
\begin{array}{c}
\end{array}
\begin{array}{c}
\end{array}
\begin{array}{c}
\end{array}
\end{array}
\]

We can also use our number line model to describe addition.

CLASS EXPLORATION: DRIVING ON THE NUMBER LINE WITH ADDITION

We can visualize adding two numbers using a car driving on the number line. The final location gives the sum. Let us practice how this works on a small scale. Use the number line you constructed in Chapter 1 from –15 to 15 as your highway. You will also need a model car or something that can represent this model car.

\[\text{\includegraphics[width=3cm]{number-line.png}}\]

**Step 1:** Place your car at the origin, 0, on the number line.

**Step 2:** If the first of the two numbers that you wish to add is positive, the car faces right, the positive direction. If the first of the two numbers is negative, the car faces left, the negative direction. Drive to the location given by the first number. Park the car.

**Step 3:** Next examine the second of the two numbers. If this number is positive, point the car to the right, the positive direction. If the second number is negative, point the car to the left, the negative direction.
Step 4: Because you are adding, move the car forward, in the direction that it is facing, the distance equal to the absolute value of the second number.

Use your car and the four-step process to compute each of the following examples. Attempt the process on your own first, and then compare your answer with the provided solution.

EXAMPLE 1

Find the sum 3 + 4, and describe how you obtain your answer using the number line model.

SOLUTION

The two numbers we are adding are 3 and 4 (which we also know as \(+3\) and \(+4\)).

Step 1: Begin with your car at 0.

Step 2: Because the first number is positive, the car faces to the right. Drive to the location \(+3\). Park the car.

Step 3: Point the car to the right because the second number, \(+4\), is positive.

Step 4: Move the car 4 units to the right. Park the car.

You are now at location 7.

EXAMPLE 2

Find the sum \(-3 + 4\). How do we start the process? In which direction does your car move first and how far? Explain how you reached your solution using a car on your number line.
SOLUTION

Step 1: Begin at 0.

Step 2: Because \(-3\) is a negative number, point the car to the left and drive 3 units. Park the car.

Step 3: Because 4, the number added to \(-3\), is positive, turn the car to face right.

Step 4: Move 4 units to the right, ending up at location 1. Park the car.

The result \((-3 + 4 = 1)\) is demonstrated below:

![Diagram showing movement of a car on a number line from -3 to 1](image)

EXAMPLE 3

Find the sum \((-3 + (-4)),\) or simply \((-3 + -4),\) using the same process.

SOLUTION

Point the car to the left and move forward 3 units. Leave the car pointing to the left because the next number is negative. Move the car forward 4 units to the location \(-7\). We have: \(-3 + (-4) = -7\).

![Diagram showing movement of a car on a number line from -3 to -7](image)

Remember, **when you are adding, the car always moves forward, in the direction that it is facing.** The signs of the integers tell us whether we face right, if positive, or left, if negative, before moving.
EXERCISES

For exercises 1–4, find each sum. You may use your car and number line.

1. a. 3 + 0       c. 0 + -5
   b. 0 + 3       d. -2 + 0

2. a. -1 + 1      c. 8 + -8
   b. 2 + -2      d. 0 + 0

3. a. 3 + 4       f. 4 + 3
   b. -2 + -7     g. -7 + -2
   c. -1 + 5      h. 5 + -1
   d. 6 + -8      i. -8 + 6
   e. 0 + -7      j. -7 + 0

4. Write rules to describe any patterns you see in problems 1–3.
   • Do you see a pattern when adding two positives?
   • adding two negatives?
   • adding a positive and a negative?
   • adding a negative and a positive?
   Explain how your rules work using a number line.

5. For this exercise, let’s pay careful attention to the order in which we add. We use parentheses to specify order. For example, (1 + 2) + 3 means first add 1 to 2, and then add the result and 3; 4 + (5 + 6) means first add 5 and 6, and then add 4 and the result. Calculate the following sums:

   a. (1 + 2) + 3   d. -3 + (0 + 3)
      1 + (2 + 3)     (-3 + 0) + 3
   b. (2 + 4) + -5  e. (-7 + 8) + (-2 + -4)
      2 + (4 + -5)   -7 + ((8 + -2) + -4)
   c. 8 + (-7 + 6)
      (8 + -7) + 6
Section 2.1 Addition of Integers

6. Predict the sign of the answer. Then find the sums. Use the number line if you need to.
   a. \(8 + 9\)  
   b. \(-7 + -5\)  
   c. \(-7 + 5\)  
   d. \(-4 + -5\)  
   e. \(-9 + -2\)  
   f. \(9 + -2\)  
   g. \(-6 + 6\)  
   h. \(4 + -5\)  
   i. \(-4 + 5\)  
   j. \(-3 + 9\)  
   k. \(7 + -4\)  
   l. \(-7 + 4\)  
   m. \(-6 + -6\)  

For exercises 7–14, write each problem as an addition problem and use positive and negative numbers where appropriate. Show your work on a number line.

7. a. Jacob observes that the temperature is \(-3 \, ^\circ\text{C}\). If it rises \(7 \, ^\circ\text{C}\) in the next two hours, what will the new temperature be?
   
   b. Marissa observes that the temperature is \(2 \, ^\circ\text{C}\). During the night, it falls \(9\, ^\circ\text{C}\). What was the low temperature that night?
   
   c. Adrian takes 8 steps forward then takes 6 steps back. How far is Adrian from where he started?
   
   d. David takes nine steps backward and then three steps forward. At what location does he end?

8. Jennifer checks the temperature and it is \(-9 \, ^\circ\text{C}\). If the temperature warms up \(8 \, ^\circ\text{C}\), what is the new temperature?

9. It was \(-5 \, ^\circ\text{C}\) in the morning. The temperature rose \(8 \, ^\circ\text{C}\). What is the temperature now?

10. Juan checks the temperature and it is \(5 \, ^\circ\text{C}\) at 5 P.M. By 10 P.M., the temperature has dropped \(8 \, ^\circ\text{C}\). What is the new temperature?

11. Chris has \(\$17\) in his bank account. If he withdraws \(\$20\), what will his balance be?

12. The temperature in St. Paul, Minnesota is \(-4 \, ^\circ\text{F}\) on a cold winter day. If the temperature falls another \(7 \, ^\circ\text{F}\), what will the new temperature be?
13. If a football player loses 6 yards in one play, loses 2 yards in another play, and then gains 4 yards in the final play, what is the net gain or loss? 

\[-6 + (-2) + 4 = -\]

14. **Ingenuity:**

Assume that, in the diagrams below, each of the small squares has sides of length one inch. Find the perimeter of each of the figures below. What surprising result do you notice?

![Diagram of figures with grids]

15. **Investigation:**

With our car model, the car moves forward when we add. What do you think we should do when we want to subtract a number from another number? Write your best guess about how to subtract two numbers.
SECTION 2.2 SUBTRACTION OF INTEGERS

With our car model, addition involves moving the car forward in the direction indicated by the signs of the numbers we are adding. What do you think we should do when we want to subtract one number from another number? One way to model subtraction is to use our number line and move the car backward, the opposite of forward.

Try each example first, and then check your answer by comparing it to the solution given. Describe each of the steps you are using in words.

EXAMPLE 1

Compute the difference $5 - 2$, and show how to model this with a number line. Use a model car and a number line to simulate your solution. What is your final location?

SOLUTION

Step 1: Place your car at the origin. Because the first number is positive, face the car to the right.

Step 2: Move the car forward 5 units to the location given by the first number. Park the car.

Step 3: Point your car to the right because the number being subtracted is positive.

Step 4: Now instead of moving forward 2 spaces, move backward 2 spaces, ending up at location 3. Remember, we move backward because we are subtracting. We can write this movement as $5 - 2 = 3$. 

\[ \begin{array}{c}
\text{Step 1: Place car at origin.} \\
\text{Step 2: Move car forward 5 units.} \\
\text{Step 3: Point car to the right.} \\
\text{Step 4: Move car backward 2 units.}
\end{array} \]
EXAMPLE 2

Compute the difference $2 - 5$. Use a number line to show how you solved the problem.

SOLUTION

Step 1: Place your car at the origin. Because the first number is positive, face the car to the right.

Step 2: Move the car forward 2 units to the location given by the first number. Park the car.

Step 3: Point your car to the right because the number being subtracted is positive.

Step 4: Now since we are subtracting, move backward 5 spaces, ending up at location $-3$. We can write this movement as $2 - 5 = -3$.

EXAMPLE 3

Compute the difference $-7 - 3$. Use a number line to show how you solved the problem.

SOLUTION

Step 1: Place your car at the origin. Because the first number is negative, the car faces left.

Step 2: Move the car forward 7 units to the location given by the first number, $-7$. Park the car.

Step 3: Point your car to the right because the number being subtracted is positive.
Step 4: Now move backward 3 spaces, ending up at location \(-10\). Be careful! This time your car was pointing to the right, so when you back up you will move backwards to the left. We can write this movement as \(-7 - 3 = -10\).

EXAMPLE 4

Compute the difference \(2 - (-5)\). Use a number line to show how you solved the problem.

SOLUTION

Step 1: Place your car at the origin. Because the first number is positive, face the car to the right.

Step 2: Move the car forward 2 units to the location given by the first number. Park the car.

Step 3: Point your car to the left because the number being subtracted is negative.

Step 4: Now move backward 5 spaces, ending up at location 7. Be careful! This time your car was pointing to the left, so when you back up you will move to the right. We can write this movement as \(2 - (-5) = 7\).

SUMMARY

In order to compute \((x - y)\), we proceed as follows:
Step 1: Place the car at 0, the origin. Then face the car in the direction of the sign of the first number x.

Step 2: Move the car $|x|$ units forward in the direction the car faces. Park the car.

Step 3: Next, face the car in the direction of the sign of the second number, y.

Step 4: Move the car $|y|$ units backward, the opposite direction from what the car faces. The car is positioned on the difference $(x - y)$.

EXERCISES

Use the car model with your number line to calculate each of the following exercises. Drive carefully.

1. a. $4 - 3$ f. $7 - 3$
   $4 + 3$ $7 + 3$
   b. $3 - 5$ g. $1 - 8$
   $3 + 5$ $1 + 8$
   c. $-2 - 4$ h. $-2 - 4$
   $-2 + 4$ $-2 + 4$
   d. $0 - 5$ i. $0 - 9$
   $0 + 5$ $0 + 9$
   e. $-3 - 2$ j. $-7 - 4$
   $-3 + 2$ $-7 + 4$

2. a. $4 - 2$ e. $6 - 7$
   $2 - 4$ $-7 - 6$
   b. $2 - 3$ f. $5 - 2$
   $3 - 2$ $-2 - 5$
   c. $-4 - 8$ g. $-1 - 8$
   $8 - 4$ $-8 - 1$
   d. $0 - 5$ h. $-3 - 2$
   $5 - 0$ $-2 - 3$

3. What patterns do you see in Exercises 1 and 2? In your own words, write a rule for each pattern that you observe.
4. Solve the following exercises using the car model and your rules from the previous exercise.
   a. \( 8 + 2 \)
   \( 8 - 2 \)
   \( 8 - 2 \)
   b. \( 2 + 6 \)
   \( 2 - 6 \)
   \( 2 - 6 \)
   c. \( -3 - 4 \)
   \( -3 + 4 \)
   \( -3 - 4 \)
   d. \( -5 + 4 \)
   \( -5 - 4 \)
   \( -5 + 4 \)
   e. \( -7 + 3 \)
   \( -7 - 3 \)
   \( -7 - 3 \)
   f. \( -6 - 6 \)
   \( -6 + 6 \)
   \( -6 + 6 \)

5. Calculate the following sums and differences. Use the car model as needed.
   a. \( 3 - 9 \)
   \( 3 + 9 \)
   \( 3 - 9 \)
   b. \( 8 - 5 \)
   \( 8 - (-5) \)
   \( 8 - (-5) \)
   c. \( -2 - 5 \)
   \( -2 + 5 \)
   \( -2 - (-5) \)
   d. \( -1 - (-5) \)
   \( -1 + 5 \)
   \( -1 + 5 \)

For Exercises 6 and 7, write a subtraction expression and compute.

6. It was 7 °C at 8 A.M. The temperature dropped 8 °C over the next four hours. What was the temperature at noon?

7. Nick opens a savings account at the bank in September. He deposits $20 in September, $35 in October, and $25 in November. He needs to withdraw $50 in December for holiday presents he wants to buy for his family. What is his balance after he makes his December withdrawal?
8. Ingenuity:

Maria has an unusual morning ritual that she performs when she goes outside to get her mail. The distance from her front door to her mailbox is 30 steps. She steps outside the front door, takes two steps forward, and then takes one step back. She then takes another two steps forward and one step back. She continues doing this until she reaches her mailbox. In all, how many steps does Maria have to take before she gets to her mailbox?

Hint: Working this out for a 30-step trip can be quite difficult. You might want to start by seeing what happens if the mailbox is closer to the front door, perhaps 5 steps rather than 30.

9. Investigation: Skip Counting and Scaling

When we initially built our number line, we used each mark to indicate one unit. The number line then corresponded to the integers 1, 2, 3, … . For larger numbers, we let the marks represent bigger lengths. So if each mark represents 5 units, then the marks correspond to the multiples of 5, and we have 5, 10, 15, 20, … using “skip counting” by 5’s.

a. Build a number line where each mark represents 3 units, skip counting by 3’s. What is the 10th number to the right of 0? What is the 10th number to the left of 0?

b. Build a number line where each mark represents 10 units, skip counting by 10’s from -40 to 40.

c. Make a table of the numbers you get when skip counting by 2’s to 20, skip counting by 3’s to 30, skip counting by 4’s to 40, … and skip counting by 10’s to 100.

d. Now skip count by 2’s, 3’s, 4’s, … up to 10’s in the opposite direction. Make a table of the numbers you get when skip counting by 2’s to -20, by 3’s to -30, by 4’s to -40, etc.

e. Do you notice any patterns when you skip count? For example, the 5th number when skip counting by 3’s is 15. This is the same as the 3rd number when you skip count by 5’s. Does this kind of symmetry always hold?
SECTION 2.3  ADDING AND SUBTRACTING LARGER NUMBERS

We continue to explore addition and subtraction with larger numbers using the patterns we observed with smaller numbers. Let’s begin with some problems using two digit numbers.

CLASS EXPLORATION: WORKING WITH LARGE NUMBERS

1. Find the following sums.
   a.  $12 + 17$  
   b.  $-12 + (-17)$
   c.  $19 + 28$  
   d.  $-19 + (-28)$

What do you observe? Is there a simple way of combining two integers that have the same sign, both positive or both negative? Write a rule that explains the process. Use your rule for the following problems:

2. Find the following sums.
   a.  $13 + 19$  
   b.  $-13 + (-19)$
   c.  $16 + 13$  
   d.  $-16 + (-13)$

Now let’s explore some problems where the numbers we are adding have opposite signs and try to modify our rules to solve these.

3. Find the following sums.
   a.  $-13 + 19$  
   b.  $13 + (-19)$
   c.  $26 + (-33)$  
   d.  $-26 + 33$

Do you see a pattern? Try writing a rule for these problems. How can you use absolute values to describe what you have done? Sketch a number line to show your rule is correct.

4. Find the following sums using what you discovered about adding a positive and a negative integer.
   a.  $28 + (-33)$
   b.  $-28 + 33$
   c.  $-45 + 32$
   d.  $45 + (-32)$
Below is a picture for the sum $-26 + 33$. Does your rule give you the answer of 7? Explain why your rule works.

**EXAMPLE 1**

Find the sum: $-103 + 94$. Explain how you obtained your answer.

**SOLUTION**

**Step 1:** We first find the absolute values $|-103| = 103$ and $|94| = 94$.

**Step 2:** Because the numbers have opposite signs, we compute the difference of the absolute values $103 - 94 = 9$.

**Step 3:** The answer is $-9$ because the number with the larger absolute value is $-103$, a negative number.

We can also find the answer by driving along the number line. The first number says to drive 103 units to the left, and the second number says to drive 94 units to the right. The result is that you will have driven 9 more units to the left than to the right, so you are still left of 0 and your final position will be $-9$. 
EXAMPLE 2

a. During a football game, David gains 13 yards on one play and gains 22 yards on the next play. What is his net yardage? **Net yardage** is the total number of yards gained or lost at the end of a series of plays.

b. On the next series of downs, he gains 16 yards on the first play and loses 9 yards the second play. What is his net yardage this time?

SOLUTION

a. His total or net yardage is calculated by addition: \(13 + 22 = 35\) yards.

b. His net gain can be calculated by \(16 - 9 = 7\) yards. We can also think of this as addition by writing the sum \(16 + (-9) = 7\). In the first play he gained 16 yards and in the second play he “gained” \(-9\) yards. Thinking of the problem in this way will help in working the following exercises. You can draw pictures or diagrams to help visualize what is going on.

EXPLORATION 1

Draw a number line from -15 to 15. Find the distance between each of the following pair of numbers. Explain how you can compute these distances without using the number line.

a. 8 and 3  
c. -8 and 3
b. 8 and -3  
d. -8 and -3

PROBLEM 1

For each pair of numbers below, sketch a number line to illustrate the distance between the two numbers. Then, show how to use subtraction to compute the distance. For example, to illustrate the distance between 22 and 8, we would draw the following number line.

\[
\begin{array}{c}
\hline
\downarrow & \downarrow & \downarrow \\
0 & 8 & 22 \\
\end{array}
\]

a. 22 and 8  
c. -14 and -26  
e. -25 and 5  
g. 0 and -29
b. -24 and 12  
d. 15 and -23  
f. -8 and -24
EXERCISES

For exercises 2-13, write and compute an expression. Label your answers.

1. Compute the following sums and differences:
   a. \(-55 + 82\)  
   b. \(55 + -82\)  
   c. \(-55 - 82\)  
   d. \(118 - 153\)  
   e. \(-118 + 153\)

2. During a football game, Francisco lost 17 yards on one play. On the next play, he gained 25 yards. Find Francisco’s net yardage for these two plays.

3. During a football game, Ramon gained 9 yards on the first play but lost 17 yards on the next play. What was his net yardage for these two plays?

4. In Fairbanks, Alaska, the temperature on February 2nd was \(-52\) °F. The next day a cold front moved in, and the temperature dropped \(21\) °F. How cold was it then?

5. In Juneau, Alaska, the temperature on January 10th was \(-23\) °F. The next day the temperature rose \(16\) °F. What was the temperature on January 11th?

6. David’s current balance is \(-\$45\). He needs to withdraw \$132 more. What will David’s balance be after he withdraws the \$132?

7. a. Alex had a sick pig. During one week the pig lost 18 pounds. The next week, the pig lost 15 pounds. How many pounds did the pig lose during these two weeks?
   b. Suppose we consider weight gained by a positive number and weight lost by a negative number. Do part a again, using this idea. In other words, he gained \(-18\) pounds the first week.

8. Our family’s pet liger Jordan lost 18 pounds during May and gained 13 pounds during June. What was the net gain during these two months?

9. The water level at Falcon Lake rose 4 feet and then dropped 7 feet from its original level. What is the change in the water level from its original position?
10. Marcus dives off a 10 m diving board and goes 3 m below the surface. What is his total change in position? What is his change in position when he dives off an 8 m platform and goes 4 m below the surface?

11. After a coal miner descends 1,582 ft, he is 798 ft from the bottom of the mine. How deep is the mine?

12. The temperature on the surface of Mercury drops 1,079 °F in a day’s revolution. If the temperature is 800 °F during the hottest part of the day, what is the coolest temperature at night?

13. A plane is flying at an altitude of 15,782 ft. For the following situations, estimate the plane’s altitude to the hundred’s place.
   a. the plane descends 2,200 ft  c. the plane ascends 2,200 ft
   b. the plane ascends 1,933 ft  d. the plane descends 1,933 ft

14. **Ingenuity:**
   a. A magic square is a three-by-three grid containing each of the whole numbers from 1 to 9 exactly once with the interesting property that the sum of the numbers on each row, the sum of the numbers on each column, and the sum of the numbers on each diagonal are all the same. What number must be in the middle of a magic square?
   b. Find a magic square with the number 1 in the upper center cell.
   c. Find any cells of a magic square that cannot contain a 1.
15. **Investigation:**

Let us explore one of the patterns you discovered in Chapter 1 more generally using the number line. Suppose \( x \) and \( y \) are two numbers such that \( x < y \). Remember, this just means that \( x \) is some number to the left of some other number on the number line.

Let us also think about where the two numbers are relative to zero. Remember, by the trichotomy property in Section 1.1, \( x \) is negative, zero, or positive. If \( x \) is negative, or to the left of zero, then \( y \) must also be to the left of zero, or negative. Do you see why? If \( x \) is zero, then \( y \) must be to the left of zero, or negative. If \( x \) is positive, we just know that \( y \) is to the left of \( x \); it can be negative, zero, or positive.

There are a total of five different cases depicted below. Remember, in every case \( x \) has to be to the right of \( y \), since \( x < y \). For each case, copy the number line, plot \(-x\) and \(-y\), and decide if \(-x\) is greater than, less than, or equal to \(-y\). Does the pattern you see match the pattern from Chapter 1? If not, try to write a new rule to describe the pattern.
In the previous sections’ exercises, you have been writing rules to describe patterns you observed about the integers. In this section, we will review some of these patterns and ways to write rules for them. We will use the number line to help demonstrate the reason the rules work.

We use variables like \( x \) to generalize the rules. Remember that these variables just represent a general number; if you were to replace every \( x \) in this section with a 4, the statements would be just as true, but less general.

Think about what happens when you add 0 to a number \( x \). You first drive \( x \) units to the location \( x \). Then you drive 0 units, remaining exactly where you were before:

In other words, adding 0 to a number doesn’t change its value. Because 0 has this property, we call 0 the **additive identity**.

<table>
<thead>
<tr>
<th>PROPERTY 2.1: ADDITIVE IDENTITY</th>
</tr>
</thead>
<tbody>
<tr>
<td>For any number ( x )</td>
</tr>
<tr>
<td>( x + 0 = x )</td>
</tr>
</tbody>
</table>

**EXAMPLE 1**

a. \( 2 + 0 = 2 \)  
b. \( -5 + 0 = -5 \)  
c. \( 0 + 0 = 0 \)
How can we find a pair of numbers on the number line that are the same distance from zero? To do this, we need to go the same distance from 0 but in opposite directions. For example, the numbers 1 and -1 are both 1 unit from 0. Similarly, 5 and -5 are the same distance from 0. We call pairs of numbers like 5 and -5 “opposites,” or additive inverses.

What happens when you add a number to its additive inverse? Beginning at the origin, you first move a certain distance in one direction, and then move exactly the same distance in the opposite direction:

Your final position is back at the origin, 0. Therefore, the sum of any number and its opposite is 0. We call this the additive inverse property.

PROPERTY 2.2: ADDITIVE INVERSE PROPERTY

For any number x, there exists a number -x, called the additive inverse of x, such that

\[ x + \text{-}x = 0 \]

EXAMPLE 2

a. \( 4 + \text{-}4 = 0 \)  
b. \( 15 + \text{-}15 = 0 \)  
c. \( 7 + \text{-}7 = 0 \)

What number is \( \text{-}(\text{-}x) \)? Because \( \text{-}(\text{-}x) \) is the opposite of \( \text{-}x \),

\[ \text{-}(\text{-}x) + \text{-}x = 0. \]

On the other hand, because \( \text{-}x \) is the opposite of x,

\[ x + \text{-}x = 0. \]
Comparing these two equations shows us that \(-(-x)\) must equal \(x\). In short, the opposite of the opposite of \(x\), \(-(-x)\), is the number \(x\) itself.

**THEOREM 2.1: DOUBLE OPPOSITE THEOREM**

For any number \(x\),

\[-(-x) = x\]

We can see this more easily on a picture:

![Diagram showing -x, 0, and -(-x) on a number line]

For example, if \(x\) is 6, the picture shows us that the opposite of the number 6 is \(-6\), and the opposite of \(-6\) is \(-(-6)\), which is the same as 6. If \(x\) is \(-9\), the opposite of \(x\) is 9, and the opposite of the opposite of \(x\) is \(-9\).

**EXAMPLE 3**

a. \(-(-12) = 12\)  
b. \(-(-3) = 3\)  
c. \(-(-56) = 56\)

Every negative number is the opposite of a positive number. If we think about negative numbers this way, the double opposite property gives us a nice way to express absolute values. If a number \(x\) is positive, then \(|x| = x\). For example, \(|5| = 5\). If a number \(x\) is negative, the opposite of a positive number, the absolute value of \(x\) will be the opposite of \(x\). So \(|-5| = -(-5) = 5\).

**DEFINITION 2.1: ABSOLUTE VALUE**

For any number \(x\), the **absolute value of** \(x\), written \(|x|\), is defined as follows:

\[|x| = x, \text{ if } x \geq 0\]
\[|x| = -x, \text{ if } x < 0\]

Compare this rule to Definition 1.5 (in Section 1.3) which says that the absolute value of a number is its distance from zero.
PROBLEM 1

Let’s try to figure out a formula for finding the distance between two points. Find the distance between each pair of points below.

a. 10 and 7
c. 14 and -3
7 and 10
-3 and 14

b. 14 and 7
7 and 14

What patterns do you notice in these examples? Try to write a rule for finding the distance between two numbers. What operation can you use to find the distance?

Try using your rule to help you find the distance between each pair of points below. Check your answers on a number line.

d. 10 and 8
8 and 10
f. -6 and -4
-4 and -6
e. 10 and -8
-8 and 10

Notice that the distance between the two numbers is the same no matter which one comes first. The distance is always nonnegative and equal in magnitude of the difference between the two numbers. This can be written as follows:

**DEFINITION 2.2: DISTANCE**

For any two numbers x and y, the distance between x and y is the absolute value of their difference; that is,

\[
\text{Distance} = |x - y|
\]

From Section 2.2, Exercise 3, you probably noticed that 4 – 3 and 4 + -3 are equivalent. This property can be summarized as follows:

**PROPERTY 2.3: SUBTRACTION PROPERTY**

For any two numbers x and y:

\[
x - y = x + -y
\]
Section 2.4 Integer Properties and Terminology

Suppose that $x$ and $y$ represent integers. Remember, to find the sum $x + y$, we started at the point 0 on the number line, moved $|x|$ units in the direction of $x$, and then moved $|y|$ units in the direction of $y$.

To find the sum $y + x$, we started at 0 and performed these two steps in the reverse order.

Reversing the order of these steps does not change the final outcome. In either case, we end up in the same place. We call this the **commutative property of addition**.

**PROPERTY 2.4: COMMUTATIVE PROPERTY OF ADDITION**

For any numbers $x$ and $y$,

$$x + y = y + x$$

**PROBLEM 2**

Compute each side of the equalities to show that addition is commutative.

a. $2 + 3 = 3 + 2$  
   b. $-5 + -10 = -10 + (-5)$  
   c. $-6 + 9 = 9 + (-6)$
Now, consider the expressions \((x + y) + z\) and \(x + (y + z)\), where \(x\), \(y\), and \(z\) are integers. To calculate the first, we first add \(x\) and \(y\) and then add \(z\); to calculate the second, we first add \(y\) and \(z\), and then add \(x\). Draw a picture using the car model for each of the expressions. Just as before, the order does not matter in determining the final value. This is called the **associative property of addition**.

**PROPERTY 2.5: ASSOCIATIVE PROPERTY OF ADDITION**

For any numbers \(x\), \(y\), and \(z\),

\[(x + y) + z = x + (y + z).\]

**PROBLEM 3**

Verify the equality.

a. \((3 + 6) + 1 = 3 + (6 + 1)\)  
b. \((-7 + 4) + 8 = -7 + (4 + 8)\)

**EXERCISES**

1. Find the opposites of the following. Remember, the opposite of \(x\) is \(-x\).
   
   a. 6  
   b. \(-7\)  
   c. 0  
   d. \(x\)  
   e. \(y\)  
   f. \(-z\)

2. Name the property associated with each of the following:
   
   a. \(3 + 2 = 2 + 3\)  
   b. \(-(-7) = 7\)  
   c. \((-5 + 4) + 6 = -5 + (4 + 6)\)  
   d. \(6 + -6 = 0\)  
   e. \(-37 + 4 = 4 + -37\)  
   f. \(879 + 0 = 879\)

3. Is there a commutative property of subtraction? In other words, is it true that for any numbers \(x\) and \(y\), \(x - y = y - x\)? Explain.

4. Evaluate the following expressions:
   
   a. i. \(|1| + |-1|\)  
      ii. \(|1 + -1|\)
   
   b. i. \(|-14 + -8|\)  
      ii. \(|-14| + |-8|\)
   
   c. i. \(|6 - (-2)|\)  
      ii. \(|6| - |-2|\)
   
   d. i. \(|15| - |27|\)  
      ii. \(|15 - 27|\)
5. Using the definition for Distance in this section, find the distances between
the following points:
   a. -8 and 4  12  
   b. 9 and -5  14  
   c. -9 and -5  4  
   d. -23 and 49  14  
   e. 34 and 2  
   f. -68 and 25  
   g. 64 and -6  
   h. -104 and -51  
6. Suppose that a, b, c, d are integers and a < b < c < d. Is it possible to
determine which of these four integers is closest to zero? Explain.
7. Estimate to the tens place:
   a. |987 – 342| 650  
   b. -|653 + 47| -700  
   c. |64 – 734| -670  
   d. -|823 – 297| -520  
   e. |43 – 362| 320  
   f. -|35 – 874| -830  
8. Locate the following integers on the number line: x + y, -x, -y, -x+ -y, and
   -(x + y).
   %\[ \begin{array}{c}
   \hline
   & 0 & x & y \\
   \hline
   \end{array} \]
9. For each of the following pictures, illustrate the distance from x to 5.
   a. %\[ \begin{array}{c}
   \hline
   & 0 & 5 & x \\
   \hline
   \end{array} \]
   b. %\[ \begin{array}{c}
   \hline
   & 0 & x & 5 \\
   \hline
   \end{array} \]
10. Ingenuity:
   a. Given a 3-liter jug, a 2-liter jug, and an unlimited supply of water, how
can you obtain exactly one liter of water?
   b. Given a 5-liter jug, a 3-liter jug, and an unlimited supply of water, how
can you obtain exactly four liters of water?
11. Investigation:
   Think about how you can model multiplication on the number line. How
would you multiply two positive numbers? A negative by a positive? Think
about what the first number and the second number mean using skip
counting.
SECTION 2.5 THE CHIP MODEL

Another way to think of positive and negative numbers is with positive and negative chips. Each positive chip counts as positive one. Each negative chip is a negative one. If you have 2 negative chips, then you “owe” the bank $2. If you have 2 positive chips then the bank “owes” you $2.

The chip model is based on the set model and the additive inverse property: \(1 + (-1) = 0\). Many times, we call \(1 + (-1)\) a zero pair because the sum is zero. We can picture this as

\[
\bigcirc + \bigcirc = \bigcirc = 0
\]

EXAMPLE 1

Model each equation with positive and negative chips:

a. 3 + 2 = 5  
b. 3 – 2 = 1  
c. 3 + -2 = 1

SOLUTION

a. \[
\begin{array}{c}
\bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \\
\end{array}
\]

b. \[
\begin{array}{c}
\bigcirc \bigcirc \\
\bigcirc \\
\bigcirc \\
\end{array}
\]

c. \[
\begin{array}{c}
\bigcirc \bigcirc \\
\bigcirc \bigcirc \\
\bigcirc \\
\bigcirc \\
\end{array}
\]
Notice that $3 - 2$ produces the same result as $3 + (-2)$ because taking away two positive chips is like adding two negative chips. Each negative chip cancels out the positive chip so that you end up with one positive chip. **Subtraction is like adding the opposite**. We can think of adding $n$ opposites as “cancelling” or taking away $n$ chips.

**EXAMPLE 2**

Represent the expression $-5 + 3$ with chips.

**SOLUTION**

![Chip representation of $-5 + 3$]

**EXAMPLE 3**

Use chips to represent $-5 - (-2)$.

**SOLUTION**

This is the same as beginning with 5 negative chips and taking away 2 negative chips. You end up with 3 negative chips. Thus $-5 - (-2) = -3$. 

![Chip representation of $-5 - (-2)$]
EXAMPLE 4

In order to model subtraction, we have assumed that we have chips in our pile to take away. But what do we do if we begin with 5 positive chips and need to take away 8 positive chips? How do we model \(5 - 8\)?

SOLUTION

Let’s first take our 5 positive chips and add in three zero pairs. Because adding zero does not change the number, we have not changed the problem, but we now have enough positive chips.

\[
\begin{align*}
\begin{array}{c}
\text{\includegraphics{positive_chips.png}} \\
\end{array}
\quad = \quad \begin{array}{c}
\text{\includegraphics{positive_chips.png}} \\
\text{\includegraphics{zero_pairs.png}} \\
\end{array}
\quad = \quad \begin{array}{c}
\text{\includegraphics{positive_chips.png}} \\
\text{\includegraphics{positive_chips.png}} \\
\text{\includegraphics{positive_chips.png}} \\
\text{\includegraphics{positive_chips.png}} \\
\text{\includegraphics{positive_chips.png}} \\
\text{\includegraphics{negative_chips.png}} \\
\text{\includegraphics{negative_chips.png}} \\
\end{array}
\end{align*}
\]

\[5 = 5 + 3 + (-3) = 8 + (-3)\]

We can now subtract 8 positive chips from our set of 8 positive chips and 3 negative chips. We are left with 3 negative chips.

\[
\begin{align*}
\begin{array}{c}
\text{\includegraphics{positive_chips.png}} \\
\text{\includegraphics{positive_chips.png}} \\
\text{\includegraphics{positive_chips.png}} \\
\text{\includegraphics{positive_chips.png}} \\
\text{\includegraphics{positive_chips.png}} \\
\text{\includegraphics{negative_chips.png}} \\
\text{\includegraphics{negative_chips.png}} \\
\end{array}
\quad = \quad \begin{array}{c}
\text{\includegraphics{negative_chips.png}} \\
\text{\includegraphics{negative_chips.png}} \\
\end{array}
\end{align*}
\]

\[8 + (-3) - 8 = -3\]

EXAMPLE 5

Use chips to model \(-3 - 2\).

SOLUTION

We begin with 3 negative chips. We now need to take away 2 positive chips. Again, we add 2 “zero pairs.”
Now we can take away 2 positive chips, and are left with 5 negative chips:

\[-5 + 0 = -5\]

**EXERCISES**

1. Find the following sums using the chip model. Check your answers using the number line model. Do your answers agree? Which model do you prefer? Why?
   a. \(5 + 3\)  
   b. \(5 + (-3)\)  
   c. \(-3 + 5\)  
   d. \(-3 + (-5)\)

2. Compute the following sums and differences using both the chip model and the number line model.
   a. \(5 - 3\)  
   b. \(5 - 8\)  
   c. \(-7 - (-9)\)  
   d. \(-7 + (-9)\)

3. When using the chip model to compute the difference \(6 - (-3)\), how do you subtract the negative? Explain what you are doing in words.

4. Sam owes Sarah $5. In the chip model, we represent a debt with negative chips. If he pays her $3, how much does he still owe her?

5. Carlos owes Claudia $5. If Claudia loans Carlos another $3, how much will he then owe?

6. Alejandro has 20 positive chips and 15 negative chips. Alma has 32 positive chips and 29 negative chips. Whose chips represent the greater sum?
7. Find 5 ways to represent the number 10 using positive and negative chips.

8. How many ways are there to represent the number 5 using positive and negative chips if you only have 6 positive chips and 6 negative chips to use?

9. How many ways are there to represent the number 5 using positive and negative chips if you have an unlimited supply? Explain.

10. Use the chip model and addition or subtraction to show how two negative numbers can result in the answer 4.

11. **Ingenuity:**
   Elena and Becky play a game in which a coin is tossed repeatedly. If the coin comes up heads twice in a row, Elena wins. If the coin comes up tails twice in a row, Becky wins. If Elena wins after five tosses, then which side of the coin came up on each toss?

12. **Investigation:**
   Using the chip model, evaluate the following sums:
   a. $1 - 2$
   b. $1 - 2 + 3 - 4$
   c. $1 - 2 + 3 - 4 + 5 - 6$
   d. $1 - 2 + 3 - 4 + 5 - 6 + 7 - 8$
   e. Do you see a pattern? Can you find a quick way to evaluate each expression below?
      $$1 - 2 + 3 - 4 + \cdots + 25 - 26$$
      $$1 - 2 + 3 - 4 + \cdots + 25 - 26 + 27$$
REVIEW PROBLEMS

1. Write a problem and its solution that represents the model: \( 6 + (-10) = -4 \)

2. Marci is snorkeling in the San Marcos River 5 feet below the surface. She dives 3 feet deeper. How many feet below the surface is Marci now?

3. Isaac has \(-\$20\) in his account. He needs to borrow \$10 more. What will his balance be now?

4. Solve each problem below, and model the operation with a number line and a chip model:
   a. \( 9 + (-2) \)
   b. \( -8 + 3 \)
   c. \( -4 + (-6) \)
   d. \( -8 + 7 + 5 \)
   e. \( 4 - 6 \)
   f. \( -3 - (-7) \)

5. A football team is on its 25-yard line. In four consecutive plays the offense rushes for \(+8, -4, -6\) and \(+16\) yards. What yard line will this team be on after these four plays?

6. Find the following sums or differences. Show all your steps and use the number line if necessary.
   a. \( 54 - 63 \)
   b. \( -31 - 24 \)
   c. \( 35 - (-22) \)
   d. \( -26 + 21 \)
   e. \( -47 + (-75) \)
   f. \( -36 - (-44) \)

7. The temperature in Anchorage, Alaska, is \(-15\)°F. The temperature rises 12°F during the day. What is the new temperature?

8. Lisa’s bank account reads \(-\$56\). If Lisa withdraws \$23 from her account, what will her balance be?
9. The temperature in Billings, Montana is 13 °F. The temperature then dips 15 °F. What is the temperature now?

10. Name the property associated with the following:
   a. \( 72 + (-72) = 0 \)    d. \( (-15 + 3) + (-8) = -15 + (3 + -8) \)
   b. \( -(-12) = 12 \)    e. \( 6 + -4 = -4 + 6 \)
   c. \( 39 + 0 = 39 \)    f. \( 8 + -10 = 8 - 10 \)
Challenge Problems

Section 2.1:
The country of Fakestan prints all of its currency in 5- and 7-dollar bills. By law, all purchases must be made with exact change and it is illegal to charge an amount that is impossible to pay. What is the largest whole dollar amount that cannot be charged?

Section 2.2:
After much public discussion, the government of Fakestan, whose currency only comes in 5- and 7-dollar bills, decided to allow purchases to be made either with exact change or with one bill in change. What is the largest whole dollar amount that cannot be charged now?

Section 2.3:
Edward tried to add the page numbers of a book, but he left one number out of the sum and got the incorrect total of 200. What number did he miss?

Section 2.4:
If each ± represents addition or subtraction, how many different values are possible for

\[ 1 \pm 2 \pm 3 \pm 4 \pm 5 \pm 6 \pm 7 \pm 8 \pm 9 \pm 10 \]

Section 2.5:
Anne and Bill play a game where Bill tries to guess what number Anne is thinking of. If Bill is correct, then he wins, otherwise Anne wins. The first round is worth one penny to the winner, and each later round is worth twice as much as the one before (the pennies come from an endless supply). Anne wins the first 9 rounds, but Bill wins the 10th round. How many more pennies does Bill have than Anne?
Chapter 3  Modeling Problems Algebraically
SECTION 3.1 VARIABLES AND EXPRESSIONS

Numbers give us a way to describe different quantities. The number 5 might be 5 marbles, or it might be 5 units on a number line. When we write 5, we have in mind a definite amount or quantity. Often, however, we will want to represent an unknown quantity. To do this, we use variables. For example, Amy has some marbles, but we don’t know how many she has. We could then write

\[ M = \text{the number of marbles Amy has}. \]

A **variable** is a letter or symbol that represents an unknown quantity. Therefore, \( M \) is a variable that represents the number of marbles Amy has, even though we do not know what that number is. Variables give us a convenient way to describe properties and ideas because variables can represent many different things.

We use numbers, variables, and mathematical operations to form expressions. For example, Lisa has one more marble than Amy. We could then write \( M + 1 \) to describe how many marbles Lisa has. **Expressions** are mathematical phrases, like \( M + 1 \), that we use to describe quantities mathematically. Expressions can be numbers, variables, or a combination using math operations. Some other examples of expressions are \( 2x + 3 \), 5 • 2, \( 2x + y - 3 \), and 15.

The following table summarizes the different ways a mathematical expression can be translated into a word phrase.
### The Language of Algebraic Expressions

<table>
<thead>
<tr>
<th>Operation</th>
<th>Expression</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Addition</strong></td>
<td>$x + 5$</td>
<td>The sum of $x$ and 5</td>
</tr>
<tr>
<td></td>
<td>$x + 5$</td>
<td>$x$ plus 5</td>
</tr>
<tr>
<td></td>
<td>$x$ increased by 5</td>
<td>$5$ added to $x$</td>
</tr>
<tr>
<td></td>
<td>$5$ greater than $x$</td>
<td>$5$ more than $x$</td>
</tr>
<tr>
<td></td>
<td>$5$ greater than $x$</td>
<td>The addition of 5 and $x$</td>
</tr>
<tr>
<td><strong>Subtraction</strong></td>
<td>$n - 10$</td>
<td>The difference of $n$ and 10</td>
</tr>
<tr>
<td></td>
<td>$n - 10$</td>
<td>10 less than $n$</td>
</tr>
<tr>
<td></td>
<td>$n - 10$</td>
<td>10 subtracted from $n$</td>
</tr>
<tr>
<td></td>
<td>$n - 10$</td>
<td>$n$ minus 10</td>
</tr>
<tr>
<td></td>
<td>$n - 10$</td>
<td>$n$ decreased by 10</td>
</tr>
<tr>
<td></td>
<td>$n - 10$</td>
<td>take away 10 from $n$</td>
</tr>
<tr>
<td><strong>Multiplication</strong></td>
<td>$12d$</td>
<td>12 times $d$</td>
</tr>
<tr>
<td></td>
<td>$12d$</td>
<td>12 multiplied by $d$</td>
</tr>
<tr>
<td></td>
<td>$12d$</td>
<td>The product of 12 and $d$</td>
</tr>
<tr>
<td><strong>Division</strong></td>
<td>$\frac{y}{4}$</td>
<td>The quotient of $y$ and 4</td>
</tr>
<tr>
<td></td>
<td>$\frac{y}{4}$</td>
<td>$y$ divided by 4</td>
</tr>
</tbody>
</table>

### EXAMPLE 1

Translate “five more than two” into a mathematical expression.

**SOLUTION**

“Five more than two” translates as $2 + 5$.

Notice that we would get the same answer if we calculated $5 + 2$, but that would really represent the expression “two more than five.”

### EXAMPLE 2

Translate “five less than two” into a mathematical expression.

**SOLUTION**

“Five less than two” translates as $2 - 5$.

In this case, this is not the same as $5 - 2$. Do you see the difference?
EXAMPLE 3

Translate “five more than x” into a mathematical expression. Illustrate this on the number line with an arbitrary point with coordinate x.

SOLUTION

“Five more than x” translates as x + 5. Notice that you get this answer no matter what the variable x represents.

Remember that the symbol < is the “less than” symbol, and the symbol > is the “greater than” symbol. So, the inequality x < 2 says that “x is less than 2.” Although we do not know what the variable x is, this inequality says that x is some number that is less than 2. On the other hand, the expression “x less than 2” is written mathematically as 2 − x.

EXAMPLE 4

Translate the expression, “x − 3” into a word phrase. Locate x − 3 on the number line in Example 3.

SOLUTION

“x − 3” translates as “3 subtracted from x.” What are other ways that this expression can be written as a word phrase? In this case, why is it wrong to say, “x less than 3”? What is the difference?

EXAMPLE 5

Translate the expression “−4 + 6” into a word phrase.
SOLUTION

"-4 + 6” translates to “the sum of -4 and 6.” What other ways can this be written as a word phrase?

EXAMPLE 6

Using the symbol >, write an inequality that says “5 is greater than x.”

SOLUTION

5 > x. Caution: “5 is greater than x” is not the same as “5 greater than x.” Do you see the difference?

EXERCISES

1. Translate and compute each of the following into a mathematical expression.
   a. 2 more than 5  b. 3 less than 6
   2 less than 5  6 less than 3

2. Translate and compute each of the following:
   a. 26 added to -14  d. 8 increased by 12
   b. the difference of -38 and 20  e. 13 greater than 9
   c. the sum of 9 and 14  f. 11 subtracted from 7

3. Translate the following expressions or inequalities into word phrases or sentences. Try to use words other than plus or minus.
   Example: -2 – 7  7 less than -2
   a. 8 – 5  
   b. -4 + 5  
   c. 4 + 2  
   d. -3 – 2  
   e. x + 2  
   f. R – 5  
   g. m < 4  
   h. m > 0
4. a. Using words, not mathematical symbols, explain the difference between the statement “2 is less than 5” and the expression “2 less than 5.”

b. Translate part a into mathematical expressions or inequalities.

5. Translate each of the following into a mathematical expression or inequality.
   a. 5 less than 8 
   b. 5 is less than 8 
   c. 7 is greater than 3 
   d. A is greater than B 
   e. A is less than B 
   f. A more than B 

6. If A is less than 5 and greater than -2, what integers could A represent?

7. Translate the equation N = 3 + 2 into a sentence.

8. Translate the sentence “A is four more than 2” into a mathematical statement.

9. Three is seven more than P. What integer does P represent?

10. Ingenuity:

Four sticks are placed end to end to form a line 52 units long. If each of the last three sticks is six units longer than the one before it, how long is the longest stick?

11. Investigation:
   b. For what value of x is 5 + x = 7?
   c. What is different about the answers to a and b? What is the difference between an expression and an equation? What verb is associated with the answer to an expression? What verb is associated with the answer to an equation?
Let’s begin with the sentence “A number is 3 more than 7.” We could figure out what this number is with relative ease, but how can we write this mathematically? We can use numbers, variables, and operations to form expressions. Then we can combine these expressions to form a mathematical sentence called an equation. An equation is a math sentence with an equality sign, =, between two expressions. As long as a math sentence contains an = sign, it is called an equation. So you could have equations like $2 + 3 = 5$ which is simply a numerical equation, or $x + 2 = 10$ which is an equation with one variable $x$, or even an equation like $A = L \cdot W$ which is a formula that can be used to find the area of a rectangle. Now, explain verbally the difference between an equation and an expression.

**EXAMPLE 1**

Translate the sentence “A number is 3 more than 7” into an equation.

**SOLUTION**

Step 1: We give the unknown number a name, $N$, and write “$N$ = the number.”

$N$ is a variable. It represents the number we are trying to find.

Step 2: We translate the sentence into an equation.

$$N = 7 + 3$$

So, the equation form of the sentence is $N = 7 + 3$.

Since $7 + 3 = 10$, we can conclude that $N = 10$. $N$ now represents a known quantity, 10, instead of an unknown quantity. Therefore, we say that we have solved the equation for $N$.

**PROBLEM 1**

Translate the sentences below where $x$ is a number.
a. $x$ less than 10 equals 8.
b. 10 less than $x$ equals 8.

**EXAMPLE 2**
What number is twice as large as six?

**SOLUTION**

**Step 1:** We define a variable to represent our number. Let $T$ be a number that is twice as large as six.

**Step 2:** The statement “A number is twice as large as six” translates as $T = 2 \times 6 = 2 \cdot 6 = (2)(6)$.

When we put the symbols 2 and 6 next to each other with parentheses around each, it is understood that we mean to multiply them. The small dot is also a symbol for multiplication. So $(2)(6) = 2 \cdot 6 = 2 \times 6$. We usually do not use the symbol $\times$ for multiplication because it could be confused with a variable $x$.

**Step 3:** We know $2 \cdot 6 = 12$, so $T = 12$.

**Step 4:** Check. Is 12 a number that is twice as large as 6? Yes.

Using variables to model problems is the beginning of learning algebra. Algebra enables us to translate problems into mathematical expressions and equations. We then use mathematics to solve for the unknowns, which provides solutions to our problems.

**PROBLEM 2**

Translate each number sentence below. Can you determine the value of the variable?

a. The product of four and a number is 64.

b. A number times seven is forty-nine.

Note: When we are able to determine the value of the variable in an equation, we say that we have **solved** the equation. The value of the variable is called a **solution** of the equation.
EXAMPLE 3

Translate the sentence “A number is 2 less than four times 10” into an equation and solve for the unknown variable. Does your answer make sense?

SOLUTION

Step 1: Let’s call our unknown number \( N \).

Step 2: \( N = (4)(10) - 2 \) is our equation for the statement above.

Step 3: The right side is equal to 38. So, \( N = 38 \) is the solution. We have solved the equation! Now let’s check our answer.

Step 4: Is 38 equal to 2 less than 4 times 10? Four times 10 equals 40, and 2 less than 40 is 38, so our answer is correct.

PROBLEM 3

Translate and solve each number sentence below.

a. Four greater than three times a number is 19.

b. Half of 36 minus six is a number.

EXAMPLE 4

We have seen how we can use numbers and variables to translate problems into equations. Consider the problem, “Jeremy is 9 years old. In how many years will Jeremy be 15 years old?”

How might you begin this problem? Did you define a variable? If so, how did you use this variable?

Here is a step-by-step approach. Do your steps resemble the following?

Step 1: Define your variable

\[ Y = \text{the number of years it takes for Jeremy to reach 15}. \]

Step 2: Translate the problem into an equation

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We know that 15 is \( Y \) more than 9, so we write \( 15 = 9 + Y \), an equation with one variable, \( Y \).

**Step 3: Solve for the unknown variable**

If you look on the number line, you’ll notice you have to move right 6 units to go from 9 to 15. So \( Y = 6 \).

**Step 4: Check your answer**

Substitute \( Y = 6 \) into the original equation to see that \( 15 = 9 + 6 \).

**PROBLEM 4**

Jacob has $73. How much more does he need if he wants to have $98?

Expressions can be evaluated by substituting in values for variables within the expression.

**EXAMPLE 5**

The statement, “Alexandra has four more marbles than Denise” can be translated into the expression: \( x + 4 \), where \( x \) is the number of marbles Denise has. If Denise has 10 marbles, how many marbles does Alexandra have?

**Solution**

Let \( x \) = the number of marbles Denise has.

\( x + 4 \) is the expression representing the number of marbles Alexandra has, relative to the number of marbles Denise has.

If \( x = 10 \), substitute 10 in for \( x \) and the expression gives us:

\( (10) + 4 = 14 \), therefore, Alexandra has 14 marbles.

**Example 6**
Ryan and Annie are downloading songs from the internet. Ryan purchased three fewer songs than Annie. If Annie bought 12 songs, how many songs did Ryan purchase?

**Solution**

Let \( s \) = the number of songs Annie purchased.

\( s - 3 \) is the expression representing the number of songs Ryan purchased, which is less than the number of songs Annie purchased.

If \( s = 12 \), substituting 12 into the expression for \( s \), gives us:

\[
(12) - 3 = 9
\]

Ryan purchased 9 songs.

**Example 7**

Eric and Alison are playing a board game. Alison is six spaces ahead of Eric. On his next turn, Eric draws a card that requires him to move back three spaces. How many spaces away from Alison is he?

**Solution**

Let \( n \) = the number of spaces Eric moves.

\( 6 - n \) is the expression representing the number of spaces Eric is away from Alison.

If \( n = -3 \), substituting -3 into the expression for \( n \) gives us:

\[
6 - (-3) = 9
\]

Eric is 9 spaces away from Alison.
EXERCISES

Try the following exercises using the four-step process. When you “solve” your problem, you should not only find the answer, but also show the way you got your answer, which is just as important.

1. Write the following statements as either an expression or an equation. Let \( N \) = the number.
   a. A number is 2 more than 43.
   b. Four less than a number.
   c. A number is twice as large as 60.
   d. The difference of a number and 4.
   e. A number is the product of 5 and 2 increased by 3.
   f. 8 more than a number.
   g. Ten less than fifteen is five.
   h. 7 subtracted from a number is thirteen.
   i. Six more than eight is fourteen.

2. Translate each of the expressions or equations below into a mathematical statement, and read your statement verbally.
   a. \( N - 2 \)
   b. \( 17 - N \)
   c. \( 2 + 4 = 6 \)
   d. \( N + 2 = 15 \)
   e. \( 3N - 2 = 13 \)
   f. \( N - M \)
   g. \( N + M \)

Do steps 1 and 2 of our four-step process for Exercises 3–6.

3. Jack is 10. In how many years will Jack be 24?
4. If Daniel will be 16 in 7 years, how old is Daniel now?
5. Jacob has $73. How much more does he need if he wants to have $98?
6. Samantha has $85 and lends George $58. How much does she have left?
Write a story problem for the equations in Exercises 6 and 7:

7. \( X + 8 = 25 \)
8. \( 64 - X = 12 \)

9. The lowest and highest points in North America are Death Valley in California and Mount McKinley in Alaska. Death Valley is below sea level. In fact, it is 282 feet below sea level! On a number line, we represent this elevation by \(-282\). Mount McKinley is 20,320 feet above sea level. There is a big difference in the two elevations. Use \( D \) to represent the height we must climb to go from the elevation of Death Valley to the elevation of Mount McKinley. The equation that models this situation is \(-282 + D = 20,320\). Solve for \( D \).

In exercises 9 and 10, define a variable, set up an equation, solve, and check. Remember to include your units of measurement, such as °F and °C, in the definition and in the answer.

10. In the morning it was a cool 65 °F. By the afternoon the temperature had reached 87 °F. What was the increase in temperature from morning to afternoon?

11. On a cold day in Canada, the temperature was \(-8\) °C at 6:00 A.M. How many degrees must it warm up to reach 5 °C?

12. Three is seven more than \( P \). What integer does \( P \) represent?

13. Emily eats three cookies and has 1 cookie left. Let \( M \) = the number of cookies Emily had at the beginning. What integer does \( M \) represent?

14. Brad was riding his bike with his father on a bike trail. Brad finished 8 minutes ahead of his father. If his father took 20 minutes to complete the course, how long did it take Brad? Write the expression for the number of minutes it took Brad to complete the course. Solve the problem.

15. Amy is bowling with her friend JD. Amy scored a total of 120 points. JD scored 15 points less than Amy. How many points did JD score? Write the expression for the number of points JD scored and solve the problem.
16. **Ingenuity:**

In a certain sequence of numbers, each term after the first is 4 greater than the previous term. We don’t know the value of the first term; but we do know that the sixth term of the sequence is 47. We want to find the value \( x \), the second term. We can represent this information as __, \( x \), __, __, __, 47, …

What is the value of \( x \)?

17. **Investigation:**

A mobile is a type of hanging sculpture in which several objects are suspended in balanced equilibrium. In the mathematical mobiles below, each shape has an associated weight, and for each horizontal beam, the total weight hanging from one side is equal to the total weight hanging from the other (we assume that the wire has no weight). For example, in the mobile on the left, the two circles together weigh as much as the rectangle; the rectangle weighs twice as much as a circle. In the mobile on the right, the circle has the same weight as the hexagon, and the rectangle has twice the weight of the hexagon.

If the rectangle has weight 4, the hexagon and circle each have weight 2. We can write this symbolically as \( \Box = 4 \), \( \bigcirc = \bigcirc = 2 \). Based on the weight given and the mobile balance property, deduce each shape’s weight.

a. 

b. 

\( \Box = 4 \)

\( \triangle = 3 \)
Chapter 3  Modeling Problems Algebraically

c. $\star = 5$
SECTION 3.3 SOLVING EQUATIONS WITH SUBTRACTION

This is a balance scale:

When we put a weight on one side of the scale, we must place the same weight on the other side in order for the scale to be balanced. If a scale is balanced and equal weights are added or subtracted from both sides of the scale, the scale will remain balanced.

In much the same way, an equation is a statement that two expressions are equal. We can think of the expressions on each side of the equality sign as representing the weight placed on each side of a balanced scale. When we add or subtract the same amount from each side of the equation, the equation will remain **balanced**.

**EXAMPLE 1**

If Jeremy was three years older, he would be the same age as his twelve-year-old sister. What is Jeremy’s age?

**SOLUTION**

We let $J$ be Jeremy’s age, and translate the sentence into an equation as follows:

<table>
<thead>
<tr>
<th>Jeremy’s age</th>
<th>three years older</th>
<th>same age as</th>
<th>twelve-year-old sister</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J$</td>
<td>$+3$</td>
<td>$=$</td>
<td>$12$</td>
</tr>
</tbody>
</table>
Now we have the equation $J + 3 = 12$. The unknown is $J$, Jeremy’s age. Pictorially, this sentence says that $J + 3$ is the same as 12, which we can show on a balance scale:

In order to solve this equation for $J$, we must find what balances $J$. To do this, we remove three “blocks” from each side of the balance scale:

This is what we have left:

We can express this algebraically as follows:

\[
\begin{align*}
J + 3 &= 12 \\
(J + 3) - 3 &= 12 - 3 \\
J + (3 - 3) &= 9 & \text{by associative property} \\
J + 0 &= 9 & \text{by additive inverse property} \\
J &= 9 & \text{by additive identity property}
\end{align*}
\]
Because we have now solved for $J$, we can go back and check the solution. Substituting $J = 9$ into the original equation $J + 3 = 12$ gives us $9 + 3 = 12$, which is correct. Jeremy’s age is 9.

**EXAMPLE 2**

If Wesley finds 5 more marbles, he will have the same number of marbles as John. John has 11 marbles. How many marbles does Wesley have?

**SOLUTION**

We can use a four-step method similar to the one in the previous section to solve the equation.

**Step 1: Define a variable.**

Let $W =$ the number of marbles that Wesley has.

**Step 2: Translate the problem into an equation.**

$\text{Wesley’s marbles finds five more same number as John (11 marbles)}$

\[
W + 5 = 11
\]

**Step 3: Solve the equation.**

Draw a balance scale which represents the problem. Solve the equation with the balance scale, and then solve it algebraically.
W + 5 = 11
(W + 5) – 5 = 11 – 5
W + (5 – 5) = 6 by associative property
W + 0 = 6 by additive inverse property
W = 6 by additive identity property

Step 4: Check the solution.
Substituting the solution W = 6 into the original equation W + 5 = 11 gives 6 + 5 = 11, which is correct. Wesley has six marbles.

EXAMPLE 3
Jeffrey checks his bank balance and finds that he has been charged a $6 fee. His balance is now -$10. What was Jeffrey’s balance before the fee was assessed?

SOLUTION
Step 1: Define a variable.
Let X = Jeffrey’s original bank balance.

Step 2: Translate the problem into an equation.
Jeffrey’s original balance with a $6 fee becomes -$10
X + -6 = -10

Step 3: Solve the equation.
We would like to use our balance scale model again, but how do we represent -6 and -10? In Section 2.5, “The Chip Model,” we used positive chips for positive numbers and negative chips for negative numbers. In the balance scale model, a gray, or filled, block on the balance beam represents +1, and a white, or empty, block represents the opposite, -1.
Section 3.3 Solving Equations with Subtraction

Step 4: Check the solution.
Substitute \( X = -4 \) into our original equation \( X + 6 = -10 \) to get \(-4 + 6 = -10\), which is true. Jeffrey’s original balance was \(-$4\).

The rule we are using to solve these problems is called the **subtraction property of equality** because in each, we are subtracting the same number (removing the same number of blocks) from both sides of an equation. The new equation we obtain is said to be **equivalent** to the original equation because the two equations have the same solution: any value for a variable that makes one of the equations balance will make the other balance as well.

**PROPERTY 3.1: SUBTRACTION PROPERTY OF EQUALITY**

If \( A = B \), then \( A - C = B - C \).
In Section 2.2 you learned that subtracting is the same as adding the opposite. The subtraction property of equality is the same as the **addition property of equality** for negative numbers.

**PROPERTY 3.2: ADDITION PROPERTY OF EQUALITY**

If \( A = B \), then \( A + C = B + C \).

This property states that a balance scale stays balanced if we add the same number of blocks to both sides. The new equation this creates is equivalent to the original. Any value for a variable that balances the new equation will balance the original equation as well.

The property is easier to see in terms of the balance scale model:

If \( A = B \),

\[
\begin{array}{c}
\text{A} \\
\hline
\end{array}
\]

Then \( A + C = B + C \).
Section 3.3 Solving Equations with Subtraction

EXERCISES

Illustrate how the equation would appear on the balance scale. Solve the problem by adding or subtracting chips to both sides of the balance. Then solve each equation algebraically. Check that your answer is correct.

1. \( B + 3 = 8 \)
2. \( 7 = 2 + A \)
3. \( 8 = N + 5 \)
4. \( x - 2 = -6 \)

Solve the following equations using the pattern seen in the problems above. Check that your answer is correct.

5. \( x + 10 = 8 \)

6. \( x - 7 = -13 \)

7. \( -3 + 5 = 2 + n \)

8. \( 3 + v - 7 = -6 \)

9. \( b + c = d \) Solve for c (in terms of b and d).

10. \( 2a + x = 3y \) Solve for x (in terms of a and y).

11. **Ingenuity:**
   What is the difference between the sum of the first 30 even positive integers (2, 4, 6, 8, \ldots) and the sum of the first 30 odd positive integers (1, 3, 5, 7, \ldots)? Try to answer this problem without calculating any large sums.

12. **Investigation:**
   We have solved some algebraic equations using the four-step model. Consider the following two equations:
   a. \( 5 - x = 2 \)
   b. \( 5 = 2 + x \)
   How can you use some of the ideas we have used previously to solve these equations?
SECTION 3.4 SOLVING EQUATIONS WITH ADDITION

Some equations use subtraction and would be difficult to solve using our balance scale model so far. The following examples show how to represent subtraction with the balance scale model. When solving equations that involve subtraction, the fact that subtracting is the same as adding the opposite is an important tool.

EXAMPLE 1

Whitney was 10 years old 5 years ago. What is Whitney’s age now?

SOLUTION

Step 1: Define a variable.
Let \( W \) = Whitney’s age now.

Step 2: Translate the problem into an equation.
\( W - 5 = 10 \)

Step 3: Solve the equation.

Method 1:
We can use “holes” to represent subtraction from a variable:

The holes in the large variable block each represent a 1 that has been subtracted, so the block with five holes means \( W - 5 \), and the entire balance scale represents our equation \( W - 5 = 10 \).
We now need to fill in the holes. To do this, we must add five blocks to both sides of the balance. So, we can use the addition property of equality to finish Whitney’s problem:

![Balance Scale Diagram]

We added five positive blocks to fill in the five holes, so the variable block is now complete and represents $W$.

We can rewrite these steps algebraically, justifying the equivalent expressions at each step using the integer properties from Section 2.4.

\[
\begin{align*}
W - 5 & = 10 \\
(W - 5) + 5 & = 10 + 5 & \text{by additive property of equality} \\
(W + \cdot 5) + 5 & = 15 & \text{by subtraction property} \\
W + (\cdot 5 + 5) & = 15 & \text{by associative property} \\
W + 0 & = 15 & \text{by additive inverse} \\
W & = 15 & \text{by additive identity}
\end{align*}
\]

**Step 4: Check the solution.**

If Whitney is 15, then Whitney’s age five years ago was $15 - 5 = 10$. 
Method 2:
We could have solved this problem a slightly different way. Because $W - 5 = W + -5$, we could have changed the equation to $W + -5 = 10$. Remember that we model this on the balance scale by using empty blocks that each represent $-1$ each.

![Balance Scale Diagram]

Now we want to remove the five negative blocks from the left. There are no negative blocks to remove on the right side, so instead we use the addition property of equality and add five positive blocks to both sides. As in the chip model, this creates five zero pairs (one positive block and one negative block each) on the left side of the balance.

![Balance Scale Diagram with Zero Pairs]

We remove the five zero pairs from the left side, and we are left with the same solution as before, $W = 15$.

![Balance Scale Diagram with Solution]

We can rewrite these steps algebraically:

\[
W - 5 = 10 \\
W + -5 = 10 \\
(W + -5) + 5 = 10 + 5
\]
EXAMPLE 2

Amanda wrote a check for $9. The balance on her checking account is now -$2. How much money did Amanda originally have in her checking account?

SOLUTION

Step 1: Define a variable.

Let $Z = \text{Amanda’s original balance}$.

Step 2: Translate the problem into an equation.

$Z - 9 = -2$

Step 3: Solve the equation.

Again, we represent subtraction from the variable by putting holes in the variable block.

To fill in the nine holes, we add nine blocks.
The holes are filled in, and on the right side of the balance we have two zero pairs.

After cancelling the zero pairs, we are left with seven blocks on the right.

\[
\begin{align*}
Z - 9 &= -2 \\
(Z - 9) + 9 &= -2 + 9 & \text{by additive property of equality} \\
(Z + (-9) + 9 &= 7 & \text{by subtraction property} \\
Z + (-9 + 9) &= 7 & \text{by associative property} \\
Z + 0 &= 7 & \text{by additive inverse} \\
Z &= 7 & \text{by additive identity}
\end{align*}
\]

**Step 4: Check the solution.**

Our solution is correct because \(7 - 9 = -2\). Amanda originally had $7 in her checking account.
EXAMPLE 3

The temperature rises 4 °C to -7 °C when the sun comes out. What was the original temperature?

SOLUTION

Step 1: Define a variable.
Let $Y$ = the temperature, in Celsius, before the sun came out.

Step 2: Translate the problem into an equation.
$Y + 4 = -7$

Step 3: Solve the equation.

We add 4 negative blocks to each side in order to pair up four positive blocks to the four negative blocks on the left hand side. This will create a zero pair on the left hand side and leave $Y$ by itself.

$$Y + 4 = -7$$
$$(Y + 4) + (-4) = -7 + (-4)$$

$Y = -11$
Step 4: Check the solution.

The solution is correct because \(-11 + 4 = -7\). The original temperature was \(-11^\circ C\).

EXAMPLE 4

Tuesday morning Amanda had $5. Later in the day, she checked and found she had only $3 left. How much had she spent?

SOLUTION

Step 1: Define a variable.

Let \(M\) = the amount of money, in dollars, Amanda spent Tuesday.

Step 2: Translate the problem into an equation.

\[5 - M = 3\]

Step 3: Solve the equation.

Because subtraction is the same as adding the opposite, we can rewrite the equation as \(5 + -M = 3\).

\[-M\]

Since we want to solve for positive \(M\), we use the addition property of equality to add \(M\) to both sides. This creates a pair of additive inverses on the left side of the balance; we can also think of our notation as shorthand for \(M\) zero pairs. 

\(-M\)
Notice that after we remove these additive inverses, we arrive at a balance representing the addition problem $5 = M + 3$, which we know how to solve. Unlike before, however, the variable is on the right side of the balance.

After removing three blocks from each side of the balance, we are left with $M$ on the right and two positive blocks on the left.

\[
5 - M = 3 \\
5 + -M = 3 \\
(5 + -M) + M = 3 + M \\
5 + (-M + M) = M + 3 \\
5 + 0 = M + 3 \\
5 = M + 3 \\
5 - 3 = (M + 3) - 3 \\
2 = M + (3 - 3) \\
2 = M + 0 \\
2 = M
\]

**Step 4: Check the solution.**

The solution is correct because $5 - 2 = 3$. Amanda spent $2 Tuesday.
EXERCISES

Solve the following equations. You may use the balance scale model first if you like, but you must show the steps algebraically. Label each step with the property that you are using. Check that your answer is correct.

1. \( x + 9 = 7 \)
2. \( -2 = T + 2 \)
3. \( x + 23 = 17 \)
4. \( 50 = H + 50 \)
5. \( -4 = D - 6 \)
6. \( -2 = M + 8 \)
7. \( -10 = 9 + x \)
8. \( t - 4 = -3 \)

In problems 9 through 11, define a variable, set up an equation, solve, and check.

9. One day three pirates found 23 gold coins. The next day there were only two pirates and 5 gold coins. How many gold coins did the third pirate steal?

10. After eating 12 cookies, Jenny only has 32 cookies left. How many cookies did Jenny start with?

11. Ingenuity:

   Helen failed the first history test of the semester. On the second test, her score was 8 points higher. Helen’s score on the third test was 28 points higher than her score on the second test. Her score on the fourth test of the semester was 16 points higher than her score on the third test. The average of her four test scores was 70. What was Helen’s score on the first test?
12. **Investigation:**
Suppose we have the following sequence of numbers: 11, 7, 3, -1, -5, …
What are the next 4 numbers in the sequence? What is the 15th number in the sequence? What is the 100th number in the sequence?
SECTION 3.5 EQUATIONS & INEQUALITIES ON NUMBER LINE

In this chapter we have used the balance scale to solve equations such as “$x + 3 = 5$” and “$x - 4 = 3$”. We can also explore equations using a number line. We begin by investigating how to visualize expressions on a number line.

EXPLORATION 1

Suppose $a$ and $x$ are numbers located on the number line as seen below. Locate and label the points that represent the indicated numbers. Use string to act out how you determine your answer.

1. Plot points that represents each of the following: $2a$, $3a$, $-a$, $-2a$, $-3a$

2. Plot points that represents each of the following: $2x$, $3x$, $-x$, $-2x$, $-3x$

3. Compare the results from parts 1 and 2. What do you notice?

EXPLORATION 2

PART A: Suppose is a number that is located on the number line as seen below. Locate and label the points that represent the indicated expressions. The numbers 0 and 1 are also labeled. Plot a point that represents each of the following expressions:

$$x + 1, x + 2, x - 1, x - 2, \frac{x}{2}, 1 - x$$
Chapter 3  Modeling Problems Algebraically

PART B: Suppose we know the location of each of the expressions as indicated on the number line below. Find the locations for a, b and c. Explain how you locate each of these points on the number line.

\[ (a+2)(b-2)(c+5) \]

In Part A in this exploration we used the location of a variable on the number line to locate expressions on the same number line. In Part B, we were given the location of an expression, such as \( a + 2 = 5 \), and used it to find the location of the variable \( a \) on the number line. We see that \( a = 3 \). In other words, we solved the equation using the number line.

PROBLEM 1

Use the number line to solve each of the following equations:

a. \( x + 3 = 5 \)

b. \( y + 5 = 2 \)

c. \( z - 4 = 2 \)

d. Discuss how solving these equations on the number line compares with the balance scale method.

Recall that an equation is a statement that two expressions are equivalent. A statement that one expression is always less than (or greater than) another is called an inequality.

EXAMPLE 1

1. The number of apples, \( x \), consumed is more than twice the number of bananas, \( y \). Thus, \( x > 2y \).
2. Bob’s age, $B$, is less than 35 years. Thus, $B < 35$.

3. The cost of three apples is less than $2.00.

   Write an inequality to represent the possible cost $A$ of one apple.
   Let $A = \text{cost of an apple in cents}$. Then $3A < 200$

Sometimes an inequality is a statement of comparison between two quantities, such as, $4 < 7$. But we can also use an inequality to describe a condition that a variable satisfies, as in Bob’s age, $B$, is less than 35. So we say $B < 35$.

We can use a number line to represent all the possible numbers that satisfy an inequality. For example, suppose $S$ is all numbers less than 3. We say that $S$ is the set of numbers that are less than 4. Another way to describe this set is

   “$S$ is the set of all numbers $x$ so that $x < 4$.”

We use the inequality “$x < 4$” to defined the set $S$. We can represent this set $S$ on the number line below.

![Number line](image)

Notice that the part of the number line to the left of 4 represents the set $S$. This means that each number to the left of 4 is in $S$ and every number of $S$ is located on the line to the left of 4. Note that the point representing 4 is not filled in to indicate that 4 does not satisfy the condition that $x < 4$.

**EXAMPLE 2**

Draw a number line and represent the set $T$ of all numbers $x$ such that $-3 \leq x$.

![Number line](image)

Notice that the point at -3 is filled in to indicate that -3 does satisfy the condition that $-3 \leq x$.

In solving an equation, such as $x + 2 = 5$, we want to find all numbers $x$ that satisfies this statement. Since the only solution is $x = 3$, we say that the solution set is $\{3\}$.
If we start with an inequality, such as \( x + 2 < 5 \), we can ask:
For what numbers \( x \) satisfy this inequality?

We can represent the inequality on the number line as shown below.

The shaded part of the number shows where the expression \( x + 2 \) could be located on the number line. If \( x + 2 < 5 \), then \( x + 1 < 4 \) and \( x < 3 \). We draw a new number line to represent where the variable \( x \) can be. The resulting number line below is called the graphical solution to the inequality \( x + 2 < 5 \).

Notice that adding or subtracting the same number to both sides of an inequality will result always preserve the inequality. An inequality is like a balance beam that is not balanced. If you add or subtract the same things from both sides of the balance, the balance beam will still be unbalanced in the same way.

If \( a < b \) then \( a + c < b + c \), and if \( a < b \) then \( a - c < b - c \). Let’s use this to solve an inequality algebraically, and then graph the solution.

**EXAMPLE 3**

Draw a graphical solution to the inequality \( x - 4 < 8 \) algebraically Then graph the solution.

**SOLUTION**

\[
x - 4 < 8 \\
x - 4 + 4 < 8 + 4 \\
x < 12
\]
EXERCISES

1. Plot a point that represents each expression:

\[ 2x, 2x + 1, 2x - 1, 3x - 1 \]

2. Plot a point that represents each expression:

\[ y + 1, y - 1, 2y, 2y + 1, 2y - 1, 2y + 8 \]

3. Solve the equation \( x + 5 = 2 \) using the number line.

4. The cost of five apples is less than $8.00.
   Write an inequality to represent the possible cost \( A \) of one apple.

5. Use a number line to solve each of the following equations:
   a. \( x + 3 = 7 \)
   b. \( x + 7 = 3 \)
   c. \( x - 5 = 7 \)
   d. \( x + 9 = 2 \)

6. Solve each of these equations. Do you need to sketch a number line to help find the solution?
   a. \( x - 10 = 6 \)
   b. \( x - 24 = -7 \)
   c. \( x + 29 = 15 \)

7. Draw a number line and represent the set of all numbers \( x \) such that \( x < 8 \).
8. Draw a number line and represent the set of all numbers $x$ such that $x < -5$.

9. Draw a number line and represent the set of all numbers $x$ such that $x > -4$.

10. Draw a number line and represent the set of all numbers $x$ such that $-6 < x$.

11. Solve each of the following inequalities and graph their solution sets:
   a. $x + 3 < 2$
   b. $x - 3 < 2$
   c. $x + 5 < 6$
   d. $x + 5 < 2$
   e. $x + 3 > 6$
   f. $x - 4 > 2$

12. Graph the solution sets for each of the following inequalities:
   a. $2 < x + 3 < 5$
   b. $0 < x - 3 < 2$
   c. $2 < x + 5 < 8$
   d. $0 < x + 4 < 3$

13. **Investigation:**

   Use a number line to solve each of the following equations:
   a. $2x - 1 = 5$
   b. $2x + 3 = 11$
   c. $2x + 8 = 4$
REVIEW PROBLEMS

1. Illustrate how the equation would appear on a balance scale. Then solve algebraically for the unknown variable. Check that your problem is correct.
   a. $x + 3 = 9$
   b. $-7 = n + -2$
   c. $z - 8 = 5$

In problems 2–6, translate the sentence into an equation and solve for the unknown variable. Does your answer make sense?

2. A number is 4 less than 11.
3. A number is 3 more than twice 15.
4. A number is the difference of 28 and 4.
5. A number is 15 subtracted from 16.
6. 8 more than a number is 15
7. 4 less than a number is -7
8. A number is the sum of 3 and another number that is half of 40.

In problems 9–18, solve the equation.

9. $3 + t = 2$
10. $p - 16 = -3$
11. $y = 20 + -283$
12. $63 + m = -18$
13. $28 + d = 32$
14. $9 = a + 8$
15. $-8 = m - 7$
16. $x = 32 - 50$
17. $54 - a = 25$
18. $17 - b = 40$
19. Artemis was born in Greece in the year 34 BCE. Her grandson was born in the year 20 AD. How old was Artemis in the year when her grandson was born?

20. Rome is said to have been founded in 753 B.C.E. How many years passed before Christopher Columbus landed in North America in 1492? Represent 753 B.C.E. by \(-753\). Write an equation which models this problem, solve it and check your answer.

21. The great Renaissance artist Michelangelo was born in 1475. If Michelangelo were still alive, how old would he be on his birthday this year? Write a mathematical equation, solve and check.

For exercises 22-24, write an equation which models each problem. Solve it and check your answer.

22. Jack has a bag of marbles. Judy gives him 27 new marbles. If Jack now has 83 marbles, how many did he have in the beginning?

23. In an Alaskan village, the temperature rose 22°F from dawn until 10 a.m. What was the temperature at dawn if we know that it is 8°F at 10 a.m.?

24. In a Canadian village, the temperature drops from 26°C from 4:00 p.m. until 7:00 p.m. If the temperature at 7:00 p.m. is -15°C, then what was the temperature at 4:00 p.m.?

In problems 25 through 27, define a variable, set up an equation, and solve.

25. Valerie bought 8 books. She now has 23 books. How many books did she start out with?

26. In 24 years, Victor will be 42 years old. How old is Victor now? 27. A city bus makes three stops and the following events take place:
   - At the first stop, 5 passengers get off and 2 passengers get on.
   - At the second stop, 8 passengers get off and 3 passengers get on.
   - At the third stop, 9 passengers get off and none get on.
After the third stop there are 14 passengers on the bus. How many passengers were on the bus initially?
28. \( 2 + a = 6 \) and \( b = 8 + a \). Solve for \( b \) and \( a \).

29. **Ingenuity:**

   Lynn is seven years older than Joan but two years younger than Bobby. How much older than Joan is Bobby?
Section 3.1:
I am older than I once was, and younger than I’ll be. That’s not unusual. Ten years from now I’ll be twice as old as I was 10 years ago. How old am I now?

Section 3.3:
The cost of 3 hamburgers, 5 milk shakes, and 1 order of fries at a certain fast food restaurant is $23.50. At the same restaurant, the cost of 5 hamburgers, 9 milk shakes, and 1 order of fries is $39.50. What is the cost of 2 hamburgers, 2 milk shakes and 2 orders of fries at this restaurant?

Bonus:
If we restrict the price of a hamburger to a whole number of dollars and \( h > m > f > 0 \), find four solutions for \((h, m, f)\). Do these values agree with your answer to the original question?

Section 3.4:
Four children came across a pile of candy. Each one takes twice as many pieces as the one before. An odd number of pieces remain. A fifth child comes to take the remaining pieces, and is happy to see that he got as many as the average of the other four. If the number of pieces taken by the first child is a power of 2, how many pieces of candy were in the original pile?
SECTION 4.1 MULTIPLICATION OF INTEGERS

In Section 2.2, we introduced skip counting. We know that skip counting by 3’s generates the list 3, 6, 9, 12, 15, 18, 21, 24, 27, …, which continues indefinitely. Skip counting provides a model for multiplication that we can represent on a number line.

When we added using the car model, we drove the distance that corresponded to each of the numbers we were adding. In order to multiply, we can think of a frog that jumps along the number line. For example, when you multiply $4 \cdot 3$

- the first factor indicates which direction the frog should face and the length of each jump;
- the second factor indicates the number of jumps.

The picture below models the multiplication $4 \cdot 3 = 12$. Notice the frog is facing in the positive direction because the first factor, 4, is positive. The frog takes 3 jumps, and each jump is 4 units long.

At each step or jump in this process, the position of the frog changes by a constant amount, 4 units.
EXPLORATION 1: FROG JUMP MULTIPLICATION

Copy and fill Table 4.1a in which each jump is 4 units long.

<table>
<thead>
<tr>
<th>Length of Jump</th>
<th>Number of Jumps</th>
<th>Frog’s Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>n</td>
<td></td>
</tr>
</tbody>
</table>

What patterns do you notice? You might recognize these numbers from a multiplication table of 4’s where the pattern is $4 \cdot 1 = 4$, $4 \cdot 2 = 8$, $4 \cdot 3 = 12$, $4 \cdot 4 = 16$. You can think of $(4)(3)$ as $(4 \text{ units per jump})(3 \text{ jumps}) = 12 \text{ units}$.

You learned how to add positive and negative integers in the first chapter. Is there a way to think about multiplying a negative integer times a positive integer? You can use the frog model to multiply $-4 \cdot 3$, as shown below.
Copy and fill the skip counting Table 4.1b as you did in Table 6.1a, but this time use jumps of directed length \(-4\).

**Table 4.1b**

<table>
<thead>
<tr>
<th>Directed Length of Jump</th>
<th>Number of Jumps</th>
<th>Frog's Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-4)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(-4)</td>
<td>1</td>
<td>(-4)</td>
</tr>
<tr>
<td>(-4)</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>(-4)</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>(-4)</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>(-4)</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>(-4)</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>(-4)</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>(-4)</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>(-4)</td>
<td>(n)</td>
<td></td>
</tr>
</tbody>
</table>

Using the pattern demonstrated in this table, compute the product \(-3 \cdot 4\), or \(-3 \cdot 4\).

The picture below models the product \(-3 \cdot 4\). The first factor tells us which direction the frog should face and the length of each jump; the second factor tells us the number of jumps.

The frog is facing left because we are modeling a jump of \(-3\) units per jump.

Use the number line to compute the following products:

a. \((-3)(6)\)  
   c. \((-3)(3)\) 
   b. \((-3)(5)\)  
   d. \((-3)(1)\)
How can we make sense of the product $(3)(-4)$? This is the first example where the second factor is negative.

The first number, 3 or $+3$, gives the length of each jump, and the direction the frog is facing. Because the number is positive, the frog faces right.

The second factor gives the number of jumps. What do we mean by the number $-4$ as the number of jumps? If we think of the jumps taking place at equal time intervals, we can imagine the frog jumping along a line.

We pick one location, call it 0 and name the time as the “0 jump.” When the frog takes its first jump, jump 1, the frog lands at location 3. When the frog takes its second jump, jump 2, the frog lands at location 6.

Let’s go back to the 0-location and ask where the frog was on the jump before it arrived at 0. We call this jump $-1$. Because the frog jumps 3 units to the right every jump, the frog must have been at location $-3$, which is 3 units to the left of 0. Two jumps before reaching 0, the frog was at location $-6$. We can now copy and fill the table below.

<table>
<thead>
<tr>
<th>Directed Length of Jump</th>
<th>Number of Jumps</th>
<th>Frog’s Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-6</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-5</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-4</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-3</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

It is now possible to answer the earlier question. What do we mean by $-4$ jumps? This means we jump backward in time, or simply jump backward.
Use the number line to compute the following products. Verify that your answers agree with the table.

a. \((3)(-6)\)

b. \((3)(-5)\)

c. \((3)(-3)\)

d. \((3)(-1)\)

Let’s summarize the frog model:

- The first factor tells us which direction the frog should face and the length of each jump.
- The second factor tells us the number of jumps and the direction of the jump. When the second factor is positive, the frog jumps forward; when the second factor is negative, the frog jumps backward.

Using the frog model, compute the product \((-3)(-4)\). The directed length of each jump is \(-3\). Determine what happens when the frog jumps backward in time.

Copy and fill the following table, starting at the bottom and working up.

**Table 4.1d**

<table>
<thead>
<tr>
<th>Directed Length of Jump</th>
<th>Number of Jumps</th>
<th>Frog’s Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-6</td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>-5</td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>-4</td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>-3</td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-3</td>
<td>1</td>
<td>-3</td>
</tr>
<tr>
<td>-3</td>
<td>2</td>
<td>-6</td>
</tr>
<tr>
<td>-3</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>
Use this table to compute the following products:

- a. \((-3)(-6)\)
- b. \((-3)(-5)\)
- c. \((-3)(3)\)
- d. \((-3)(-1)\)

**EXERCISES**

1. Use the frog model on the number line to compute the following products. As you multiply, visualize the process to verify the accuracy of the products.

- a. \(-3 \cdot 2\)
- b. \(-5 \cdot 3\)
- c. \(-12 \cdot 3\)
- d. \(8 \cdot -5\)
- e. \(11 \cdot -6\)
- f. \(30 \cdot -2\)

2. Use the number line to demonstrate \((-2)(-4)\). Do the same for \((-3)(-7)\).

3. Use the number line frog model to compute the following products:

- a. \((6)(7)\)
- b. \((-5)(8)\)
- c. \((-6)(7)\)
- d. \((-6)(-7)\)
- e. \((5)(-8)\)
- f. \((5)(8)\)

4. Suppose \(n\) and \(m\) are positive integers. Using Exercise 3, how is the product \((n)(m)\) related to the product \((-n)(m)\)? \((n)(-m)\)? \((-n)(-m)\)?

5. We know from experience that when we multiply a positive number by another positive number we will always get a positive number. Find a rule for the product of a negative number and a positive number. Find a rule for the product of a positive number and a negative number. Find a rule for the product of two negative numbers.

6. Evaluate the following products:

- a. \(15(-13)\)
- b. \(22(-35)\)
- c. \((-8)9\)
- d. \(8(-19)\)
- e. \((-12)(-11)\)
- f. \((-118)(-3)\)
- g. \(217(-12)\)
- h. \((-7)8\)
- i. \(7(-8)\)
7. Pedro has 17 bags of tater tots that have approximately 80 tots in each bag. Predict whether Pedro has fewer than 100 tots, between 100 and 500 tots, between 500 and 1000 tots, between 1000 and 2000 tots, or more than 2000 tots in all. Show how you arrived at your prediction.

8. Juan has a checking account with a balance of 0 dollars. He then makes 8 withdrawals of $6 each. What is the new balance in Juan's checking account? Work this as an addition problem.

9. Andrew has an account with a balance of $10. He then makes 3 withdrawals of $8 each. What is the new balance in Andrew's checking account?

10. On a November day, a cold front blew into town. The temperature was 70 °F before the temperature dropped an average of 4 °F an hour. What was the temperature after 7 hours?

11. One evening in San Antonio, the temperature drops for five hours, from 5:00 P.M. to 10:00 P.M. The temperature drops an average of 3 °F per hour. The temperature at 10:00 P.M. is 88 °F. What was the temperature at 5:00 P.M.?

12. On a cold winter day in Roanoke, Virginia, the temperature at 6:00 P.M. is 10 °F. The temperature decreases an average of 4 °F for each of the next four hours. What is the temperature at 10:00 P.M.? Write this problem as an addition problem. Rewrite it as a combination multiplication and subtraction problem.

13. A bee flies by Tommy traveling east at 6 feet per second. Assuming the bee flies in a straight line, how far is the bee from Tommy after 3 seconds? After 5 seconds? How far west of Tommy was the bee at -3 seconds?

14. Ingenuity:

In Spring Branch, Texas, population 1,920, there are an average of 6 televisions for every 8 people. How many more televisions would there be if Spring Branch had 8 televisions for every 6 people?
15. **Investigation:**

Fernando the Frog is resting at the point 0 on the number line. He can make two kinds of jumps: “leaps” 6 units long and “hops” 4 units long. He can move in either a positive or negative direction.

a. Fernando wants to catch a fly that is buzzing around point 24 on the number line. His tongue isn’t long enough to reach the fly unless he is at point 24 as well. What sequence of moves can Fernando make to end at point 24?

b. Make a table of solutions. Do you notice any patterns? Find more solutions if possible.

c. Find one pair of integers $x$ and $y$ such that $6x + 4y = 24$. How does this help you answer the previous problem?
SECTION 4.2 AREA MODEL FOR MULTIPLICATION & THE DISTRIBUTIVE PROPERTY

In the previous section, we explored multiplication using the frog model of skip counting on the number line or repeated addition. This is also called the linear model for multiplication. In addition to the linear model, we can represent multiplication as area.

To multiply 4 by 6, consider the picture of the rectangle below. The area of a rectangle is the number of units, or $1 \times 1$ squares, that it takes to cover the figure with no overlaps and no gaps. What is the area of the rectangle below assuming that each square in the grid has area 1 square unit?

Dimensions are measured in one direction: the length, width, or height of a figure. The dimensions of this rectangle, which is 4 units wide and 6 units long, are 4 units in width and 6 units in length. The area can be computed by summing the areas of the columns: $4 + 4 + 4 + 4 + 4 + 4 = 4 \cdot 6 = 24$. We can also think of this area as the sum of the area of the rows: $6 + 6 + 6 + 6 = 6 \cdot 4 = 24$. The rectangle is called a 4 by 6, or a 6 by 4, rectangle because the area is computed as the product $4 \cdot 6 = 6 \cdot 4 = 24$. Remember, in the frog model, it is the same as when a frog jumps 6 times on a number line, with each jump 4 units long. The visual representation of area above is the area model that describes multiplication.

<table>
<thead>
<tr>
<th>PROPERTY 4.1: COMMUTATIVE PROPERTY FOR MULTIPLICATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \cdot B = B \cdot A$</td>
</tr>
</tbody>
</table>
PROPERTY 4.2: ASSOCIATIVE PROPERTY FOR MULTIPLICATION

For any numbers x, y and z

\[(xy)z = x(yz)\]

EXAMPLE 1

The Elliots are constructing a small building that is one room wide and two rooms long. Each room is 5 meters wide. The front room is 4 meters long, and the back room is 6 meters long. What is the floor space of each room? What is the floor space of the building? How are the areas of the two rooms related to the area of the building? The floor plan below shows the situation:

```
  4 m   6 m

  5 m
```

SOLUTION

The area of the room on the left is calculated by \((5m)(4m) = 20\) square meters, or 20 sq. m. The area of the room on the right is \((5m)(6m) = 30\) square meters. The total area is the sum of the areas of the two rooms:

\[20 \text{ square meters} + 30 \text{ square meters} = 50 \text{ square meters}.\]

Another way to compute the total area is to consider the larger rectangle and its width and length:

\[(5 \text{ meters})(4 \text{ meters} + 6 \text{ meters}) = (5 \text{ meters})(10 \text{ meters}) = 50 \text{ square meters}.\]

Notice that this is the same area.
PROBLEM 1

a. Compute the area of the larger rectangle by computing the areas of the inner rectangles.

\[
\begin{array}{c}
\text{8} \\
\text{4} \\
\text{7}
\end{array}
\]

b. Compute the area of the outer rectangle. Compare to answer in part a.

EXAMPLE 2

Now suppose the dimensions of the Elliots’ building have not been decided yet. We need a formula for the areas. Call the width of the building \(n\) feet and the lengths of rooms 1 and 2, \(k\) and \(m\) feet respectively. Find the area of each room and the building’s total area.

\[
\begin{array}{|c|c|}
\hline
k \text{ ft} & m \text{ ft} \\
\hline
n \text{ ft} & \\
\hline
\end{array}
\]
SOLUTION

The area of room 1 is \((n \text{ ft})(k \text{ ft}) = n \cdot k\) square ft.
The area of room 2 is \((n \text{ ft})(m \text{ ft}) = n \cdot m\) square ft.
The area of the building is \(n(k + m)\) square ft.

Remember, the area of the building can also be computed as the sum of the areas of the two rooms, \((n \cdot k + n \cdot m)\) square ft.

So, \(n(k + m) = n \cdot k + n \cdot m\). We call this relationship the distributive property. This property tells us how addition and multiplication interact.

PROBLEM 2

Use the distributive property to write the following product as a sum: \(a(b + c)\).

<table>
<thead>
<tr>
<th>PROPERTY 4.1: DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION</th>
</tr>
</thead>
<tbody>
<tr>
<td>For any numbers (k), (m), and (n),</td>
</tr>
<tr>
<td>(n(k + m) = n \cdot k + n \cdot m).</td>
</tr>
</tbody>
</table>

You have already learned to multiply two-digit and three-digit numbers. Now you can use the area model and the distributive property to explore this process carefully. Begin by modeling the product of a one-digit number and a two-digit number. To multiply \(6 \cdot 37\), use place value to write the product \(6 \cdot 37\) as \(6(30 + 7)\). By the distributive property, \(6 \cdot 37 = 6(30 + 7) = 6 \cdot 30 + 6 \cdot 7 = 180 + 42 = 222\).

\[
\begin{array}{ccc}
30 & & 7 \\
6 & | & 180 & | & 42 \\
\end{array}
\]

You can extend the same process to multiply 43 by 27 using the distributive property,

\[
43 \cdot 27 = (40 + 3)(20 + 7)
\]

\[
= 40(20 + 7) + 3(20 + 7)
\]

\[
= 40 \cdot 20 + 40 \cdot 7 + 3 \cdot 20 + 3 \cdot 7
\]

\[
= 800 + 280 + 60 + 21 = 1161
\]
Visualize the product as area with the picture below:

Area of A = 20 \cdot 40 = 800;
Area of B = 20 \cdot 3 = 60;
Area of C = 7 \cdot 40 = 280;
Area of D = 7 \cdot 3 = 21.

The total area is 800 + 60 + 280 + 21 = 1161.

EXAMPLE 3

Pat has a rectangle with a length that is 3 units more than x, a positive length. The width of the rectangle is 4 units. What is the area of the rectangle?

SOLUTION

Begin by drawing a picture of the rectangle. How do you draw a side of length of x? Any of the following pictures could represent the rectangle.
Each of these visual representations helps us find the area. The area is the product of the length and width $4(x + 3)$. Using the distributive property, the total area is equal to the sum of the areas of the two smaller rectangles: $4 \cdot x + 4 \cdot 3 = 4x + 12$. Either way, the area is $4x + 12$. Note that this is an expression for the area of the outer rectangle.

**EXAMPLE 4**

Luz sees an advertisement for $3 off the regular price of DVDs in her favorite store. She buys 4 DVDs, all of which have the same regular price of more than $3. Represent this situation with an algebraic expression and with an area model representation.

**SOLUTION**

First of all, notice that you do not know the regular price of a DVD. Represent this unknown with the variable $x$, where $x = \text{ the regular price of a DVD}$. Then, $x - 3$ represents the sale price, and $4(x - 3)$ is the total cost of Luz’s four DVDs.

The previous example shows us a way to visualize $x + 3$ or $3$ more than $x$. To represent $3$ less than $x$ visually using the area model, we can rewrite the problem as adding the opposite rather than subtracting: $x - 3 = x + (-3)$. The rectangle then looks like this:
The outer rectangle has area 4x, the original cost of 4 CDs. The 4 x 3 rectangle represents the discount. So the sale price is 4x – 12, which is the same as 4(x – 3). Therefore, 4(x – 3) = 4x – 12.

In computing algebraic expressions, we often end up with several terms that look alike, such as 3x + 2x. In the following Exploration, we will model this sum in several ways.

**EXPLORATION 1**

Use the number line below to locate and label 2x, 3x, and 3x + 2x.

```
0   x
```

**EXPLORATION 2**

Compute the area of each of the rectangles below. Explain how to use the distributive property. Show the connection between the area of rectangle A and B to that of rectangle C.

```plaintext
A  B  C
```

```
x     x    x
3     4    7
```
EXERCISES

1. Use the distributive property to simplify the following expressions. Compare the linear and area models for each.
   a. 2(3) + 7(3)  
   b. 3x + 4x  
   c. 2a + 5a  
   d. 7x – 3x  
   This is also called combining like terms.

2. How do you apply the distributive property to the multiplication problem 3(x – 2)? If necessary, draw a picture in which the product is the area of a rectangle. It might be useful to rewrite the subtraction factor (x – 2) as the sum (x + -2).

3. Draw a picture model that uses the product 2(x – 4) as the area of the shaded interior of a rectangle.

4. Compute the area of shaded rectangle below. The dimensions of the large rectangle are 3 units wide and x units long.

   ![Rectangle Diagram]

5. Draw a rectangle model for the following problems and then compute the following products using the distributive property:
   a. 2(y + 5)  
   b. 3(x + 2)

6. Rewrite the following products using the distributive property. Draw a rectangle model for the first 2 problems, if necessary.
   a. 2(x + 2)  
   b. 3(x + 4)  
   c. 2(x + 5)  
   d. 2(x – 5)  
   e. 2(3x + 4)  
   f. 2(3x – 4)

7. Mike has a certain number of baseball cards, and Jill has 5 more cards than Mike. Ramon has three times as many cards as Jill. How many cards does Ramon have if Mike has:
   a. 10 cards?  
   b. 35 cards?  
   c. x cards?
8. Eddie and Sam like to play marbles. Eddie has $x$ marbles. He buys 3 more marbles. Now Sam has twice as many marbles as Eddie. Express the number of marbles Sam has, in terms of $x$.

9. Eddie’s friend, Ilya, has $x$ marbles. Sam’s sister, Natasha, has twice as many marbles as Ilya. Natasha buys three more marbles. How many marbles does Natasha have now?

10. Sophia has two nephews, Juan and Ted. Ted is two years older than Juan. Sophia is three times older than Ted. If Juan is $J$ years old, what is Sophia’s age?

11. Gloria has two nieces, Sara and Jane. Sara is three years younger than Jane. Gloria is twice as old as Sara. If Jane is $J$ years old, how old is Gloria?

12. Ronnie has a large rectangular pen that is fenced to create 4 smaller rectangular compartments. The plans below show the dimensions of the pen.

<table>
<thead>
<tr>
<th>2</th>
<th>4</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
</table>
   x |

   a. What is the area of the large pen?
   b. How is this area related to the areas of the smaller compartments?

13. Compute the area of the rectangle below.

<table>
<thead>
<tr>
<th>3</th>
</tr>
</thead>
</table>
   a |
   4 |
14. Compute the area of the large rectangle below as well as the areas of the smaller rectangles. Explain your reasoning. Note that the horizontal length of the rectangle is \((c + d)\) units and the vertical width is \((a + b)\) units. Then compute the total area using the formula: \(\text{Area} = (\text{width})(\text{length})\).

What is another way to compute the total area?

15. Draw a rectangle with area \(ac + 3a + 2c + 6\).

16. Develop a formula for the area of each of the rectangles below. Write your answer in simplest terms.

   a. $x \times 3$

   b. $x \times 5$

   c. $x - 3 \times x = 4$
17. **Ingenuity:**

Numbers like 536 and 712 are divisible by 4, while 378 is not. Make a conjecture for a rule to determine whether a number is divisible by 4.

18. **Investigation:**

Use the area model to explain how to divide 183 by 14. What do the quotient and the remainder represent?
SECTION 4.3 APPLICATIONS OF MULTIPLICATION

In the previous sections, you explored models for multiplication including the area model. You will now explore properties of rectangles.

EXPLORATION

Of rectangles A, B, C, and D, which is the biggest? Explain your answer.

DISCUSSION

There are several ways to think about what “biggest” means. One way to measure “biggest” is to find the area by counting the number of unit squares that are needed to cover each figure.

1. What are the areas of rectangles A, B, C, and D?
2. Which one has the largest area?
3. Does this agree with the rectangle you chose?

Another way to measure the size of a rectangle is to add the lengths of all the sides. This sum is called the **perimeter**. Its name comes from the Greek words peri, meaning “around,” and metron, meaning “measure.”

4. What are the perimeters of the 4 rectangles?
5. Which one has the largest perimeter?
6. Does this agree with the rectangle you chose?
1. Calculate the area and perimeter of each rectangle on the following grid. Make a table like the one that follows showing the length, width, area, and perimeter of each rectangle. Assume the length is the horizontal distance.

<table>
<thead>
<tr>
<th>Rectangle</th>
<th>Length</th>
<th>Width</th>
<th>Area</th>
<th>Perimeter</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>G</td>
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<td></td>
</tr>
<tr>
<td>H</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>J</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What patterns do you notice in the values in the table? Do you see any relationships between the four categories that you can state as a rule?
2. Using a sheet of grid paper, make as many different rectangles of area 24 square units as possible. Make a chart like the one below to record information about these rectangles. Did you find more than four different rectangles?

<table>
<thead>
<tr>
<th>Rectangles of area 24 square units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
</tr>
<tr>
<td>--------</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
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<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

Looking at the information in the table, what do you notice?

3. Using another sheet of grid paper, make and label as many different rectangles of area 36 square units as possible. Make another chart to organize the information about these rectangles. How is the perimeter of a rectangle related to the area of the rectangle?

4. Jim wants to build a rectangular pen for his pet chicken. A friend says that chickens need 24 square meters to move around comfortably. What dimensions would be best for this pen? Why?

5. Draw as many different rectangles as you can that have perimeter 24 units.

6. What is the area and perimeter of a square with side length 2? What if it has side length 3, 4, or 5? Make a chart of these side lengths, areas, and perimeters, and look for patterns. If a square has side length $s$, what are formulas for the area and the perimeter of the square? Why is "$s^2$" called "s squared?"
7. Draw two different rectangles, each with area 6 square units. Use the corner of a sheet of grid paper or a large grid. Document your work for this exercise on the grid.
   a. Double the dimensions of the rectangles. Check to see if the rule or pattern you observed in exercise 6 still works. What happens to the area and the perimeter?
   b. Triple the dimensions of the rectangles with area 6 square units. Does your rule predict the area and perimeter of these new rectangles correctly?

8. Explore the effect of changing length and width.
   a. When the dimensions, the length and the width, of a rectangle double, what happens to the area? What happens to the perimeter?
   b. What if the dimensions of a rectangle triple?
   c. Write rules expressing any patterns you notice.

9. The perimeter of a rectangle is 42 cm, and the length is twice the width. What are the dimensions of the rectangle?

10. If the perimeter of a rectangle is 48 cm, and the length is three times the width, what are the dimensions of the rectangle?

11. Kenny wants to buy enough carpet to cover the floor in a rectangular room that is 12 feet by 15 feet. The carpet costs $2 per square foot.
   a. How much does one square foot of carpet cost? Two square feet? Three square feet? How much does it cost to buy N square feet of carpet?
   b. What is the total area of carpet that Kenny will need?
   c. How much will it cost for Kenny to carpet his room?
   d. Kenny has another rectangular room that is 15 feet long and has area 315 square feet. What is the width of this room?
12. **Ingenuity:**

Joseph is a computer user with terrible security habits: every password he creates is a string of ones and zeros. There are only two passwords of length 1 that fit this description: 0 and 1. There are four passwords of length 2 that Joseph could use: 00, 01, 10, and 11.

a. How many such passwords are there of length 3? length 4?

b. Complete the following table:

<table>
<thead>
<tr>
<th>Length of password</th>
<th>Number of passwords</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

c. Did you notice a pattern in part b? How many such 6-digit passwords do you think there are? How many such n-digit passwords do you think there are? Why does the number of passwords follow this pattern?

13. **Investigation:**

Raylene wants to pour a sidewalk around her rectangular garden. The garden is 6 feet by 8 feet, and the sidewalk will be 3 feet wide. What will the area of this sidewalk be? Which has greater area, the garden or the sidewalk? How much greater?
SECTION 4.4 THE LINEAR MODEL FOR DIVISION

Just as with multiplication, we will explore the operation of division. We will start by looking at some models to better understand how division works.

ACTIVITY: MODELS FOR DIVISION

Your teacher will now lead the class in an activity that reviews models of division.

You divided a class of 21 by 3 in the activity. One method involved subtracting 3 objects at each step from the original group and counting the number of times it took to distribute all 21 objects. Another way to think about this problem is to add groups of 3 until you have 21. Skip count by 3’s to accumulate objects until you have the desired number, 21. The number of skips that it takes to get to 21 is the result 21 divided by 3.

To skip count by 3’s, count 3, 6, 9, 12, 15, 18, 21, and so on. You know that 21 = 3 · 7 because we must skip count 7 steps by 3’s to get to 21. The inverse is 21 ÷ 3 = 7, which means when you divide 21 by 3, the result is 7 because 21 is decomposed into 7 skips of 3 units per skip. This is equivalent to 7 groups of 3. We call 3 the divisor, the quantity by which another quantity, the dividend, is to be divided. We call 7 the quotient, the end result of a division problem, and 21 the dividend, a quantity to be divided by the divisor.

When the divisor divides evenly into the dividend, or the remainder is zero, the word factor is used interchangeably with divisor. Dividing 21 by 3 is the same as looking for the missing factor x that satisfies 3 · x = 21. The x that satisfies this equation is called the quotient and represents the number of skips of length 3 it takes to reach 21. We call this the missing factor model. It is the reverse of the multiplication process.
We should note here that the word “factor” can be a noun that means divisor, as above where 3 is a factor of 21. It can also be a verb. When we say, “Factor 21,” we mean write 21 as a product of two or more positive integers. In this case, write $21 = 3 \cdot 7$ to factor 21 into a product of two numbers, 3 and 7.

How would you model the division problem $8 \div -2$? We interpret this as: How many jumps of length -2 does it take to reach the location of 8? With 1 jump, we would land at -2 and with 2 jumps at -4. This is the wrong direction. But with a -1 jump, we land on 2 and with a -2 jump, we land on 4. So, with a -4 jump, we land on 8. Thus, the quotient of $8 \div -2 = -4$.

Using the missing factor model, this division problem is the same as looking for the missing factor $x$ that satisfies the equation: $-2x = 8$. We see that if $x$ is -4, then $-2(-4) = 8$. This pattern can be summarized by the following rule:

**RULE 4.1: POSITIVE DIVIDEND AND NEGATIVE DIVISOR**

A positive number divided by a negative number yields a quotient that is negative.

Similarly, if we divide a negative number by a positive number, the quotient (the number of jumps required) must also be a negative number of jumps. For example, $-8 \div 2 = -4$ because the missing factor model demonstrates that $2(-4) = -8$. This gives us another rule that states:

**RULE 4.2: NEGATIVE DIVIDEND AND POSITIVE DIVISOR**

A negative number divided by a positive number yields a quotient that is negative.

Also, notice $-2(4) = -8$, and so $x = 4$ is the quotient of $-8 \div -2$. Therefore:

**RULE 4.3: NEGATIVE DIVIDEND AND NEGATIVE DIVISOR**

A negative number divided by a negative number yields a quotient that is positive.
PROBLEM 1
Use a number line with the appropriate scale and the skip counting model to compute the following quotients:

a. $56 \div 7$  
   b. $91 \div 13$  
   c. $210 \div 15$  

d. $-15 \div 3$  
   e. $12 \div -4$  
   f. $-18 \div -6$

EXAMPLE 1

Robin has 47 feet of ribbon on a roll. She wants to cut this roll into 4-foot strips for decorations. How many 4-foot strips of ribbon can she make? How much ribbon will be left over, if any?

SOLUTION

In order to make the 4-foot strips, Robin rolls out all of the ribbon and marks off 4-foot lengths. She then skip counts the number of pieces she needs to cut and finds that $4 \cdot 11 = 44$. Therefore, 47 feet divided into 4-foot pieces equals 11 pieces with 3 feet of ribbon left. In other words, $47 \div 4$ is 11 with a remainder of 3.

![Number line](image)

Notice that when using the remainder, the solution is $47 = 4 \cdot 11 + 3$. For now, any number left after dividing, you may leave as a remainder.

PROBLEM 2

Solve each of the following equations:

a. $4x = 12$  
   b. $3x = 15$  
   c. $4x = -20$  
   d. $-2x = 8$
EXPLORATION

Mr. Garza has 20 pieces of candy. He wants to divide the candy equally among 6 children. How should he distribute the candy?

One way to distribute the candy is to think of this process in steps. In step 1, give each child 1 piece of candy. This means Mr. Garza has $20 - 6 = 14$ pieces of candy left. In step 2, Mr. Garza gives each child a second piece of candy. He now has $14 - 6 = 8$ pieces of candy left. In step 3, Mr. Garza gives each child a third piece of candy. He now has $8 - 6 = 2$ pieces of candy left. He can no longer give an equal number of pieces to each of the 6 children, so he stops. It took 3 steps to equally distribute as many pieces of candy as Mr. Garza could. That means each child received 3 candies. Write this as $20 = 3 \cdot 6 + 2$. Picture this as a linear model by skip counting to divide 20 by 6, which corresponds to the counting 3 skips of length 6: $3 \cdot 6 = 18$; 2 units short of 20.

In division, the problem involves the dividend and the divisor, and the task is to compute the quotient. In the linear model, the dividend is the total length. There are two possible cases:

1. Know the length of each jump, and call it the divisor. Find the quotient, which, in this case, is the number of jumps that equal the total length.
2. Know the number of jumps, and call it the divisor. Find the quotient, which, in this case, is the length of each jump.

In multiplication, start with the length of each jump and the number of jumps. The answer is the accumulated length of all the jumps. Again, division is the reverse of the multiplication process.

EXERCISES

1. Evaluate the following, and write the associated multiplication fact. Use the linear model or long division, if needed.
   a. $32 \div 8$
   b. $12 \div 2$
   c. $25 \div 5$
   d. $42 \div 6$
   e. $35 \div 1$
   f. $24 \div 2$
   g. $264 \div 8$
   h. $627 \div 11$
   i. $1728 \div 24$
2. Write the associated multiplication fact, making the remainder as small as possible.
   a. 33 ÷ 4   d. 35 ÷ 7   g. 514 ÷ 8
   b. 24 ÷ 7   e. 43 ÷ 10  h. 679 ÷ 13
   c. 14 ÷ 6   f. 63 ÷ 12  i. 4003 ÷ 33

3. Use the missing factor method to compute the following quotients:
   a. (-3) ÷ 1   d. 84 ÷ (-12)  g. (-318) ÷ (-6)
   b. (-12) ÷ 2  e. 54 ÷ (-3)  h. (-705) ÷ (-15)
   c. (-16) ÷ 8  f. 88 ÷ (-22)  i. (-1848) ÷ (-42)

4. Predict whether the quotient is between 1 and 10, between 10 and 100, or between 100 and 1000 by estimating the number of jumps.
   a. 54 ÷ 6   c. 264 ÷ 4   e. 1254 ÷ 6
   b. 195 ÷ 13  d. 1080 ÷ 18  f. 972 ÷ 6

5. Peter has 528 apples. If a sack of apples contains 24 apples, predict whether Peter can make between 1 and 10 sacks, between 10 and 50 sacks, or between 50 and 100 sacks. Show how you made this prediction.

6. Olivia has invited 6 friends to a small dinner party. Earlier this morning, Olivia picked 34 wildflowers. She wants to give each of her friends an equal number of flowers. How many flowers should she give each friend? How many flowers, if any, will be left to add to the centerpiece?

7. Erica and her friends are making macaroni necklaces. It takes exactly 11 pieces of macaroni to make one necklace. They have 103 pieces of macaroni. How many necklaces can they make? How much macaroni, if any, will be left over?

8. Madison and two of her friends decide to play a game of cards. Madison has a standard deck of 52 cards. She deals the cards so that all three have an equal number of cards. What is the maximum number of cards each of them can be dealt? How many cards, if any, will be left in this case?
9. There are 23 students in Mr. Scott’s math class. One day Mr. Scott comes to class with a bag full of pens. He gives an equal number of pens to each student and discovers that he has 11 pens left. He knows that his bag originally held fewer than 100 pens. What are the possible numbers of pens he could have originally had? In each case, how many pens did he give each student?

10. Annie Paul is installing a sprinkler system in her yard. The system requires 14-inch sections of PVC pipe. She has one long 9-foot piece of PVC. How many 14-inch sections can she cut from this long piece? How much will be left after she cuts the sections?

11. Solve each of the following equations:
   a. $2x = 12$
   b. $3x = 12$
   c. $4x = 12$
   d. $4x = 20$

12. Calculate the following, if possible. If it is not possible, explain why.
   a. $0 ÷ 4$
   b. $0 ÷ (-35)$
   c. $4 ÷ 0$
   d. $-35 ÷ 0$

13. Evaluate the following using the missing factor model:
   a. $(2 \cdot 14) ÷ 2$
   b. $(4 \cdot 37) ÷ 4$
   c. $(15 \cdot 43) ÷ 15$
   d. $(29 \cdot 367) ÷ 29$

14. Evaluate the following using the missing factor model:
   a. $(4 \cdot 17) ÷ 2$
   b. $(8 \cdot 17) ÷ 4$
   c. $(16 \cdot 17) ÷ 8$
   d. $(12 \cdot 17) ÷ 6$
   e. $(36 \cdot 17) ÷ 18$
   f. $(108 \cdot 17) ÷ 54$

15. Evaluate:
   a. $(18 + 18 + 18) ÷ 3$
   b. $(3081 + 3081 + 3081 + 3081) ÷ 4$
   c. $(29 + 29 + 29 + 29 + 29 + 29 + 29 + 29) ÷ 9$
   Do you notice any patterns?
   d. $(258 + 258 + 258 + 258) ÷ 2$
   e. $(123 + 123 + 123 + 123 + 123 + 123) ÷ 2$

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16. **Ingenuity:**

The **factorial** of a non-negative integer $n$ is the product of all positive integers less than or equal to $n$. We use the notation $n!$, which is read as “$n$ factorial,” to represent this product. For example, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$, and $6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$. How many zeros are at the end of $30!$?

17. **Investigation:**

Use the skip counting model and number line to solve the following equations:

a. $6x + 2 = 26$  
   b. $2a + 1 = 11$  
   c. $4n - 3 = 19$  
   d. $2y + 3 = -4$
SECTION 4.5 THE DIVISION ALGORITHM

Another way of thinking of division is the area model. This is similar to the missing factor model. To divide 24 by 4, draw a length of 4 and ask what the width x is to equal a total area of 24. So what you are doing is looking for the missing factor: $24 = 4 \cdot \text{(what?)}.  

\[ \begin{array}{c|c} 4 & 24 \\ \hline \end{array} \]

You know that division is the reverse operation for multiplication, just as subtraction is the reverse operation for addition. What do we mean by this? Begin with the number 12. Add 3 to get 15. To undo the addition, you need to subtract 3 from 15 and return to the original number 12. Similarly, in the example above, you found the number 6. Multiply by 4 to obtain 24. That is, $24 = 6 \cdot 4$. To undo this multiplication, divide 24 by 4 and return to the start because $24 \div 4 = 6$.

EXAMPLE 1

Using the area model, what is $20 \div 3$?
SOLUTION

Begin with a length of 3 on the y-axis. If we mark off a length of 6 on the x-axis, the area of the rectangle is 18. We compute this as $18 = 3 \cdot 6$. To get an area of 20, we must add 2 more square units to the end of the rectangle. That means $20 \div 3$ has quotient 6 with remainder 2 because this corresponds to the calculation $20 = 3 \cdot 6 + 2$.

Why is the quotient 6 and the remainder 2? Why not say the quotient is 5 and the remainder is 5? Why not say the quotient is 4 and the remainder is 8 because $20 = 3 \cdot 4 + 8$? How do we decide between the different quotients and remainders?

If we think of 20 as $3 \cdot 5 + 5$, the picture shows that we could break up the last column into pieces of lengths 3 and 2. Adding this extra 3 to the rectangle is represented by the calculation $3 \cdot 6 + 2$. The picture on the next page shows we can break up that last column into two pieces of length 3 and another piece of length 2. By adding these extra pieces of length 3 to the rectangle, we have the same calculation $3 \cdot 6 + 2$. 
When using the area model for division, say \( a \div b \), we write \( a \) in the form of the calculation \( a = b \cdot q + r \) where \( b \) is the height of the rectangle, \( q \) is the length of the rectangle along the \( x \)-axis and \( r \) is the remainder or the height of the column added to equal \( a \). This is the division algorithm where \( b \) is the divisor, \( q \) is the quotient and \( r \) is the remainder.

Are there any restrictions on \( r \) and \( b \)? Yes! When examining the above example, 20 can be written in several different ways:

\[
\begin{align*}
20 &= 3 \cdot 4 + 8 \\
20 &= 3 \cdot 5 + 5 \\
20 &= 3 \cdot 6 + 2
\end{align*}
\]

The smaller the values of \( r \) are, the closer we are to seeing if 20 is a multiple of 3. Only when we get to \( r = 2 \), do we see that 20 is not a multiple of 3. There will be some remainder, namely 2. So, one condition to put on \( r \) and \( b \) is that \( r < b \). If \( r \) is not less than \( b \), then we have the situation shown in the picture above.

Is this enough to ensure exactly one answer in the division algorithm? Returning to the above example, \( 20 = 3 \cdot 7 + (-1) \) also fits the division algorithm with \( r < b \) as the only restriction on the remainder. Remember, \( b \), the divisor in the division algorithm, is like \( h \), the height in the area model. From this we realize we must add one final restriction to the remainder: \( r \geq 0 \).

The divisor \( b \) must be positive because \( r \) is not negative and \( b \) is greater than \( r \). We write this with our inequalities as follows: Because \( r < b \), then \( b > r \). And because \( r \geq 0 \), then \( b > r \geq 0 \) and \( b > 0 \). With this added restriction, we write \( 20 = 3 \cdot 6 + 2 \).
We state the formal division algorithm:

**THEOREM 4.1: DIVISION ALGORITHM**

Given two positive integers \(a\) and \(b\), we can always find unique integers \(q\) and \(r\) such that \(a = bq + r\) and \(0 \leq r < b\).

We call \(a\) the **dividend**, \(b\) the **divisor**, \(q\) the **quotient** and \(r\) the **remainder**.

In our previous example with \(20 = 3 \cdot 6 + 2\), the dividend \(a = 20\), the divisor \(b = 3\), the quotient \(q = 6\) and the remainder \(r = 2\).

Compute the following division problems by writing the corresponding division algorithm and sketching a picture that explains what the algorithm represents.

a. \(43 \div 6\)  

b. \(87 \div 12\)  

c. \(148 \div 16\)

**EXERCISES**

1. Compute the following using the division algorithm: See TE.
   
a. \(32 \div 7\)  
   b. \(24 \div 5\)  
   c. \(179 \div 14\)  
   d. \(37 \div 4\)  
   e. \(48 \div 9\)  
   f. \(100 \div 8\)

2. Calculate the following: See TE.
   
a. \(27 \div 3\)  
   b. \(270 \div 3\)  
   c. \(2700 \div 3\)  
   d. \(27000 \div 3\)  
   e. \(24 \div 4\)  
   f. \(240 \div 4\)  
   g. \(33 \div 11\)  
   h. \(3300 \div 11\)

   i. Write rules to describe any patterns you noticed in a–h. What causes these patterns?

3. Using the area model, predict whether the quotient is between 1 and 10, between 10 and 100, or between 100 and 1000 by estimating the length of the rectangle’s base.
   
a. \(861 \div 41\)  
   b. \(217 \div 31\)  
   c. \(1452 \div 12\)
4. If each mp3 file takes up 4 MB of space, how many mp3 files can you fit on a 700 MB CD? Will you have any space left over?

5. Model $35 \div 6$ using the area model. Use graph paper.

6. Alice has 53 quarters. She goes to the bank and trades them for dollar bills. How many dollar bills will she get? Will she have any quarters left?

7. We have 40 square tiles, each 1 foot by 1 foot. We would like to construct a path that is 3 feet wide. How long can we make the path? Will there be any tiles left?

8. Dan wants to fill 33 bags with candy. If Dan has 1221 pieces of candy, predict whether the number of pieces of candy in each bag is between 1 and 10, between 10 and 50, between 50 and 100, or more than 100. Show how you made your prediction.

9. Given two positive integers $a$ and $b$, explain why the area and linear models of division for $a \div b$ give the same results.

10. **Ingenuity:**
Evaluate the following expressions. (Hint: you may use the missing factor method.)

   a. $(3 \cdot 7) \div 3$
   
   b. $(5 \cdot 3 \cdot 7) \div (5 \cdot 3)$
   
   c. $(8 \cdot 2 \cdot 3) \div 8$
   
   d. $(8 \cdot 2 \cdot 3) \div (8 \cdot 2)$
   
   e. $(10 \cdot 12 \cdot 11) \div (12 \cdot 11)$
   
   f. $(10 \cdot 11 \cdot 12) \div (12 \cdot 10 \cdot 11)$
   
   g. Write rules to describe any patterns you noticed in a–f? What causes these patterns?
   
   h. Compute $(12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6) \div (11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6)$.
11. **Investigation:**
Find the quotients and remainders using the division algorithm.

a. $7 \div 5$  
   $11 \div 5$  
   $(7 + 11) \div 5$

b. $7 \div 3$  
   $10 \div 3$  
   $(7 + 10) \div 3$

c. $13 \div 4$  
   $22 \div 4$  
   $(13 + 22) \div 4$

What patterns do you notice?

d. $13 \div 6$  
   $17 \div 6$  
   $(13 + 17) \div 6$

e. $18 \div 7$  
   $24 \div 7$  
   $(18 + 24) \div 7$

What patterns do you notice with the remainders in parts d and e? Explain why d and e are different from a–c.
SECTION 4.6 SOLVING EQUATIONS

Now that you have seen different operations in Chapters 3 and 4, we summarize the order in which mathematical operations are performed below.

Order of Operations

- Compute the numbers inside the parentheses
- Compute any exponential expressions
- Multiply and divide as they occur from left to right
- Add and subtract as they occur from left to right

EXPLORATION 1

When you go to Game Go and purchase three video games of equal cost, your total is $84. What is the purchase price per game?

How could we write this problem as an equation and solve the equation using the Four Step Process to Solving Equations in Section 3.2 Exploration: Charting the Process?

EXPLORATION 2

Consider the following expressions:

a. $9 + 2 \cdot 3 - 2$  
b. $9 + 2 \cdot (3 - 2)$  
c. $(9 + 2) \cdot 3 - 2$

1. What similarities and differences do you notice between expressions a and b?  
2. What similarities and differences do you notice between expressions a and c?  
3. Evaluate a, b, and c using the order of operations.  
4. Why did you get different results?
Section 4.6 Solving Equations

In earlier chapters, we learned about equations and how to solve equations like \( x + 4 = 6 \). In section 4.4, we studied the equation \( 3x = 21 \) using the missing factor model. We showed how to use the operation of division to solve this equation: \( x = 21 \div 3 = 7 \).

Now let us apply the knowledge we have of equations and order of operations to some real life applications.

**EXPLORATION 3**

When you go to Game Go and purchase three video games of equal cost, your total is $84. What is the purchase price per game?

How could we write this problem as an equation and solve the equation using the Four Step Process to Solving Equations in Section 3.2 Exploration: Charting the Process?

**EXAMPLE 1**

A store sells CDs for $9 each. Sandy buys some CDs for $54. How many CDs did she buy? Write an equation for this problem and solve.

**SOLUTION**

**Step 1: Define the variable.**

Let \( x \) = the number of CDs Sandy bought.

**Step 2: Translate the problem to an equation.**

We know that each CD costs $9

\[ 9x = \text{total cost of the CDs} \]

\[ 9x = 54 \]

**Step 3: Solve for the unknown.**

\[ 9x = 54 \]

\[ 9x \div 9 = 54 \div 9 \] or equivalently \( x = \frac{9x}{9} = \frac{54}{9} = 6 \)

\( x = 6 \)

**Step 4: Check your answer.**

Check. Is 9 multiplied by 6 equal to 54? Yes.
PROBLEM 1

Solve each of the following equations. Do you see a pattern?

a. \(2x = 8\)  
   \(2x = 9\)  
   \(2x = 10\)

b. \(3x = 12\)  
   \(3x = 13\)  
   \(3x = 14\)

c. \(4a = 16\)  
   \(4a = 17\)  
   \(4a = 18\)  
   \(4a = -18\)

EXAMPLE 2

Terry, Aissa, and Steve went to the snack bar to buy some snacks. They bought three slushies and a bag of Flaming Hot Chips which costs $1. If they spent $7 for all of this what was the cost of each slushie at the snack bar?

SOLUTION

Using the Four Step Process, we write an equation for this problem situation and solve it.

Step 1: Define the variable.
Let \(c\) = the cost of each slushie.

Step 2: Translate the problem to an equation.
We know that they spent $7. The Flaming Hot Chips cost $1 added to the 3 slushies which each cost \(c\).
\[3c + 1 = \text{cost of 3 slushies and chips}\]
\[3c + 1 = 7\]

Step 3: Solve for the unknown.
This is an equation so we can use a balance scale. So the equation can be modeled as
Section 4.6 Solving Equations

Step 4: Check your answer.

Check. Is 2 multiplied by 3 plus 1 equal to 7? Yes.

Notice that it took two steps to solve the equation: $3c + 1 = 7$. The first step was to subtract 1 from each side of the equation. Explain why. This results in the equivalent equation: $3c = 6$. The next step used the missing factor method to determine: 3 times what number equals 6.
PROBLEM 2

Show how to use this two step process to solve each of the following equations. Draw balance scale models for a and b.

a. \(3n + 5 = 11\)  
b. \(4a + 1 = 33\)  
c. \(2x + 3 = 4\)  
d. \(2x - 3 = 11\)  
e. \(3y - 2 = 13\)

As in section 3.5, we can use a number line model to show another way to think about order of the two step process in solving the equation \(3c + 1 = 7\). Consider the number line below.

Note that the location of the value \(3c + 1\) is equivalent to the location of the number 7. Where is the location \(3c\)? It is 1 unit to the left of 7, which is the location 6.

This is equivalent to the process of subtracting 1 from \(3c + 1\). The new equation is \(3c = 6\). Knowing the location of \(3c\), how do we find the location of the value of \(c\)? Using the linear model for multiplication, we can think of this situation in two ways:

a. How many jumps of length 3 does it take to reach 6? The answer is 2 jumps. So \(c = 2\).

b. What length of jump would it take to reach 6 in 3 jumps? The answer is a jump length of 2. This model is nice because it produces the following picture:
PROBLEM 3

Use this number line method to show how to solve the equations in Problem 2, parts a, c, and d.

Number line models can also be used to solve inequalities. On the number line below, $x + 1 > 5$ can be shown as

Note that the location of $x + 1$ is above the number 5, but because $x + 1$ is greater than 5, the circle is open and not filled in. The value of $x$ is one unit to the left of 5. Since $x$ represents all values greater than 4, the model shows the solution as an open circle with a bold arrow pointing to the right.

PROBLEM 4

How would the solution to the inequality $x + 1 < 5$ differ from the solution to the above example? Model the solution on a number line.

PROBLEM 5

Use the number line method to model and solve the inequality $2x + 3 < 7$.

To check that our answer is correct, we can pick a point on the graph of the inequality and plug it in. If the inequality remains true, your answer is in the set of possible solutions.
EXERCISES

1. Write an equation to represent each statement, then compute each equation.
   a) five times a number is forty
   b) a number times negative three is eighteen
   c) six is equal to a number divided by four
   d) the quotient of a number and three is negative ten
   e) seven less than twice a number equals nineteen
   f) one half of a number is fifty

2. Solve each equation. Check your answer and show your work through substitution. Use the balance scale or number line model for problems a-c.
   a. \(3x = 12\)   h. \(8 = 3r - 1\)
   b. \(4k = 56\)    i. \(5a - 3 = 17\)
   c. \(11y = 55\)   j. \(4x + 3 = 17\)
   d. \(-9 = 3z\)    k. \(6x + 4 = 25\)
   e. \(14 = 6m + 2\)  l. \(4a + 3 = -7\)
   f. \(4(x + 1) = 8\)  m. \(3a - 5 = 8\)
   g. \(4a + 3 = 11\)  n. \(3(x - 2) = 12\)

3. Fred buys 14 baseballs for $63. How much does each ball cost? Write an equation for this problem and solve.

4. Devyn is half the height of her older brother Jordan who is 5 feet 4 inches. How many inches tall is Devyn? Write an equation and show how you solved.

5. Amy, Sandy, Michelle, and Christina all ended up with the same amount of Halloween candy after trick or treating. Amy snuck into the pantry and took 12 extra pieces. Now they have a total of 52 pieces. How many pieces did each girl originally have? How many did Amy have including the pieces she found? Write an equation and solve.
6. For each of the inequalities below, determine which number(s) in the list \([-4, -2, 0, 3, 5]\) satisfies the inequality:
   a. \(x + 3 < 5\)       c. \(2x + 3 < 7\)
   b. \(x - 2 > -1\)       d. \(3x - 2 < 7\)

7. Solve each of these inequalities. Represent the solutions.
   a. \(x + 3 < 5\)       c. \(2x + 3 < 7\)
   b. \(x - 2 > 4\)       d. \(3x - 2 < 7\)

8. Mike can buy a discount card for laser tag for $10. This allows him to play laser tag for $4 per game. After one year, Mike has spent $146. Set up an equation to solve for how many games Mike played and solve this equation.

9. Cynthia’s cell phone plan costs $35 per month with 200 texts included. She is charged 10 cents for extra texts. Her bill for the month of January was $95. How many extra texts did she send? Write an equation and solve.

10. Shelly is saving up for a new car for college. She has $2580 saved already, but the car costs $4100. She is working hard at Sandy’s Sandwich Shop and saving an extra $40 per week. Set up an equation and solve for how many weeks it will take Shelly to buy this car.

11. Todd has $30 to spend at the fair. It costs $5 to get in, and roller coaster rides are $2 a piece. Write and solve an inequality to show the possible number of rides Todd can take. Model the solution to the inequality on a number line.

12. Melissa needs an average of at least 80 on her five math tests. She scored 95, 60, 85, and 90 on her four tests. What is the minimum grade Melissa needs on her fifth test?

13. **Ingenuity:**
   Write and solve a real-world problem corresponding to the inequality \(3y + 10 \leq 46\).

14. **Investigation:**
   Solve the following two equations:
   a. \(\frac{x + 2}{3} = 7\)       b. \(\frac{a - 1}{2} = 4\)
REVIEW PROBLEMS

1. Draw the frog model on the number line to compute the following products.
   a. -7 • 2  
   b. 5 • 3  
   c. -6 • 4

2. Evaluate the following products.
   a. 17 (-13)  
   b. -56 (74)  
   c. -281 (-34)

3. Suppose the temperature outside changes -3 ° every hour. How much will the temperature change in 7 hours?

4. A submarine is at -132 meters. If the submarine dives 4 times its initial depth, at what depth will the submarine be?

5. The temperature at 6:00 p.m. is 11° F. The temperature decreases an average of 3° F for each of the next 5 hours. What is the temperature at 11:00 p.m.?

6. Write an expression that represents the area model below.

   \[
   \begin{array}{c|c|c}
   \hline
   x & 7 \\
   \hline
   5 &  &  \\
   \hline
   \end{array}
   \]

7. Use the distributive property to write the following product as a sum. Simplify if necessary. Draw a rectangle model.
   a. 5 (12)  
   b. 3 (x + 4)  
   c. 2 (6 + x)

8. Rewrite the expressions using the distributive property. Simplify if necessary.
   a. 4 (6) + 4 (3)  
   b. 7y + 11y  
   c. 9a - 2a

9. List all the possible dimensions (using only integer pairs) of rectangles with a perimeter of 18 units.

10. List all the possible dimensions (using only integer pairs) of rectangles with area of 36 square units.

11. The dimensions of a rectangle are doubled:
    a. What happens to the perimeter?
    b. What happens to the area?
12. Use the missing factor method to compute the following quotients.
   a. \((-18) \div 6\)  
   b. \((-65) \div 5\)  
   c. \(56 \div (-8)\)  
   d. \(46 \div (-2)\)  
   e. \((-77) \div (-11)\)  
   f. \((-48) \div (-12)\)

13. Shelly has 26 books that she is placing on her bookshelf. Her bookshelf has 4 shelves. If Shelly puts the same number of books on the shelves, how many books will each shelf have? How many books, if any, will be remaining?

14. Model \(47 \div 5\) using the area model.

15. Compute the following division problems by writing the corresponding division algorithm.
   a. \(67 \div 9\)  
   b. \(107 \div 20\)  
   c. \(14 \div 3\)

16. Solve these equations:
   a. \(4x + 7 = 3\)  
   b. \(3y – 5 = 13\)  
   c. \(2x + 8 = 3\)  
   d. \(2x – 8 = 3\)  
   e. \(-2x = 12\)  
   f. \(4 – 2x = 10\)

17. Write an equation to represent each statement, solve, and check your work by substitution. Make a balance scale for problems a and b and a number line model for problems c and d.
   a. the sum of twice a number and 3 is 7  
   b. negative 8 is 5 less than three times a number  
   c. the product of 3 times a number, increased by 6, is 12  
   d. two more than 5 times a number is 17

18. Joan is a member of the Rent a DVD Club and pays a monthly membership fee of $32. Each DVD rental is an additional $2. How many DVDs did Joan rent if she paid $68 for the month of June? Write an equation and solve.
CHALLENGE PROBLEMS

Section 4.1:
The integers 1 through 12 appear on the circular face of a clock. Alex crosses out one of the numbers, then moves around the circle clockwise crossing out every fourth number that has not yet been crossed out. If 7 is the last number he crosses out, what was the next-to-last number?

Section 4.2:
A carpenter needs to tile a 6 square foot rectangular area, but he only has square tiles that are all the same size. By using one whole tile and half of another (making only one cut) he can fill the area. How long was the tile he cut (before cutting, in feet)?

Section 4.3:
Eleven pirates find a treasure chest. When they split up the coins in it, they find that there are 5 coins left over. They put the coins back, throw one pirate overboard, and split the coins again, only to find that there are 3 coins left over. So, they throw another pirate over and try again. This time, the coins split evenly. What is the least number of coins there could have been?

Section 4.4:
When a number $n$ is divided by 11, the quotient is 11 with a possible remainder. When $n$ is divided by 10, the quotient is 12 with a possible remainder. When $n$ is divided by 9, the quotient is 13 with a possible remainder. What is the quotient when $n$ is divided by 8?
SECTION 5.1 GRAPHING ON THE COORDINATE PLANE

We use the number line to represent numbers as locations. To each point on the number line we associate a number or coordinate, which is the location of that number.

For example, the number line above shows points P and Q. To graph or plot a point P with coordinate 3 on the number line, we graph the point 3 units to the right of 0. Because point Q has coordinate −2, we graph the point 2 units to the left of 0 on the number line.

If we draw our number line horizontally as above, then positive numbers are located to the right of 0 and negative numbers are located to the left of 0.

Suppose instead that we draw our number line vertically like a thermometer. In this case, positive numbers are above zero, and negative numbers are below zero.

On the vertical number line, we could locate a point R with coordinate 3 by graphing the point 3 units above 0. The point S with coordinate −2 would be located 2 units below 0.
On a number line, each point corresponds to a number. If we want to plot points on a plane, we will need to use two numbers, called coordinates, to locate the point. A \textit{coordinate plane} is constructed as follows:

We begin by drawing a horizontal number line and locating the zero point, which is called the \textit{origin}:

![horizontal number line with zero point labeled as origin]

Next, draw a vertical number line through the origin of the horizontal number line, so that the two zero points coincide:

![coordinate plane with x-axis and y-axis]

The horizontal number line is called the \textit{horizontal axis} or the \textit{x-axis}; the vertical number line is called the \textit{vertical axis} or the \textit{y-axis}.
Remember, the point at which the two number lines, or *axes*, meet is called the origin and is the zero point on both number lines.

The *x*- and *y*-axes divide the plane into four regions. Because there are four of them, we call each region a *quadrant*. By convention, we number the quadrants *counterclockwise*, starting with the upper-right quadrant. The axes don’t belong to any quadrants, but rather are their boundaries.
The coordinates for the points in the coordinate plane are always ordered pairs of numbers. The first coordinate is called the \textit{x-coordinate}; the second is the \textit{y-coordinate}.

Consider the point \(P\) on the coordinate plane above. To identify point \(P\), we begin at the origin. First we move 4 units to the right on the \(x\)-axis, then we move up 3 units. We arrive at the point \(P\). Therefore, the point \(P\) is identified as \((4, 3)\), where 4 represents the \(x\)-coordinate and 3 represents the \(y\)-coordinate.

Notice that the coordinates of the origin are \((0, 0)\) because the origin lies at the zero point of both number lines. The points that contain integers, like \((4, 3)\) or \((5, -2)\), are called \textit{lattice points} because they fall on the cross grids, which look like a lattice.

This coordinate system with horizontal and vertical axes is called a \textit{Cartesian coordinate system}. It is named after René Descartes, the French mathematician and philosopher who invented it.
EXERCISES

For most of these exercises you will need coordinate planes to draw on.

1. Write the coordinates for each of the points A to L shown on the coordinate plane below.

2. Plot the following points on the coordinate plane.

- M (2, 5)
- N (5, 2)
- O (-2, 5)
- P (5, -2)
- Q (-2, -5)
- R (-5, 2)
- S (1, 1)
- T (-1, 1)
- U (0, 3)
- V (4, 0)
- W (-3, 0)
- X (0, -3)
Chapter 5 Patterns and Functions

3. Find and plot 3 points that meet the following conditions:
   a. Each point has a negative $x$-coordinate and a positive $y$-coordinate.
   b. Each point has a positive $x$-coordinate and a negative $y$-coordinate.
   c. Each point has positive $x$- and $y$-coordinates.
   d. Each point has negative $x$- and $y$-coordinates.
   e. What do you notice about the points in each situation?

4. Find and plot 3 points that satisfy the following conditions:
   a. Each point has the $x$-coordinate equal to 0 and a positive $y$-coordinate.
   b. Each point has the $x$-coordinate equal to 1 and a negative $y$-coordinate.
   c. Each point has the $x$-coordinate equal to 0, but is different from part a.
   d. Each point has the $y$-coordinate equal to 0.
   e. What do you notice in each situation?

5. Find and plot 5 points that satisfy the following conditions:
   a. Each point has a $y$-coordinate that is double the $x$-coordinate.
   b. Each point has a $x$-coordinate that is double the $y$-coordinate.
   c. What do you notice about the points in each situation?

6. Find and plot 5 points that meet the following conditions:
   a. Each point has the $y$-coordinate equal to 1.
   b. Each point has the $y$-coordinates greater than 1.
   c. Each point has the $y$-coordinate less than 1.
   d. What do you notice?

7. Find and plot 5 points that meet the following conditions:
   a. Each point has the $x$-coordinate equal to $-3$.
   b. Each point has the $x$-coordinates equal to $-1$.
   c. Each point has $x$-coordinates greater than $-3$ and less than $-1$.
   d. What do you notice?
8. Find and plot 5 points that satisfy the following conditions:
   a. Each point has the same y-coordinate as the x-coordinate.
   b. Each point has the y-coordinate larger than the x-coordinate.
   c. Each point has the y-coordinate smaller than the x-coordinate.
   e. What do you notice?

9. Plot each of the following lattice points. For each of these points, locate and label the four nearest lattice points.
   a. $(0, 0)$  b. $(-2, 1)$  c. $(3, -2)$  d. $(-4, -2)$  e. $(3, 4)$

10. Suppose $(x, y)$ is a lattice point. What are the four nearest lattice points? See TE.

11. Ingenuity:
    In a number game a move is described by the following rule: a player on a point $(a, b)$ moves to the point $(2a, -b)$. If a player starts at $(3, 1)$, where will she be after 3 moves?

12. Investigation:
    Look at a globe. Lines of latitude and longitude help us locate places on the earth. This forms a coordinate system.
    a. Find the coordinates of your city?
    b. Find another city on the same latitude as your city?
    c. Coordinate systems are often called Cartesian coordinate systems. Why?
    d. Do longitude and latitude form a Cartesian coordinate system? Why?
    e. What do you notice that is the same?
    f. Do you notice anything that is different? Why?
SECTION 5.2 TRANSLATIONS AND REFLECTIONS

Points on the coordinate system can seem very fixed once you locate or place them. However, it is possible to “move” the points around the plane in very systematic ways. For example, the points move to the right by one unit or the points move up by 3 units. Or we move all the points an equal distance across the y-axis. You will examine two such ways in this section.

EXPLORATION 1

On a piece of grid paper, draw the vertical and horizontal axes approximately centered on the paper. Locate and label the point P (4, -2).

a. Move this point to the location (3, 2) in the first quadrant with both coordinates as positive integers. Label the new location P’ (read as P prime) to distinguish that this is where P has moved to. Describe carefully how you did it, using directional terms (for example; up, down, right, left).

b. Now move point P’ to (-3, 1) in the second quadrant and call it P” (read as P double prime). Move P” to the location (-1, -3) and label it P”’. Restrict the movement from grid points to grid points, not just horizontally or vertically.
You probably observed from Exploration 1 that one way to move a point, like \( P \), is to shift or slide it by adding to or subtracting from the coordinates. For example, in the figure below, one way to slide \( P \) to the first quadrant is to add 6 to the \( y \)-coordinate. Notice point \( S \) is point \( P \) translated 6 units up. The translation of \( P \) to \( S \) adds 6 to the \( y \)-coordinate. In the coordinate system, translations are described as a specific horizontal motion and vertical motion.

Pick another point of your choosing on the above coordinate system and label it \( A \). Translate \( A \) using the same translation rule as the one from \( P \) to \( Q \) in Exploration 2 and label it \( B \). Now translate the line segment \( AP \) from the coordinate system above in the same way you did point \( P \). Draw \( BQ \) and explain what you did.
EXAMPLE 1

Translate triangle ABC, written ΔABC, below using the rule “adding 3 to the x-coordinate and subtracting 1 from the y-coordinate.”

SOLUTION
EXAMPLE 2

Describe the translation rule if rectangle $ABCD$ translates to $A'B'C'D'$:

![Graph of the translation]

**SOLUTION**

The translation subtracts 4 from the $x$-coordinate and adds 2 to the $y$-coordinate. Another way to describe this translation is that each point $(x, y)$ is translated to $(x - 4, y + 2)$.

A *translation* is a transformation that slides a figure a certain distance along a line in a specified direction. Notice that the original shape and the translated shape are identical except in their location in the plane. Two shapes are congruent if their size and shapes are the same.
EXPLORATION 2

a. Determine the distance from P (4, -2) to the x-axis. Locate and label a point Q in the first quadrant with first coordinate 4 that is the same distance from the x-axis as the point P (4, -2). Describe how you chose the location for Q.

b. Determine the distance from P to the y-axis. Move from point P across the y-axis to the third quadrant to a new point R that is the same distance away from the y-axis as P. Describe how you chose the location for R.

You can describe the moving of the point P (4, -2) to the point Q (4, 2) as a reflection across the x-axis. In this case, the x-axis is said to be a line of reflection. The point P(4,-2) is reflected across the y-axis to the point R (-4,-2)

Finally, because the line or axis also acts like a mirror, the movement described in Exploration 2 is called a reflection.

EXAMPLE 3

Draw a reflection for each of the shapes below across the specified line of reflection.

a. Reflect the triangle below across the x-axis

![Triangle Diagram](image)
EXPLORATION 2

a. Determine the distance from P (4, -2) to the x-axis. Locate and label a point Q in the first quadrant with first coordinate 4 that is the same distance from the x-axis as the point P (4, -2). Describe how you chose the location for Q.

b. Determine the distance from P to the y-axis. Move from point P across the y-axis to the third quadrant to a new point R that is the same distance away from the y-axis as P. Describe how you chose the location for R.

You can describe the moving of the point P (4, -2) to the point Q (4, 2) as a reflection across the x-axis. In this case, the x-axis is said to be a line of reflection. The point P(4,-2) is reflected across the y-axis to the point R (-4, -2)

Finally, because the line or axis also acts like a mirror, the movement described in Exploration 2 is called a reflection.

EXAMPLE 3

Draw a reflection for each of the shapes below across the specified line of reflection.

a. Reflect the triangle below across the x-axis
b. Notice that each of the corresponding vertices is the same distance away from the y-axis:

Let us look at lines of reflection without a coordinate grid. For example, the letter A is symmetric about a vertical line through the point at the top of the letter. If you reflect the left part of the A across this vertical line, you will exactly get the other half of the letter A.
You can now examine shapes that contain the property of reflection.

**EXPLORATION 3**

Examine all the capital letters in the English alphabet. Explore the letters that have **lines of symmetry**, that is, half of the figure is the mirror image of the other half. For each letter, determine how many lines of symmetry the letter has. Identify and draw these symmetries.

**EXPLORATION 4**

Examine flags from different countries. Include the flags of the US, Japan, South Korea, Israel, Mexico, Australia, France, Brazil, Switzerland and Egypt. You may add more, if you wish. Determine which flags have lines of symmetry or symmetries, how many symmetries they have, and where the lines are situated in the figure.

**EXAMPLE 4**

Examine the following examples of the indicated shapes to determine whether they have lines of symmetry, how many they have, and where the lines are situated.

- a. Equilateral triangle: 
- b. Scalene right triangle: 
- c. Rectangle: 
- d. Parallelogram: 
- e. Trapezoid: 

SOLUTION

a. The equilateral triangle has 3 lines of symmetry:

[Diagram of an equilateral triangle with lines of symmetry drawn]

b. The scalene right triangle has 0 lines of symmetry:

[Diagram of a scalene right triangle]

c. The rectangle has 2 lines of symmetry:

[Diagram of a rectangle with lines of symmetry drawn]

d. The parallelogram has 0 lines of symmetry:

[Diagram of a parallelogram]

e. The trapezoid has 1 line of symmetry:

[Diagram of a trapezoid with a line of symmetry drawn]

Translations and reflections are two transformations that move shapes to other locations in the plane. When you see two identical shapes in different parts of the plane, ask how the shapes are related and what transformations relate one to the other.

EXERCISES

1. Plot the following points and translate each by using the rule “add 2 to the x-coordinate and subtract 2 from the y-coordinate.”
   a. (3, 3)  b. (-1, 5)  c. (5, 0)

2. Plot the following points and translate each by using the rule “add -4 to the x-coordinate and add 3 to the y-coordinate.”
   a. (2, 3)  b. (-1, -1)  c. (2, -1)

3. Plot the following points and reflect each point about the y-axis.
   a. (2, 3)  b. (1, -4)  c. (-2, 4)  d. (-3, -2)
4. Reflect the points from Exercise 3 about the x-axis.

5. Draw $\triangle ABC$ with the vertices $A (2, 2), B (3, 5)$ and $C (5, 1)$.
   a. Translate $\triangle ABC$ using the rule "adding +4 to the x-coordinate and adding -2 to the y-coordinate." What are the new vertices?
   b. Translate $\triangle ABC$ by translating each point $(x, y)$ to the point $(x - 6, y + 2)$. What are the new vertices?
   c. Reflect $\triangle ABC$ about the x-axis. What are the new vertices?
   d. Reflect $\triangle ABC$ about the y-axis. What are the new vertices?

6. Describe the translation rule that transforms $\triangle ABC$ to $\triangle A'B'C'$ where the vertices of $\triangle ABC$ are $A (-3, 1), B (0, 5)$ and $C (1, -1)$ and the vertices of $\triangle A'B'C'$ are $A' (1, -1), B' (4, 3)$ and $C' (5, -3)$.

7. Draw $\triangle ABC$ with the vertices $A (0, 3), B (-2, 3)$ and $C (-2, 0)$.
   a. Reflect $\triangle ABC$ about the line $x = 1$. What are the new vertices? Draw this transformed triangle $\triangle A'B'C'$'. What do you notice about the relationship between $\triangle ABC$ and $\triangle A'B'C'$?
   b. Reflect the original $\triangle ABC$ about the line $y = -2$. Draw and label the new vertices $\triangle A''B''C''$.

8. Perform the following transformations.
   a. Reflect the point $(3, -2)$ about the x-axis then translate the point by adding 5 to the x-coordinate and adding 2 to the y-coordinate.
   b. Do the two transformations return the point $(3, -2)$ to the same ending point?

9. Find 5 capital letters of the alphabet that have lines of symmetry. Find 2 letters that do not have lines of symmetry.

10. Perform the following transformations.
    a. What transformation reflects the point $(x, y)$ to $(x, -y)$? Explain.
    b. What transformation reflects the point $(x, y)$ to $(-x, y)$? Explain.
    c. What transformations reflect the point $(x, y)$ to $(-x, -y)$? Explain.
11. On a coordinate plane,
   a. Find 3 points whose distances from the origin are identical to the distance from the point (5, 3) to the origin.
   b. Find 3 points whose distances from the origin are identical to the distance from the point (4, 4) to the origin. Explain whether it is possible to find more points.

12. Draw a square and find all of the lines of symmetry.

13. How many lines of symmetry does a circle have?

14. Determine all the lines of symmetry for the figures below:
   a. \[ \begin{array}{c}
   \end{array} \]
   b. \[ \begin{array}{c}
   \end{array} \]
   c. \[ \begin{array}{c}
   \end{array} \]

15. Identify the figures that have lines of symmetry. For the figures which have symmetry draw the appropriate line(s) of symmetry.
   a. \[ \begin{array}{c}
   \end{array} \]
   c. \[ \begin{array}{c}
   \end{array} \]
   d. \[ \begin{array}{c}
   \end{array} \]
   b. \[ \begin{array}{c}
   \end{array} \]

16. Draw all the lines of symmetry for the figure below.
17. **Ingenuity:**

How many 3-letter code words, like "AHA," have a vertical line of symmetry? For simplicity, assume all the letters are capitalized.

18. **Investigation:**

   a. Graph the point $P(4, 5)$ and the line $y = 3$ on a coordinate plane.
   
   b. Find and label a point $P'$ that is the reflection of point $(4, 5)$ about the line $y = 3$.
   
   c. Repeat part b for the points $A(-2, 4)$, $B(0, 7)$ and $C(-4, 6)$.
   
   d. What does every pair of points have in common?
SECTION 5.3 FUNCTIONS

In our daily lives, we often encounter situations in which we receive a set of instructions and then perform certain tasks based on those instructions. In mathematics, this is the role of a function. A function is a rule that assigns a unique output value to each number in a set of input values.

EXPLORATION 1

Sarah builds model airplanes. She makes two airplanes each day. How many airplanes will she make in 4 days? 10 days? Organize the information to reveal a pattern in the number of airplanes she makes in a given number of days.

How did you organize information in the exploration above? Do you see a pattern in the number of airplanes she can make in a given number of days?

One way to organize such information is to build a table such as the one to the right. Notice that the first column is the number of days, and the second column is the total number of airplanes that she can make in the corresponding number of days.

What do you notice about this table? Why is this a good way to organize the information? If you are given an input of x, what is an equation for the corresponding output y?

This is an example of a function. There is a rule, or function, to determine how many planes Sarah has produced based on the number of days she has worked. We can think of a function as a machine with inputs and outputs. The input is the number of days Sarah has worked. The output is the number of planes produced.
Now that we have an idea of what a function does, let’s go ahead and make a more formal definition.

**DEFINITION 5.1: FUNCTION**

A function is a rule which assigns to each member of a set of inputs, called the **domain**, a member of a set of outputs, called the **range**.

Functions are often expressed by equations, like \( y = 2x \) from Exploration 1. The input \( x \) is called the **independent** variable, and the output \( y \), which depends on the value of \( x \), is called the **dependent** variable. These terms can be a little confusing, since given an output \( y \), we could solve for \( x \) and find that \( x = \frac{y}{2} \). However, this would be like running our function machine backwards. In function form, \( y \) traditionally takes the place of the function \( f(x) \) and is set equal to an expression involving \( x \).

For example, consider again Sarah’s function from Exploration 1. The domain is the set of non-negative integers 0, 1, 2, 3, … and the range is the set of even non-negative integers 0, 2, 4, 6… .

<table>
<thead>
<tr>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>⋮</td>
<td>⋮</td>
</tr>
</tbody>
</table>

Notice that a function produces pairs of numbers. Let’s call Sarah’s function \( F \). From the table and the picture above, we can see that (0, 0), (1, 2), (2, 4), (3, 6), and (5, 10) are some of the pairs that belong to the function \( F \), where the \( x \), or first-coordinate is, the input, and the \( y \), or second-coordinate, is the output. In other words, the ordered pairs are of the form (input, output) or \((x,F(x))\). Each
pair of numbers can be thought of as a point on the coordinate system, so we can also talk about the graph of a function. The graph of the function is the pictorial representation of the function.

In mathematics, a notation is a technical system of symbols used to represent unique objects. We can write “The function $F$ pairs the number 1 with 2” symbolically as “$F(1) = 2$.” We read this as “$F$ of 1 equals 2.” This means $F$ sends the input 1 to the output 2. Note: We write $F(x)$ to denote the value of the output with input $x$. So $F(1) = 2$ since $F$ represents 1 to 2. Be careful, because $F(x)$ is NOT $F \cdot x$. $F$ is a function, not a variable. Similarly, because the function $F$ pairs the number 2 in our domain with the number 4 in the range to give us the pair $(2, 4)$, we write “$F(2) = 4$.”

So $F(x) = y$ the number of planes that can be produced in $x$ days. We can express this rule in general as $F(x) = 2x$. We also say the function has the equation $y = 2x$.

**EXPLORATION 2:**

Juan sells peanuts for $4 per pound. Fill in the table below:

<table>
<thead>
<tr>
<th>lbs of peanuts</th>
<th>Cost in dollars</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>26</td>
<td></td>
</tr>
<tr>
<td>$x$</td>
<td>$C(x) = y =$</td>
</tr>
</tbody>
</table>

a. What is an equation for the cost $C$?
b. Why is $x = -1$ not an input for this test function?
c. Is $x = 1.5$ a possible input? What is the value of $C(1.5)$?
d. What is the domain of this cost function?
e. What is the independent variable in the cost table?
f. What is the dependent variable in the cost table?
PROBLEM 1

Mary pays $5 to get into the carnival. It costs $2 per ride. Make a table with the number of rides as inputs and the total cost as the output. What is the equation for the cost function?

EXERCISES

1. This problem deals with function $F$ from Exploration 1.
   a. What is the value of $F(4)$? $F(6)$?
   b. To represent $F$ by a table of values, copy and fill in the adjacent table.
   c. Rewrite each row of the adjacent table as an ordered pair, of the form ($x$, $F(x)$), and plot the corresponding points on a coordinate plane. Produce three more pairs that belong to this list.
   d. What is the independent variable in the cost variable?
   e. What is the dependent variable in the cost table?
   f. What is the relation between the two variables?

\[\begin{array}{|c|c|}
\hline
x & F(x) \\
\hline
0 & F(0) = \\
1 & F(1) = \\
2 & F(2) = \\
3 & F(3) = \\
5 & F(5) = \\
10 & F(10) = \\
x & F(x) = \\
\hline
\end{array}\]

2. Sally sells pencils for 6 cents each. If Sally sells 3 pencils, then she earns 18 cents. In other words, Sally’s revenue for selling 3 pencils is 18 cents.
   a. Let $R(x) =$ Sally’s revenue for selling $x$ pencils. Make a table showing $R(0)$, $R(1)$, $R(2)$, $R(3)$, $R(5)$, $R(10)$, and $R(x)$.
   b. Make a list of the first six ordered pairs from part (a), and plot these points on the coordinate plane.
   c. What are the domain and range for $R(x)$?
3. In order to sell lemonade, Juan paid $20 for a sign and a table. It then cost him $4 for each gallon of lemonade he made.

a. What was Juan’s total cost for making 2 gallons, including the $20 set-up cost?

b. Complete the adjacent cost table, where \( C(x) \) = the total cost to produce \( x \) gallons of lemonade.

c. Plot these ordered pairs on the coordinate plane.

d. Suppose that we do not know how many gallons Juan made, but we know the cost. Can we determine the number of gallons that can be produced at that cost? Complete the adjacent table.

e. How many gallons of lemonade can be made for $68?

f. How many gallons of lemonade can be made for $30?

4. Consider the function \( H \) given by the rule \( H(x) = 3x + 1 \). Compute the following:

a. \( H(0) \)

d. \( H(-1) \)

b. \( H(1) \)

e. \( H(5) \)

c. \( H(2) \)

f. \( H(3) \)

g. Complete the table to the right using the function \( H(x) \) as given above.
5. Consider the function \( J \) given by the rule \( J(x) = -2x \). Compute the following:
   a. \( J(0) \)
   b. \( J(-4) \)
   c. \( J(8) \)
   d. \( J(-3) \)
   e. \( J(12) \)
   f. \( J(-9) \)

6. Terry has 9 M&M’s. He eats them very slowly; in fact, he takes 1 minute to eat each one.
   a. Make a table for the number of M&M’s Terry has left after \( x \) minutes. Use zero for the starting time.
   b. Consider the table of inputs and outputs as a table of points \((x,y)\) where \( x = \) minutes and \( y = \) M&Ms left after \( x \) minutes. Graph the points from this table.
   c. Find the function \( M(x) \) that gives the number of M&M’s Terry has left after \( x \) minutes. What do you notice?
   d. When will Terry run out of M&M’s?
   e. What are the domain and range for \( M(x) \)?

7. Let the function \( h \) be defined by \( h(x) = x + |x| \) for all integers \( x \). Compute the following:
   a. \( h(0) \)
   b. \( h(2) \)
   c. \( h(4) \)
   d. \( h(-4) \)
   e. \( h(-6) \)
   f. \( h(-2) \)
   g. Plot points from parts a-f on a coordinate plane.
   h. Compute \( h(y) \), where \( y \) is a positive integer.
   i. Compute \( h(y) \), where \( y \) is a negative integer.
   j. What is the domain and range for \( h(x) \)?

8. Let the functions \( f \) and \( g \) be defined by \( f(x) = x + 5 \) and \( g(x) = 4x \). Compute each of the following:
   a. \( f(0) + g(0) \)
   b. \( f(1) + 2 \cdot g(3) \)
   c. \( f(2) \cdot g(2) \)
9. Find the rule that can be used to determine the value of any term described in the table below.

<table>
<thead>
<tr>
<th>Position</th>
<th>Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>13</td>
</tr>
<tr>
<td>4</td>
<td>17</td>
</tr>
<tr>
<td>x</td>
<td>?</td>
</tr>
</tbody>
</table>

10. Jason is playing miniature golf. He must pay $2.00 to rent the putter and $7.00 for each game he plays. Fill in the outputs in the following table. Determine a rule to calculate the cost of playing $x$ games, including the putter rental.

<table>
<thead>
<tr>
<th>Number of Games ($x$)</th>
<th>Total Cost $C(x) = y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>$x$</td>
<td></td>
</tr>
</tbody>
</table>

a. What is the independent variable in the table?
b. What is the dependent variable in the table?
c. What is the relation between the two variables?

11. Suppose that the functions $f$ and $g$ are defined by $f(x) = |x| + 5$ and $g(x) = |x + 5|$ for all integers $x$. What are the domain and range of $f$? What are the domain and range of $g$?

12. Let the function $j$ be defined by $j(x) = x - |x|$ for all integers $x$. Find the domain and range of $j$.

13. **Ingenuity**: 
   
a. Write a formula of a function that has domain equal to all integers, and the range is all integers greater than or equal to 4.
   
b. Write a formula for a function that has domain equal to all integers and, the range is all integers less than or equal to $-7$. 

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14. **Investigation 1:**
In Peoria, Illinois, the temperature drops an average of 3 °F per hour over a twelve-hour period from 1 P.M. to 1 A.M. The temperature at 7 P.M. is 13 °F. Let \( T(x) \) represent the temperature in Peoria \( x \) hours after 7 P.M.

a. What does \( x = 0 \) and \( x = -2 \) mean in the context of this problem? What is \( T(0) \), \( T(1) \), and \( T(-2) \)?
b. What \( x \)-value corresponds to 11 P.M.? What is the temperature at 11 P.M.?
c. Start to make a table of the time \( x \) from 1 P.M. to 1 A.M. and the corresponding temperature \( T(x) \).
d. Write the relationship between temperature and time using function notation. In other words, find an equation for \( T(x) \).
e. When is the temperature equal to 4 °F? 28 °F?
f. Is it possible to use multiplication to help determine the temperatures in parts a and b? If so, explain how.

15. **Investigation 2:**
For each positive odd integer \( n \), let \( s(n) \) be equal to the sum of all the first \( n \) odd positive integers. Calculate the following:

a. \( s(1) \)  
b. \( s(2) \)  
c. \( s(3) \)  
d. \( s(4) \)  
e. \( s(5) \)

f. What patterns do you notice about this function? Write a formula for the function \( s \).
g. Represent this function in a geometric way that reinforces the pattern.
SECTION 5.4  GRAPHING FUNCTIONS

In section 5.3 we plotted points on a coordinating system where the first coordinate was the input and the second coordinate was its corresponding output. The following exploration will help you develop a set of good graphing techniques.

EXPLORATION 1: GRAPHING POINTS

To graph the following data, follow a logical process:

<table>
<thead>
<tr>
<th>x-coordinate</th>
<th>y-coordinate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
</tr>
<tr>
<td>7</td>
<td>35</td>
</tr>
<tr>
<td>8</td>
<td>40</td>
</tr>
</tbody>
</table>

1. What are the inputs (domain)? What are the outputs (range)?

2. Based on your answer to question 1, which quadrants are necessary for the graph? What will your graph look like? Sketch an example.

3. Based on your answer to question 1, will all the points fit on a 10 x 10 graph if the scale is 1 for each axis? If not, what should each graph unit be equal to, on the horizontal axis? on the vertical axis? Notice that the two axes need not have the same scale.

4. Now that you have carefully considered an appropriate scale for the axes, graph the points given in the data.

Now that we can plot points on the coordinate plane, let’s look at more complex structures.
EXPLORATION 2

Consider the line below:

A. Identify the coordinates of the two labeled lattice points. Find three other lattice points that are also on the line. Make a list of all five points. How can you find more lattice points on the line without looking back at the graph?

B. One way to organize this list is to make a table. Make a table and fill in the points from your list.

C. Did you put the points in your table in a certain order? Without looking at the graph, find three more points on the line using your table. Check that these points are actually on the line.

D. Describe any patterns you see in the points of this line.

E. What point on the line has 40 as its first coordinate? What point has 40 as its second coordinate?

F. If the point \((x, y)\) is on this line, what is the relationship between \(x\) and \(y\)? Write this as an equation.

G. In the graph above, what is the independent variable?

H. What is the dependent variable?
EXPLORATION 3

A. Identify the coordinates of the four labeled lattice points on lines $a$ and $b$. Find three other lattice points that are also on each line. Make a list for each of these two lines. How can you find more lattice points on each line without looking back at the graph?

B. One way to recognize this list is to make a table. Make a table for each line and fill in the points from your list for each line.

C. Did you put the points in your table in a certain order? Without looking at the graph, find three more points on the line using your table. Check that these points are actually on the lines.

D. Describe any patterns you see in the points of these lines.

E. What point on line $a$ has 20 as its first coordinate? What point on line $b$ has -10 as its second coordinate?

F. If the point $(x, y)$ is on line $a$, what is the relationship between $x$ and $y$? Write this as an equation.

G. If the point $(x, y)$ is on line $b$, what is the relationship between $x$ and $y$? Write this as an equation.
GRAPHING CALCULATOR ACTIVITY

Objective: To explore the graphing calculator as a tool to study lines. We will use the calculator to graph the lines given by the equations:

\[ y = x \quad y = x + 2 \quad y = 2x \]

1. Turn on the calculator and press the "y=" button. Type in each formula using the first 3 slots. That is, the "y1=", "y2=" and "y3=".
2. Next, push the "zoom" button on the top row and use the down arrow to move the cursor to #6. Push "enter". Push "graph". You should have three graphs on your screen.
3. What do you notice about these lines? What are the similarities? What are the differences?
4. Explore these lines given by these formulas: \[ y = 2x + 3 \quad y = x + 3 \quad y = -x + 3 \]
   What do you notice about these lines? What are the similarities? What are the differences?
5. We will now explore the first 3 lines using the table button. Type the equations in again if you cleared them. Be sure to disable or clear the other equations.
6. First push "2nd" button and then the "tableset" button. Make sure the "tblstart=" slot has the value 0, which tells the table to start at 0. Next, make sure the button with a " tbl=" has the value of 1. This will have the first coordinate column ("x") change by 1 unit as you move down the table.
7. Next, push the "table" button. Notice that these are columns of numbers just like the tables we have made. The first column is called the "x" column and it contains the first coordinates. The other columns are labeled by "y1=", "y2=" and "y3=". Using this table, what similarities and differences do you notice about these lines? Compare these observations with what you noticed about their graphs. Explore the table by using the up and down arrow and the right and left arrow. Can you find the values for the line given by the equation in the "y3=" slot?
8. Push the "tableset" button again. Type in the number 0 in the "tblstart=" slot. Now change the " tbl=" to 0.5 so that the first coordinates will change by only \( \frac{1}{2} \) as we move up or down the "x" column.
9. Explore the tables if you push the "tableset" button and reset the " tbl=" to 0.1.
10. Explore these different formulas that produce curves. For example, $y_1 = x \cdot x = x^2$, $y_2 = x^2 + 4x$, $y_3 = \frac{1}{x}$, $y_4 = x^3$, $y_5 = x^4 - 4x$, $y_6 = 4x - x^2$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y_4$</th>
<th>$y_5$</th>
<th>$y_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>
EXERCISES

Plot the points (1, 2) and (3, 4) on a coordinate plane. Carefully draw a straight line through these two points. Label this line L. Exercises 1–4 deal with line L. Remember that the x-coordinate is the first-coordinate. The y-coordinate and the second-coordinate are also interchangeable terms.

1. a. Write a list of four more points on line L.
   b. Include these points in the table to the right and fill in the second coordinates that are missing.

2. a. What point has first coordinate 30? (30, ___)
   b. What point has second coordinate 30? (___, 30)

3. If (x, y) is a point on this line, write an equation defining y in terms of x.

4. a. If p is the first coordinate of a point on the line, what is the second coordinate of the point in terms of p? (p, _____)
   b. If q is the second coordinate of a point on the line, what is the first coordinate of the point in terms of q? (___, q)

Plot the points (2, 6) and (4, 8) on a coordinate plane. Draw a straight line through these two points and label this line M. We will explore points on line M in Exercises 5–7.

5. a. Write a list of four points on line M.
   b. Expand your list to a table like the one above.
   c. What point has first coordinate 10? (10, ___)
   d. What point has second coordinate 24? (___, 24)

6. a. What point has first coordinate x? (x, _____)
   b. What point has second coordinate y? (_____, y)
7. What is the equation that describes line \( M \)? Write your equation in the same form as your answer to Exercise 3, in terms of a general point \((x, y)\).

8. Plot the graph line that satisfies the equation \( y = 4x \). For \( b-e \), how could you find this point using the table, equation, or graph?
   a. What point on the line has 5 as the first coordinate?
   b. What point on the line has 24 as the second coordinate?
   c. What point on the line has 56 as the second coordinate?
   d. What point on the line has 92 as the second coordinate?
   e. What point on the line has -36 as the second coordinate?
   f. In your graph, what are the independent and dependent variables?

9. Plot the graph line that satisfies the equation \( y = -6x \).
   a. What point on the line has 5 as the first coordinate?
   b. What point on the line has -18 as the second coordinate?
   c. What point on the line has -42 as the second coordinate?
   d. What point on the line has 12 as the second coordinate?

Plot the points \((1, 5)\) and \((4, 2)\) on a coordinate plane. Draw a straight line through these two points and label this line \( N \). Exercises 10 and 11 deal with this line.

10. a. Write a list of 4 more points that lie on line \( N \).
    b. Include these points in the table to the right and fill in the second coordinates that are missing.

11. a. Use the pattern you see in the table to write an equation for this line involving a typical point \((x, y)\), solving for \( y \).
    b. Use this equation to find the point that has 12 as its first coordinate.
    c. Use this equation to find the point that has 10 as its second coordinate.
12. For the line given by the equation \(y = -3x\), make a table of points with first coordinates ranging from \(-6\) to \(+6\). Plot these points on a coordinate plane.

13. a. Make a table of points for the line given by the equation \(y = -3x + 13\), with first coordinates ranging from \(-3\) to \(+6\). Show how to compute the second coordinates for these points.
   b. Plot these points on a coordinate plane.
   c. What is the difference between the table in this question and the table in problem 10?
   d. What is the \(x\)-coordinate, or the first coordinate, when the \(y\)-coordinate is \(-17\)? Is there a way to solve for the \(x\)-coordinate using the table? If so, how?

14. For the line given by the equation \(y = -2x + 12\),
   a. Make a table of points with first coordinates ranging from \(-8\) to \(+8\).
   b. Plot these points on a coordinate plane.
   c. What is the \(x\)-coordinate, or the first coordinate, when the \(y\)-coordinate is \(30\)? Show how to solve for the \(x\)-coordinate.

15. Perform the following transformations.
   a. Reflect the line given by the equation \(y = 3\) about the \(x\)-axis.
   b. Take the result from part a and reflect it about the \(y\)-axis.
   c. Describe what happens.

16. **Ingenuity:**
Heather has $0.30 in her pocket in pennies and nickels.
   a. Make a table of all the possible combinations of coins Heather could have, with number of pennies, \(P\), in the left column and number of nickels, \(N\), in the right column.
   b. Plot these points on a graph, using number of pennies \(P\) as the first coordinate. Find an equation which relates \(N\) and \(P\).
18. **Investigation:**
   a. Draw the lines represented by the equations $y = x$, $y = 2x$ and $y = 4x$ on the same coordinate plane.
   b. What are the similarities between these lines? What are the differences? What causes these differences?
   c. What do you think the line $y = 8x$ would look like? What about the line $y = 0 \cdot x$ or $y = 0$?
   d. Check your work with your graphing utility.
EXPLORATION 1

a. Fill in the table for each of these function given in the table below. Plot these prints and draw the lines.
b. What do the graphs of these lines have in common? 
c. What do their tables have in common?

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>f(x)=y=2x</td>
<td>g(x)=2x+2</td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The rules that produce straight line graphs all have a common form:

\[ y = mx + b \]

The functions are called linear functions, that is, they have equations of the form \( y = mx + b \).

Questions:
1. In Exploration 1, what is the value of \( m \) for each of the lines?
2. What effect does the number \( m \) have on the table for each line?
3. What effect does the number \( m \) have on the graph for each line?
4. What role does the number \( b \) and the y-intercept play for each line?

Notice that the value of the number \( m \) in each of the linear functions above is the amount of change in the second coordinate when the first coordinate increases by 1 unit. This number \( m \) is the constant rate of change of \( y \). The value of \( m \) determines how fast the value of \( y \) increases or decreases in the table and on the graph.
PROBLEM 1

Consider the following three linear function:

a. \( y = 3x - 2 \)
b. \( y = 3x + 4 \)
c. \( y = 3x + 11 \)

Predict how their graphs will be related. Use a graphing calculator to check your predictions.

The following exercises use the mathematics you have learned and apply them to practical problems. Before you begin, recall the word cost. You may have thought about the cost of a new pair of shoes, or some candy, or even a movie ticket.

People or companies that produce, make, or sell these items also have costs. An important idea that we use in the exercises is that cost can have two parts: a fixed cost and a variable cost. By fixed cost, we mean the part of the cost that does not depend on how many items a company or person produces. Variable cost, on the other hand, is the cost that depends on the number of items the company or person makes. Usually, the more items you produce the more it costs. In the following problems, we consider the Total cost = Variable cost + Fixed cost.

Another important idea is the amount of money that a company or person makes for selling the items. We call this amount the revenue. Keep these ideas in mind as you work on the following problems that include making and selling lemonade and birdhouses.

EXPLORATION 2

Functions \( F \) and \( G \) are defined as \( y_1 = F(x) = 4x \) and \( y_2 = G(x) = x + 4 \).

a. Fill in the table for \( F \) and \( G \) with \( x = \{0, 1, 2, 3, 4, 5\} \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( F(x) = y_1 )</th>
<th>( y \div x )</th>
<th>( G(x) = y_2 )</th>
<th>( y \div x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Section 5.5 Applications of Linear Functions

b. Is there a pattern for the third column? For the fifth column? Explain.

If a function has a rule in the form \( y = Kx \), then for any input \( x \neq 0 \), the quotient of \( \frac{y}{x} \) will always have the value \( K \). The number \( K \) is called the constant of proportionality, and the function is proportional.

EXERCISES

Sarah opens a lemonade stand. It costs her $18 to set up the stand and $1 for the ingredients to make each gallon of lemonade.

Problems 1 through 6 deal with Sarah’s lemonade stand.

1. a. How much does it cost Sarah to make 1 gallon of lemonade? How much does it cost Sarah to make 2 gallons? 0
b. Copy and complete the following chart.
c. How many gallons of lemonade can she make for $18?

2. a. Plot the data from Problem 1 as points on a coordinate plane with the first coordinate representing the number of gallons and the second coordinate representing the cost in dollars. Label the \( x \)-axis as the “number of gallons” and the \( y \)-axis as the “cost in $.”
b. What patterns do you notice in the points plotted in part a?
c. How would you describe the way the points appear to someone who cannot see the graph?

3. a. If she made \( x \) gallons, what is her cost \( y \)? Write an equation for the cost \( y \) in terms of gallons \( x \). What is the constant rate of change of the linear function?
b. If she has $30 to spend on making lemonade, how many gallons can she make?
c. If she has $46 to spend on making lemonade, how many gallons can she make?
4. Sarah wants to sell the lemonade for $3.00 per gallon. Copy and complete the table below. How much money will she collect selling \( x \) gallons of lemonade?

<table>
<thead>
<tr>
<th>Number of Glasses</th>
<th>Sales (Revenue) in $</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00</td>
</tr>
<tr>
<td>1</td>
<td>3.00</td>
</tr>
<tr>
<td>2</td>
<td>6.00</td>
</tr>
<tr>
<td>3</td>
<td>9.00</td>
</tr>
<tr>
<td>5</td>
<td>15.00</td>
</tr>
<tr>
<td>7</td>
<td>21.00</td>
</tr>
<tr>
<td>10</td>
<td>30.00</td>
</tr>
<tr>
<td>15</td>
<td>45.00</td>
</tr>
<tr>
<td>20</td>
<td>60.00</td>
</tr>
<tr>
<td>30</td>
<td>90.00</td>
</tr>
</tbody>
</table>

a. If she sells 10 gallons of lemonade, what is the amount of her sales or revenue? If she sells 20 gallons? 30 gallons?

b. If she wants her revenue to be $36, how many glasses does she need to sell?

c. Describe the relationship between the first and second coordinates of these points.

d. Write an equation for this line, the graph of the revenue function. What is the constant rate of change of the linear function?

5. Determine whether each of the following linear functions has a constant of proportionality. Explain the differences in the graphs that lead to your decision.

a. \( y = 3x \)

b. \( y = x + 1 \)

c. \( y = -2x \)

d. \( y = x - 2 \)
6. Sarah wants to compare her cost of production with her revenue. So, to compare, plot these data points on the coordinate plane from Exercise 2. What does each coordinate represent for the new points? Compare the graphs of the cost function with the revenue function. Where does the graph of the revenue function intersect the graph of the cost function? What do the coordinates of this intersection point represent?

7. Rent-It-All charges $25 for the deposit on a tool rental plus $15 an hour.
   a. Write the function of the total cost $y$ in terms of the number of hours $x$.
      What is the constant rate of change of the linear function?
   b. Fred and Jane rent a sander to redo their floors. They check the sander out at 10 A.M. and return it at 5 P.M. How much do they pay?

Juan decided to build birdhouses to sell. He made an initial investment of $4 to purchase tools. He then went to the store to buy wood to make his birdhouses. It cost him $2 for the wood for each birdhouse, after his initial investment. Juan decided to sell his birdhouses for $3 each. Exercises 8, 9, 10, and 11 deal with Juan's birdhouse business.

8. Begin by making a chart of how much it would cost Juan to make various numbers of birdhouses. There are two types of cost we want to consider: variable costs and fixed costs. The variable cost is the cost of making the birdhouses without counting Juan's initial $4 investment. The fixed cost is the $4 that Juan spent initially for tools. The total cost is the sum of the variable cost and the fixed cost.

   a. Copy and complete the adjacent table.

<table>
<thead>
<tr>
<th>Number of Houses</th>
<th>Variable Cost in $</th>
<th>Total Cost in $</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>14</td>
</tr>
<tr>
<td>10</td>
<td>20</td>
<td>24</td>
</tr>
<tr>
<td>20</td>
<td>40</td>
<td>44</td>
</tr>
<tr>
<td>50</td>
<td>100</td>
<td>104</td>
</tr>
<tr>
<td>100</td>
<td>200</td>
<td>204</td>
</tr>
<tr>
<td>$x$</td>
<td>$2x$</td>
<td>$2x + 4$</td>
</tr>
</tbody>
</table>

   b. Plot these data for the variable cost as points on a coordinate system with the $x$ coordinate representing the number of birdhouses and the $y$ coordinate representing the cost in dollars. Label the $x$-axis as the "number of birdhouses" and the $y$-axis as the "cost in $"."
c. Now plot the points that show the total cost on the same coordinate plane.

d. Write an equation for the total cost, \( T \), to build \( x \) birdhouses. Is \( T \) a linear function? What is \( m \)? What is \( b \)?

e. How many bird houses can Juan build for \$160? \$70

9. a. Make a table of revenue collected for selling different quantities of birdhouses. Complete the table below.

<table>
<thead>
<tr>
<th>Number of Houses</th>
<th>Revenue R in $ \ y = R(x)</th>
<th>y ÷ x</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. Write an equation involving \( R \) and \( x \), where \( R \) is the revenue from selling \( x \) birdhouses. Is \( R \) a linear function? What is \( m \)? What is \( b \)? Does \( R \) have a constant of proportionality?

c. Graph your revenue line on the coordinate plane you used in the previous problem.

10. a. Now consider the total cost and the revenue. At what point do the two graphs intersect?

b. The point where the cost and revenue intersect is called the break-even point. Why do you think it’s called this?

c. What does this point represent?

d. Can you find the intersection of the two lines algebraically?

11. How is Juan’s profit represented on the graph?
12. For each of the following linear functions, determine the y-intercept and if there is a constant of proportionality, what is its value?
   a. \( y = 4x \)
   b. \( y = 4x + 3 \)
   c. \( y = 6x - 2 \)
   d. \( y = -3x \)

13. **Ingenuity:**
   The three lines represented by the equations \( y = 6x + 3 \), \( y = -x + 24 \), and \( y = 3x + k \) intersect at the same point. What is the value of \( k \)?

14. **Investigation:**
   In mathematics, a palindrome is a number that reads the same backward or forward. For example, 12321, 5665 and 11 are all palindromes.
   a. Find all the two digit palindromes. How many are there?
   b. How many three digit palindromes are there? Try to list them, using a pattern.
   c. How many four digit palindromes are there?
SECTION 5.6 PATTERNS AND SEQUENCES

There are many areas of mathematics in which we study lists of numbers like those in Exploration 1. We call these lists of numbers sequences. Looking at the table of a function where the inputs start at 0 or 1 and increase by 1 unit, the second column is a list of outputs which can be written horizontally in a list. We have looked for patterns in these lists of outcomes in order to find the rule for the function. When the inputs of a function are restricted to positive integers or non-negative integers, the function is called a sequence. Most of the sequences that we will study have a pattern that can be described by a rule.

EXPLORATION 1

For the following list of numbers, determine what you think is the next number in the list. Explain how you made your choice.

a. 4, 8, 12, 16, __, ...

b. 1, 3, 5, 7, __, ...

c. 3, 6, 12, 24, __, ...

EXAMPLE 1

Consider the sequence defined by the rule that for each input n, representing the place in the list, the output is given by $y = 2n$. Making a horizontal table of outputs, we get

<table>
<thead>
<tr>
<th>Input</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>10</th>
<th>25</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2n</td>
</tr>
</tbody>
</table>

Thus the sequence can also be written as the list: 2, 4, 6, 8, ... . Each member of a sequence is called a term of the sequence. The three dots at the end of the list means that the sequence continues for all positive integers. It is common to
think of the first term in the list as the output you get with input of n = 1. If we use function notation, then the first term is denoted by \(a(1)\) or \(a_1\). Organizing the sequence vertically, we get

\[
\begin{align*}
  a(1) &= a_1 = 2 \\
  a(2) &= a_2 = 4 \\
  a(3) &= a_3 = 6 \\
  a(4) &= a_4 = 8 \\
  a(n) &= a_n = 2n \\
\end{align*}
\]

Another way to represent the information above is:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_1)</td>
<td>(a_2)</td>
<td>(a_3)</td>
<td>(a_4)</td>
<td>(a_n)</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>2n</td>
</tr>
</tbody>
</table>

The input tells us the position of the output in the list. Thus, \(a(3) = 6\) means that the third number in the sequence is 6. Some typical questions about this sequence are:

a. What are the next 3 terms? That is, what are \(a_5\), \(a_6\), and \(a_7\)? The answers are 10, 12, and 14.

b. What is the value of the 25th term of the sequence, that is, what is \(a_{25}\)? The answer is \(a(25) = a_{25} = 2 \cdot 25\).

c. What is the \(n\)th member of the sequence? This is another way of asking what the rule for the sequence is. The answer is \(a(n) = a_n = 2n\).

Of course the question in part c is obvious because we gave the rule in the beginning. But we are often given a sequence of a list of numbers with the first 3 to 5 terms in the list. In this case, the question in part c can be challenging.

You probably noticed that given a term of this sequence, you can determine the next term by adding 2. In other words, we can write
Method 1
\[ a_2 = a_1 + 2 = 4 = 2 \cdot 2 \]
\[ a_3 = a_2 + 2 = 6 = 2 + 2 = 2 \cdot 3 \]
\[ a_4 = a_3 + 2 = 8 = 2 + 2 + 2 = 2 \cdot 4 \]
\[ a_5 = 2 \cdot 5 \]
and so on.

Method 2
\[ 2, 4, 6, 8, \ldots \]
\[ +2 +2 +2 \]

**DEFINITION 5.2: ARITHMETIC SEQUENCE**

A sequence \( a_1, a_2, a_3, a_4, \ldots \) is an **arithmetic sequence** if there is a number \( c \) such that for each positive integer \( n \), \( a_{n+1} = a_n + c \), that is, \( a_{n+1} - a_n = c \).

Thus, our sequence above is an arithmetic sequence because for each positive integer \( n \), \( a(n+1) = a(n) + 2 \).

**PROBLEM 1**

Based on the portion of the sequences shown below, write the next two terms. Use either method 1 or method 2 from above to determine which of the sequences is an arithmetic sequence and, if it is, what is the constant difference between any two consecutive terms.
Section 5.6 Patterns and Sequences

a. 3, 6, 9, 12, ...
b. 4, 7, 10, 13, ...
c. 2, 6, 10, 14, 18, ...
d. 2, 4, 5, 7, 8, 10, 11, 13, ...

How do we determine a rule or formula for the sequence in part a. in Problem 1?
One way to discover the rule is to list the sequence carefully and look for a pattern.

For the sequence 3, 6, 9, 12, ..., we make a vertical list:

\[
\begin{align*}
a_1 &= 3 \\
a_2 &= 3 + 3 = (2) \cdot 3 = 3 \cdot 2 \\
a_3 &= 3 + 3 + 3 = 3 \cdot 3 \\
a_4 &= 3 + 3 + 3 + 3 = 3 \cdot 4 \\
\text{What is } a_{10}? & \text{ The answer is } a_{10} = 3 \cdot 10. \text{ So, what is the rule for } a_n? \text{ The answer is } a_n = 3n.
\end{align*}
\]

PROBLEM 2

Determine a rule for each of the sequences in parts b. and c. from Problem 1.

\[
\begin{array}{|c|c|}
\hline
\text{Input} & \text{Output} \\
\hline
1 & a_1 = 3 \\
2 & a_2 = 3 + 3 = (2) \cdot 3 = 3 \cdot 2 \\
3 & a_3 = 3 + 3 + 3 = 3 \cdot 3 \\
4 & a_4 = 3 + 3 + 3 + 3 = 3 \cdot 4 \\
10 & a_{10} = 3 \cdot 10 \\
n & a_n = 3n \\
\hline
\end{array}
\]
PROBLEM 3

Use the pattern blocks to make the next two designs. Sketch the designs and record your work in the table on your answer sheet. Write a function that describes the sequence.

Design 1 Design 2

<table>
<thead>
<tr>
<th>Design Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Blocks</td>
<td>7</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

EXERCISES

1. For each of the following rules for a function, write out the first 5 terms of the corresponding sequence.
   a. \( a_n = a(n) = 2n + 3 \)
   b. \( a_n = a(n) = 3n - 1 \)
   c. \( a_n = a(n) = 4n - 2 \)
Section 5.6 Patterns and Sequences

2. For each of the following sequences, write the next two terms. Use either method 1 or method 2 from above to determine which of the sequences is an arithmetic sequence and, if it is, what is the constant difference between any two consecutive terms.

a. 3, 6, 9, 12, 15 ...

b. 7, 15, 21, 28, ...

c. -4, -8, -12, -16, ...

d. 1, 4, 7, 10, ...

e. 3, 8, 13, 18, ...

f. 2, 8, 14, 20, 26, ...

g. Determine a rule for the \( n^{th} \) term for each of these sequences.

3. For each of the following sequences, write the next 2 terms. Determine a rule for each sequence.

a. 5, 10, 15, 20, ...

b. 9, 14, 19, 24, ...

c. 5, 7, 9, 11, ...

d. 5, 9, 13, 17, ...

4. Suppose you make the following sequence of figures with tooth picks. Let \( S_n \) be the number of squares at the \( n^{th} \) stage.

```
  □  □ □  □ □ □
```

a. Write out the first 5 terms of this sequence.
b. Write a formula for \( S_n \).
c. Write a sequence for the number of tooth picks \( T_n \) used at the \( n^{th} \) stage.
5. Suppose you make the following sequence of figures with tooth picks. Let $S_n$ be the number of squares at the $n$th stage.

a. Write out the first 5 terms of this sequence.
b. Write a formula for $S_n$.
c. Write a sequence $P_n$ for the number of tooth picks on the outside (perimeter).
d. Write a formula for $P_n$. 
REVIEW PROBLEMS

1. Plot the following points on a coordinate plane and determine their quadrants.
   
   A (1,5)  B (4.5, -4)  C (3,11)
   D (-2, 0.5)  E (-6, -2)  F (1,0)

2. a. On the another coordinate plane, draw the line that passes through points A (1,5) and C (3,11).
   b. Label 3 other points on the line.
   c. Make a chart of the points below that lie on the same line by filling in the table below.
   d. Using the table, describe the pattern in the points.
   e. Write an equation for this line in terms of x, solving for y.
   f. Find the point that has 11 as its x-coordinate.
   g. Find the point that has -13 as its y-coordinate.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>10</td>
</tr>
<tr>
<td>-3</td>
<td>7</td>
</tr>
<tr>
<td>-1</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>20</td>
<td>P</td>
</tr>
</tbody>
</table>

3. Function g is defined as \( g(x) = 2x + 5 \).
   Compute the following:
   a. \( g(1) \)
   b. \( g(-3) \)
   c. \( g(-2) \)
   d. \( g(2) \)
   e. \( g(-1) \)
   f. \( g(-4) \)
   g. Graph all the ordered pairs \((x, g(x))\) from parts a through f.
4. Manuel decided to build jewelry boxes. He purchased tools for $6. He went to the hardware store for wood for the jewelry boxes. After the initial investment, Manuel discovered it cost him $3 for the wood necessary to make each jewelry box. So he decided to sell each jewelry box for $5.

a. Make a table for the variable $x$, the number of jewelry boxes, and the total cost $T(x)$ in dollars. Fill in three values for $x$ and $T(x)$.

b. Write a rule that computes the total cost, $T(x)$, for making $x$ jewelry boxes.

c. Plot the data for the total cost on a coordinate system.

d. Make a second table for revenue collected from selling jewelry boxes.

e. Write a rule that computes the revenue $R(x)$ for selling $x$ jewelry boxes.

f. Plot the data for revenue earned on the same coordinate system as part (c). At what point do they intersect?

g. What is the domain and range of $T(x)$?

5. Victor decides to make cakes. He invests $14 for startup supplies. He went to the store to buy ingredients for the cakes. After the initial investment, Victor discovers it cost him $1 to make each cake. He decides to sell each cake for $3.

a. Write a rule for the total cost to make $x$ items.

b. Write a rule for the revenue earned by selling $x$ items.

c. How many cakes can he make for a cost of $30$?

d. How many cakes does he need to make to produce a revenue of $48$?

e. Plot the total cost and revenue.

f. Where do the 2 graphs intersect? What does the point of intersection mean in the problem?

6. Let the function $f$ be defined by $f(x) = |x| - x$ for all integers $x$. Compute the following:

a. $f(0)$

b. $f(4)$

c. $f(1)$

d. $f(-3)$

e. $f(-6)$
7. Logan rents a boat from Rent-It Boats. They charge a flat fee of $75 plus $55 per hour.
   a. Write the function of the cost, \( y \), in terms of the number of hours \( x \).
   b. Logan goes deep-sea fishing from 8 A.M. to 2 P.M. How much does he pay to rent the boat?
   c. If they are charged $625, how long did they use the boat?

8. Plot the following points and translate each by using the rule “add 5 to the \( x \)-coordinate and add -2 to the \( y \)-coordinate.”
   a. (4, 4) b. (0, 4) c. (-4, 1)

9. Plot the following points and reflect each point about the \( x \)-axis.
   a. (4, 4) b. (3, -6) c. (-1, 4) d. (-2, -2)

10. Determine all lines of symmetry for the figures below.

11. For each of the following rules for a function, write out the first 5 terms of the corresponding sequence.
   a. \( a_n = a(n) = n + 5 \)
   b. \( a_n = a(n) = 3n - 2 \)
   c. \( a_n = a(n) = 5n + 8 \)

12. For each of the following sequences, write the next 2 terms and determine a rule for each.
   a. 4, 7, 10, 13, ...
   b. 2, 5, 8, 11, 14, ...
   c. 6, 10, 14, ...
CHALLENGE PROBLEMS

Section 5.1:
Alan, an ant, starts at the origin in the coordinate plane. Every minute he can crawl one unit to the right or one unit up, thus increasing one of his coordinates by 1. How many different paths can Alan take to the point (4,3)?

Section 5.2:
Let f be a function whose domain is S = {1, 2, 3, 4} and whose range is contained in S. How many such functions do not map more than two domain values to the same range value?

Section 5.3:
Alan, an ant, is back at the origin. Every minute he can crawl one unit to the right or one unit up, but after the first step he will not cross the line $y = x$ (he can touch the line, though). How many such paths can Alan take to the point (5,5)?

Section 5.4:
Phone company A charges 23 cents per minute. Company B charges 11 cents per minute with a $10 monthly fee. Company C charges a flat rate of $30 per month with unlimited calls. What are the minimum and maximum number of minutes you can use each month such that company B will be the cheapest?
SECTION 6.1  DECIMALS

We have all been concerned about the cost of an item when we shop or go out to eat. For example, a cheeseburger might cost $3.85 at a restaurant, a pair of jeans might cost $18.97 at a store or the entrance fee to an amusement park might cost $27.00. All of these prices use decimal notation, a common way of writing numbers that include parts that are less than 1. Instead of $27.00, we can write $27 instead. This means 2 tens and 7 ones. However, when we write $27.00, the zeros to the right of 7 give us more information: that the price is exact and includes no cents. In the number 43.26, the 3 is in the one’s place which is one-tenth the ten’s place occupied by the 4. Then 2 is in the tenth’s place which is one-tenth the one’s place. In an example with money, notice that the 8 in the cost of the $3.85 cheeseburger is in the dime’s place or the tenths-of-a-dollar place and 5 is in the penny’s place or the hundredth’s place. It takes 10 dimes to make a dollar and it takes 10 pennies to make a dime. Therefore it takes 100 pennies to make a dollar.

EXPLORATION:  LOCATING DECIMAL NUMBERS ON A NUMBER LINE.

If we think of 1 on the number line as $1.00, where would we locate half a dollar or $0.50? Because there are 10 dimes in a dollar, where would $0.10 be located on the number line? $0.20? $0.30? Can you locate $0.01, or more simply 0.01, on the number line, knowing that there are 10 pennies in a dime?

We know when we write 0.30 that there is another way that this decimal can be written. Thirty cents, or hundredths, can be written as 0.3. How could you show the two numbers are really equivalent to each other on the number line? We know three dimes or 0.3 has the same value as 30 pennies or 0.30.
Use a number line like the one below to estimate the locations of the following decimal numbers. Notice that 0 and 1 are labeled on the number line.

a. 0.4  c. 0.68  e. 0.374  g. 0.397
b. 0.27  d. 0.7  f. 0.307

One strategy that you can use to compare decimals is to write the numbers so that they have the same smallest place value. For example, 0.2 and 0.27 can be written as 0.20 and 0.27. When we compare the hundredths place, clearly 27 hundredths is more than 20 hundredths because 27 is greater than 20.

For each pair of numbers, determine which is greater. Justify your answer using the number line.

a. 0.68 and 0.7  b. 0.34 and 0.339  c. 0.268 and 0.271

**ADDITION AND SUBTRACTION OF DECIMALS**

Betty is about to take a trip. She fills her car with gas for $29.90 and buys a map for $3.49, a drink for $1.09 and a pack of gum for $0.99. Estimate the cost of her purchase before taxes. Is $40.00 enough to pay for the purchase, excluding tax?

If you calculated $30.00 + $3.50 + $1.00 + $1.00 to get $35.50 you had a good estimate of her cost. To get the actual cost, however, you must add $29.90 + $3.49 + $1.09 + $0.99 or take the estimated cost and subtract the excess of 10¢ + 1¢ + 1¢ from the estimated cost, then add in 9¢ for the underestimation of $1.09. You overestimated by 3¢ so you should subtract 3¢ from $35.50 for the actual cost.
When you subtract 3¢ from $35.50, you are really subtracting $0.03 from $35.50 to get $35.47. As with addition, it is important to keep in mind the place value and subtract the hundredths from the hundredths, the tenths from the tenths, and so forth. You might have heard the phrase “line up the decimals.” This vertical method assures that the place values also line up to do the calculation.

One way to find a sum of money is to first add the hundredths or pennies, to get 27 pennies or 0.27 dollars, then add the tenths, dimes, to get 22 dimes or 2.2 dollars and finally add the 33 dollars. So, $0.27 + $2.2 + $33 = $35.47. Explain how this is the same as the traditional method of stacking the numbers to line up the decimal points and adding from right to left.

**Linear Model:**

How do you use the number line to add decimal numbers? Compute the following sums using the number line, and then add using the traditional stacking method.

- a. 0.2 + 0.06
- b. 0.38 + 0.47
- c. 0.23 + 0.54
- d. 0.26 + 0.31

Explain how the number line can help to estimate a sum before you calculate the actual total.

How do you use the number line to subtract decimals? Compute the following differences using the number line, and then subtract using the traditional stacking method.

- a. 0.63 – 0.47
- b. 0.2 – 0.06

There are times when we need to write really big numbers but want to avoid writing too many zeros. For example, we write 2 million instead of 2,000,000 or 30 million instead of 30,000,000. If we write 700 thousand, we mean 700,000.

- a. Write 8.9 billion as a whole number.
- b. What is another way of writing 1.5 million?
- c. What is another way of writing 0.5 million?
EXERCISES

1. Order the following sets of four numbers by locating them on a small segment of the number line.
   a. 0.7, 0.08, 0.69, 0.16
   b. 0.109, 0.098, 0.23, 0.228
   c. 4.08, 3.16, 4.2, 3.61

2. In working each of the following exercises, be careful to scale your number line appropriately.
   a. Draw and label 0.1 and 0.3 on a number line. Plot and name 4 additional points between these two points.
   b. Draw and label 0.29 and 0.307 on a number line. Plot and name 4 additional points between these two points.
   c. Draw and label -0.11 and -0.24. Is -.0109 on your number line? Is it between -0.11 and -0.24? Explain. Next, plot and label 4 additional points.

3. Convert the following measurements:
   a. 235 centimeters to meters
   b. 54.672 meters to centimeters

4. Compute the following, using the car model from Chapter 2, if necessary:
   a. 0.31 + 0.45
   b. 0.3 + 0.78
   c. 1.307 + 2.46
   d. 0.74 + -0.57
   e. 2.3 – 1.06
   f. 0.603 – 0.045

5. Determine which of the following pairs of numbers is closer together.
   a. 0.4 and 0.5 or 0.23 and 0.27
   b. 0.57 and 0.61 or 0.592 and 0.608
   c. 0.12 and 0.096 or 0.089 and 0.11

6. Bill is making cookies from a European recipe. The recipe calls for 336 grams of sugar. Bill discovers if he is short 1.7 grams of sugar or if he adds 1.7 grams of sugar too much, the taste of the cookies is unchanged. Between what amounts of sugar will the taste remain unchanged?
7. Solve the following equations.
   a. \( x + 0.35 = 0.46 \)
   b. \( y + 1.2 = 0.7 \)
   c. \( m + 4.03 = 3.565 \)
   d. \( z + 0.02 = 0.64 \)

8. Simplify the following expressions:
   a. \( 0.2x + 0.3x \)
   b. \( 0.3x - 0.1x \)

9. Ms. Trent has the same number of quarters as she has dimes. Write an expression that describes the total amount of money Ms. Trent has.

10. Sandra has a plant that is 1.354 meters tall after a growth spurt. If it grew 0.296 meters, how tall was it before the growth spurt?

11. Draw a coordinate plane and plot and label the following points:
   a. \((0.5, 0.5)\)
   b. \((-1, 0.5)\)
   c. \((-1.25, 0.75)\)
   d. \((7.5, -5.5)\)
   e. \((-4.5, -3.25)\)
   f. \((-7.5, -5.5)\)

12. Use graph paper to draw the graph of the line \( y = 2x + 1.5 \). If necessary, set up a table of values.

13. Ricky fills a 12 liter bottle in the lab by adding 9.37 liters of hydrochloric acid. How much acid did he have in the bottle in the beginning?

14. Bobby is conducting two independent chemistry experiments. To conduct these experiments, he needs 0.887 grams of silver nitrate for Experiment A and a total of 1.438 grams of silver nitrate for both Experiment A and B. How much silver nitrate does he need for Experiment B?

15. Jon is feeling sick. He decides to go to the doctor who finds that his temperature is 102.37 °F. If normal body temperature is 98.6 °F, what was the increase in his body temperature?

16. Ingenuity:

   Find the midpoint between each of the following pairs of numbers, showing your work on the number line.
   a. 0.3 and 0.4
   b. 0.12 and 0.15
   c. 0.3 and 0.37
17. **Investigation:**

Use a graphing calculator for the following using the table function. Scale by .5.

Draw the graph of the line \( y = 2x \).

a. Find the first coordinate of a point on this line whose second coordinate is 1 by looking at the graph. Check using a table on your calculator.

b. Find the first coordinate of a point on this line whose second coordinate is 3 by looking at the graph. Check using a table on your calculator.

c. Find the first coordinate of a point on this line whose second coordinate is 5 by looking at the graph. Check using a table on your calculator.

d. Find the first coordinate of a point on this line whose second coordinate is \(-1\) by looking at the graph. Check using a table on your calculator.
EXPLORATION 1

Use the linear model to show how to compute the following products:

a. $3 \cdot 2$

b. $(0.3) \cdot 2$

c. $(0.3) \cdot (0.2)$

The first product is simply 2 jumps of length 3 which yields a product of 6. The second product is 2 jumps of length 0.3, as shown below:

Two jumps of length 0.3 gives us the location of 0.6.

However, modeling the third product, $(0.3)(0.2)$, is not clear. What do we mean by 0.2 jumps? For this product, the area model may be more helpful.

Using the area model for $(3)(2)$, we find the area of a 3 by 2 rectangle. How do we use the area model for $(0.3)(2)$?

Draw a rectangle that has length 0.3 and width 2. When drawing this rectangle it is helpful to use grid paper and choose an appropriate scale. In this case, we need to measure both 0.3 and 2. Using a grid, assign each small square a length of 0.1
Outline a 1 by 1 square using the 0.1 grid. How many 0.1 by 0.1 small squares are in a 1 by 1 square? Now outline a 0.3 by 2 rectangle on the grid. With a picture of the rectangle and its dimensions, you can see that the area is made up of 60 small squares, each with an area of $0.1 \times 0.1 = 0.01$. So the product is $(0.3)(2) = (60)(0.01) = 0.60 = 0.6$.

We can rearrange the model above to look like the following:

Thus, $(60)(0.01) = 0.60$ since there are 60 hundredths shaded above. However, we can write 0.60 as 0.6, or 6 tenths, which is modeled below.
How do we compute the product $(0.3)(0.2)$ using the area model? Consider the grid below. Each side of the square has length 1. Note that the length of each little square is 0.1. Use this grid to model the product $(0.3)(0.2)$.

One way to show the product $(0.3)(0.2)$ is to shade a rectangle within the grid that is 0.3 long (horizontally) and 0.2 wide (vertically). The result is a small rectangle with 6 little squares. What is the area of each little square? Since the large square has area 1 and there are 100 little squares, the area of each little square is 0.01 or $1/100$. So the area of 6 little squares is 0.06 or $6/100$.

**PROBLEM 1**

Compute the following products using the grid.

a. $(0.4)(0.7)$

b. $(0.8)(0.9)$

c. $(0.6)(0.3)$

d. $(0.5)(4)$
PROBLEM 2

Compute the following groups of products. Look for patterns that can help us multiply decimal numbers. You may use a calculator.

<table>
<thead>
<tr>
<th>(4)(1)  =</th>
<th>(0.4)(2)  =</th>
<th>(0.8)(7)  =</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4)(0.1) =</td>
<td>(0.4)(0.2) =</td>
<td>(0.8)(0.7) =</td>
</tr>
<tr>
<td>(4)(0.01) =</td>
<td>(0.4)(0.02) =</td>
<td>(0.8)(0.07) =</td>
</tr>
<tr>
<td>(4)(0.001) =</td>
<td>(0.4)(0.002) =</td>
<td>(0.8)(0.007) =</td>
</tr>
<tr>
<td>(0.4)(0.1) =</td>
<td>(0.04)(0.2) =</td>
<td>(0.08)(0.07) =</td>
</tr>
</tbody>
</table>

What patterns do you notice?

PROBLEM 3

Compute the following groups of products. Look for patterns that can help us multiply decimal numbers. You may use a calculator.

<table>
<thead>
<tr>
<th>(2.4)(3.1) =</th>
<th>(562)(7) =</th>
<th>(0.483)(27) =</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.24)(3.1) =</td>
<td>(562)(0.7) =</td>
<td>(4.83)(27) =</td>
</tr>
<tr>
<td>(0.024)(3.1) =</td>
<td>(56.2)(0.7) =</td>
<td>(48.3)(27) =</td>
</tr>
<tr>
<td>(0.24)(0.31) =</td>
<td>(5.62)(0.7) =</td>
<td>(4.83)(2.7) =</td>
</tr>
<tr>
<td>(24)(0.31) =</td>
<td>(0.562)(7) =</td>
<td>(4.83)(0.27) =</td>
</tr>
</tbody>
</table>

a. What patterns do you notice?
b. How many decimal places does each factor have?
c. How many decimal places are in each product?
d. What is the connection between these two for each product?
PROBLEM 4

Compute the following groups of products.

<table>
<thead>
<tr>
<th>(0.2)(0.35)</th>
<th>(6.2)(4)</th>
<th>(0.48)(0.25)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.35)(0.2)</td>
<td>(4)(6.2)</td>
<td>(0.25)(0.48)</td>
</tr>
</tbody>
</table>

a. What patterns did you notice?
b. State a Commutative property for all numbers.

EXPLORATION 2

Explore the change in the y values in the graph of the line \( y = 8x \) using the table of a graphing calculator with an increment of 1, then 0.5, then 0.25 and finally 0.125. What did you find?

When the increment is 1, the y values change by 8. If the increment is 0.5, the change in y values is 4. There is a 0.25 increment with associated change of 2, and a 0.125 increment with a change of 1.

EXPLORATION 3

a. What is the area of the shaded rectangle with sides of length 0.2 and \( x \)? What is the area of the shaded rectangle with sides of length 0.3 and \( x \)? What is the total area of both rectangles?
b. Compute the following sum: \(0.3x + 0.2x\). Draw a picture to illustrate how this answer makes sense.

In the Exploration above, the area of each shaded rectangle is computed by the formula of length times width. The area of the first rectangle is \((0.2)(x) = 0.2x\) and the area of the second rectangle is \((0.3)(x) = 0.3x\). The total area of the two rectangles is the sum of these two areas: \(0.2x + 0.3x\). Do you see that this is the same as \(0.5x\)?

**PROBLEM 5**

Compute the area of each of the following rectangles. Then, compute the total area of both rectangles.

![](image)

**EXERCISES**

1. Use the area model and the distributive property to compute the following products. Indicate the area of each interior part in your model.
   a. \((3)(5.2)\)
   b. \((4.7)(3.2)\)
   c. \((0.26)(4)\)
   d. \((37)(0.24)\)

2. Predict whether the product is between 0.01 and 0.1, between 0.1 and 1, between 1 and 10 or between 10 and 100. Explain your reasoning.
   a. \((0.73)(0.44)\)
   b. \((0.245)(0.2)\)
   c. \((31.7)(0.77)\)

3. Compute the following:
   a. \((0.2)(3.3)\)
   b. \((1.28)(2.4)\)
   c. \((0.512)(0.36)\)
4. Compute the following sums:
   a. $0.2x + 0.7x$
   b. $0.7a + 0.5a$
   c. $x + 0.3x$
   d. $0.8b - 0.3b$
   e. $3A - 0.6A$
   f. $x - 0.2x$
   g. $y + 0.4y$

5. Draw rectangle $A$ with length 2.5 cm and width 12 cm. Draw rectangle $B$ with length 25 cm and width 1.2 cm. Explain why these rectangles have the same area.

6. Ramses has 38 framed posters he plans to sell for $24.49 each. Estimate the amount of money Ramses will make.

7. Draw a number line from -5 to 5 divided into tenths and use it to find the following products:
   a. $(0.3)(4)$
   b. $(0.3)(-6)$
   c. $(1.2)(3)$
   d. $(-0.4)(3)$
   e. $(-1.2)(4)$
   f. $(3.6)(-7)$

8. Predict whether each of the following products is between 0.1 and 1, between 1 and 10, between 10 and 100 or between 100 and 1,000. Explain how you made your decision.
   a. $(23)(37)$
   b. $(0.012)(9)$
   c. $(13.75)(4)$
   d. $(0.47)(110)$
   e. $(0.023)(5)$
   f. $(0.002)(412)$

9. For each group of products, compute the first product and check your answer with a calculator. Then fill in the other products. Check your answers with a calculator.

<table>
<thead>
<tr>
<th>Product 1</th>
<th>Product 2</th>
<th>Product 3</th>
<th>Product 4</th>
<th>Product 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0.7)(0.9)$</td>
<td>$(0.5)(8)$</td>
<td>$(83)(67)$</td>
<td>$(56)(9)$</td>
<td></td>
</tr>
<tr>
<td>$(0.7)(9)$</td>
<td>$(5)(8)$</td>
<td>$(0.83)(6.7)$</td>
<td>$(56)(0.9)$</td>
<td></td>
</tr>
<tr>
<td>$(7)(0.9)$</td>
<td>$(0.5)(0.8)$</td>
<td>$(8.3)(6.7)$</td>
<td>$(56)(0.09)$</td>
<td></td>
</tr>
<tr>
<td>$(0.7)(0.09)$</td>
<td>$(0.5)(0.08)$</td>
<td>$(83)(0.67)$</td>
<td>$(0.56)(9)$</td>
<td></td>
</tr>
<tr>
<td>$(7)(0.9)$</td>
<td>$(0.05)(8)$</td>
<td>$(0.83)(0.67)$</td>
<td>$(5.6)(0.9)$</td>
<td></td>
</tr>
<tr>
<td>$(70)(0.09)$</td>
<td>$(0.05)(0.08)$</td>
<td>$(0.083)(67)$</td>
<td>$(0.056)(9)$</td>
<td></td>
</tr>
</tbody>
</table>
10. Brandon went to the Farmer’s Market and purchased 3.2 pounds of bananas that cost $0.48 per pound. How much does he pay for these bananas?

11. Donna bought a toy for each of her 17 cats. Each toy costs $2.35. What was the cost of these toys before tax?

12. Juanita averaged 6.5 minutes per mile in a race that covered 8.5 miles. How long did it take her to run this race?

13. Sophia bought 4.75 kilos of beans from the store at the price of $1.29 per kilo. How much did she pay for the beans?

14. Fill in the outputs in the table for the function given by the equation \( y = 3x \).

<table>
<thead>
<tr>
<th>Input</th>
<th>-0.8</th>
<th>-0.6</th>
<th>-0.4</th>
<th>-0.2</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

15. Fill in the outputs in the table for the function given by the equation \( y = 0.4x \).

<table>
<thead>
<tr>
<th>Input</th>
<th>-2</th>
<th>-1.5</th>
<th>-1</th>
<th>-0.5</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

16. Explore the change in the \( y \) values in the graph of the line \( y = 8x \) using the table of a graphing calculator with an increment of 1, then 0.5, then 0.25 and finally 0.125. What did you find?

17. **Investigation:** A bird refuge is in the shape of a rectangle 3.6 miles long and 2.9 miles wide. Draw a visual representation of this refuge on grid paper, using a 0.1 scale, and use it to determine the area of this rectangle. Explain how you use the grid to compute the area. Multiply 3.6 miles by 2.9 miles using the traditional algorithm only after you have an answer using the visual representation.
SECTION 6.3  LONG DIVISION

We have seen how closely related multiplication and division are. For example, we know $8 \div 4 = 2$ because $4 \times 2 = 8$. Also recall that in the long division form, the multiplication fact is rewritten as

The first product is simply 2 jumps of length 3 which yields a product of 6. The second product is 2 jumps of length 0.3, as shown below:

We have the dividend 8 "under" the quotient 2, and the divisor 4 is to the left of the dividend.

By changing the dividend to 9, our problem becomes $4 \overline{)9}$. Because $8 \div 4 = 2$, $9 \div 4$ must be more than 2. In the long division form,

$$
\begin{array}{c}
4 \overline{)9} \\
-8 \\
1
\end{array}
$$

The area model looks like this:

The quotient is 2 and the remainder is 1.

Consider the problem $4 \overline{)80}$, where the dividend is not simply 8 but 8 tens. The quotient is then 2 tens or 20 because $4 \times 20 = 80$. In the long division form,

$$
\begin{array}{c}
4 \overline{)80} \\
-80 \\
0
\end{array}
$$

The area model is:

Use the long division form to evaluate the problem $4 \overline{)800}$ with 8 hundreds. Do you agree the answer is 2 hundreds or 200? In the long division form,

$$
\begin{array}{c}
4 \overline{)800} \\
-200 \\
200
\end{array}
$$

The area model is:
Note that the above picture is not to scale. If you were to draw it to scale and leave the height unchanged, imagine how long the rectangle would be.

A more complex problem is \( 4 \div 84 \). One way to think of this problem is to notice the place values and observe that \( 84=80+4 \). You know that

\[
\begin{align*}
&\frac{20}{4} \div 80 & &\frac{1}{4} \div 4 \\
&4 \div 84
\end{align*}
\]

Putting these together gives

\[
\frac{21}{4} \div 84
\]

Here is the area model for this problem:

![Area Model](image)

or

\[
\begin{align*}
&\frac{1}{4} \div 80 \\
&\frac{20}{4} \div 84 \\
&-80 \\
&4 \\
&-4 \\
&0
\end{align*}
\]

This is called the **scaffolding** method because the different partial quotients are first computed and stacked, then combined, much as a scaffold is used in constructing a building.
EXAMPLE 1

Use scaffolding method to compute the division problem $567 \div 12$.

SOLUTION

There is more than one way to implement the scaffolding method in division. Here is one way:

\[
\begin{array}{c}
2 \\
5 \\
20 \\
20 \\
12) 567 \\
-240 \\
87 \\
-60 \\
27 \\
-24 \\
3 \\
\end{array}
\]

so $20 + 5 + 2 = 27$. Therefore, $567 \div 12 = 27$ with remainder 3.

PROBLEM 1

Use this scaffolding method to compute the following quotients. You may sketch a picture of the corresponding area model if it helps.

a. $380 \div 14$ 
   b. $950 \div 6$

In division, we know that it is more common to start with the largest place value to determine the first digit of the quotient and then gradually include the smaller place values. We will now compute the following division problem using both the traditional method side by side with the scaffolding method. Compare the two approaches.

EXAMPLE 2

Use both the scaffolding method and area model to compute the division problem $552 \div 15$. 

215
Solution

Scaffolding Method:  

Traditional Method:

\[
\begin{array}{cccc}
1 & \hspace{2cm} & \hspace{2cm} & 36 \\
5 & & & 15 \overline{352} \\
10 & & & -45 \\
20 & & & 102 \\
15 \overline{352} & & & \\
-300 & & & -90 \\
252 & & & 12 \\
-150 & & & \\
102 & & & \\
-75 & & & \\
27 & & & \\
-15 & & & \\
12 & & & \\
\end{array}
\]

PROBLEM 2

Use both the scaffolding and traditional models to compute the following division problem:  \(3528 \div 13\)

EXAMPLE 3

Sara spent $3 on 6 Hershey bars. How much did each candy bar cost?

SOLUTION

This is a typical division problem where the cost of each candy bar is \(3 \div 6\). You might expect rouble because the divisor is greater than the dividend. Using the linear skip counting model, how long does each skip need to be to travel a distance of 3 units, or dollars in this case, in 6 skips?
You can see that each jump is $0.50 or half a dollar. This represents the fact that each candy bar costs $0.50. If necessary, verify this using the calculator by computing 3÷6 or adding six skips 0.50 long. The long division method gives us the same result, because there is a decimal point before the 5. You might write the problem like step 1 below. The divisor is greater than the dividend, so modify the long division process by placing a decimal point. Add the two zeros in the dividend because we are working with money and we know that $30=0\$3.00 where $0.00 represents no cents. Where does the decimal place appear in the quotient? Why does this make sense?

**Step 1:**

\[
\begin{array}{c}
6)
\hline
3.00
\end{array}
\]

**Step 2:**

\[
\begin{array}{c}
0.50 \\
6)
\hline
3.00 \\
-3.00 \\
\hline
0.00
\end{array}
\]

**PROBLEM 3**

Compute the following division problems by using an abbreviated number line from 0 to 2, like the one below. Find the quotient using the skip-counting method. Then use the scaffolding method to verify your answer. Make sure the decimal point in the quotient makes sense in the context of the problem. Then use the calculator to confirm your work, if necessary.

a. \(1 \div 5\)  
   b. \(1.60 \div 8\)  
   c. \(1.20 \div 4\)

When long division involves two-digits numbers, the skip-counting model becomes more difficult. We will now use the traditional method (or the scaffolding method) to compute division problems. At this point knowing the multiplication facts and how to use them make life much simpler.
PROBLEM 4

Mr. Garza has some money in his pocket that he intends to divide equally among his four nephews. How much will each nephew receive if he has

a. $26 in his pocket,

b. $27.40 in his pocket

Recalling our earlier problem 12\overline{567} let us consider a related problem.

Draw a representation for this problem to see the long division process. Then use the long division algorithm to see how the quotient is obtained numerically as well.

Find the quotient 12\overline{56.7} and include a visual representation, although you might have to think small. Do the process and your picture correspond?

Notice the magnitude of your quotient changes by a power of 10. What do you think the quotient of 12\overline{0.567} is?

What about 12\overline{5670}? What changes between the two quotients and what remains the same?

EXERCISES

1. Compute the following quotients using scaffolding long division, if necessary. Verify your answer using multiplication. You might also want to check using your calculator, if you are unsure.

<table>
<thead>
<tr>
<th>a. 72 ÷ 8</th>
<th>b. 868 ÷ 14</th>
<th>c. 736 ÷ 23</th>
</tr>
</thead>
<tbody>
<tr>
<td>720 ÷ 8</td>
<td>8680 ÷ 14</td>
<td>7360 ÷ 23</td>
</tr>
<tr>
<td>7.2 ÷ 8</td>
<td>8.68 ÷ 14</td>
<td>7.36 ÷ 23</td>
</tr>
</tbody>
</table>
2. Compute the following quotients using long division and check your answer with multiplication.
   a. \(1 \div 4\)  
      \(2 \div 8\)  
      \(3 \div 12\)  
      \(4 \div 16\)
   b. \(4 \div 10\)  
      \(16 \div 40\)  
      \(20 \div 50\)  
      \(28 \div 70\)

3. Compute the following quotients using the method of long division and check with a calculator, if necessary.
   a. \(1 \div 5\)  
   b. \(2 \div 5\)  
   c. \(3 \div 5\)
   d. Predict what \(4 \div 5\) equals.
   e. Plot and label these quotients on a number line like the one below:

4. Compute the following quotients and check your answer:
   a. \(2 \div 5\)  
   b. \(7 \div 5\)  
   c. \(12 \div 5\)
   d. Predict what \(17 \div 5\) equals.

5. a. Donna paid $60 for five CD’s of equal cost. How much did each CD cost?
   b. Shirley paid $6 for five hot dogs. How much did she pay for each hot dog?
   c. Gary paid $0.60 for five pieces of candy. How much did he pay for each piece?

6. Mindy is shopping for groceries at the store. The Jones Brand rice sells for $2.40 for 40 ounces and the Martel Brand rice sells for $2.50 for 50 ounces. Which brand is a better value and why?
7. A small grocery store sells coffee for $4.00 per pound. As you did in previous sections, let \( x \) be the number of pounds of coffee and \( y \) be the cost of \( x \) pounds of coffee. The equation \( y = 4x \) describes the relationship between pounds of coffee \( x \) and the cost \( y \).

a. Copy and fill the adjacent table for the line given by the equation \( y = 4x \). Plot the points in this table on a coordinate system.

<table>
<thead>
<tr>
<th>Pounds of Coffee ( (x) )</th>
<th>Cost ( (y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>1.5</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>2.5</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>3.5</td>
<td>14</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>4.5</td>
<td>18</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
</tr>
</tbody>
</table>

b. Use the table to determine the costs of 1 lb., 2 lbs., 3 lbs., and 4 lbs. of coffee. What is the cost of 3.5 pounds of coffee?

c. How much coffee can you buy with $32.00?

d. How much coffee can you buy with $9.00?

   Where would this appear in the table?

e. How much coffee can buy with $43.00?

8. Compute the following quotients using division and check the answer with a calculator, if necessary.

a. \( 1 \div 10 \)

b. \( 1 \div 20 \)

c. \( 2 \div 20 \)

d. \( 3 \div 20 \)

e. Use the pattern to predict what \( 4 \div 20 \) is.

f. Plot these quotients on a number line like the one below. Label each point plotted.
9. A farmer sells his pecans for $5 per pound. Assign \( x \) as the number of pecans and \( y \) the price of \( x \) pounds of pecans. The equation \( y = 5x \) describes the relationship between pounds of pecans and the cost in dollars. Enter the equation \( y = 5x \) in the graphing utility.

a. Compute the prices of 2 lb., 4 lb., 6.5 lb. and 7.2 lb. of pecans. Check your answers by using the table function on the calculator or by tracing the graph of the line.

b. How many pounds of pecans can a customer buy with $7.50?

c. How many pounds of pecans can a customer buy with $9.30?

d. How many pounds of pecans can a customer buy with $22.85?

10. Use long division on the following division problems until you see a pattern:

a. \( 1000 \div 8 \)  
   b. \( 100 \div 8 \)  
   c. \( 10 \div 8 \)  
   d. \( 1 \div 8 \)

   Use the number line in Exercise 8, part c, to plot and label the following quotients.

   e. \( 1 \div 8 \)  
   f. \( 2 \div 8 \)  
   g. \( 3 \div 8 \)  
   h. \( 4 \div 8 \)  
   i. \( 5 \div 8 \)

11. Compute the following:

a. \( 2 (1 \div 2) \)  
   d. \( 8 (1 \div 8) \)  
   g. \( (4 \div 5) (5 \div 4) \)

b. \( 4 (1 \div 4) \)  
   e. \( 10 (1 \div 10) \)  
   h. \( (5 \div 8) (8 \div 5) \)

c. \( 5 (1 \div 5) \)  
   f. \( 10 (1 \div 20) \)  
   i. \( (10 \div 4) (4 \div 10) \)

12. **Ingenuity:**

   Consider the following game that is played with two people and a row of 30 toothpicks. The rules are that on each turn a player can take away at most 4 toothpicks, but must take away at least 1. The person who takes away the last toothpick loses.

a. Play a few rounds with a friend.

b. Suppose there are two toothpicks left and it is your turn. How many toothpicks should you take away? What should you do if there are five toothpicks left? If you follow your strategy are you guaranteed a victory?

c. What should you do if there are six toothpicks left? If your opponent knows your strategy from part b can you still win?

d. What should you do if there are 7, 8, 9, or 10 toothpicks left? Who will win?

e. Can you think of a strategy for any number of toothpicks? Are there some numbers where you are guaranteed a victory?
SECTION 6.4 DIVISION OF DECIMALS

In previous sections, we have used visual models of division to reexamine the process of long division. All of the division problems we have examined have had divisors that were integers.

EXPLORATION

A. What is the effect of increasing the dividend by a factor of 10?

Complete the table below by performing the division indicated. For the last two calculations in each column, if you see a pattern, fill in the answer and check with the calculator, if necessary.

<table>
<thead>
<tr>
<th>Calculation</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 ÷ 4$</td>
<td>0.25</td>
</tr>
<tr>
<td>$10 ÷ 4$</td>
<td>2.5</td>
</tr>
<tr>
<td>$100 ÷ 4$</td>
<td>25</td>
</tr>
<tr>
<td>$1000 ÷ 4$</td>
<td>250</td>
</tr>
</tbody>
</table>

What is the pattern?

B. What is the effect of increasing the divisor by a factor of 10 in the following sequence of division problems?

Complete the table below by performing the division indicated. For the last two calculations in each column, if you see a pattern, fill in the answer and check with the calculator, if necessary.

<table>
<thead>
<tr>
<th>Calculation</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 ÷ 1$</td>
<td>2</td>
</tr>
<tr>
<td>$2 ÷ 10$</td>
<td>0.2</td>
</tr>
<tr>
<td>$2 ÷ 100$</td>
<td>0.02</td>
</tr>
<tr>
<td>$2 ÷ 1000$</td>
<td>0.002</td>
</tr>
</tbody>
</table>

What pattern do you notice?
C. What is the effect of decreasing the dividend by a factor of 10?

Complete the table below by performing the division indicated. For the last two calculations in each column, if you see a pattern, fill in the answer and check with the calculator

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$112 \div 7 = 16$</td>
<td>$3276 \div 14 = 234$</td>
</tr>
<tr>
<td>$11.2 \div 7 =$</td>
<td>$327.6 \div 14 =$</td>
</tr>
<tr>
<td>$1.12 \div 7 =$</td>
<td>$32.76 \div 14 =$</td>
</tr>
<tr>
<td>$0.112 \div 7 =$</td>
<td>$3.276 \div 14 =$</td>
</tr>
</tbody>
</table>

What pattern do you notice? How does this pattern compare to the patterns you observed in parts A and B?

D. What is the effect of decreasing the divisor by a factor of 10?

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$117 \div 9 = 13$</td>
<td>$450 \div 18 = 25$</td>
</tr>
<tr>
<td>$117 \div 0.9 =$</td>
<td>$450 \div 1.8 =$</td>
</tr>
<tr>
<td>$117 \div 0.09 =$</td>
<td>$450 \div 0.18 =$</td>
</tr>
<tr>
<td>$117 \div 0.009 =$</td>
<td>$450 \div 0.018 =$</td>
</tr>
<tr>
<td>$117 \div 0.0009 =$</td>
<td>$450 \div 0.0018 =$</td>
</tr>
</tbody>
</table>

The trickiest long division problems are those from part D in which the divisor is a decimal number. It might not be clear where to place the decimal point in the quotient. There is a connection between division by a decimal number and the patterns you observed in parts A, B, C, and D.

From the patterns you have observed, the quotient $0.36 \div 0.2$ is related to the quotient $3.6 \div 2 = 18$. In part B, increasing the divisor by a factor of 10 decreases the quotient by a factor of 10. In part A, increasing the dividend by a factor of 10 increases the quotient by a factor of 10. When both the dividend and the divisor increase by a factor of 10, the quotient remains the same as the original division problem.

223
You have previously learned that a fraction is equivalent to division. In the following tables, we will use the fractional way to represent division to help us see the pattern of where the decimal point is located in the quotient.

<table>
<thead>
<tr>
<th>342</th>
<th>360</th>
<th>5508</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>80</td>
<td>34</td>
</tr>
<tr>
<td>34.2</td>
<td>36</td>
<td>5508</td>
</tr>
<tr>
<td>0.9</td>
<td>0.8</td>
<td>0.08</td>
</tr>
<tr>
<td>34.2</td>
<td>36</td>
<td>5508</td>
</tr>
<tr>
<td>0.09</td>
<td>0.8</td>
<td>0.08</td>
</tr>
<tr>
<td>3420</td>
<td>36</td>
<td>5508</td>
</tr>
<tr>
<td>90</td>
<td>0.36</td>
<td>0.08</td>
</tr>
</tbody>
</table>

For the following division problems, find an appropriate transformed division problem by multiplying both the original dividend and divisor by the same power of 10. Then compute the answer to the transformed problem using division and check you answer for both with a calculator, if necessary.

<table>
<thead>
<tr>
<th>Original division problem with decimal divisor</th>
<th>Transformed division problem with whole number divisor</th>
</tr>
</thead>
<tbody>
<tr>
<td>48 ÷ 0.4 =</td>
<td></td>
</tr>
<tr>
<td>192 ÷ 1.2 =</td>
<td></td>
</tr>
<tr>
<td>0.324 ÷ 3.6 =</td>
<td></td>
</tr>
<tr>
<td>14 ÷ 0.25 =</td>
<td></td>
</tr>
<tr>
<td>4.452 ÷ 0.84 =</td>
<td></td>
</tr>
</tbody>
</table>

You might have noticed that all of the long division problems that we have considered have had a very nice property in common: at the end, every remainder is 0. The quotients terminate or stop. This type of decimal number is called a **terminating decimal**. Of course, in real computations this is seldom the case. Look at the following division problems:

a. 1 ÷ 3  

b. 2 ÷ 3  

c. 5 ÷ 6
Using the long division process,

\[
\begin{array}{ccc}
0.3333\ldots & 0.6666\ldots & 0.8333\ldots \\
3) 1.0000\ldots & 3) 2.0000\ldots & 6) 5.0000\ldots \\
-0 & -0 & -0 \\
1.0 & 2.0 & 5.0 \\
-0.9 & -1.8 & -4.8 \\
0.10 & 0.20 & 0.20 \\
-0.09 & -0.18 & -0.18 \\
0.010 & 0.020 & 0.020 \\
-0.009 & -0.009 & -0.018 \\
0.0010 & 0.0020 & 0.0020 \\
-0.0009 & -0.0018 & -0.0018 \\
0.0001 & 0.0002 & 0.0002 \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\]

As you can see, the long division process in the three problems above never has a remainder of 0. The decimals forms of the quotients of the division problems above are

\[
1 \div 3 = 0.333\ldots = 0.\overline{3} = 0.3 \\
2 \div 3 = 0.666\ldots = 0.\overline{6} = 0.6 \\
1 \div 6 = 0.1666\ldots = 0.1\overline{6} = 0.1\overline{6}
\]

These decimals number are called repeating decimals. When dividing 1 by 2, the decimal form of the quotient is \(1 \div 2 = 0.500\ldots\). However, we do not include the zeros and not call 0.5 a repeating decimal because the last remainder is 0. For what positive integers \(m\) and \(n\) will the quotient \(m \div n\) equal a repeating decimal? You can explore this question in the Repeating Decimal Game.
EXERCISES

1. Complete the table below by performing the division indicated. For the last two calculations in each column, if you see a pattern, fill in the answer and check with the calculator, if necessary.

<table>
<thead>
<tr>
<th>350 ÷ 10 =</th>
<th>160 ÷ 3.2 =</th>
<th>63 ÷ 0.75 =</th>
</tr>
</thead>
<tbody>
<tr>
<td>350 ÷ 100 =</td>
<td>160 ÷ 32 =</td>
<td>63 ÷ 7.5 =</td>
</tr>
<tr>
<td>350 ÷ 1000 =</td>
<td>160 ÷ 320 =</td>
<td>63 ÷ 75 =</td>
</tr>
<tr>
<td>350 ÷ 1000 =</td>
<td>160 ÷ 3200 =</td>
<td>63 ÷ 750 =</td>
</tr>
</tbody>
</table>

2. Compute the following quotients. Explain the results in each set of problems.
   a. 72 ÷ 8
   b. 868 ÷ 14
   c. 736 ÷ 23
   720 ÷ 80
   8680 ÷ 140
   7360 ÷ 230
   7.2 ÷ 0.8
   86.8 ÷ 1.4
   73.6 ÷ 2.3
   0.72 ÷ 0.08
   8.68 ÷ 0.14
   7.36 ÷ 0.23

3. Compute the following quotients. Leave the quotient in decimal form and check your answers with a calculator, if necessary. What do you notice about each set of problems? How can this help you compute the quotients quickly?
   a. 2 ÷ 5
   b. 1 ÷ 3
   4 ÷ 10
   2 ÷ 6
   8 ÷ 20
   3 ÷ 9
   16 ÷ 40
   4 ÷ 12

4. For the following division problems, first find an appropriate transformed division problem. Then, perform division on the transformed problem and check your answer with a calculator.

<table>
<thead>
<tr>
<th>Original division problem with decimal divisor</th>
<th>Transformed division problem with whole number divisor</th>
</tr>
</thead>
<tbody>
<tr>
<td>84 ÷ 2.4 =</td>
<td></td>
</tr>
<tr>
<td>56 ÷ 4.5 =</td>
<td></td>
</tr>
<tr>
<td>37.6 ÷ 0.16 =</td>
<td></td>
</tr>
<tr>
<td>52 ÷ 1.25 =</td>
<td></td>
</tr>
<tr>
<td>1.86 ÷ 0.027 =</td>
<td></td>
</tr>
</tbody>
</table>

5. Compute the following quotients and check with a calculator, if necessary.
   a. 1 ÷ 6
   b. 2 ÷ 6
   c. 3 ÷ 6
   d. Predict what 4 ÷ 6 and 5 ÷ 6 are.
6. Write an equation for each problem and solve it.
   a. Sally bought $64 worth of fabric. Each yard cost $2.50. How many yards of fabric did Sally purchase?
   b. Maria paid $6.40 for some computer cable. The computer cable cost $2.50 per meter. How many meters of cable did she buy?
   c. Gary paid $0.64 for a bag of candy. The candy sold for $2.50 per pound. How many pounds of candy did he buy?

7. A small grocery store sells cheddar cheese for $4.00 per pound.
   a. Let \( x \) be the number of pounds of cheese and \( y \), the cost of \( x \) pounds of cheese. What is the equation for \( y \) that describes the relationship between pounds of cheese and cost?
   b. What is the cost of 1.25 pounds of cheese? Where would you find this cost in a table for the line given by the equation in part a?
   c. Copy and fill the adjacent table. Plot these points on a coordinate system.
   d. If you shop at the store, how much cheese can you buy with $1.50? $5.00? $7.00?
   e. How much cheese can you buy with $8.84?

8. A store sells candy bars for $0.65 each and small bags of chips for $0.49 each. Sandy has $17.00 to spend.
   a. How many candy bars can she buy? What is her change?
   b. How many bags of chips can she buy? What is her change?

9. Solve the following equations:
   a. \( 25x = 1500 \)  
   b. \( 4x = 160 \)  
   c. \( 5x + 600 = 900 \)
   \[ 0.25x = 15 \]  
   \[ 0.4x = 16 \]  
   \[ 0.05x + 6 = 9 \]

10. **Ingenuity:**
    The quotient \( 1 \div 7 \) is the repeating decimal \( 0.142857142857142857 \ldots \) and has a 6-digit cycle. Explain why the quotient \( 1 \div 7 \) could not be a repeating decimal with a cycle longer than 6. If \( n \) is a positive integer, what is the longest possible cycle \( 1 \div n \)?
11. **Investigation:**

In this section, you discovered that the quotient $10 \div 3$ is ten times the quotient $1 \div 3$. If we let $x = 0.33\ldots$, then $10x = 3.33\ldots$. Subtracting equal quantities from both sides,

\[
\begin{align*}
10 x &= 3.333\ldots \\
-x &= 0.333\ldots \\
9 x &= 3.000\ldots
\end{align*}
\]

So $9x = 3$. Remember the missing factor form, which says that $x$ is the quotient $3 \div 9$. When we divide $3$ by $9$, we compute the quotient $3 \div 9 = 0.333\ldots$. Using the same technique, find a division problem that equals the repeating decimal.

a. 0.555\ldots  

b. 0.2525\ldots  

c. 0.1212\ldots  

d. 0.999\ldots
REVIEW PROBLEMS

1. Predict whether the product is between 0.01 and 0.1, between 0.1 and 1, between 1 and 10, between 10 and 100, or between 100 and 1000. Then, compute each and describe the pattern.
   a. \((0.24)(3.8)\)
   b. \((0.024)(3800)\)
   c. \((24)(0.038)\)
   d. \((2.4)(380)\)

2. Jimmy filled his car last Monday. Gas cost $3.95 per gallon and he bought 7.4 gallons. How much did Jimmy pay?

3. Tim is at the fish market. Grouper sells for $5.60 a pound and Tim buys 2.4 pounds of it. How much does Tim pay for the fish?

4. Use the linear model to find the quotient that makes the remainder as small as possible:
   a. \(21 \div 3\)
   b. \(120 \div 11\)
   c. \(143 \div 4\)
   d. \(94 \div 28\)

5. Use the area model to find the quotient that makes the remainder as small as possible:
   a. \(24 \div 6\)
   b. \(71 \div 9\)
   c. \(44 \div 12\)
   d. \(210 \div 24\)

6. Find the quotient and remainder using scaffolding.
   a. \(216 \div 3\)
   b. \(819 \div 32\)
   c. \(39218 \div 73\)

7. Find the quotient using scaffolding.
   a. \(210 \div 4\)
   b. \(989 \div 40\)
   c. \(78821 \div 80\)

8. Find the quotient and remainder using long division.
   a. \(311 \div 3\)
   b. \(2122 \div 13\)
   c. \(34121 \div 317\)

9. Find the quotient using long division.
   a. \(611 \div 5\)
   b. \(3412 \div 12\)
   c. \(12022 \div 48\)
10. Mary has an 11-foot piece of ribbon. She is making a number of hair bows each 14 inches long. How many hair bows can she make?

11. Alex pays $1.35 for a bag of candy. The candy sold for $3.60 a pound. How many pounds of candy did he buy?

12. Solve the following equations:
   a. \(4x = 27\)
   b. \(6x - 7 = 92\)
   c. \(6.25x = 1.275\)
   d. \(3.18x - 1.5 = 6.45\)
   e. \(2.3x = 20.93\)
   f. \(-4x = 19.1\)

13. Transform the division problems into problems with whole number divisors and then compute the quotient.
   a. \(210 \div 4.2\)
   b. \(56 \div 1.5\)
   c. \(37.6 \div .32\)
   d. \(40 \div 1.25\)
   e. \(3.72 \div 0.054\)
   f. \(0.064 \div 5.12\)

14. Determine which of the following quotients is equivalent to the quotient \(42.65 \div 3.84\).
   a. \(4.265 \div 3.84\)
   b. \(4.265 \div .384\)
   c. \(426.5 \div 384\)
   d. \(426.5 \div 38.4\)
Section 6.3:

While performing a trick of long division, a mathemagician made some of his digits disappear. Alas, he cannot reconstruct the missing digits... until he remembers that one of them is a 7. Fill in the blanks to complete the calculation.

```
   4.
_ )  1 3.0

  ==
  - -
   2
  - -
  ==
  0
```

Section 6.4:

If \( \frac{1}{a} = 0.041\overline{d} \) for a positive integer \( a \) and a single digit \( d \), find \( a \) and \( d \).
One of the most important concepts in mathematics is the idea of divisibility. Suppose you have 14 marbles, and you want to give the same number of marbles to each of three friends. Is it possible to give each friend the same number and have none left?

You can let each person have 4 marbles, but there are two left. This process is called division. You divide 14 by 3 to get the quotient 4 with remainder 2. This is equivalent to the calculation $14 = 3 \cdot 4 + 2$. In building the multiplication table for 3, your skip counting by 3, starting at 0, does not list 14.

On the other hand, if you have exactly 12 marbles, you can give each friend 4 marbles, and everybody has an equal number of marbles. This process corresponds to $12 = 3 \cdot 4$.

What does this have to do with divisibility? We know that 14 objects cannot be divided equally among 3 people. Another way to say this is, “14 is not divisible by 3,” or “3 is not a factor of 14.” Note that this is equivalent to “14 is not a multiple of 3.” On the other hand, we can divide 12 things equally among 3 people. Mathematically, “12 is divisible by 3” means “3 is a factor of 12.” The last statement is equivalent to “12 is a multiple of 3.” Although there is no integer that you can multiply by 3 to equal 14, there is the integer 4 that you can multiply by 3 to equal 12.

**DEFINITION 7.1: DIVISIBILITY**

Suppose that $n$ and $d$ are integers, and that $d$ is not 0. The number $n$ is **divisible** by $d$ if there is an integer $q$ such that $n = d \cdot q$. Equivalently, $d$ is a **factor** of $n$ and $n$ is a **multiple** of $d$. 
For example, we know that 12 is divisible by 3 because $12 = 3\cdot 4$. We can show this using our marble example:

\[ \begin{array}{c c c c c}
\odot & \odot & \odot & \odot & \odot \\
\odot & \odot & \odot & \odot & \odot \\
\odot & \odot & \odot & \odot & \odot \\
\end{array} \]

\[ \text{Factor} \cdot \text{Factor} = \text{Product} \]

\[ 3 \cdot 4 = 12 \]

Notice that the twelve marbles are in a neat, rectangular array. This suggests another way for us to see the divisors of a positive integer. We’ll experiment with this in the following activity.

**EXPLORATION 1: THE POSSIBLE RECTANGLE MODEL**

**Materials:** You will need graph paper for this activity.

1. For each positive integer $n$ from 1 to 20, make as many rectangles with integer side lengths as you can that have area equal to $n$ square units.
2. Organize the data in a table provided by your teacher. In the first column, write the positive integer \( n \). In the second column, write the number of rectangles possible with area \( n \). In the third column, list all the possible dimensions of the rectangles. In the fourth column, list all the possible lengths of sides of the rectangles, in increasing order. For example, we have filled in the results for the value of \( n = 4 \) on the table.

<table>
<thead>
<tr>
<th>( n )</th>
<th>Number of Possible Rectangles</th>
<th>Possible Rectangle Dimensions</th>
<th>Possible Side Lengths (in increasing order)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1 \times 4, 2 \times 2</td>
<td>1, 2, 4</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. What do you notice in the table so far?

4. Without drawing rectangles, continue the table for \( n \) from 21 to 40.

5. Looking at the extended table, do the patterns continue?
6. Looking at a given number \( n \), what do you notice about the numbers in the last column for this value of \( n \)?

7. What do we call the numbers in the last column in relation to \( n \)? For each rectangle, the dimensions form a factor pair, such as 3 and 6 for \( n = 18 \). If you put all the factors in the last column in order, such as 1, 2, 3, 4, 6, 12 for \( n = 12 \), how do the factor pairs line up?

8. What do you notice about the number 1? Find any other numbers that have this same property, if possible.

9. Circle the values of \( n \) that generate only one rectangle.
   How many factors does each of these have?
   How would you describe the circled numbers, excluding 1?

10. Use a different color pen or marker to box the values of \( n \) that have an odd number of positive divisors.

Notice that all of the factors in our chart are positive. Generally when talking about factors, we just mean the positive factors.

Let’s explore some strategies for finding all the factors or divisors of a given positive integer \( n \). In particular, if \( n \) is a positive integer and \( k \) is a positive integer, how do we determine whether \( k \) is a factor of \( n \); or equivalently, whether \( n \) is a multiple of \( k \)? The method used in the Possible Rectangle Activity works well for small numbers but does not work as well for larger numbers. So we want to find a method that can be used when we have to deal with large numbers.

**EXAMPLE 1**

Is 13 a factor of the number 798? Equivalently, is 798 a multiple of 13?

**SOLUTION**

Starting with the second question, we could skip count by 13 to determine whether 798 is a multiple of 13. However, this is not an efficient strategy. Instead, ask if there is a positive integer \( q \) such that 798 = 13 \( \cdot q \). You can answer this using long division, which results in a quotient of 61 and a remainder of 5. This means
that $798 = 13 \cdot 61 + 5$. The goal is to find an integer $q$ so that $798 = 13q$. If you skip counted by 13, you would not land on 798. That is, there is no integer $q$ for which $798 = 13q$. So 13 is not a factor of 798.

**PROBLEM 1**

a. Is the number 825 divisible by 15?

b. Is every number divisible by 15 also divisible by 3 and 5? Explain.

What distinction in the remainders did you notice between the examples of the divisible case and the not divisible case? Check with a few other examples to confirm that the distinction holds in those cases.

**EXPLORATION 2: SIEVE OF ERATOSTHENES**

This exploration is based on an ancient method attributed to a famous Greek mathematician, Eratosthenes of Cyrene. The process involves letting a certain kind of number pass through the sieve leaving only another kind of number left in the sieve. Try the exploration and see for yourself.

1. Use the grid of the first 100 natural numbers in the rows of ten handout.

2. Mark out the number 1. We will see why in the next section.

3. Using a colored pencil or marker, circle the number 2 and then mark out every remaining multiple of 2 until you have gone through the whole list. What is a mathematical term for the marked out numbers?

4. From the beginning, with a different colored pencil or marker, circle the first number that is not marked out and not circled. Then mark out all remaining multiples of that number.

5. Repeat this process until you have gone all the way through the list.

6. Make a new ordered list of all the circled numbers. What do these numbers have in common? How is this list of numbers related to patterns from the possible rectangle activity?
7. You might have noticed that in the third round, some of the multiples of 3 were already crossed out in the second round. Find 3 such numbers. Why did this happen?

8. What kind of numbers did you mark out exactly twice? How many factors do these numbers have?

9. What kind of numbers did you mark out once?

10. After what number do we just circle the rest of the numbers on the list?

EXERCISES

1. In each of the following problems, values of \( n \) and \( d \) are given. Determine whether \( d \) is a factor of \( n \). Explain how you use any patterns or prior knowledge that help you answer the questions. Use long division only on the starred items.

<table>
<thead>
<tr>
<th>( d ) (factor)</th>
<th>( n )</th>
<th>Is ( d ) a factor of ( n )? Explain.</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. 5</td>
<td>34</td>
<td></td>
</tr>
<tr>
<td>*b. 19</td>
<td>152</td>
<td></td>
</tr>
<tr>
<td>c. 4</td>
<td>68</td>
<td></td>
</tr>
<tr>
<td>*d. 13</td>
<td>300</td>
<td></td>
</tr>
<tr>
<td>e. 9</td>
<td>81</td>
<td></td>
</tr>
<tr>
<td>f. 4</td>
<td>86</td>
<td></td>
</tr>
<tr>
<td>g. 3</td>
<td>57</td>
<td></td>
</tr>
<tr>
<td>h. 99</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>*i. 17</td>
<td>788</td>
<td></td>
</tr>
<tr>
<td>j. 1</td>
<td>44</td>
<td></td>
</tr>
</tbody>
</table>

2. Find the largest factor of each of the following numbers, excluding the number itself.
   a. \( n = 255 \)
   b. \( n = 134 \)
   c. \( n = 1,001 \)
   d. For a-c, sketch and label the sides of a rectangle in which \( n \) represents area, and the largest factors are the lengths. Using these rectangles, find the corresponding factor to determine the width.
3. a. What is the least multiple of 5 that is greater than 23?  
b. What is the least multiple of 7 that is greater than 59?  
c. A small boat is used to cross a river. The boat can carry 5 passengers. If 23 people want to cross, how many trips does the boat need to take? 5  
d. Buses are used for school field trips. Each bus can fit 22 people. How many buses are needed if 317 students want to go on the field trip? 15  

4. a. What is the least multiple of 22 that is greater than 317?  
b. What is the least multiple of 18 that is greater than 350?  

5. Sally wrote two number patterns, as shown below.  
Set R = {2, 4, 6, 8, 10, ....}  
Set T = {4, 8, 12, 16, 20, ....}  
If these patterns continue, which of the following numbers would appear in both Set R and Set T? Select the best choice and explain your answer.  
a. 46  
b. 30  
c. 52  
d. 70  

6. The numbers in Set R share a common characteristic.  
Set R = {48, 54, 66, 12, 24}  
The numbers in Set S do not share this characteristic with set R.  
Set S = {9, 20, 39, 15, 63, 27, 44}  
Which best describes the characteristic that only the numbers in Set R share?  
Select the best choice and explain your answer.  
a. Numbers less than 70  
b. Numbers greater than 5  
c. Numbers that are composite  
d. Numbers divisible by 6  

7. a. How many positive multiples of 3 are less than or equal to 30? 10  
b. How many positive multiples of 4 are less than or equal to 30? 8  
c. Suppose 0 < x < 30. How many positive multiples of x are less than or equal to 30? 30/x
8. a. Serena has a large number of nickels in her purse. Can she pay with exact change for a pack of gum that costs 65 cents? If so, how many nickels does she need? Explain your answer in terms of factors and/or multiples.

b. Serena has only dimes instead of nickels in her purse. Can she use exact change for the same pack of gum? If so, how many dimes does she need? If not, what is the problem? Explain your answer in terms of factors and/or multiples.

c. Explain your answers to these problems in terms of factors and multiples.

9. Ms. Ellis has 72 feet of red ribbon and 48 feet of green ribbon. She wants to make a wall decoration using pieces of ribbon all cut the same whole number length of both colors. What lengths of ribbon are possible if she does not want any ribbon left over? Are these possible lengths factors or multiples of 72 and 48?

10. A teacher has 32 students in her class. She wants to put the students into groups so that each group has the same number of students. Which of the following does NOT represent the number of students she could put into groups? Select the best choice and explain your answer.

   a. 4  b. 10  c. 8  d. 16

11. A high school newspaper is printing the names of students who are on the math team. The editor of the newspaper wants to divide the names into a certain number of columns so that each column contains the same number of names. There are 60 students on the math team, and the number of columns has to be fewer than 10. What are the different formats, by rows and columns, the newspaper can use?

12. Thomas is baking cookies for a party. Alice, Brad, Carlos and Diana have told Thomas that they definitely plan to attend. Eric has told Thomas that he would like to attend, but he is not sure he can make it. Thomas wants to bake enough cookies so that he and all his guests can have the same number of cookies, whether or not Eric shows up. What is the smallest number of cookies that Thomas can bake for his party?
13. a. Using your data from the Possible Rectangle Exploration, write all the
factors of 30.

b. There is a natural way to pair up the positive factors of 30. Explain how
to do this in terms of the rectangle model. What pairs do you get?

14. Is it possible to pair up the positive factors of 36? Explain.

15. a. Consider the statement "m is a multiple of 4". Write an algebraic
expression for m.

b. Consider the statement "3 is a factor of n". Write an algebraic expression
for n.

c. Suppose 6 is a factor of n. Write an algebraic expression for n. Is 3 a
factor of n? Explain.

16. If \( p \) is a factor of \( q \), and \( q \) is a factor of \( r \), show that \( p \) is a factor of \( r \). Try this
with numbers. Prove it in general.

17. The numbers 1, 2, 4, 5, 7 and 10 are six factors of 140 in numerical order.
How can you use this information to find larger factors of 140? Find all those
factors.

18. **Ingenuity:**

Five brothers want to divide 100 cookies among themselves. They decide to
divide the cookies like this:

a. The oldest brother suggests a way to divide the cookies among the 5
siblings.

b. The 5 brothers vote on the proposal. If 50% or more agree with the
proposal, the cookies will be divided as the oldest brother proposed.

c. If fewer than 50% agree with the proposal, the oldest brother will leave
the game, can get no cookies, and can no longer vote. Then the next
oldest brother will suggest a way to divide the cookies.

d. This process is repeated until a proposal is accepted and the cookies are
divided among the brothers.

Assuming that all the brothers are extremely clever, greedy, and want at least
one of the cookies, what would the oldest brother suggest? The brothers
have no choice in their behavior and must act according to these qualities.
19. **Investigation:**

A band of five pirates finds a chest containing a number of gold coins. The pirates try to divide the stash equally, but one pirate ends up with one coin more than each of the other pirates. The other pirates become jealous, take his coins, and toss him overboard. The remaining four pirates again try to divide the stash equally, but one pirate ends up with three more coins than the other three pirates, so the three take his coins and toss him overboard. The three remaining pirates are able to divide the stash equally. If the pirates have between 40 and 60 coins all together, how many coins do they have?
SECTION 7.2 PRIME AND COMPOSITE NUMBERS

In the Possible Rectangle Model and the Sieve of Eratosthenes Exploration in Section 7.1, you examined factors and multiples. In the Possible Rectangle Model Exploration, you found the positive factors of the integers from 1 to 40. The results are below. You might have noticed that some numbers had many positive factors, while some had only two. The integers with two factors are highlighted in the table.

<table>
<thead>
<tr>
<th>n</th>
<th>Number of rectangles of area n</th>
<th>Dimensions of the rectangles</th>
<th>Side lengths of rectangles</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1 × 1</td>
<td>1</td>
</tr>
<tr>
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<td>1</td>
<td>1 × 2</td>
<td>1, 2</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1 × 3</td>
<td>1, 3</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1 × 4, 2 × 2</td>
<td>1, 2, 4</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1 × 5</td>
<td>1, 5</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>1 × 6, 2 × 3</td>
<td>1, 2, 3, 6</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1 × 7</td>
<td>1, 7</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>1 × 8, 2 × 4</td>
<td>1, 2, 4, 8</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>1 × 9, 3 × 3</td>
<td>1, 3, 9</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>1 × 10, 2 × 5</td>
<td>1, 2, 5, 10</td>
</tr>
<tr>
<td>11</td>
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<td>1 × 11</td>
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<td>12</td>
<td>3</td>
<td>1 × 12, 2 × 6, 3 × 4</td>
<td>1, 2, 3, 4, 6, 12</td>
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<td>2</td>
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<td>1, 3, 5, 15</td>
</tr>
<tr>
<td>16</td>
<td>3</td>
<td>1 × 16, 2 × 8, 4 × 4</td>
<td>1, 2, 4, 8, 16</td>
</tr>
<tr>
<td>17</td>
<td>1</td>
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</tr>
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<td>18</td>
<td>3</td>
<td>1 × 18, 2 × 9, 3 × 6</td>
<td>1, 2, 3, 6, 9, 18</td>
</tr>
<tr>
<td>19</td>
<td>1</td>
<td>1 × 19</td>
<td>1, 19</td>
</tr>
<tr>
<td>20</td>
<td>3</td>
<td>1 × 20, 2 × 10, 4 × 5</td>
<td>1, 2, 4, 5, 10, 20</td>
</tr>
<tr>
<td>21</td>
<td>2</td>
<td>1 × 21, 3 × 7</td>
<td>1, 3, 7, 21</td>
</tr>
<tr>
<td>22</td>
<td>2</td>
<td>1 × 22, 2 × 11</td>
<td>1, 2, 11, 22</td>
</tr>
</tbody>
</table>
Chapter 7  Number Theory

<table>
<thead>
<tr>
<th>n</th>
<th>Number of rectangles of area n</th>
<th>Dimensions of the rectangles</th>
<th>Side lengths of rectangles</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td>1</td>
<td>$1 \times 23$</td>
<td>1, 23</td>
</tr>
<tr>
<td>24</td>
<td>4</td>
<td>$1 \times 24, 2 \times 12, 3 \times 8, 4 \times 6$</td>
<td>1, 2, 3, 4, 6, 8, 12, 24</td>
</tr>
<tr>
<td>25</td>
<td>2</td>
<td>$1 \times 25, 5 \times 5$</td>
<td>1, 5, 25</td>
</tr>
<tr>
<td>26</td>
<td>2</td>
<td>$1 \times 26, 2 \times 13$</td>
<td>1, 2, 13, 26</td>
</tr>
<tr>
<td>27</td>
<td>2</td>
<td>$1 \times 27, 3 \times 9$</td>
<td>1, 3, 9, 27</td>
</tr>
<tr>
<td>28</td>
<td>3</td>
<td>$1 \times 28, 2 \times 14, 4 \times 7$</td>
<td>1, 2, 4, 7, 14, 28</td>
</tr>
<tr>
<td>29</td>
<td>1</td>
<td>$1 \times 29$</td>
<td>1, 29</td>
</tr>
<tr>
<td>30</td>
<td>4</td>
<td>$1 \times 30, 2 \times 15, 3 \times 10, 5 \times 6$</td>
<td>1, 2, 3, 5, 6, 10, 15, 30</td>
</tr>
<tr>
<td>31</td>
<td>1</td>
<td>$1 \times 31$</td>
<td>1, 31</td>
</tr>
<tr>
<td>32</td>
<td>3</td>
<td>$1 \times 32, 2 \times 16, 4 \times 8$</td>
<td>1, 2, 4, 8, 16, 32</td>
</tr>
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<td>33</td>
<td>2</td>
<td>$1 \times 33, 3 \times 11$</td>
<td>1, 3, 11, 33</td>
</tr>
<tr>
<td>34</td>
<td>2</td>
<td>$1 \times 34, 2 \times 17$</td>
<td>1, 2, 17, 34</td>
</tr>
<tr>
<td>35</td>
<td>2</td>
<td>$1 \times 35, 5 \times 7$</td>
<td>1, 5, 7, 35</td>
</tr>
<tr>
<td>36</td>
<td>5</td>
<td>$1 \times 36, 2 \times 18, 3 \times 12, 4 \times 9, 6 \times 6$</td>
<td>1, 2, 3, 4, 6, 9, 12, 18, 36</td>
</tr>
<tr>
<td>37</td>
<td>1</td>
<td>$1 \times 37$</td>
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</tr>
<tr>
<td>38</td>
<td>2</td>
<td>$1 \times 38, 2 \times 19$</td>
<td>1, 2, 19, 38</td>
</tr>
<tr>
<td>39</td>
<td>2</td>
<td>$1 \times 39, 3 \times 13$</td>
<td>1, 3, 13, 39</td>
</tr>
<tr>
<td>40</td>
<td>4</td>
<td>$1 \times 40, 2 \times 20, 4 \times 10, 5 \times 8$</td>
<td>1, 2, 4, 5, 8, 10, 20, 40</td>
</tr>
</tbody>
</table>

The numbers that have only two positive factors play a special role in mathematics and have a special name.

**DEFINITION 7.2: PRIME AND COMPOSITE**

A **prime** number is an integer $p$ greater than 1 with exactly two positive factors: 1 and $p$. A **composite** number is an integer greater than 1 that has more than two positive factors. The number 1 is the **multiplicative identity**; that is, for any number $n$, $n \cdot 1 = n$. The number 1 is neither a prime nor a composite number.
Your work with the Possible Rectangle Table (PR Table) allows you to see the relationships between a given number, \( n \), the number of rectangles possible with \( n \) as its area, and the number of factors \( n \) has. In particular, you can see from your PR Table that the prime numbers between 1 and 40, in increasing order, are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31 and 37. These are the only integers greater than 1 and less than or equal to 40 that had exactly one rectangle possible and hence exactly two factors. Let’s consider a larger number and determine whether it is prime or not.

**PROBLEM 1**

Is 119 prime or composite?

Here is another approach that is not as geometric, but it is systematic.

**EXAMPLE 1**

Is the number 171 prime?

**SOLUTION**

To see if there are any other factors of 171 between 1 and 171, begin to divide 171 by numbers less than 171, in increasing order beginning with 2.

<table>
<thead>
<tr>
<th>Divide by</th>
<th>Quotient</th>
<th>Remainder</th>
<th>Factor?</th>
</tr>
</thead>
<tbody>
<tr>
<td>171 ÷ 2</td>
<td>85</td>
<td>1</td>
<td>No</td>
</tr>
<tr>
<td>171 ÷ 3</td>
<td>57</td>
<td>0</td>
<td>yes</td>
</tr>
</tbody>
</table>

Dividing 171 by 3 gives quotient 57 and remainder 0, showing that 3 is a factor of 171. This is enough information to conclude that 171 is composite because 171 has not just 1 and itself as factors, but it also has 3 and 57 as factors.

You might have noticed that your Investigation in Section 7.1 involving the Sieve of Eratosthenes allowed you to find the prime numbers less than 100 quickly. Can you explain how the process worked? At what point is it possible to know that all the numbers left are prime?
The number 171 does not appear on the Sieve of Eratosthenes chart. Instead of using the possible rectangle model, we approached the problem by finding factors. Why was that smart? We could have shortened our search even more. Because 171 is odd, we know that 2 is not a factor of 171.

**PROBLEM 2**

Is the number 127 prime?

**EXPLORATION 1**

- How do you know that a number is divisible by 2?
- How do you know that a number is divisible by 5?
- How do you know that a number is divisible by 10?
- How do you know that a number is divisible by 3?
- How do you know that a number is divisible by 6?

<table>
<thead>
<tr>
<th>A number is divisible by:</th>
<th>If</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

**EXPLORATION 2**

In small groups or individually, determine whether the following numbers are prime or composite. Try to devise as many time-saving strategies as you can, so you don’t have to check every integer between 1 and the target number.

a. 87  

b. 131  

c. 323
EXERCISES

1. For each of the following numbers, find all of the factors of the number. Circle the prime factors in each. You may want to use the data you collected in Exploration 1 in this chapter or the Sieve of Eratosthenes.
   a. 6   c. 18   e. 27   g. 41   i. 60
   b. 11  d. 20  f. 30  h. 55  j. 65

2. Are there any numbers given in Exercise 1 that are prime? If so, how did you determine they are prime?

3. Valerie used square tiles to make 4 designs, as shown below.

   Design 1
   Design 2
   Design 3
   Design 4

   Which design is composed of a prime number of tiles? Design 2

4. For each of the following numbers, find all the factors of N. List them in order from least to greatest. Draw a line above the list from one factor to its factor pair.
   a. N = 36
   b. N = 48
   c. What is different about the list of factors for 36 and 48?
   d. Find the factors for N = 64. Compare this list to the lists for 48 and 36. What do you notice?

5. S is the set {4, 8, 12, 16, 20, 24, ...}, and T is the set {2, 4, 6, 8, 10, 12, 14, ...}. Remember, the ellipsis .. means that the set continues infinitely in the same way it started. Because of that understanding, what is the next number not shown in S? In T? Draw a Venn Diagram showing the relationship between S and T. Is S a subset of T? Is T a subset of S? Explain.

7. Try to find a prime between the following pairs of numbers. Verify that your choice is a prime.
   a. 100 and 110       b. 110 and 150       c. 150 and 200
   d. 200 and 250       e. 250 and 300       f. 300 and 350

8. Determine whether the following are prime or composite numbers. Explain the strategies used to determine your solution.
   a. 125      b. 321      c. 122      d. 127

9. Factor each of the following numbers using only prime factors.
   a. 30       b. 66       c. 28       d. 26       e. 24

10. In determining the factors of 503, what possible divisors do you need to check to determine whether it is prime or composite?

11. Determine as efficiently as possible whether each of the following numbers is prime or composite.
   a. 151      b. 117      c. 107      d. 121      e. 143      f. 211

12. Write each of the following numbers as products of only prime factors: 10, 100, 1,000, 10,000. Find the prime factors of each. What do you notice? Show that 50 and 25 have the same prime factors as the numbers in the first list. Ignore multiplicity, or repetition of a factor. For example, since 2 is a factor of 24 and $24 = 2 \cdot 2 \cdot 2 \cdot 3$ we say that 2 is a factor of 24 with multiplicity of 3. Using multiplicity, how are the factors of 50 different from the factors of the numbers in the list?

13. Numbers like 15 and 140 are divisible by 5. State a conjecture about what numbers divisible by 5 look like.


15. Investigation:

   The numbers 9, 99, 999 and 9,999 are divisible by 9. One way to see why is to convert the number into a sum in which each term is a multiple of 9. For example, $99 = 90 + 9$ and $9,999 = 9,000 + 900 + 90 + 9$. You can use this pattern to determine whether other numbers are multiples of 9. Explain why each of the numbers below is a multiple of 9 by observing the way in which it is decomposed.
Section 7.2 Prime and Composite Numbers

a. $261 = 2 \cdot 100 + 6 \cdot 10 + 1$
   $= 2(99 + 1) + 6(9+1) + 1$
   $= 2 \cdot 99 + 2 + 6 \cdot 9 + 6 + 1$
   $= 2 \cdot 99 + 6 \cdot 9 + 9$

b. $846 = 8 \cdot 100 + 4 \cdot 10 + 6$
   $= 8(99 + 1) + 4(9+1) + 6$
   $= 8 \cdot 99 + 8 + 4 \cdot 9 + 4 + 6$
   $= 8 \cdot 99 + 4 \cdot 9 + 18$
   $= 8 \cdot 99 + 4 \cdot 9 + 2 \cdot 9$

c. Use the same method to determine whether each of the following numbers is divisible by 9: 342, 83, 954, 276 and 2,574.

d. Make a conjecture for a rule to determine whether or not a number is divisible by 9.

16. Ingenuity:
   a. Using a process similar to that in the investigation, to show why the rule to determine whether or not a number is divisible by 3 works.

17. p and q are primes.
   a. List all of the factors of p. 1 and p
   b. Let $n = pq$. List all of the factors of n. 1, p, q, and pq

18. Copy and complete the following addition table, with each row and each column headed by a prime number. Then list all the integers that appear as sums in your addition table. Make as many observations as possible.

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>11</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
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<td>13</td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>
19. Copy and complete the following multiplication table, in which each row and each column is headed by a prime number. Are there any products in the table that appear only once? Are there any products that appear in the table more than twice?

<table>
<thead>
<tr>
<th>x</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>11</th>
<th>13</th>
</tr>
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<tbody>
<tr>
<td>2</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

20. **Ingenuity:**

In the process of determining whether a number $n$ is prime or composite, we discussed how checking for any prime factors less than $n$ is sufficient. Let's explore if it is sufficient to check a smaller set of numbers. For example, consider the number 103 and explore whether it is necessary to check the entire list of numbers less than 103 to determine if it is prime or whether it is sufficient to check a smaller set of numbers. Explain your conclusion. Consider what you might be able to conclude about checking whether any number $n$ is or is not prime. What set of numbers must one check to determine whether $n$ is prime?

21. **Investigation:**

a. The number 152 is the sum of 120 and 32. Both 120 and 32 are multiples of 4. Is 152 a multiple of 4?

b. The number 231 is the sum of 140 and 91. Both 140 and 91 are multiples of 7. Is 231 a multiple of 7?

c. Make a conjecture based on the pattern in these two examples. Show why your conjecture always works.
SECTION 7.3  EXPONENTS AND ORDER OF OPERATIONS

In Section 4.1, we modeled multiplication by repeatedly adding an integer to itself. There are also situations in which it is useful to multiply a number repeatedly by itself.

EXAMPLE 1

Escherichia coli bacteria are more commonly known as E. coli. One of the bacteria lives in a petri dish. The number of bacteria in the dish doubles each hour. How many bacteria are living in the dish after 1 hour? 2 hours? 3 hours? 5 hours? n hours?

SOLUTION

Because the number of bacteria doubles each hour, after 1 hour there will be 2 bacteria. After 2 hours there will be $2 \cdot 2 = 4$ bacteria. We can write this information using exponential notation as

1st hour = $2 = 2^1$
2nd hour = $2 \cdot 2 = 2^2 = 4$
3rd hour = $2 \cdot 2 \cdot 2 = 2^3 = 8$

Continuing this pattern, 5th hour = $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32$ bacteria are living in the dish after 5 hours. If we let $n =$ number of hours, there are $2^n$ bacteria after $n$ hours.

DEFINITION 7.3: EXPONENTS AND POWERS

Suppose that $n$ is a whole number. Then, for any number $x$, the $n^{th}$ power of $x$, or $x$ to the $n^{th}$ power, is the product of $n$ factors of the number $x$. This number is usually written $x^n$. The number $x$ is usually called the base of the expression $x^n$, and $n$ is called the exponent.
Continuing Example 1, use a calculator to answer the following questions:
1. How many bacteria will there be after 10 hours?
2. We need at least 10,000 bacteria for an experiment. When can we harvest this many bacteria?

EXPLORATION 1

By using the definition of exponential notation and multiplication, we see that:
\[ 3^4 \cdot 3^6 = (3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) = 3^{10} = 3^{4+6}. \]

Compute the following products, showing all your work.

a. \[ 3^2 \cdot 3^3 \]

b. \[ 2^2 \cdot 2^3 \]

c. \[ 3^3 \cdot 3^2 \]

d. \[ 10^3 \cdot 10^5 \]

Does this calculation agree with your rule? Use this pattern to check your answers in Exploration 1.

Do you see a pattern for multiplying numbers in exponential form with the same base? Use your observation to determine the product in the problem below.

PROBLEM 1

Compute the product: \[ 2^5 \cdot 2^4. \]

The pattern leads to the multiplication property for exponents.

<table>
<thead>
<tr>
<th>PROPERTY 7.1: MULTIPLICATION OF POWERS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Suppose that ( x ) is a number and ( a ) and ( b ) are whole numbers. Then</td>
</tr>
<tr>
<td>( x^a \cdot x^b = x^{a+b} )</td>
</tr>
</tbody>
</table>

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Section 7.3 Exponents and Order of Operations

PROBLEM 2

Use this property to rewrite the following products and then compute them. Use a calculator to compute both equivalent forms.

a. \(4^3 \cdot 4^5\) \(= 65536\)  
   c. \(10^3 \cdot 10^4\) \(= 10000000\)

b. \(2^5 \cdot 2^5\) \(= 1024\)  
   d. \(3^2 \cdot 4^3\) \(= 576\)

Special Cases: What do \(4^1\) and \(4^0\) equal?

We note that \(4 \cdot 4 = 4^2 = 4^{1+1} = 4^1 \cdot 4^1\), so \(4^1\) must be the same as 4. We can use the same process for any number \(x\): \(x \cdot x = x^2 = x^{1+1} = x^1 \cdot x^1\), so \(x^1 = x\).

What does \(4^0\) equal? Because \(4 \cdot 4^0 = 4^1 \cdot 4^0 = 4^{1+0} = 4^1 = 4 = 4 \cdot 1\), we see that multiplying by \(4^0\) is the same as multiplying by the number 1. We therefore assume that for any positive integer \(n\), \(n^0 = 1\).

Place Value

Consider the number 4,638. Using place value and our notation of exponents, we can rewrite 4,638 in the following way:

\[4 \cdot 10^3 + 6 \cdot 10^2 + 3 \cdot 10^1 + 8 \cdot 10^0\]

Start with the expression \(4 \cdot 10^3 + 6 \cdot 10^2 + 3 \cdot 10^1 + 8 \cdot 10^0\) or in calculator notation, \(4 \times 10^3 + 6 \times 10^2 + 3 \times 10^1 + 8 \times 10^0\). In what order can we perform the calculations in this expression so the sum equals 4,638?

PROBLEM 3

Compute the following:

a. \(7 \cdot 10^2 + 3 \cdot 10^1 + 2 \cdot 10^0\) \(= 732\)

b. \(6 \cdot 10^3 + 5 \cdot 10^2 + 9 \cdot 10^1 + 3 \cdot 10^0\) \(= 6,593\)

c. \(6 \cdot 10^2 + 9 \cdot 10^1 + 3 \cdot 10^0\) \(= 6,093\)
ORDER OF OPERATIONS

To compute $2^4$ on a calculator, we enter $2^4$. To multiply 3 by 5, we enter $3 \times 5$. Suppose we enter the following into a calculator:

a. $1 + 2^3$

b. $3 + 2 \times 5$

c. $2 \times 4^3$

What will the results be? Can you explain what the calculator is doing? You might wonder why the calculator does not perform these calculations from left to right, as we read them. We can see that the order the calculator uses, which is called the order of operations, is natural by examining our place value system.

Remember from Section 4.6, we have the order in which mathematical operations are performed, as shown below.

Order of Operations

- Compute the numbers inside the parentheses
- Compute any exponential expressions
- **Multiply** and **divide** as they occur from left to right
- **Add** and **subtract** as they occur from left to right

There are different forms of grouping symbols: parentheses, ( ), brackets, [ ], and braces, { }. Absolute value symbols are treated as a type of grouping symbol.

Why do these two problems have different solutions?

a. $7 \cdot 8 - 6 \div 2$

b. $7 \cdot (8 - 6) \div 2$

EXPLORATION 2

Compute the following, showing all your work.

$20 - 10 \div 2 + 3^3 - 9$

There are different forms of grouping symbols: parentheses, ( ), brackets, [ ], and braces, { }. Absolute value symbols are treated as a type of grouping symbol.
PROBLEM 4

Compute the following:

a. \( 6 \div 2(4^2 + 7) \)  
   c. \( 4 + 2^3 \times 3 - (17 - 5) \times 3 + (17 - 5) \div 2 \)

b. \( 4 \cdot |7 - 3| \div 2 \)  
   d. \( 10 + (5 - 2^3) \cdot |-9 + 8| \)

EXERCISES

1. Expand and compute the answer of the following. Decide which is greater or if the two numbers are equal.

   a. \( 5^3 \) or \( 3^5 \)  
   c. \( 5^4 \) or \( 10^3 \)

   b. \( 9^4 \) or \( 3^6 \)  
   d. \( 5^3 \) or \( 2^7 \)

2. Evaluate the following expressions:

   a. \( 2^3 + 2 \)  
   c. \( 2^3 \times 5^3 \)

   b. \( 3^2 + 4^2 \)  
   d. \( 2^3 \times 2 \)

3. Evaluate the following expressions:

   a. \( 2^3 \times 4^2 \) and \( 2^7 \)  \( 128 \)

   b. \( 3^3 \times 3^2 \) and \( 3^5 \)  \( 243 \)

   c. \( 2^3 \times 3^2 \)  \( 72 \)

   d. \( 2^3 \times 3^3 \) and \( 6^3 \)  \( 216 \)

4. Evaluate the following numerical expressions using Order of Operations.

   a. \( 17 - 2^3 + 6 \)  
   d. \( 4 - 8 + 5 \times 3 \)

   b. \( 17 - (2^3 + 6) \)  
   e. \( 7 + 5 \times 3 - 16 \times 3^2 \)

   c. \( 4 - (8 + 5) \)  
   f. \( 28(5 + 3)^2 - 3(2^4) \)
5. Rhonda has one sheet of paper. She cuts it into thirds and stacks the three sheets. If she completes this process a total of 5 times, how many sheets thick will the resulting stack be? By the way, she only has to complete the process 27 times before the stack reaches the moon.

6. Calculate the following:
   a. \(10^1\)
   b. \(10^2\)
   c. \(10^3\)
   d. \(10^8\)
   e. Explain how you would calculate \(10^{200}\)

7. Evaluate the following numerical expressions using Order of Operations.
   a. \(3 \cdot 10^2 - 204 + (8 \cdot 10^0)\)
   b. \(8,034 + (4 \cdot 10^2) \cdot 10 - 10^2 \div 10\)
   c. \(3 \cdot 10^2 + 7 \cdot 10 + 8 \cdot 10^0\)
   d. \(4 \cdot 10^3 + 5 \cdot 10^2 + 9 \cdot 10^1 + 2 \cdot 10^0\)
   e. \(4 \cdot 10^1 + 9 \cdot 10^0 + 2 \cdot 10^0\)

8. What is the value of the expression below? Select the best choice and explain your answer.
   \[5 + 5(9 \div 3)^2\]
   a. 35  b. 90  c. 50  d. 230

9. Six students decide to have a candy eating contest. The first student eats 1 piece of candy. Each student must then eat 5 times as many pieces of candy as the previous student. How many pieces of candy must the sixth student eat?

10. Evaluate the following numerical expressions using Order of Operations.
    a. \(2 \times 8 + 48 \div 3 - 4^2 \times 2\)
    b. \(4^3 + 100 - 4(7 - 4)^3 \div 18\)
11. Compute the following:
   a. \(4 - (8 + 5)\)  
   f. \(6 \cdot 10^2 + 4 \cdot 10^1 + 2 \cdot 1\)  
   b. \(4 - 8 + 5 \cdot 3\)  
   g. \(4 \cdot 10^3 + 8 \cdot 10^2 + 3 \cdot 10^1 + 7 \cdot 10^0\)  
   c. \(8(8 + 3) \div 2^2\)  
   h. \(5 \cdot 10^1 + 9 \cdot 10^0 + 4 \cdot 10^0\)  
   d. \(5^2 + 6(-10 + 4)\)  
   e. \(4 + |9 - 1| \div 2^2 + 3\) 

12. What is the value of the expression below? Select the best choice and explain your answer.
   \(10 + 7 \cdot 8^2 \div 2\)
   a. 61  
   b. 234  
   c. 544  
   d. 66

13. Compute the following:
   a. \((3 + 4)^3\)  
   d. \((3 \cdot 4)^3\)  
   b. \((3 + 5)^3\)  
   e. \((2 \cdot 5)^3\)  
   c. \((2 + 3)^4\)  
   f. \((2 \cdot 3)^4\)

14. Part I: A mistake was made in simplifying the expression below.
   Simplify: \(3 + 4(5 + 2) - 2^3\)
   Step 1: \(3 + 4(7) - 2^3\)
   Step 2: \(7(7) - 2^3\)
   Step 3: \(49 - 2^3\)
   Step 4: \(49 - 8\)
   Step 5: 41
   In which step did the first mistake occur?
   a. Step 1  
   b. Step 2  
   c. Step 3  
   d. Step 4

Part II: What was the mistake made?
15. Look back at Example 1. If the number of bacteria initially is $P$, how many bacteria will there be after 2 hours? After 3 hours? After $n$ hours?

16. **Ingenuity:**

If $2^6$ cookies are divided evenly between two people, how many will each person receive? Explain how you reached your answer.

17. **Investigation:**

We can get $x^{10}$ from $x$ in 5 multiplications of previously generated powers, e.g. $x \rightarrow x^2 \rightarrow x^4 \rightarrow x^8 \rightarrow x^{16} \rightarrow x^{10}$. What is the minimum number of such multiplications required to get $x^{100}$ from $x$?
SECTION 7.4 SQUARE NUMBERS AND SQUARE ROOTS

SQUARE NUMBERS AND SQUARE ROOTS

Identify the dimensions of the squares below and their areas.

You found that the:

1x1 square has area $1^2 = 1$ square unit
2x2 square has area $2^2 = 4$ square units
3x3 square has area $3^2 = 9$ square units
4x4 square has area $4^2 = 16$ square units
5x5 square has area $5^2 = 25$ square units

Square numbers or perfect squares are whole numbers, n, that can be written as the square of a whole number, b: That is $n = b^2$.

9 is a perfect square because $9 = 3^2$. Identify other perfect squares less than 9. How many perfect squares are there less than or equal to 100? Identify and locate these perfect squares on the number line.

Create a table similar to the one below, and identify the perfect squares associated to squares of dimensions 1x1 to 25x25.

<table>
<thead>
<tr>
<th>Area</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1x1</td>
<td></td>
</tr>
<tr>
<td>2x2</td>
<td></td>
</tr>
<tr>
<td>3x3</td>
<td></td>
</tr>
<tr>
<td>4x4</td>
<td></td>
</tr>
<tr>
<td>5x5</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
</tr>
<tr>
<td>25x25</td>
<td></td>
</tr>
</tbody>
</table>
Just as we say the numbers 1, 4, 9, 16 etc. are perfect squares, 1, 2, 3, 4, are their corresponding square roots. We use the radical notation \( \sqrt{n} = b \), read square root of \( n \) is \( b \), to mean that \( b \) when squared equals \( n \).

For example, \( \sqrt{16} = 4 \) because \( 4^2 = 16 \). What is \( \sqrt{25} \)?

It would be correct that \( \sqrt{25} = 5 \) because \( 5 \cdot 5 = 25 \). Do you see a relationship between 5 and 25? Between 4 and 16? Find all the square roots of numbers less than or equal to 100 that are whole numbers. How does your table above relate perfect squares and square roots? Explain.

What patterns do you observe about the square numbers, the square roots of numbers, and square numbers and their corresponding square roots? (Odd square numbers have odd square roots, even square numbers have even square roots; difference between two consecutive square numbers is an odd number).

**PROBLEM 1**

Draw a square that illustrates each of the following and explain:

a. A side with a length of \( \sqrt{144} \) units (\( \sqrt{144} = 12 \), therefore draw a square with dimensions 12x12).

b. A side with a length of \( \sqrt{625} \) units (\( \sqrt{625} = 25 \), therefore draw a square with dimensions 25x25).

c. An area of 289 square units (\( \sqrt{289} = 17 \), therefore draw a square with dimensions 17x17).

You identified perfect squares such as 1, 4, 9, 16, 25, 36, 49, 64, 81, 100. In addition, you identified square roots such as \( \sqrt{1} \), \( \sqrt{4} \), \( \sqrt{9} \), \( \sqrt{16} \), \( \sqrt{25} \), \( \sqrt{36} \), \( \sqrt{49} \), \( \sqrt{64} \), \( \sqrt{81} \), \( \sqrt{100} \) and saw a relationship between the perfect squares and their square roots. Notice that these are very special whole numbers. As an extension to this topic, you may wish to consider what it would mean to talk about \( \sqrt{2} \) or \( \sqrt{3} \).

**SCIENTIFIC NOTATION**

In studying the real world, we often have to use large numbers. For example, the earth’s circumference is 40,000,000 meters. There are even larger numbers that play an important role in understanding the world we live in. The earth’s
mass is approximately 5,973,600,000,000,000,000,000 metric tons. We can use exponents to help us write and compare such large numbers. For example, we can write the following numbers in equivalent ways:

\[ 3,500 = 3.5 \times 1000 = 3.5 \times 10^3 \]
\[ 35,000 = 3.5 \times 10,000 = 3.5 \times 10^4 \]
\[ 35,000,000 = 3.5 \times 10^7 \]

We call writing numbers in this form scientific notation. What does the exponent in each of these represent? How do you determine the exponent for each of these numbers? Notice that the first part of a number written in scientific notation is always greater than or equal to 1 and less than 10. The advantage of converting a number in standard notation to scientific notation is apparent in converting the earth’s mass from standard notation to scientific notation: \( 5.9736 \times 10^{21} \) metric tons.

**PROBLEM 2**

Write the distances from each of the following planets to the sun using scientific notation.

<table>
<thead>
<tr>
<th>Planet</th>
<th>Distance in kilometers</th>
<th>Distance from sun in kilometers using scientific notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>57,900,000 km.</td>
<td>5.79 \times 10^7</td>
</tr>
<tr>
<td>Venus</td>
<td>108,200,000 km.</td>
<td>1.082 \times 10^8</td>
</tr>
<tr>
<td>Earth</td>
<td>149,600,000 km.</td>
<td>1.496 \times 10^8</td>
</tr>
<tr>
<td>Mars</td>
<td>227,900,000 km.</td>
<td>2.279 \times 10^8</td>
</tr>
<tr>
<td>Jupiter</td>
<td>778,300,000 km.</td>
<td>7.783 \times 10^8</td>
</tr>
<tr>
<td>Saturn</td>
<td>1,427,000,000 km.</td>
<td>1.427 \times 10^9</td>
</tr>
<tr>
<td>Uranus</td>
<td>2,871,000,000 km.</td>
<td>2.871 \times 10^9</td>
</tr>
<tr>
<td>Neptune</td>
<td>4,497,100,000 km.</td>
<td>4.4971 \times 10^9</td>
</tr>
</tbody>
</table>

In some situations, we need to work with very small numbers. The pattern of converting a number such as 0.00035 into scientific notation is to count the number of decimal places that you need to shift in the number 0.00035 to get to the form 3.5. So,

\[ 0.00035 = \frac{0.00035 \times 10^4}{10^4} = \frac{35}{10^4} = 3.5 \times 10^{-4} \]
An application of this type of conversion is in converting the width of an electron into scientific notation. The width of an electron (one of the particles that form an atom) is .0000000000000028 meters. So,

\[
0.0000000000000028 = \frac{0.0000000000000028 \times 10^{15}}{10^{15}} = \frac{2.8}{10^{15}} = 2.8 \times 10^{-15}
\]

**PROBLEM 3**

Fill in the chart by converting the numbers to scientific notation or standard notation.

<table>
<thead>
<tr>
<th>Standard Notation</th>
<th>Scientific Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000378</td>
<td></td>
</tr>
<tr>
<td>0.00000000024</td>
<td></td>
</tr>
<tr>
<td>4.973 \times 10^{-5}</td>
<td></td>
</tr>
<tr>
<td>9.831 \times 10^{-3}</td>
<td></td>
</tr>
</tbody>
</table>

**EXERCISES**

1. Draw a square with side lengths given below. Write the dimensions as whole numbers and also indicate the area of each square.
   a. \( \sqrt{169} \)  
   b. \( \sqrt{49} \)  
   c. \( \sqrt{400} \)

2. Which of the numbers below are perfect squares? Explain your reasoning.
   a. 100  
   b. 200  
   c. 300  
   d. 400

3. What is another way of writing \( \sqrt{0} \)? Explain.  
   0, because \( 0^2 = 0 \)

4. Draw a model for the following squares with:
   a. area 1 square unit  
   b. area 441 square units
c. side length \( \sqrt{144} \) units
d. area \( x^2 \) square units, for a positive number \( x \).
e. area \( 9x^2 \) square units, for a positive number \( x \).
f. side length \( 2x \) units, for a positive number \( x \). What is the area?

5. The model below is a square with an area of 144 square units.

Which of these equations can be used to determine \( s \), the side length of this model in units?

a. \( s = \sqrt{12} \)  
b. \( s = 144 \)  
c. \( s = 12^2 \)  
d. \( s = \sqrt{144} \)

6. A concert needs to set up 625 chairs on the floor level. If the chairs are placed in a square arrangement, how many should be in each row? 25

7. If the area of a square is 289 square inches. What is the side length of the square? 17

8. Estimate the following to the nearest tenth. Check your solution with a calculator.

a. \( \sqrt{23} \)  
b. \( \sqrt{66} \)  
c. \( \sqrt{103} \)  
d. \( \sqrt{376} \)

9. \( p \) is a prime number.

a. List all of the factors of \( p^2 \). 1, \( p \), \( p^2 \)

b. List all of the factors of \( p^3 \). 1, \( p \), \( p^2 \), \( p^3 \)

c. What is the square root of \( p^6 \)? \( \sqrt{p^6} = p^3 \)
10. Write the following number using scientific notation:
   a. 45,700
   b. 507,300,000,000
   c. 289,000,000,000,000,000
   d. 0.00036
   e. 0.285
   f. 0.0000000000000000003

11. Convert the following numbers in scientific notation to standard notation:
   a. 6.437 \times 10^8
   b. 4.7 \times 10^{12}
   c. 6 \times 10^{14}
   d. 5.3 \times 10^{-4}
   e. 7.9 \times 10^{-6}
   f. 8.3 \times 10^{-12}

12. Ingenuity:
   Consider the following sequence: 1, 3, 7, 15, ...
   a. what are the next three terms? 1, 3, 7, 15, ___, ___, ___
   b. What is a formula for the nth term? a_n = _________
SECTION 7.5 UNIQUE PRIME FACTORIZATION

One reason we are so interested in prime numbers is that they are the building blocks of the integers. In the previous section, we learned that a prime number is a positive integer greater than 1 that can be written as a product of two positive integers in only one way. For example,

\[ 13 = 13 \cdot 1 = 1 \cdot 13. \]

We cannot write 13 as a product of two positive integers without using the number 13 itself. In this way the number 13 cannot be divided into smaller equal whole parts. We can, however, use the number 13, together with other prime numbers, to form many other numbers:

\[ 13 \cdot 2 = 26 \]
\[ 13 \cdot 3 = 39 \]
\[ 13 \cdot 5 = 65 \]
\[ 13 \cdot 7 = 91 \]

Primes are combined in various ways to form different positive integers. In some cases, you might use a certain prime factor more than once when building a number:

\[ 4 = 2 \cdot 2 = 2^2 \]
\[ 8 = 2 \cdot 2 \cdot 2 = 2^3 \]
\[ 12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3 \]
\[ 36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2 \]

In a similar way, every positive integer greater than 1 that is not prime can be written as the product of prime factors. In other words, each positive integer can be identified by its prime factors and the number of times each of these factors occurs. For example, \( n \) is a positive integer that is composed of 3 factors of 2, 1 factor of 3 and 2 factors of 5. What is the exact value of \( n \)? Does it matter if \( n \) is \( 2 \cdot 3 \cdot 5 \cdot 2 \cdot 5 \cdot 2 \) or \( 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 \)? Is there an accepted way to organize these factors as a product of \( n \)?

We can answer these questions with the following:
Chapter 7  Number Theory

THEOREM 7.1: FUNDAMENTAL THEOREM OF ARITHMETIC

If \( n \) is a positive integer, \( n > 1 \), then \( n \) is either prime or can be written as a product of primes

\[ n = p_1 \cdot p_2 \cdot \cdots \cdot p_k \]

for some prime numbers \( p_1, p_2, \ldots, p_k \) such that \( p_1 \leq p_2 \leq \cdots \leq p_k \).

In fact, there is only one way to write \( n \) in this form.

The Fundamental Theorem of Arithmetic, or FTA, tells us that there is only one way to decompose a given integer into prime factors with the factors written in order. That is, if two different people correctly express a positive integer as a product of prime factors, their products always contain exactly the same prime factors, whether the order is the same or not. Using the FTA, however, the prime factors should be in increasing order.

The prime factorization of an integer gives us useful information about its factors. Factoring a number into primes usually takes some trial and error, but there is a technique that makes the process easier. Let’s look at an example:

EXAMPLE 1

Find the prime factors of 60.

SOLUTION

When looking for prime factors of a positive integer, it is useful to have a list of prime numbers. Look at the Sieve of Eratosthenes from the activity in Section 7.1 to confirm that the first few primes are

\[ 2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots \]

Work your way through these primes to see if any of them are factors of 60. Divide 60 by 2, to get a quotient of 30 and a remainder of 0. So 2 is a factor of 60, with

\[ 60 = 2 \cdot 30. \]
It is tempting to go on to the next prime in our list, but remember that a prime might appear more than once in a prime factorization. So before you continue, notice that 30 is even and realize that 2 is a factor of 30. Dividing, you will find that $30 = 2 \cdot 15$. So

$$60 = 2 \cdot 2 \cdot 15.$$ 

Using the multiplication facts, divide 15 by 3 to find that $15 = 3 \cdot 5$. So

$$60 = 2 \cdot 2 \cdot 3 \cdot 5.$$ 

Because each of the factors 2, 2, 3, and 5 is prime, you are through. You have written 60 as a product of prime factors. A useful way to write your result is to use exponents:

$$60 = 2^2 \cdot 3 \cdot 5.$$ 

The process of finding the prime factors of a number is called **prime factorization**. We also use the same term to describe the result of the process. For example, the prime factorization of 60 is $2^2 \cdot 3 \cdot 5$.

**EXAMPLE 2**

Find the prime factors of 672.

**SOLUTION**

If you want, use the same step-by-step process you used in the previous example. But there is a faster way to write the information. You can track the prime factors using a **tree diagram**. You know that 2 is a prime factor of 672 because 672 is even, so start by dividing 672 by 2:
Continue factoring by 2 until the remaining number is odd:

![Factor Tree](image)

Remember that $7 \cdot 3 = 21$, so you know that the last 2 factors in the process are 7 and 3. The prime factorization of 672 is $2^5 \cdot 3 \cdot 7$.

Notice that the processes used in Examples 1 and 2 are related. The second process is the same as the first, except that it organizes the factoring process more visually using a tree diagram, which can be helpful when working with larger numbers. Another useful tool in factoring is the Divisibility Table you came up with in Section 7.2, as shown below.

<table>
<thead>
<tr>
<th>A number is divisible by:</th>
<th>If</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>The last digit is 0, 2, 4, 6, or 8.</td>
</tr>
<tr>
<td>3</td>
<td>The sum of the digits is divisible by 3.</td>
</tr>
<tr>
<td>5</td>
<td>The last digit is 0 or 5.</td>
</tr>
<tr>
<td>6</td>
<td>The number is divisible by 2 or 3</td>
</tr>
<tr>
<td>9</td>
<td>The sum of the digits is divisible by 9</td>
</tr>
<tr>
<td>10</td>
<td>The last digit is 0.</td>
</tr>
</tbody>
</table>
EXERCISES

1. Factor each of the following integers into primes. Write the integer as a product of its prime factors, using exponents when there are repeated prime factors.
   a. 6   c. 16   e. 36   g. 897   i. 1,225   k. 6,292
   b. 9   d. 23   f. 252   h. 819   j. 3,195   l. 6,300

2. Factor each of the following numbers into its prime factorization.
   a. 10   b. 100   c. 1,000
   What pattern do you notice?
   Use this pattern to factor the following numbers:
   a. 10,000   b. 100,000

3. Factor each of the following numbers into its prime factorization:
   a. 21   b. 210   c. 2,100
   What pattern do you notice?
   Use this pattern to factor the following numbers:
   a. 21,000   b. 210,000

4. Which of the following is the prime factorization of 350? Select the best choice and explain your answer.
   a. \(2^2 \cdot 5 \cdot 7\)   b. \(2^2 \cdot 5 \cdot 35\)   c. \(2 \cdot 5 \cdot 7\)   d. \(2^2 \cdot 5 \cdot 7\)

5. Which of the following is the prime factorization of 440? Select the best choice and explain your answer.
   a. \(2^2 \cdot 5 \cdot 11\)   b. \(2^2 \cdot 5 \cdot 11\)   c. \(2^2 \cdot 5 \cdot 11\)   d. \(2^2 \cdot 5 \cdot 55\)

6. The prime factorization of a number can be used to find a perfect square, which is the square of an integer. Look at the numbers given in Exercise 1. Are any of these perfect squares? If so, which ones? How can you use their prime factorization to determine whether each is a perfect square?

7. A perfect cube is an integer \(n\) that can be written in the form \(n = k^3\), where \(k\) is an integer. Some examples of perfect cubes are
   \[0^3 = 0, \ 1^3 = 1, \ 2^3 = 8, \ 3^3 = 27, \ 4^3 = 64, \ \ldots\]
   How can you use the prime factors of a number to determine whether it is a perfect cube?
8. What is the smallest positive integer that has four different prime factors?

9. Write the prime factorization for each of 1,224 and 1,225. What do you notice about their prime factorizations?

10. 30 = 2 · 3 · 5
   a. List all the factor pairs of 30.
   b. Find the prime factorization of each of the factors of each factor pair. For example, 24 = 2 · 12 = 2 · (2^2 · 3). What do you notice?

11. Use the prime factorization, 40 = 2^3 · 5, to find the number of rectangles you can make with integer side lengths and area equal to 30 square units.

12. Use the prime factorization, 2 · 2 · 3 · 5 = 60, to find the number of rectangles you can make with integer side lengths and area equal to 60 square units.

13. p and q are primes and n = p^3q
   a. List all the factor pairs of n. (hint: look at exercise above)
   b. How many factors does n have?

14. Find the prime factorization for all whole numbers from 140 to 150. Write the prime factorization in exponential notation. Explain any patterns you may have noticed.

15. Find the prime factorization for the integers 333, 444, 555, and 888. Write the prime factorization in exponential notation. Explain any patterns you may have noticed.

16. Ingenuity:
   In general, if p and q are primes, how many factors are there for p^4? For q^3? For p^4 · q^3? What about p^m · q^n?
REVIEW PROBLEMS

1. Determine whether $g$ is a factor of $h$.

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<tbody>
<tr>
<td>$g$</td>
<td>$h$</td>
<td>Explain</td>
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<td>8</td>
<td>112</td>
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<tr>
<td>3</td>
<td>123</td>
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</table>

2. What is the greatest multiple of 7 that is less than 80?

3. If there are 24 students in your class, what are the different equal size groups that can be arranged? Explain.

4. What are the possible dimensions of all the possible rectangles with area 72?

5. Determine and explain whether the following numbers are prime or composite.
   a. 47  b. 21  c. 35  d. 71  e. 111

6. For the following numbers, find all the prime factors of each number.
   a. 100  b. 22  c. 49  d. 60  e. 125

7. Determine the prime factorization of each of the following integers, using exponents when necessary. Indicate if and why any of the integers are perfect squares.
   a. 20  b. 36  c. 50  d. 144  e. 504

8. Write the following using exponents.
   a. $2 \cdot 2 \cdot 2 \cdot 2$  d. $2 \cdot 7 \cdot 7 \cdot 7 \cdot 7$  
   b. $2 \cdot 2 \cdot 2 \cdot 3$  e. $x \cdot x \cdot x \cdot y \cdot y \cdot y$  
   c. $3 \cdot 3 \cdot 5 \cdot 5 \cdot 5$

9. Write the following in prime factorizations without exponents:
   a. $2^3 \cdot 3^2$  b. $7^2 \cdot 11^3$  c. $2^2 \cdot 5 \cdot 7^3$  
   d. $a^4 \cdot b^2$
10. Compare the following numbers using $>$, $<$ or $=$:
   a. $9^2$   $4^3$
   b. $2^3$   $3^3$
   c. $11^2$   $5^3$
   d. $8^2$   $4^3$

11. Compute:
   a. $12^2$
   b. $4^4$
   c. $4^3$
   d. $5^5$

12. Compute:
   a. $7 + 6 \cdot 82$
   b. $7 \cdot 10^3 + 6 \cdot 10^2 + 2 \cdot 10 + 3$
   c. $10 + 6 \cdot 82 - 12$
   d. $5 - 4^3 \div (17 - 9) \cdot 2$
   e. $-7 + 6 \cdot 8^2$
   f. $4 \cdot 10^4 + 8 \cdot 10^2 + 7 \cdot 10$
   g. $9 \cdot 10^3 + 4 \cdot 10^2 + 8$

13. Write the following numbers using scientific notation:
   a. 145,000
   b. 34,000,000,000
   c. .0034
   d. .00000527

14. Write the following numbers using standard notation:
   a. $2.5 \cdot 10^7$
   b. $8.367 \cdot 10^4$
   c. $5.421 \cdot 10^{-5}$
   d. $4 \cdot 10^{-6}$
15. Write an expression using exponents that could be used to find the number of small squares.

16. Evaluate:
   a. $\sqrt{289}$   b. $\sqrt{625}$   c. $\sqrt{196}$   d. $\sqrt{441}$
Chapter 7  Number Theory

CHALLENGE PROBLEMS

Section 7.1:
A man has three children, whose ages he can’t remember. He remembers that the product of their ages is 168, and he remembers the sum of their ages, but he still can’t figure out how old they are. What are all possible ages of his oldest child?

Section 7.2:
Find the smallest prime number p such that \( x^2 + y^2 + 1 = p \) where x and y are both multiples of 8.

Section 7.3:
We can get \( x^{10} \) from x in 5 multiplications of previously generated powers, e.g. . What is the minimum number of such multiplications required to get \( x^{100} \) from x?

Section 7.4:
Find the smallest positive integer whose prime factorization uses each odd digit exactly once.
SECTION 8.1  GCF AND EQUIVALENT FRACTIONS

We know that when we multiply 3 and 8 to obtain the product 24, the numbers 3 and 8 are factors of 24. Notice that 2 · 12 also equals 24, so 2 and 12 are factors of 24. You have also discovered several other numbers that are factors of 24. In this section, we examine how we can use what we know about factors to determine factors common to two or more numbers. When is it necessary to find common factors? Let’s examine the following situation to see.

EXPLORATION 1

To prepare for a frog-jumping contest, Fernando decided to train a group of his fellow frogs. Each frog was trained to jump a certain length along a number line starting at 0. He trained a 1-frog to jump a distance of 1 unit in each hop. He also trained a 2-frog to jump 2 units, a 3-frog to jump 3 units and so on. The frogs always start at the zero point on the number line. Now Fernando wants to know which frogs will land on certain locations on the number line.

1. Which of his frogs will land on both the locations 24 and 36?

2. Which is the longest jumping frog that will land on both 24 and 36? Explain why this answer makes sense.

3. What is the longest jumping frog that will land on both 20 and 32?

4. What is the longest jumping frog that will land on both 24 and 25?

If a frog lands on 24, then the length of its jump is a factor of 24. So, if a frog lands on both 24 and 36, then the length of its jump is a factor of 24 and 36. Another way to ask the question in part 2 above is: “What is the greatest number that is a factor of both 24 and 36?”
DEFINITION 8.1: COMMON FACTOR AND GCF

Suppose $m$ and $n$ are positive integers. An integer $d$ is a common factor of $m$ and $n$ if $d$ is a factor of both $m$ and $n$. The greatest common factor, or GCF, of $m$ and $n$ is the greatest positive integer that is a factor of both $m$ and $n$. We write the GCF of $m$ and $n$ as GCF($m$, $n$).

You saw that in question 2, the GCF of 24 and 36 is 12. From question 3, the GCF of 20 and 32 is 4, and the GCF of 24 and 25 is 1.

There are several different ways to calculate the GCF of two numbers. Here is one way that reinforces the term Greatest Common Factor.

**EXAMPLE 1**

Find the GCF of 30 and 36.

**SOLUTION**

Find the GCF of two numbers by first listing all the factors of each of the numbers in a chart. Find all the common factors of the two numbers, then choose the greatest.

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<tr>
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<th>30</th>
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<th>36</th>
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<tbody>
<tr>
<td>1</td>
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<td>2</td>
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<td>2</td>
<td>18</td>
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<td>6</td>
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</tbody>
</table>

So the common factors of 30 and 36 are 1, 2, 3, and 6, and the GCF of 30 and 36 is 6 because it is the largest of the four common factors.
EXAMPLE 2

Find the GCF of 27 and 32.

SOLUTION

Use the same method you used in the previous example. First, list the factors of each number.

<table>
<thead>
<tr>
<th></th>
<th>27</th>
<th></th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>27</td>
<td>1</td>
<td>32</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>2</td>
<td>16</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
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</table>

In this case, there is only one common factor: 1. Therefore, the GCF of 27 and 32, written also as GCF(27, 32), is 1. There is a term to describe a relationship between numbers whose GCF is 1.

**DEFINITION 8.2: RELATIVELY PRIME**

Two integers \( m \) and \( n \) are **relatively prime** if the GCF of \( m \) and \( n \) is 1.

Based on the definition above, the numbers 27 and 32 are relatively prime. Notice that neither 27 nor 32 are prime numbers. If we consider two prime numbers like 3 and 7, what is their GCF? Check a few more examples. Make a generalization about the GCF of any two prime numbers.

Find the GCF of a pair of larger numbers like 108 and 168 using the process of first finding factors, then the common factors, and eventually the greatest common factor.

First, list all the factors of 108 in order. Then, list all the factors of 168 in order. There are many factors to find. If you do not have 12 factors for 108 and 16 factors for 168, go back and find them all.
Determine all the factors the two numbers have in common. You should find that the common factors are 1, 2, 3, 4, 6, 12. From this list, you can see that the greatest common factor of both 108 and 168 is 12. As you discovered, this method for finding the GCF works well. However, the more factors the numbers have, the more time it takes to make the list of factors for each number. Fortunately, prime factorization makes finding the GCF of two numbers easier.

Here is an example of the efficiency of prime factorization.

**EXAMPLE 3**

Find the GCF of 108 and 168 using prime factorization.

**SOLUTION**

Start by finding the prime factors of 108 and 168.

Use factor tree diagrams to do this:
Recall that the prime numbers are the building blocks of the integers. If you want to find the GCF of two integers, find the building blocks, or prime factors, the two numbers have in common. Remember, when doing this, it is helpful to write out the prime factors of each number in an organized way using exponents.

\[ 108 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = 2^2 \cdot 3^3 \]
\[ 168 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 = 2^3 \cdot 3 \cdot 7 \]

You have written the prime factors of 108 and 168 with and without exponents. It is clear which factors the two numbers have in common:

\[ 108 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = 2^2 \cdot 3^3 \]
\[ 168 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 = 2^3 \cdot 3 \cdot 7 \]

The prime factors have two 2’s and one 3 in common. So \(2 \cdot 2 \cdot 3 = 12\) is the greatest common factor of 108 and 168. It is also possible to check the common factors using long division. If there were a larger common factor, the quotients would have even more common prime factors, or higher powers of common prime factors, or both. However, the quotient after dividing 108 by 12 is 9 and the quotient after dividing 168 by 12 is 14. It is easy to check that 9 and 14 have no common factor other than 1, and therefore are relatively prime.

**PROBLEM 1**

Use the prime decomposition (factorization) to find the GCF for each of the following pairs of integers:

a. 72 and 84  

b. 48 and 58  

c. \(x^2y^3\) and \(x^3yz\)

A visual representation called a Venn Diagram might help you to see how the GCF of numbers like 108 and 168 is constructed from the prime factorization. To make the Venn Diagram, draw a circle for each number you are considering. Inside each circle write all its prime factors. If there are common prime factors, then the circles will intersect and the common prime factors will be in the area common to both, or in the intersection of the circles. If the circles have no common prime factors,
then you conclude that the only common factor is 1. Remember 1 is always a common factor for any pair of integers, and it is the greatest common factor if there are no other common factors.

The Venn Diagram will look like this.

\[
\begin{array}{c}
\text{3} \\
\text{2} \\
\text{7}
\end{array}
\quad
\begin{array}{c}
\text{3} \\
\text{2} \\
\text{7}
\end{array}
\]

\[
108 \quad \text{168}
\]

GCF = \(2 \times 3 \times 1 = 12\)

**PROBLEM 2**

Compute the GCF of 196 and 210. Use a Venn Diagram and the method from Example 3, and decide which you prefer.

**EXAMPLE 4**

Compute the GCF of 8, 12, and 18.

**SOLUTION**

We shall solve this problem using the prime factorization method. The factor trees for these 3 numbers are shown below.

Thus, it is clear which factors the three numbers have in common:
Section 8.1  GCF and Equivalent Fractions

\[
\begin{align*}
8 &= 2 \cdot 2 = 2^3 \\
12 &= 2 \cdot 2 \cdot 3 = 2^2 \cdot 3 \\
18 &= 2 \cdot 3 \cdot 3 = 2 \cdot 3^2
\end{align*}
\]

Therefore, the greatest common factor of 8, 12, and 18 is 2.

Now that we know how to find the GCF, let us look at how to use it. T-charts, prime factorization, and Venn Diagrams will be used later to simplify fractions.

Review of Fractions

We use two numbers in writing a fraction, the **numerator** and the **denominator**. The numerator is above the denominator. The denominator tells into how many parts the whole is divided. "Denominator" comes from the same root as "name." It names the parts into which the whole is divided, like halves or fourths. "Numerator" comes from the same root as "number." It counts the number of parts.

ACTIVITY: FOLDING PAPER

**Materials:** You will need several sheets of paper for this activity. Each sheet represents one whole.

**Step 1:** Fold the paper to represent the number \( \frac{1}{2} \). Write \( \frac{1}{2} \) on each of the two parts of the folded paper. Is there more than one way to represent \( \frac{1}{2} \)?

**Step 2:** Use a new sheet to create \( \frac{1}{4} \). How many parts equal to \( \frac{1}{4} \) are there in the whole sheet of paper? What fraction represents three of these parts? What represents two parts of the paper?

**Step 3:** Use a new sheet of paper to make a folded piece that has eight equal parts. Identify and make a list of as many fractions involving the denominator 8 as you can. Which of these fractions represent the same fractional part as the fractions in Step 1 and 2 with different denominators?

**Step 4:** Fold this same sheet of paper once more to make sixteenths. How many times did you fold the paper?
Use the area model to draw the fractions below. Draw the whole as a rectangle because it is easier to divide into equal pieces. For each fraction, identify the numerator and denominator. Then write the fraction mathematically.

a. Three-fourths  

b. Two-fifths  

c. One-fifth  

d. Three-tenths

It is possible for two fractions with different numerators and denominators to represent the same amount. Consider the following example:

\[
\frac{1}{2} = \frac{2}{4} = 
\]

If we divide a whole into 4 equal parts, 2 of the 4 parts can be written as the fraction

\[
\frac{2}{4} = 
\]

Are these both a representation of \(\frac{2}{4}\)? They do not look the same. So what is the difference?

Shading 2 equal parts out of 4 is equivalent to shading 1 part out of 2. This means the fractions \(\frac{1}{2}\) and \(\frac{2}{4}\) are equivalent. Another way to show that the fraction \(\frac{1}{2}\) is equivalent to the fraction \(\frac{2}{4}\) is to take the picture representing \(\frac{1}{2}\) and draw a horizontal slice as shown below:

\[
\frac{1}{2} = 
\]

The horizontal slice doubles the numerator and also doubles the number of parts into which the whole is divided, the denominator.

Suppose we make three horizontal cuts in the original rectangular model for \(\frac{1}{2}\) to form equal sized pieces. What fraction is shaded?
Like the example above, the picture represents both \( \frac{1}{2} \) and \( \frac{4}{8} \). These fractions are equivalent. We write \( \frac{1}{2} = \frac{2}{4} = \frac{4}{8} \) because the fractions represent the same part of a whole.

Remember, the “whole” is not just a geometric shape that represents one whole, like a circle or a rectangle, subdivided into equal parts. For example, the whole might be a class that has 8 girls and 8 boys. What fraction of the class is female? Male?

**EXAMPLE 5**

A class of 12 consists of 4 boys and 8 girls. We know that the fraction of the class that is male can be written \( \frac{4}{12} \), and the fraction of the class that is female can be written \( \frac{8}{12} \). Write another way to express the fraction of the class that is boys as well as the fraction of the class that is girls, using equivalent fractions.

**SOLUTION**

If you said \( \frac{1}{3} \) as another way to express the fraction of the class that represents the boys, then you were correct. In the original fraction, the denominator 12 represented the number of people in the class, and the numerator 4 represented the number of boys in the class.

Remember, two fractions that represent the same part of a whole are called **equivalent fractions**.

In general, we can find equivalent fractions by multiplying the numerator and denominator by the same number. For example,

\[
\frac{1}{4} = \frac{(2)(1)}{(2)(4)} = \frac{(1)(2)}{(4)(2)} = \frac{2}{8}.
\]
Chapter 8 Adding and Subtracting Fractions

Pictorially,

\[
\frac{1}{4} = \text{Diagram}
\]

Multiplying the numerator and denominator by 2 has the effect of doubling the number of slices:

\[
\frac{1}{4} \times \frac{2}{2} = \frac{2}{8} = \text{Diagram}
\]

Multiplying the numerator and denominator by the same number changes the number of shaded parts and the total number of parts by the same factor, yielding an equivalent fraction.

PROPERTY 8.1: EQUIVALENT FRACTION PROPERTY

For any number \(a\) and nonzero numbers \(k\) and \(b\):

\[
\frac{a}{b} = \frac{k \cdot a}{k \cdot b} = \frac{a \cdot k}{b \cdot k} = \frac{ak}{bk}
\]

We have generated equivalent fractions using the area model by dividing a given representation into smaller equal pieces, as seen in the diagram, by converting 1 part out of 4 parts into 2 parts out of 8 parts. Notice that the new denominator was always a multiple of the original denominator. But many times we will want to find an equivalent fraction with a smaller denominator, if possible. We call this process simplifying a fraction. We will do this by using Property 8.1 in reverse or by using the GCF. For example, to simplify the fraction \(\frac{6}{10}\), we first recognize that \(\frac{6}{10} = \frac{2(3)}{2(5)}\). The numerator 6 and the denominator 10 have a greatest common factor, 2. Dividing both the numerator and the denominator by the common factor of 2 produces an equivalent fraction. Using the Equivalent Fractions Property, we see that \(\frac{6}{10}\) is equivalent to \(\frac{3}{5}\). So, we have simplified \(\frac{6}{10}\) to the form \(\frac{3}{5}\). A fraction is in simplest form if the numerator and denominator have no common factors except 1.
EXERCISES

1. Find the GCF of each pair of integers using one of the following strategies: T-chart, prime factorization, or Venn Diagram.
   a. 12 and 10   f. 44 and 84
   b. 12 and 15   g. 45 and 81
   c. 15 and 22   h. 65 and 90
   d. 80 and 81   i. 66 and 90
   e. 24 and 36   j. 120 and 195

2. John has 12 jars of strawberry jam, 16 jars of grape jam, and 24 jars of pineapple jam. He wants to place the jars into the greatest possible number of boxes so that each box has the same number of jars of each kind of jam. How many boxes does he need, if every jar of jam is used?

3. Sarah is making candy bags for her birthday party. She has 24 lollipops, 12 candy bars, and 42 pieces of gum. She wants each bag to have the same number of each kind of candy. What is the greatest number of bags she can make if all the candy is used? How many pieces of each kind of candy will be in each bag?

4. Mrs. Blackburn wrote the following riddle on the board for her mathematics class. We are 2-digit numbers. Our greatest common factor is 14. Our difference is 42. Our sum is 98. What are the 2 numbers of the riddle?
   a. 14 and 42 because their greatest common factor is 14.
   b. 28 and 70 because their difference is 42, their greatest common factor is 14, and their sum is 98.
   c. 14 and 56 because their difference is 42, and their greatest common factor is 14.
   d. 42 and 84 because their difference is 42.

5. Which of the following is the greatest common factor of 9, 27, and 36? Select the best choice and explain your answer.
   a. 3       b. 24       c. 18       d. 9
6. Which of the following is the greatest common factor of 12, 18, and 36? Select the best choice and explain your answer.

   a. 6       b. 1       c. 12       d. 24

7. Draw a $5 \times 7$ grid. Is it possible to tile this grid with squares larger than $1 \times 1$? What about tiling a $9 \times 15$ grid with larger square tiles? How does this exercise involve common divisors? Without drawing the grid, determine the dimensions of the largest size square that tiles a $24 \times 36$ grid.

8. What is the GCF of two prime numbers $p$ and $q$? Explain your reasoning.

9. Find three equivalent fractions for each of the fractions below. You may use paper folding or any other model to determine the equivalent fractions.

   a. $\frac{8}{10}$   b. $\frac{1}{7}$   c. $\frac{3}{6}$   d. $\frac{4}{8}$   e. $\frac{12}{15}$   f. $\frac{1}{x}$

10. For each of the following fractions, find a common factor in the numerator and denominator. Then, simplify the fraction.

    a. $\frac{24}{30}$       c. $\frac{24}{36}$       e. $\frac{25}{35}$

    b. $\frac{14}{21}$       d. $\frac{32}{48}$       f. $\frac{51}{72}$

11. Use common factors in both numerator and denominator to rewrite each of the following fractions in simplest form. If the fraction is already in its simplest form, explain why it is.

    a. $\frac{108}{160}$       b. $\frac{27}{32}$       c. $\frac{200}{625}$       d. $\frac{340}{663}$       e. $\frac{380}{885}$       f. $\frac{210}{221}$

    g. $\frac{59}{83}$       h. $\frac{124}{125}$       i. $\frac{p^2q}{pq^3}$       j. $\frac{xy^2}{wx^2y^2}$
12. What fraction is represented by the shaded portion of each figure below?

a. 

b. 

c. 

d. 

e. Simplify each of your fractions in a – d, if possible.

13. Determine the fraction that represents each of the labeled regions assuming the large square represents the whole or 1.

14. Jeremy practiced juggling for 40 minutes and Amy practiced for 45 minutes. For what fraction of an hour did each practice? Simplify each fraction.

15. A class has 12 boys and 18 girls. What fraction of the class is male? What fraction of the class is female? Simplify each of these fractions.

16. If the time is 2:50 P.M., what time will it be in half an hour? What time will it be in \( \frac{1}{4} \) of an hour?

17. Find an equivalent fraction for \( \frac{8}{12} \) that has a larger denominator. Find 2 equivalent fractions for \( \frac{8}{12} \) that have smaller denominators.
18. Determine whether each of the following fractions has an equivalent fraction with a denominator that is less than the denominator in the given fraction. That is, determine whether you can simplify the fractions. If you can simplify, do so.

   a. \( \frac{15}{18} \)    
   b. \( \frac{24}{53} \)    
   c. \( \frac{12}{25} \)    
   d. \( \frac{18}{23} \)    

   e. What properties of the numerator or denominator helped you to answer parts a–d?

19. Ingenuity:
A positive integer \( d \) has the following properties:
   - When 100 is divided by \( d \), the remainder is 2.
   - When 145 is divided by \( d \), the remainder is 5.

   a. What are all the possible values of \( d \)?
   b. What is the greatest possible value of \( d \)?

20. Investigation:
Using grid paper, draw a large \( 1 \times 1 \) square. Shade an area of the square that is the same shape and \( \frac{1}{4} \) of the original area. What are the dimensions of the smaller square? Repeat this process for several other rectangles, each time shading a smaller rectangle that is the same shape and \( \frac{1}{4} \) of the original rectangle’s area. What do you notice?
SECTION 8.2 UNIT FRACTIONS AND MIXED NUMBERS

In Chapter 4, we developed a model for multiplication based on repeated addition. Apply this model to fractions like $\frac{1}{3}$, which are called unit fractions. Using the frog model, if the length of each jump is $\frac{1}{3}$ and the frog takes 3 jumps, it will land on 1. In other words, $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{1}{3} \cdot 3 = 1$.

If the length of each jump is $\frac{1}{4}$ and the frog takes 4 jumps, it will land on 1. We write this as $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{1}{4} \cdot 4 = 1$. In a similar way, we can show that $\frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} \cdot 5 = 1$, and $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{8} \cdot 8 = 1$.

DEFINITION 8.3: MULTIPLICATIVE INVERSE (RECIPROCAL)
The number $x$ is called the multiplicative inverse or reciprocal of the positive integer $n$ if $x \cdot n = 1$.

For example, you say $\frac{1}{7}$ is the multiplicative inverse of 7 because $\frac{1}{7} \cdot 7 = 1$ and 6 is the multiplicative inverse of $\frac{1}{6}$ because $6 \cdot \frac{1}{6} = 1$.

THEOREM 8.1: UNIT FRACTION
For any positive integer $n$, the multiplicative inverse or reciprocal of $n$ is the unit fraction $\frac{1}{n}$.

A unit fraction always has 1 in the numerator. The denominator is a positive integer. For example, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and so on are all examples of unit fractions.
Chapter 8  Adding and Subtracting Fractions

Now extend the addition of unit fractions to make another connection to multiplication. You have seen several models that represent the fraction $\frac{3}{5}$. In the area model, $\frac{3}{5}$ represents three $\frac{1}{5}$‘s of a whole. This means $\frac{3}{5}$ is the sum $\frac{1}{5} + \frac{1}{5} + \frac{1}{5}$. In the frog model, this is the same as taking 3 jumps of length $\frac{1}{5}$. That is, $\frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{1}{5} \cdot 3 = \frac{3}{5}$. This understanding can be extended to all fractions. For example, the fraction $\frac{5}{9}$ is the same as the sum of 5 copies of $\frac{1}{9}$:

$$\frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{9} \cdot 5 = \frac{5}{9}.$$ 

Write the sum of 8 copies of $\frac{1}{5}$. Using the frog model, what is the result of 8 jumps of length $\frac{1}{5}$ each?

Using math with recipes can be fun not only for learning but for eating, too! Usually recipes involve quantities of ingredients and cooking directions. Here is a chocolate chip cookie recipe:

- 2 $\frac{1}{4}$ cups flour
- $\frac{3}{4}$ cup sugar
- $\frac{3}{4}$ cup brown sugar
- 12 oz chocolate chips
- 1 tsp vanilla
- 1 tsp salt
- 1 tsp baking soda

This makes approximately 6 dozen cookies.

Each of the sugar quantities is $\frac{3}{4}$ of a cup. This quantity is called a proper fraction because it is less than 1.

Notice that one easy way to check if a fraction is proper is to make sure the numerator is less than the denominator. A fraction that is not proper is called improper. An improper fraction is a fraction that is greater than or equal to 1. For a fraction $\frac{a}{b}$ to be improper, the numerator must be greater than the denominator, or $a > b$.

Look at the first ingredient in our recipe. Did you think “two and one-fourth cups?” Remember, “and” means adding two cups to one-fourth of a cup. Quantities like $2 \frac{1}{4}$ are called mixed numbers because they consist of an integer like 2, in addition to a fractional part that is less than a whole, like $\frac{1}{4}$. It is customary
to write the fractional part in simplified form. The mixed number $2 \frac{1}{4}$ is actually the sum $2 + \frac{1}{4}$. The rest of the recipe contains both fractional parts of cups or teaspoons and numbers of ounces and dozens.

Look at the mixed number $2 \frac{1}{4}$. If you have only a quarter-cup measure, describe how you can measure the correct amount with the quarter cup.

Did you find $2 \frac{1}{4}$ equivalent to $\frac{9}{4}$? In fact, what you have found are two ways to write the same quantity: as a mixed number, $\frac{1}{4}$, and as an improper fraction, $\frac{9}{4}$. How would you describe improper fractions? Why do you think they are called improper?

**Problem 1**

State the difference between proper and improper fractions. What is the advantage of using an improper fraction or using a mixed number?

If $a$ is positive ($a > 0$) what is the value of $ax$?

There are three possibilities:

1. $x < 1$. For example, $x = \frac{1}{3}$. Multiplying $\frac{1}{3} \cdot a$ is the same as taking $\frac{1}{3}$ of $a$, which is less than $a$.
2. $x = 1$. Then $1 \cdot a = a$.
3. $x > 1$. For example $x = 3$. Multiplying $3 \cdot a$ is greater than $a$.

If $x \cdot a > a$, what is the value of $x$? There are 3 possibilities:

$x > 1$, $x = 1$, or $x < 1$.

The value $x$ cannot be 1 or $x$ less than 1 because in both cases, $x \cdot a$ is not greater than $a$.

So if $x \cdot a > a$, then $x > 1$.

Similarly, if $x \cdot a = a$, then $x = 1$, and if $x \cdot a < a$, then $x < 1$. 

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A ruler is a type of number line people use daily. Determine where $2\frac{1}{4}$ inches is located on a ruler. Use the ruler to explain why $\frac{9}{4}$ is an equivalent fraction to $2\frac{1}{4}$. What does the 4 represent in the fraction?

A recipe for pancakes calls for $1\frac{3}{4}$ cups of flour. Locate this point on the number line. Describe the equivalent, improper form for the mixed number and use the number line to explain the value in the numerator.

Jack has three identical pans of brownies and decides to divide each pan into 12 equal pieces. How many brownie pieces does he have in all? Because Jack was very hungry, he ate 2 of the pieces. If you assume each brownie pan represents 1 or a whole, express the amount of brownies that remain in terms of the whole and pieces.

If he takes half of the uneaten brownies to a party, what quantity will he take? Using the area model, draw the brownie quantities he will take and leave. Be sure to include the fact that each pan is divided into 12 pieces.

PROBLEM 2
Write each of the following mixed numbers as improper fractions:

a. $1\frac{2}{5}$  
b. $3\frac{3}{4}$  
c. $2\frac{1}{3}$

PROBLEM 3
Write each of the following mixed numbers as improper fractions:

a. $\frac{5}{2}$  
b. $\frac{7}{4}$  
c. $\frac{11}{3}$
Section 8.2 Unit Fractions and Mixed Numbers

EXERCISES

1. Verify that the multiplicative inverse rule is true by substituting the equivalent decimal for each of the unit fractions below to perform the multiplication indicated. For example, \( \frac{1}{2} \cdot 2 = (0.5) 2 = 1.0 = 1 \).
   
a. \( \frac{1}{5} \cdot 5 = 1 \)  
b. \( \frac{1}{8} \cdot 8 = 1 \)  
c. \( \left( \frac{1}{25} \right) 25 = 1 \)  
d. \( \left( \frac{1}{40} \right) 40 = 1 \)

2. A type of computer chip is \( \frac{5}{32} \) inch long. In assembling the motherboard of a new computer, a technician lines up 4 of the chips.
   
a. How long is this row of chips with no gap between them?
   
b. If this part of the motherboard has only 1 inch of space, how many chips can fit in one row?
   
c. How much space is left?

3. Rewrite these sums as an improper fraction and as a mixed number:
   
a. \( \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \)
   
b. \( \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \)
   
c. \( \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \)
   
d. \( \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \)

4. Convert each of these improper fractions to a mixed number. Sketch a number line from 0 to 5 with fourths marked and locate each mixed number.
   
   \( \frac{7}{4}, \frac{19}{4}, \frac{10}{4}, \frac{14}{4}, \frac{17}{4}, \frac{11}{4}, \frac{12}{4} \)

5. Write the improper fraction \( \frac{12}{5} \) as a mixed number. Explain your process

6. Convert each of these improper fractions to a mixed number.

   Then order the fractions from least to greatest.
   
   \( \frac{5}{3}, \frac{8}{3}, \frac{11}{3}, \frac{17}{5}, \frac{23}{7}, \frac{14}{8}, \frac{18}{4}, \frac{28}{8}, \frac{36}{10}, \frac{30}{20}, \frac{24}{9} \)

7. Convert the mixed number \( 2 \frac{5}{6} \) to an improper fraction. Explain your process

8. Convert each of these mixed numbers to an improper fraction.
Chapter 8 Adding and Subtracting Fractions

\[
2 \frac{4}{5}, 3 \frac{4}{7}, 4 \frac{1}{3}, 2 \frac{7}{15}, 3 \frac{1}{7}, 8 \frac{1}{2}, 8 \frac{1}{6}, 3 \frac{5}{16}
\]

9. How many one-eighth cups of flour are in \(1 \frac{3}{4}\) cups of flour?

10. Which is greater: \(2 \frac{2}{3}\) or \(\frac{15}{6}\)? Explain your reasoning.

11. Which is greater: \(\frac{23}{7}\) or \(3 \frac{1}{4}\)? Explain.

12. Uncle Jim brings home \(3 \frac{1}{3}\) pounds of shrimp. He estimates he will need an average of \(\frac{1}{3}\) pound of shrimp per serving. How many people can he serve?

13. Johnny has a candy cane that is \(3 \frac{3}{4}\) inches long. He cuts it in two equal pieces and gives one to his sister. How long is each piece?

14. Draw a number line with 0 and \(\frac{1}{x}\) as shown below. Plot \(\frac{2}{x}, \frac{3}{x}, \) and \(\frac{4}{x}\) on your number line.

\[\begin{align*}
\text{0} & \\
\end{align*}\]

15. For each of the following questions, make a copy of the picture below and use it as a linear model for a fraction bar.

For each scenario below, determine the numbers that each of the other labeled points represent. Write the values below each point

a. If the point A represents the number 1
b. If the point B represents the number 1
c. If point C represents the number 1
d. If point D represents the number 1
e. If point E represents the number 1
f. If point F represents the number 1
16. Let $a = 12$. Compute $ax$ for each value of $x$ below, and determine if the value is greater than, equal to, or less than $a$. Explain how to do this without computation.
   a. $x = 2$
   b. $x = 3$
   c. $x = 1/2$
   d. $x = 1/3$

17. **Ingenuity:**
   Suppose $x$ is a positive integer. Convert the following mixed numbers to improper fractions.
   a. $x = 2 \frac{5}{9}$
   b. $x = 7 \frac{4}{8}$

   Assume that $n$ is a nonzero integer.
   c. First, suppose $n$ is positive. Plot $n$, $\frac{n}{2}$, $\frac{n}{3}$, $\frac{n}{4}$ and $\frac{n}{5}$ on a number line. Now make another number line representing the case where $n$ is negative.
   d. Find a fraction between $\frac{n}{3}$ and $\frac{n}{4}$ and then another between $\frac{-n}{3}$ and $\frac{-n}{4}$.
   e. The number $n$ is a nonzero integer. Which fraction is greater, $\frac{n}{3}$ or $\frac{n}{4}$?
   Explain your answer.

18. **Investigation:**
   Copy the rectangle to the right for each problem and suppose that its area is given in each of the problems below. Determine and shade an area of 1 square unit for each of the rectangles.
   a. $1 \frac{2}{3}$
   b. $2 \frac{2}{3}$
   c. $\frac{5}{4}$
   d. $\frac{2}{3}$
SECTION 8.3  COMMON MULTIPLES AND THE LCM

Have you noticed that hot dogs often come in packages of eight, and hot dog buns come in packages of twelve? When people plan to cook hot dogs, they tend to buy one package of hot dogs and one package of buns. But if they do this, they are left with four extra buns.

Some people who pay for the extra hot dog buns don’t want to waste them. What can they do? They could buy another package of eight hot dogs:

But now there are four extra hot dogs without buns. If they buy more buns:

There are eight buns without hot dogs, even more extra buns than the first time. Will this process ever end? Try buying one more package of hot dogs:
Aha! We have finally reached a point where we have exactly the same number of hot dogs and buns. Of course, in order to get there, the consumers had to buy two packages of buns and three packages of hot dogs.

What happened mathematically with the hot dogs and buns? One way to organize the number of hot dogs and the number of buns is

| Hot dogs: | 0 | 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 |...
|-----------|---|---|----|----|----|----|----|----|----|----|---
| Buns:     | 0 | 12| 24 | 36 | 48 | 60 | 72 |...

Look for some positive integer \( N \) where anyone could buy exactly \( N \) hot dogs and \( N \) buns. Finding the number gives the consumer the number of hot dogs and the number of buns to buy so that they come out even.

It is easy to see that 24 is the smallest positive integer that is in both lists. So, the smallest value of \( N \) for both items is 24. Notice that the numbers in the first list are the multiples of 8. This makes sense, because it is only possible to buy 8 hot dogs at a time. The numbers in the second list are the multiples of 12, because buns come only in packages of 12. Thus, 24 is the smallest positive integer that is a multiple of both 8 and 12. Mathematicians have a term for this:

![Definition 8.4: Common Multiple and LCM](image)

Notice that 48 and 72 are also common multiples of 8 and 12 but not the least.

As we found with the GCF, there are several ways to find the LCM of two numbers. Try some examples:

**EXAMPLE 1**

Find the LCM of 5 and 7.
SOLUTION

One way to find the least common multiple of two numbers is to list the positive multiples of each number in increasing order until you find an integer that is in both lists. For example, start by writing the first fourteen positive multiples of 5 and the first 10 positive multiples of 7:

Multiples of 5:

5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, …

Multiples of 7:

7, 14, 21, 28, 35, 42, 49, 56, 63, 70, …

Notice that 35 is in both lists. 70 is also on both lists. In fact, if we continued listing the multiples, we would find other common multiples of 5 and 7. However, the smallest positive integer that is a multiple of both 5 and 7 is 35. That means 35 is the LCM of 5 and 7.

As you have seen, with very large numbers, it might be necessary to write many multiples to find their LCM. This can be very time-consuming, so it would be better to have a more efficient method for computing the LCM of two numbers, just as you did when looking for the GCF. Thankfully, prime factorization comes to the rescue again.

EXAMPLE 2

Find the LCM of 54 and 63.
SOLUTION

First, factor 54 and 63, using prime factorization:

\[
\begin{align*}
54 &= 2 \cdot 3^3, \\
63 &= 3^2 \cdot 7.
\end{align*}
\]

The prime factorizations are 54 = 2 · 3³, and 63 = 3² · 7. Again, it is useful to express the prime factors with the exponents, aligning like factors:

\[
\begin{align*}
54 &= 2 \cdot 3^3 = 2 \cdot 3 \cdot 3 \cdot 3, \\
63 &= 3^2 \cdot 7 = 3 \cdot 3 \cdot 7.
\end{align*}
\]

To find a number that is a multiple of both 54 and 63, the multiple must have the prime building blocks that both 54 and 63 have. Before continuing, consider how you might be able to construct a multiple of 54 and 63 using the prime factors for each of the numbers. How can you construct the smallest such multiple?

Now, look at the method for finding the LCM of numbers using their prime factorization. By examining the two sets of prime factors, you can see that a common multiple must include 2, 3, and 7. However, 2 · 3 · 7 = 42 is not a multiple of 54 or 63. Because 54 has three factors of 3 and 63 has two factors of 3, a common multiple must have three factors of 3. Why won’t two factors of 3 be enough? Now, multiply the factors 2, 3³, and 7.

\[
2 \cdot 3^3 \cdot 7 = 378
\]
Notice that $378 = 2 \cdot 3 \cdot 3 \cdot 3 \cdot 7 = (2 \cdot 3 \cdot 3) \cdot 7 = 54 \cdot 7$ which is a multiple of 54.
Also, $378 = 2 \cdot 3 \cdot 3 \cdot 3 \cdot 7 = (2 \cdot 3) \cdot (3 \cdot 3 \cdot 7) = 6 \cdot 63$ which is a multiple of 63.
This is the smallest integer that contains all of the building blocks or prime factors in both sets of prime factors. Therefore, 378 is the LCM of 54 and 63.

**EXPLORATION 1**
Describe a method for finding the LCM of two numbers using their prime decompositions. Use it to find the LCM for $m$ and $n$ without computing the two numbers.

\[
m = 2^3 \cdot 3^6 \cdot 5 \cdot 7^2 \cdot 11^3 \cdot 17^2
\]
\[
n = 2^6 \cdot 3 \cdot 5^4 \cdot 7 \cdot 13^2 \cdot 17
\]

Another approach to solving for the LCM is to look at the Venn diagram, as you did in Section 8.1, when you worked with GCFs. Examine the prime factors of 6 and 8. The Venn diagram includes the prime factors for each number in the respective circles. Note the common factors in the overlapping part of the circles. This is the GCF of 6 and 8. Note that the product of $6 \cdot 8 = (2 \cdot 3)(2 \cdot 2 \cdot 2)$ is a common multiple of 6 and 8.

But it is the least common multiple. A multiple of 6 requires a factor of 2 and 3; a multiple of 8 requires 2, 2, and 2. So, a multiple of 6 and 8 requires one 3 and three 2s as factors. The shortest list of factors that will produce a multiple of both is $2^2 \cdot 2^2 \cdot 3$. The greatest common factor is 2 and we avoid double counting it in finding the LCM. To compute the LCM \{6,8\}, you compute the product of all the prime factors in the Venn Diagram, $3 \cdot 2 \cdot 2 \cdot 2$, which avoids using factors in the overlapping part twice. Thus, to get the LCM of 6 and 8, take the product of the highest power of all the factors that occur in either number, that is $2^2 \cdot 3$, to get the LCM of 24. *Remember to use the highest power of any prime for the LCM just as you used the lowest power of any common prime for the GCF.*
You will find that the Venn diagram is more practical when the numbers are larger and there are three numbers or more. For example, to find the least common multiple of 70, 36, and 60, write all the prime factors and notice which factors are common.

\[
\begin{align*}
70 &= 2 \cdot 5 \cdot 7 \\
36 &= 2^2 \cdot 3^2 \\
60 &= 2^2 \cdot 3 \cdot 5
\end{align*}
\]

We can represent this information as a Venn diagram:

![Venn Diagram](image)

Both 70 and 60 have a common factor of 5, so 5 is the intersection of the 70 circle and the 60 circle, but not the 36 circle. The factors \(2^2\) and 3 are in both 60 and 36, so \(2^2 \cdot 3\) is the intersection of the 60 circle and the 36 circle. Factor common to 70 and 36 is 2

After you have separated the factors into the different regions in the Venn diagram by multiplying all the numbers in the circles and their intersections, you will have the LCM of 70, 36, and 60. Did you get 1260? If so, you are correct.

To summarize these investigations, the LCM of two integers \(a\) and \(b\) is equal to the product of all the primes that occur in the prime factorizations, raised to the highest exponent that appears in either factorization.
The GCF is useful in finding the simplest equivalent fraction. The LCM can also be useful when working with fractions. For example, consider the fractions \( \frac{1}{6} \) and \( \frac{3}{8} \). Find equivalent fractions for both fractions that have the same denominator. How might this information help you to find equivalent fractions with a common denominator? The LCM is useful in adding and subtracting fractions.

EXERCISES

1. Find the LCM of the following pairs of numbers by listing multiples of the two numbers until you find the first multiple common to both lists.
   a. 4 and 6  
   b. 5 and 10  
   c. 5 and 6  
   d. 3 and 11  
   e. 10 and 14  
   f. 13 and 7

2. Find the LCM of the given numbers either by listing multiples of the two numbers until you find the first multiple common to both lists or by using the prime factorization of each number.
   a. 15 and 18  
   b. 32 and 20  
   c. 5, 6, and 7  
   d. 30 and 20  
   e. 120 and 16  
   f. 8, 9, and 12

3. Were there any pairs of integers in the previous exercise where the integers were relatively prime? Explain whether you have to check every prime factor of both integers to find that the two integers are relatively prime.

   Based on the evidence in the previous exercises, if \( p \) and \( q \) are different prime numbers, what is the LCM of \( p \) and \( q \)? Use what you know about the common factor of two relatively prime numbers to explain why your previous answer makes sense.
Section 8.3 Common Multiples and the LCM

4. Find the LCM for each of the following pairs of numbers:
   a. 12 and 60   b. 6 and 72   c. 45 and 15
   d. \( pq \) and \( pq^2 \)
   e. Look for a pattern in computing the LCM for a-c. Make a conjecture that explains this pattern.

5. For each pair of integers below, use prime factorization to find the GCF and LCM.
   a. 108 and 180   c. 80 and 110
   b. 77 and 72   d. 1000 and 625

6. Terry rides his bike every 5 days, and Max rides his bike every 7 days. If they start at the same time and same place on a Sunday morning, how many days will it be before they ride together again?

7. Teresa and Vanessa like to go swimming every few days to keep in shape. Each time Teresa goes swimming, she waits exactly six days before swimming again. For example, if she goes swimming on a Monday, the next time she goes swimming is Sunday. Each time Vanessa goes swimming, she waits exactly nine days before swimming again. Each time the two go swimming on the same day, they carpool to the swimming pool. If Teresa and Vanessa carpool on June 18, what will be the next two days they carpool? What date will they next carpool if Vanessa waits ten, rather than nine, days every time she goes swimming?

8. At Happy Days Day Care, the director will give one juice box and one blueberry breakfast bar as a morning snack to each child. Juice comes in packs of 6 and breakfast bars in packs of 4. If she wants to buy enough so that there are the same number of juice boxes and breakfast bars with none left over, what is the fewest number of each she will have to buy? What is the number of packs of each she will have to buy?

9. Mrs. Tolento wants to give pencils and erasers as gifts to her students. Pencils come in packs of 10 and erasers in packs of 8. How many students can she reward with exactly one pencil and one eraser with none left over? How many packages of each will she need to buy?
10. Randy is baking cookies for a family get-together at his house. He wants to bake enough cookies so that each person at the party, including himself, gets the same number of cookies. Randy knows that there are seven family members who will definitely come to the party. In addition, Randy has an aunt, an uncle, and two cousins who might or might not come. Because they are in the same family, either all four of these people will show up, or none of them will. What is the smallest number of cookies that Randy can cook if he wants to guarantee that everyone gets the same number of cookies, with no leftovers?

11. Use a Venn diagram to find the GCF and LCM for 15, 24, and 36.

12. What is the LCM of 23,000,000 and 37,000,000?

13. You are planning a party and want to invite 48 people. You need to buy invitations and party favors. Invitations come in packages of 12, and party favors come in packages of 24. What is the least number of packs of invitations and party favors you should buy to have one for each person and none left over?

14. **Investigation:**
   
The table below lists two variables \( m \) and \( n \). Copy and fill out this table. In the column labeled GCF \((m, n)\), write the GCF of \( m \) and \( n \). In the column labeled LCM \((m, n)\), write the LCM of \( m \) and \( n \). As you fill out the table, what do you notice?

<table>
<thead>
<tr>
<th>( m )</th>
<th>( n )</th>
<th>GCF ((m, n))</th>
<th>LCM ((m, n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>6</td>
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<td>15</td>
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<td>8</td>
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<td>77</td>
<td>81</td>
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<td></td>
</tr>
<tr>
<td>96</td>
<td>100</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
15. **Ingenuity:**
   a. Find a pair of numbers \( m \) and \( n \) for which \( \text{LCM}(m, n) \) is 4. How many such pairs \((m, n)\) are there?
   b. Find a pair of numbers \( m \) and \( n \) for which \( \text{LCM}(m, n) \) is 12. How many such pairs \((m, n)\) are there?
   c. Find a pair of numbers \( m \) and \( n \) for which \( \text{LCM}(m, n) \) is 120. How many such pairs \((m, n)\) are there?
SECTION 8.4 ADDITION AND SUBTRACTION OF FRACTIONS

Adding 1 foot to 2 feet equals 3 feet. Combining 1 apple with 2 apples gives 3 apples. In each case, both numbers and units are important. Given these two examples, it seems reasonable to say that the sum of 1 fifth and 2 fifths is 3 fifths. More precisely, in Section 8.2, the linear skip counting model demonstrated that \( \frac{3}{5} \) is \( \frac{1}{5} + \frac{1}{5} + \frac{1}{5} \). Using skip counting, it is easy to see that

\[
\frac{2}{5} + \frac{1}{5} = \left( \frac{1}{5} + \frac{1}{5} \right) + \frac{1}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5}.
\]

In general, for each positive integer \( m \) and \( n \), the fraction \( \frac{m}{n} \) is the sum of \( m \) unit fractions in the form \( \frac{1}{n} \). For the rest of this chapter, assume all possible denominators are positive integers.

PROBLEM 1

Compute the sum of \( \frac{3}{8} \) and \( \frac{2}{8} \). Explain your answer.

Now look at the area model. How is the sum \( \frac{1}{5} + \frac{2}{5} \) computed using the area model? Use a candy bar model. Betsy had \( \frac{1}{5} \) of a candy bar, and her friend had \( \frac{2}{5} \) of a candy bar like Betsy’s. Together, they have \( \frac{3}{5} \) of a candy bar. Express this as \( \frac{1}{5} + \frac{2}{5} = \frac{3}{5} \).

Write rules to generalize the previous discussion of adding fractions.
RULE 8.1: SUMS WITH LIKE DENOMINATORS

The sum of two fractions with like denominators, \( \frac{a}{n} \) and \( \frac{b}{n} \), is given by

\[
\frac{a}{n} + \frac{b}{n} = \frac{a + b}{n}
\]

The same principle applies when subtracting fractions.

PROBLEM 2

Compute \( \frac{7}{9} - \frac{4}{9} \) and explain how to obtain the answer.

Describe how to subtract fractions with like denominators. What is the difference \( \frac{a}{n} - \frac{b}{n} \)? How does your method compare to the addition rule above?

EXAMPLE 1

If you eat \( \frac{2}{3} \) of a candy bar, how much of the candy bar is left? How can you use subtraction of fractions to answer this question?

SOLUTION

Using mathematical fractions, the problem looks like this: \( 1 - \frac{2}{3} \). In order to perform this calculation, begin by drawing a picture of a candy bar and divide it into three pieces. First convert 1 into the fraction \( \frac{3}{3} \):

\[
1 = \frac{3}{3} = \\
\]

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What happens when you subtract $\frac{2}{3}$ of the candy bar? Shading the portions that are subtracted, the difference is

$$1 - \frac{2}{3} = \frac{1}{3} = \boxed{\text{ }}$$

Write this as $1 - \frac{2}{3} = \frac{3}{3} - \frac{2}{3} = \frac{1}{3}$.

Another way to think of this subtraction uses the linear model. Like the model for subtracting integers, move 1 unit to the right and then back up a distance of $\frac{2}{3}$ to land on the number $\frac{1}{3}$. This model represents $1 - \frac{4}{3}$.

For integers, the subtraction problem $n - m$ is equivalent to the addition problem $n + (-m)$. Similarly, the subtraction problem $1 - \frac{2}{3}$ is equivalent to the addition problem $1 + (-\frac{2}{3})$. What is the car model picture for this addition problem?

**PROBLEM 3**

Compute the difference $2 - \frac{1}{4}$ and illustrate the process with either the area model or the linear model.
Now that you’ve explored adding and subtracting fractions with like denominators, explore finding the sum of an integer and a fraction, like the addition problem \(2 + \frac{3}{5}\). Applying a similar process for this sum as you did for the last sum, there are several ways to combine these two quantities:

\[
2 + \frac{3}{5} = 1 + \frac{3}{5} = 1 + \frac{5}{5} + \frac{3}{5} = 1 + \frac{8}{5}, \text{ or }
\]

\[
2 + \frac{3}{5} = \frac{10}{5} + \frac{3}{5} = \frac{13}{5} \text{ or }
\]

\[
2 + \frac{3}{5} = 2 \frac{3}{5}.
\]

The first method trades one addition problem for another addition problem. The second results in an answer that is an improper fraction. The third way results in the mixed number \(2 \frac{3}{5}\). In Section 8.2, you learned how to convert mixed numbers to improper fractions and vice versa. Now you see that a mixed number can also be thought of as a sum.

**EXAMPLE 2**

Explore how to use the ideas just learned to compute the sum of two fractions when the denominators are not the same.

Use the area model to compute the sum \(\frac{1}{2} + \frac{1}{3}\).

**SOLUTION**

Begin by looking at a visual representation.

\[
\frac{1}{2} = \begin{array}{c}
\text{\includegraphics{}} \\
\end{array} \\
\frac{1}{3} = \begin{array}{c}
\text{\includegraphics{}} \\
\end{array}
\]

Notice that each outer rectangle is the same size because they represent the whole or 1. Also notice that one fraction is modeled with a horizontal cut and the other with a vertical cut. Why is this helpful?
Is it possible to combine the shaded amounts? Modify the picture above to display equivalent divisions of the whole.

To do this, divide the first model horizontally to represent $\frac{1}{2}$ as $\frac{3}{6}$ as 3 parts out of 6 parts. Then, divide the second model vertically to represent $\frac{1}{3}$ as $\frac{2}{6}$ as 2 parts out of 6 parts. It is easy to see from the model that $\frac{1}{2} = \frac{3}{6}$ and $\frac{1}{3} = \frac{2}{6}$. It is also easy to see how to add the two fractions in their equivalent forms. Using the rule for adding fractions with like denominators, the sum is

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

In order to add the fractions, find a common-sized piece so that the two fractions can be written with the same or common denominator.

Using the equivalent fractions property, transform the two fractions to fractions with the same denominator.

$$\frac{1}{2} = \frac{1 \cdot 3}{2 \cdot 3} = \frac{3}{6} \text{ and } \frac{1}{3} = \frac{1 \cdot 2}{3 \cdot 2} = \frac{2}{6}$$

The most important thing to remember when adding fractions is to ensure that you have a common denominator.

**EXPLORATION**

Compute the sum $\frac{1}{3} + \frac{1}{4}$ by first using the area model and then the equivalent fractions property to convert the fractions into equivalent fractions with like denominators.

Find the pattern to add the fractions $\frac{1}{a}$ and $\frac{1}{b}$ and show the process.
PROBLEM 4

Find three common denominators for the fractions $\frac{1}{6}$ and $\frac{1}{4}$. Write each fraction in equivalent forms using the three denominators. What do you notice about these common denominators? Which denominator would be the best choice for computing the sum $\frac{1}{6} + \frac{1}{4}$? Why?

Look at all the denominators you created in Problem 4. What is the relationship between each of the common denominators and the original denominators? For which choice of denominator did you not have to simplify? Now combine the discoveries about common denominators in Problem 4 with those about common multiples from Section 8.3.

PROBLEM 5

For each of the following sums: (1) find a common multiple for both denominators, (2) use it to find equivalent fractions for each fraction, (3) compute their sum and (4) simplify your answer, if necessary.

a. $\frac{1}{9} + \frac{1}{12}$  
b. $\frac{3}{8} + \frac{5}{12}$  
c. $\frac{7}{12} - \frac{5}{18}$

DEFINITION 8.5: LEAST COMMON DENOMINATOR

The least common denominator of the fractions $\frac{p}{n}$ and $\frac{k}{m}$ is the least common multiple of $n$ and $m$.

In adding or subtracting fractions, the LCM of the denominators produces the least common denominator or LCD.
Chapter 8 Adding and Subtracting Fractions

In Example 1, you saw that subtracting a fraction is equivalent to adding the negative of that fraction. What is the connection between \(-\frac{2}{3}\) and \(\frac{2}{3}\)? The number \(-\frac{2}{3}\) is the opposite of \(\frac{2}{3}\), that is, \(\frac{2}{3} + \frac{-2}{3} = 0\) and \(-\frac{2}{3}\) is located the same distance from 0 and on the opposite side from \(\frac{2}{3}\):

\[
\begin{array}{cccccccc}
& & -\frac{4}{3} & -1 & -\frac{2}{3} & -\frac{1}{3} & 0 & \frac{1}{3} & \frac{2}{3} & 1 & \frac{4}{3} & \\
\end{array}
\]

What is \(-\frac{2}{3}\) equivalent to and where is it located on the number line? First, think of \(-\frac{2}{3}\) as the quotient \(-2 \div 3\). Using the missing factor model, \(-\frac{2}{3} = x\) means \(3 \cdot x = -2\). The number \(x\) must be negative because the product of 3 and \(x\) is \(-2\).

Using the frog method, if \(x\) is the length of each jump and 3 is the number of jumps, the frog lands on \(-2\). To do that, the frog must divide \(-2\) into 3 equal jumps:

As you can see from the number line above, the directed length of each jump is \(-\frac{2}{3}\). So by the missing factor model, the quotient \(-\frac{2}{3} = -2 \div 3\) is the fraction \(-\frac{2}{3}\). In general, if each of \(a\) and \(b\) is a positive number, then \(\frac{-a}{b} = \frac{-a}{b} = \frac{a}{-b}\).

Now that you can plot rational numbers on a number line, you can also locate points in the coordinate plane with rational coordinates.

**PROBLEM 3**

Locate the points below in the coordinate plane:

A. \((\frac{2}{3}, \frac{1}{2})\)
B. \((-\frac{2}{3}, \frac{1}{2})\)
C. \((5,0)\)
D. \((-4, -10\frac{1}{3})\)
E. \((3\frac{1}{2}, -3\frac{1}{2})\)
F. \((-\frac{2}{3}, \frac{1}{4})\)
Section 8.4 Addition and Subtraction of Fractions

EXERCISES

1. Locate the following numbers on the number line:
   \[ \frac{2}{3}, \frac{1}{5}, 0, -2, \frac{1}{2}, -5 \frac{1}{3} \]

2. Graph the points below.
   \[ A = (2 \frac{1}{2}, 2 \frac{1}{2}) \]
   \[ B = (-3 \frac{1}{3}, 5 \frac{1}{4}) \]
   \[ C = (-3 \frac{1}{3}, -5 \frac{1}{4}) \]
   \[ D = (-2 \frac{1}{2}, 0) \]
   \[ E = (4 \frac{1}{5}, 3 \frac{2}{5}) \]

3. In the graph below, what are the coordinates of the points A, B, C, D, E.

4. Add or subtract the following fractions. Write your answers in simplest form.
   a. \[ \frac{4}{9} + \frac{2}{9} \]
   b. \[ \frac{3}{8} + \frac{1}{8} \]
   c. \[ \frac{4}{7} + \frac{6}{7} \]
   d. \[ \frac{2}{a} + \frac{3}{a} \]
   e. \[ 1 - \frac{4}{7} \]
   f. \[ \frac{3}{8} + \frac{5}{8} \]
   g. \[ 3 - \frac{3}{7} \]
   h. \[ \frac{8}{y} - \frac{3}{y} \]
   i. \[ \frac{8}{3} - \frac{2}{3} \]
   j. \[ \frac{10}{3} + \frac{5}{3} \]
   k. \[ \frac{1}{x} + \frac{1}{x} \]
   l. \[ \frac{3}{z} - \frac{3}{z} \]
5. Compute the sums or differences. Write your answers in simplest form.

a. \( \frac{2}{7} + \frac{3}{5} \)  
   f. \( \frac{1}{6} - \frac{1}{7} \)  
   k. \( \frac{7}{10} - \frac{1}{6} \)

b. \( \frac{1}{4} + \frac{5}{8} \)  
   g. \( \frac{3}{8} + \frac{5}{12} \)  
   l. \( \frac{5}{24} + \frac{3}{8} \)

c. \( \frac{1}{4} + \frac{1}{5} \)  
   h. \( \frac{8}{21} - \frac{3}{14} \)  
   m. \( \frac{5}{18} + \frac{4}{15} \)

d. \( \frac{5}{14} + \frac{3}{7} \)  
   i. \( \frac{1}{12} - \frac{1}{24} \)  
   n. \( \frac{1}{x} + \frac{1}{y} \)

e. \( \frac{8}{9} - \frac{1}{6} \)  
   j. \( \frac{2}{9} + \frac{3}{8} \)  
   o. \( \frac{2}{a} + \frac{3}{b} \)

6. Compute the following sums or differences. Write your answers in simplest form.

a. \( \frac{1}{2 \cdot 3 \cdot 5} + \frac{1}{2 \cdot 3 \cdot 7} \)  
   c. \( \frac{7}{90} + \frac{25}{168} \)

b. \( \frac{4}{3 \cdot 5 \cdot 7^2} + \frac{8}{3 \cdot 5 \cdot 7} \)  
   d. \( \frac{99}{700} - \frac{77}{4500} \)

7. Jennifer mixes \( \frac{2}{3} \) of a gallon of milk with \( \frac{1}{8} \) of a gallon of chocolate syrup. How much chocolate milk will she make?

8. Find the sum or difference of the following algebraic fractions. Simplify your answer, if necessary.

a. \( \frac{2}{x} + \frac{1}{3x} \)  
   f. \( \frac{1}{4x} - \frac{1}{6y} \)  
   k. \( \frac{2x}{10} + \frac{5x}{6} \)

b. \( \frac{1}{4a} + \frac{1}{2a} \)  
   g. \( \frac{x}{8} + \frac{x}{12} \)  
   l. \( \frac{a}{4b} - \frac{1}{6b} \)

c. \( \frac{1}{4x} + \frac{1}{5y} \)  
   h. \( \frac{A}{2} - \frac{A}{10} \)  
   m. \( \frac{x}{y} + \frac{y}{x} \)

d. \( \frac{2}{3n} + \frac{1}{2n} \)  
   i. \( \frac{a}{12} + \frac{b}{16} \)  
   n. \( \frac{1}{x} + \frac{3}{xy} \)

e. \( \frac{5}{6x} - \frac{1}{3x} \)  
   j. \( \frac{a}{b} + \frac{c}{3b} \)  
   o. \( \frac{7}{ab} + \frac{3}{b} \)

9. Julie bakes a carrot cake. She plans to take half of it to her grandmother and serve \( \frac{1}{3} \) of the cake to her two brothers for a snack. How much of the cake will be left?
10. There are two identical bars of chocolate. Bar A is cut into \(x\) equal-sized pieces. Bar B is cut into twice as many equal-sized pieces. If Earl eats one piece from each chocolate bar, what fraction of a chocolate bar has he eaten, in terms of \(x\)?

11. Jill has a jar of marbles. Fred estimates that \(\frac{3}{5}\) of the marbles are blue and Rene estimates that \(\frac{7}{10}\) of them are blue. Jill knows that exactly \(\frac{2}{3}\) of her marbles are blue. Use subtraction to determine whose estimate was closest to the correct fraction, and by how much.

12. On a one-inch ruler, the \(\frac{1}{8}\) mark is labeled point A, the \(\frac{5}{16}\) mark is labeled point B, the \(\frac{1}{2}\) mark is labeled point C and the \(\frac{3}{4}\) mark is labeled point D. Show how subtraction can be used to calculate these distances.
   a. What is the distance from point A to point B?
   b. What is the distance from point A to point C?
   c. What is the distance from point A to point D?
   d. What is the distance from point B to point C?
   e. What is the distance from point C to point D?

13. Compute and simplify.
   a. \(\frac{2}{9} + \frac{1}{3} + \frac{1}{2}\)
   b. \(\frac{1}{4} + \frac{1}{2} + \frac{1}{8}\)
   c. \(\frac{1}{4} + \frac{1}{5} + \frac{3}{10}\)
   d. \(\frac{1}{3} + \frac{1}{8} + \frac{5}{12}\)
   e. \(\frac{1}{24} + \frac{1}{3} + \frac{1}{2}\)
   f. \(\frac{3}{4} + \frac{3}{5} + \frac{3}{8}\)
   g. \(\frac{1}{3} + \frac{1}{4} + \frac{1}{5}\)
   h. \(\frac{1}{2} + \frac{1}{8} + \frac{1}{12}\)
   i. \(\frac{5}{8} + \frac{2}{9} + \frac{1}{12}\)

14. Compute the following sums. Express each as a simplified mixed fraction.
   a. \(\frac{2}{3} + \frac{3}{2}\)
   b. \(\frac{4}{5} + \frac{5}{4}\)
   c. \(\frac{3}{7} + \frac{7}{3}\)
   d. Check to see that your general answer from Exercise 4, part m agrees with the answers using arithmetic in parts a, b and c.

15. Both \(p\) and \(q\) are prime numbers. For each of the following sums, find the LCD of each pair of fractions and use it to compute the sum:
   a. \(\frac{1}{p^2q} + \frac{1}{pq^2}\)
   b. \(\frac{1}{p^3q} + \frac{1}{p^2q^2}\)
16. **Ingenuity:**

Write the following fractions in simplest form:

a. \( \frac{x^2 + x}{x} \)

b. \( \frac{a^2 b + ab}{ab} \)

c. \( \frac{x^2 y^2 + x^2 y}{x^2 y} \)

17. **Investigation:**

If one fraction is \( \frac{1}{a^2 b} \) and another fraction is \( \frac{1}{ab} \), what common denominator(s) do the fractions have? Explain how to decide which denominator to use when adding the fractions. Then compute the addition problem. Perform the addition using another common denominator and check to see how it compares with your first answer.
In Section 8.4, the discussion of adding two fractions involved finding a common denominator. In many problems, you probably used the least common denominator (LCD), the LCM of the given denominators.

In the Exploration in Section 8.4 you discovered a rule for adding two fractions with unknown and unlike denominators:

\[
\frac{1}{a} + \frac{1}{b} = \frac{1 \cdot b}{a \cdot b} + \frac{1 \cdot a}{b \cdot a} = \frac{b + a}{a \cdot b}
\]

**EXAMPLE 1**

Assume \( p \) and \( q \) are primes. Compute the following two sums using the pattern from above. Notice that the first sum is a special case of the second, where \( p = 2 \) and \( q = 3 \).

a. \( \frac{1}{6} + \frac{1}{9} \) 

b. \( \frac{1}{pq} + \frac{1}{q^2} \)

**SOLUTION**

Common Denominator Method:

As in the rule above, you can create a common denominator for each sum by multiplying the two given denominators:

\[
\frac{1}{6} + \frac{1}{9} = \frac{1 \cdot 9}{6 \cdot 9} + \frac{1 \cdot 6}{9 \cdot 6} \\
\frac{1}{pq} + \frac{1}{q^2} = \frac{1 \cdot q^3}{pq \cdot q^3} + \frac{1 \cdot pq}{q^3 \cdot pq}
\]

\[
= \frac{9 + 6}{6 \cdot 9} = \frac{15}{54} \\
= \frac{q^3 + pq}{pq^3}
\]
Are these fractions simplified? Do the numerator and denominator have any common factors? The numerator can be factored using the distributive property:

\[
\frac{15}{54} = \frac{3 \cdot 5}{2 \cdot 3 \cdot 3 \cdot 3} = \frac{5}{18} \quad \frac{q^3 + pq}{pq^3} = \frac{q(q + p)}{pq^3} = \frac{q + p}{pq^2}
\]

Notice that this approach did not involve finding the LCD.

**LCD Method:**

Another approach is to first find the LCD of the fractions in each sum or, equivalently, the LCM of the denominators. Look at the prime factorizations of the denominators of the fractions: \(6 = 2 \cdot 3\), \(9 = 3^2\), and because \(p\) and \(q\) are primes, \(pq\) and \(q^2\) are their own prime factorizations. Remember the rule for finding the LCM of two numbers from their prime factorizations: take the product of each prime raised to its larger exponent. So, the LCDs are \(\text{LCM}(6, 9) = 2 \cdot 3^2\) and \(\text{LCM}(pq, q^2) = pq^2\).

Now, in computing the sums \(\frac{1}{6} + \frac{1}{9}\) and \(\frac{1}{pq} + \frac{1}{q^2}\), multiply the numerator and denominator of each fraction by a factor that will make the denominator the LCD:

\[
\frac{1}{6} + \frac{1}{9} = \frac{1}{2 \cdot 3} + \frac{1}{3^2} \quad \frac{1}{pq} + \frac{1}{q^2} = \frac{1}{pq} + \frac{1}{q^2}
\]

\[
= \frac{1 \cdot 3}{(2 \cdot 3) \cdot 3} + \frac{1 \cdot 2}{3^2 \cdot 2} \quad = \frac{1 \cdot q}{pq \cdot q} + \frac{1 \cdot p}{q^2 \cdot p}
\]

\[
= \frac{3 + 2}{2 \cdot 3^2} = \frac{5}{18} \quad = \frac{q + p}{pq^2}
\]

Notice that the final answer in each sum is already simplified.

**PROBLEM 1**

Use the process just developed to compute the LCD for the fractions and then compute the sum:

a. \(\frac{1}{40} + \frac{1}{50}\)  
   b. \(\frac{3}{8} + \frac{5}{12}\)  
   c. \(\frac{7}{10} + \frac{4}{9}\)
Formulate a written procedure that describes the process of:

- Finding the LCD for any two fractions
- Rewriting fractions equivalently using the LCD
- Computing sums and differences of two fractions.

**EXPLORATION 1**

Silvia is baking six sheet cakes for a party. The recipe she is using calls for $3\frac{1}{6}$ pounds of refined sugar and $5\frac{1}{4}$ pounds of unrefined sugar. First, use the linear model to give an estimate of how much sugar Silvia needs. Then, compute how many pounds of sugar Silvia needs. Explain your process for both the estimation and the calculation. Can you use the same process to add other mixed numbers?

**EXAMPLE 2**

Compute the sum $6\frac{3}{5} + 3\frac{5}{7}$.

**SOLUTION**

There are at least three ways to compute this sum.

1. **Improper Fractions**:

   One approach is to treat this as an ordinary fraction addition problem by converting from mixed numbers to improper fractions and back again. First, convert the mixed numbers to improper fractions:

   $6 \frac{3}{5} = 6 + \frac{3}{5} = \frac{6 \cdot 5}{5} + \frac{3}{5} = \frac{33}{5}$

   and

   $3 \frac{5}{7} = 3 + \frac{5}{7} = \frac{3 \cdot 7}{7} + \frac{5}{7} = \frac{26}{7}$
Then, find the LCD and compute the sum. Note that in this case the denominators are relatively prime, so the LCD is their product.

\[
\frac{33}{5} + \frac{26}{7} = \frac{33 \cdot 7}{5 \cdot 7} + \frac{26 \cdot 5}{7 \cdot 5}
\]

\[
= \frac{231}{35} + \frac{130}{35}
\]

\[
= \frac{361}{35}
\]

Finally, convert the improper fraction to a mixed number and simplify. Because the largest multiple of 35 less than 361 is 350, convert 361 to 

\[
35 \cdot 10 + 11 = 350 + 11
\]

or 

\[
361 \div 35 = 10 \text{ with a remainder of } 11.
\]

\[
\frac{361}{35} = \frac{350 + 11}{35}
\]

\[
= \frac{350}{35} + \frac{11}{35}
\]

\[
= 10 + \frac{11}{35}
\]

\[
= 10 \frac{11}{35}
\]

2. Combining Like Parts:

The improper fractions approach can be cumbersome because it involves working with relatively large numbers. Another approach is to consider each mixed number as the sum of an integer and a proper fraction and regroup, using the Commutative and Associative Properties of Addition:

\[
6 \frac{3}{5} + 3 \frac{5}{7} = (6 + \frac{3}{5}) + (3 + \frac{5}{7}) = (6 + 3) + (\frac{3}{5} + \frac{5}{7})
\]
This leads to the sum of proper fractions:
\[
\frac{3}{5} + \frac{5}{7} = \frac{3 \cdot 7}{5 \cdot 7} + \frac{5 \cdot 5}{7 \cdot 5}
\]
\[
= \frac{21}{35} + \frac{25}{35}
\]
\[
= \frac{46}{35}
\]
\[
= 1 \frac{11}{35}
\]

Combining these results, the original sum is
\[
6 \frac{3}{5} + 3 \frac{5}{7} = (6 + \frac{3}{5}) + (3 + \frac{5}{7})
\]
\[
= (6 + 3) + \left(1 + \frac{11}{35}\right)
\]
\[
= 10 + \frac{11}{35}
\]
\[
= 10 \frac{11}{35}
\]

As you can see, in computing the sum of mixed numbers, it is often easier to regroup the mixed numbers as whole parts and fractional parts, add each group and then combine these two partial sums.

**3. Vertical Addition:**

There is another way to organize and write this same process vertically:

\[
\begin{align*}
6 & \frac{3}{5} + 3 & \frac{5}{7} + 3 & \frac{5}{7} + \frac{3}{9} + \frac{21}{35} + \frac{3}{9} + \frac{25}{35} \\
& \frac{21}{35} & \frac{46}{35} & \frac{11}{35}
\end{align*}
\]

\[
= 9 + \left(1 + \frac{11}{35}\right)
\]
\[
= 10 + \frac{11}{35}
\]
\[
= 10 \frac{11}{35}
\]

How would finding the difference between two mixed numbers be different?
EXAMPLE 3

Compute the following differences:

\[ \begin{align*}
\text{a.} & \quad 8 \frac{4}{5} - 5 \frac{3}{10} \\
\text{b.} & \quad 6 \frac{3}{5} - 3 \frac{5}{7}
\end{align*} \]

SOLUTION

\[ \begin{align*}
\text{a.} & \quad \text{Use the vertical method from the previous example:} \\
& \quad \begin{align*}
8 & \frac{4}{5} \\
\underline{-} & \frac{3}{10} \\
& 5 \frac{3}{10}
\end{align*} \\
& \quad \begin{align*}
8 & \frac{4}{5} \\
\underline{-} & \frac{3}{10} \\
& 5 \frac{3}{10} \\
& = 3 + \frac{5}{10}
\end{align*} \\
& \quad \text{Notice that the fraction } \frac{5}{10} \text{ in the solution is simplified to its equivalent fraction } \frac{1}{2}.
\]

\[ \begin{align*}
\text{b.} & \quad \text{Again, use the vertical method. However, a complication arises when attempting to subtract } \frac{5}{7} \text{ from } \frac{3}{5}. \text{ } \frac{5}{7} \text{ is greater than } \frac{3}{5}. \\
& \quad \begin{align*}
6 & \frac{1}{3} \\
\underline{-} & \frac{3}{5} \\
& 3 \frac{5}{7}
\end{align*} \\
& \quad \begin{align*}
6 & \frac{1}{3} \\
\underline{-} & \frac{3}{5} \\
& 3 \frac{25}{35}
\end{align*}
\]

To avoid the negative fraction, rename 6 as 5 + 1 and associate 1, or \( \frac{35}{35} \), with the fraction

\[ \begin{align*}
& \quad \begin{align*}
5 & \quad 1 + \frac{21}{35} \\
\underline{-} & \frac{25}{35} \\
& \frac{31}{35}
\end{align*} \\
& \quad \begin{align*}
5 & \quad \frac{35}{35} + \frac{27}{35} \\
\underline{-} & \frac{25}{35} \\
& \frac{31}{35}
\end{align*} \\
& \quad \begin{align*}
5 & \quad \frac{56}{35} \\
\underline{-} & \frac{25}{35} \\
& \frac{31}{35}
\end{align*}
\]

\[ \begin{align*}
\text{= } & \quad 2 + \frac{31}{35}
\end{align*} \]

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Section 8.5  Common Denominators and Mixed Numbers

EXERCISES

1. Compute the following sums and differences by using the prime factorization of each denominator to find the LCD. Show all the steps in the process.
   a. \(\frac{3}{28} + \frac{5}{12}\)  
   b. \(\frac{3}{4} + \frac{5}{16}\)  
   c. \(\frac{7}{24} + \frac{5}{36}\)  
   d. \(\frac{8}{15} + \frac{3}{28}\)  
   e. \(\frac{11}{60} - \frac{1}{126}\)  
   f. \(\frac{1}{20} - \frac{1}{75}\)  
   g. \(\frac{108}{7} + \frac{72}{5}\)  
   h. \(\frac{19}{21} - \frac{71}{84}\)  
   i. \(\frac{27}{28} - \frac{1}{28}\)  
   j. \(\frac{2}{45} + \frac{3}{40}\)  
   k. \(\frac{12}{25} + \frac{3}{4}\)  
   l. \(\frac{7}{8} + \frac{7}{12}\)

2. Compute the following sums of mixed numbers using either the horizontal or vertical method. Show all the steps in the process.
   a. \(3 \frac{1}{8} + 1 \frac{5}{8}\)  
   b. \(6 \frac{2}{5} + 2 \frac{3}{7}\)  
   c. \(5 \frac{1}{3} + 2 \frac{2}{3}\)  
   d. \(9 \frac{2}{15} + 5 \frac{11}{12}\)  
   e. \(3 \frac{5}{6} + 5 \frac{3}{4}\)  
   f. \(6 \frac{3}{8} + 2 \frac{11}{12}\)

3. Compute the following differences of mixed numbers using the stacking method. Show all the steps in the process.
   a. \(3 \frac{1}{8} - 1 \frac{5}{8}\)  
   b. \(5 \frac{1}{3} - 2 \frac{2}{3}\)  
   c. \(4 - 3 \frac{3}{7}\)  
   d. \(9 \frac{8}{15} - 5 \frac{11}{12}\)  
   e. \(5 \frac{5}{24} - 1 \frac{7}{36}\)  
   f. \(4 \frac{1}{30} - 2 \frac{1}{48}\)

4. Assume that each variable in this problem is a prime number. For each of the following sums, find the LCD of the denominators and use it to compute the sum. Show all the steps in the process.
   a. \(\frac{1}{x^2y} + \frac{1}{x^3y}\)  
   b. \(\frac{1}{xyz} + \frac{1}{x^2}\)  
   c. \(\frac{1}{pqr} + \frac{1}{p^2q^2r^2}\)  
   d. \(2 + \frac{1}{x}\)  
   e. \(3 + \frac{a}{xy}\)  
   f. \(\frac{3a}{x^2z} + \frac{4b}{yz}\)

5. Susie has \(3 \frac{3}{8}\) gallons of honey. Her friend John has \(6 \frac{1}{5}\) gallons of honey. How much honey do they have together?

6. Gabriel ran \(2 \frac{5}{8}\) miles while Miguel ran \(5 \frac{1}{4}\) miles. How much further did Miguel run?

7. Sophia is making a cake that requires \(2 \frac{1}{3}\) cups of milk. If she pours her milk out of a four-cup milk container, how much will she have left in the container?
8. Adam cut $\frac{2}{3}$ foot off of a $5\frac{1}{4}$ foot wire. How much wire was left?

9. Lydia drinks $3\frac{2}{3}$ cups of milk, and $2\frac{3}{5}$ cups of juice. How much liquid does she consume?

10. On Monday it rained $2\frac{1}{4}$ inches. On Tuesday it rained $1\frac{2}{3}$ inches. On Wednesday it rained $3\frac{3}{5}$ inches. What was the total rainfall for these three days?

11. There are two fifth grade classes in a school. Class A is one-fourth female and Class B is three-fourths female. Both classes have PE together. Class A has 20 students and Class B has 32 students. What fraction of the combined PE class is female?

12. **Ingenuity:**
Jack and Jill are house painters. Jack can paint $\frac{8}{10}$ of a standard-sized house in a day. Jill can paint $\frac{6}{10}$ of a standard-sized house in one day. Estimate about how long it might take Jack and Jill to paint a standard house working together.

13. **Investigation:**
   a. The students in the sixth grade are all in two classes, Class C and Class D. If Class C has three times as many students as Class D, what fraction of the sixth grade students are in Class C? Draw a picture.
   b. In the same sixth grade class, Class C is one-third girls and Class D is one-half girls. What fractional part of the sixth grade is made up of girls?

14. **Investigation:**
   In Example 2, the process of adding two mixed numbers was written horizontally. Formulate a similar horizontal process for computing the difference $6\frac{3}{5} - 3\frac{5}{7}$. 
REVIEW PROBLEMS

1. Find the greatest common factor of each pair of integers. Show your work with the Venn Diagram method or the unique factorization method.
   a. 20 and 25  
   b. 45 and 65  
   c. 12 and 36  
   d. 16 and 17  
   e. 60 and 84  
   f. 378 and 420  
   g. 14, 35, and 56  
   h. 105, 120, and 135  
   i. $x^2y^3$ and $xy^5$

2. Find an equivalent fraction for each of the following:
   a. $\frac{2}{5}$  
   b. $\frac{6}{10}$  
   c. $\frac{2}{3}$  
   d. $\frac{1}{4}$  
   e. $\frac{8}{12}$  

3. Label the shaded fractional parts of the following and order from least to greatest:
   a.  
   b.  
   c.  
   d.  
   e.  
   f.  

4. If Isabel worked on her math homework for 20 minutes, what fraction of an hour did she work on her homework? Write this fraction in its simplest form.

5. Rewrite the following fractions in simplest form.
   a. $\frac{20}{25}$  
   b. $\frac{45}{65}$  
   c. $\frac{12}{36}$  
   d. $\frac{16}{17}$  
   e. $\frac{63}{72}$  
   f. $\frac{126}{264}$
6. Simplify each of the following fractions:
   a. \( \frac{56}{72} \)  
   c. \( \frac{126}{210} \)  
   e. \( \frac{225}{330} \)
   b. \( \frac{90}{120} \)  
   d. \( \frac{77}{81} \)

7. Your friend eats \( \frac{1}{3} \) of a pizza and you eat \( \frac{1}{2} \) of the same pizza. Who ate the most pizza? How much of the pizza is left over?

8. Dora is planning to have a party. Cookies are sold in packages of six. Juice drinks are sold in packages of eight. She is trying to figure out how many friends she is able to invite so that each friend gets exactly one cookie and one drink with nothing left over. How many friends can Dora invite to her party?

9. Christopher is having friends over for a fun day of water games. He has 12 water guns and 18 water balloons. If all the water guns and water balloons are handed out and each person has the same number of water guns and water balloons, how many people were playing? How many water guns and water balloons did each person have?

10. Write the following shaded areas as an improper fraction and a mixed number.
   a. 
   b. 
   c.
11. Convert each improper fraction to a mixed number.
   a. $\frac{12}{7}$
   b. $\frac{23}{4}$
   c. $\frac{55}{6}$
   d. $\frac{10}{3}$
   e. $\frac{79}{12}$

12. Convert each mixed number to an improper fraction.
   a. $3 \frac{2}{3}$
   b. $5 \frac{4}{5}$
   c. $8 \frac{1}{8}$
   d. $12 \frac{5}{6}$

13. Find the least common multiple of each pair of integers. Show your work with the Venn Diagram method or the unique factorization method.
   a. 8 and 12
d. 15 and 8
   b. 10 and 12
e. 15 and 20
   c. 14 and 21
f. 25 and 36
   g. 4, 16, and 8
h. 8, 12, and 20
   i. $a^2b$ and $xy^2$

14. Find the greatest common factor and least common multiple for each of the following. Show your work with the Venn Diagram method or the unique factorization method.
   a. $2 \cdot 2 \cdot 2 \cdot 2$ and $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
d. 48 and 60
   b. $2 \cdot 2 \cdot 2 \cdot 3$ and $2 \cdot 3 \cdot 5$
e. 16 and 15
   c. $2 \cdot 2 \cdot 3 \cdot 5$ and $2 \cdot 2 \cdot 3$
f. 180 and 420

15. Add or subtract the following fractions. Write your answers in simplest form.
   a. $\frac{4}{7} + \frac{2}{7}$
   b. $\frac{5}{9} + \frac{1}{9}$
c. $\frac{5}{6} - \frac{2}{6}$
   d. $\frac{8}{12} - \frac{3}{12}$

16. Compute the sums or differences. Write your answers in simplest form.
   a. $\frac{3}{10} + \frac{2}{5}$
   b. $\frac{5}{9} + \frac{1}{3}$
c. $\frac{5}{6} - \frac{2}{15}$
   d. $\frac{1}{4} + \frac{2}{5}$
e. $\frac{5}{8} + \frac{1}{6}$
   f. $\frac{7}{8} + \frac{3}{10}$

17. Marina is mixing cookie dough. She would like the dough to include $\frac{1}{3}$ cup chocolate chips, $\frac{3}{5}$ cup peanut butter and $\frac{1}{5}$ cup pecans. How many total cups of these ingredients will the cookie dough have?
18. Nancy and Leslie are sewing buttons to a quilt. Nancy would like $\frac{1}{8}$ of the buttons to be pink while Leslie would like $\frac{1}{6}$ of the buttons to be green. Both Nancy and Leslie agree that $\frac{1}{4}$ of the buttons should be white. How many more white buttons will be on the quilt than green buttons? How many more green buttons will there be than pink buttons?

19. Compute the following. Write your answer in simplest form.
   
   a. $\frac{3}{4} + \frac{7}{10}$  
   b. $\frac{3}{8} + \frac{15}{16}$  
   c. $\frac{5}{8} + 3\frac{1}{2}$  
   d. $10\frac{1}{3} - 6\frac{2}{3}$  
   e. $8\frac{2}{3} - 6\frac{1}{2}$  
   f. $7\frac{1}{4} - 2\frac{2}{3}$  
   g. $\frac{23}{30} \cdot \frac{3}{12}$  
   h. $\frac{9}{16} \cdot \frac{5}{24}$

20. Abigail has $2\frac{2}{3}$ feet of ribbon. Brianna has $3\frac{1}{3}$ feet of ribbon. How much ribbon do they have all together? How much more ribbon does Brianna have than Abigail?

21. Nicole babysat her younger cousins for $2\frac{1}{4}$ hours on Monday. She babysat for $1\frac{1}{2}$ hours on Wednesday and for $3\frac{3}{4}$ hours on Friday. How much time did she spend babysitting her cousins?
CHALLENGE PROBLEMS

Section 8.3:
The LCM of 1, 2, 3, …, 98, 99 ends in how many zeros?

Section 8.4:
If 0.abab… is a repeating decimal with digits \( a \neq b \), what are the possible values of the denominator of the reduced fraction?

Section 8.5:
If the length of a rectangle is increased by 25%, by what percentage must the width be decreased to keep the area from changing?

Section 8.6:
Let \( a, b, \) and \( c \) denote single digits, not all 0. Find all possible values of

\[
\frac{0.\text{abc} + 0.\text{acb} + 0.\text{bac} + 0.\text{bca} + 0.\text{cab} + 0.\text{cba}}{a + b + c}
\]
SECTION 9.1 MULTIPLICATION OF FRACTIONS

We used the linear model to understand addition and subtraction of fractions. Initially, the linear model is also useful in understanding multiplication of fractions. We will begin by reviewing the linear model.

PROBLEM 1

What would $\frac{1}{3}$ of 6 be? In other words, what is the product $\frac{1}{3} \cdot 6$? Illustrate the process on the number line below and represent the product in words.

![Number line diagram]

EXAMPLE 1

Jane has $\frac{1}{2}$ a yard of ribbon and needs to cut $\frac{1}{3}$ of its length. To do this, she finds out what $\frac{1}{3}$ of $\frac{1}{2}$ is. Use a number line to show how much ribbon she cuts.
SOLUTION

To find how much ribbon Jane cuts, use the linear model to calculate \( \frac{1}{3} \cdot \frac{1}{2} \):

The arithmetic statement is \( \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} \). If each jump is \( \frac{1}{3} \) yard and the frog makes a \( \frac{1}{2} \) of a jump, it travels \( \frac{1}{6} \) of a yard.

With the linear model it is important to be very exact when drawing the picture. To see the advantage of the area model, look at the problem above: \( \frac{1}{3} \cdot \frac{1}{2} \). To begin the process using the area model, draw \( \frac{1}{3} \) as a shaded part of the whole rectangle with area 1.

One way to represent \( \frac{1}{2} \) of the shaded area is to cut the rectangle representing \( \frac{1}{3} \) in half by cutting vertically. But this is the same process as the linear model. Instead, we cut the rectangle into 2 equal pieces by cutting horizontally.

One of the 2 pieces from the second cut is shaded to represent \( \frac{1}{2} \) of the original \( \frac{1}{3} \) rectangle. What part of the whole rectangle is the double-shaded area?
**EXPLORATION 1**

a. Translate \( \frac{1}{2} \) of \( \frac{1}{5} \) into a multiplication problem and draw the corresponding picture to find the product.

b. Translate \( \frac{1}{4} \) of \( \frac{1}{3} \) into a multiplication problem and draw the corresponding picture to find the product.

c. Predict what is \( \frac{1}{5} \) of \( \frac{1}{7} \) without drawing a model.

d. Explain why multiplication of unit fractions is commutative, that is

\[
\frac{1}{m} \cdot \frac{1}{n} = \frac{1}{n} \cdot \frac{1}{m}.
\]

**PROBLEM 2**

What is \( \frac{1}{2} \) of \( \frac{3}{4} \)?

![Fraction Illustration](image)

**EXPLORATION 2**

What is the product of the fractions \( \frac{\bar{a}}{\bar{b}} \) and \( \frac{\bar{c}}{\bar{d}} \), where \( a, b, c, \) and \( d \) are positive integers with \( b \) and \( d \) not zero? Use the area model to compute the following products:

a. What is \( \frac{1}{2} \) of \( \frac{2}{3} \)? Use the area model to illustrate and find the answer.

b. What is \( \frac{1}{3} \) of \( \frac{2}{3} \)? Use the area model to illustrate and find the answer.

c. What is \( \frac{1}{3} \) of \( \frac{3}{4} \)? Use the area model to illustrate and find the answer.

d. Predict what \( \frac{3}{4} \) of \( \frac{5}{7} \) is without drawing a model.

What do you notice about the product of two proper fractions?
Summarizing this pattern of multiplying fractions,

**RULE 9.1: MULTIPLYING FRACTIONS**

The product of two fractions $\frac{a}{b}$ and $\frac{c}{d}$, where $b$ and $d$ are non-zero, is

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

**PROBLEM 3**

Multiply the following fractions:

a. $\frac{1}{2} \cdot \frac{3}{4}$  
   c. $\frac{3}{5} \cdot \frac{5}{3}$  
   e. $\frac{5}{9} \cdot \frac{3}{4}$

b. $\frac{3}{5} \cdot \frac{2}{7}$  
   d. $\frac{3}{4} \cdot \frac{4}{5}$

In the linear model, we see that:

$$\frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{1 \cdot 5}{5} = 1.$$  
This leads us to the following definition:

**RULE 9.2: RECIPROCAL OF AN INTEGER**

If $n$ is a number, other than 0, then the multiplicative inverse, or reciprocal, of $n$ is the fraction $\frac{1}{n}$. The product $n \cdot \frac{1}{n}$ is $n \cdot \frac{1}{n} = 1$. 

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What is the relationship between the rule for Reciprocals of an Integer and the rule for Multiplying Fractions? Each positive integer \( n \) can be written as the fraction \( \frac{n}{1} \). Substituting \( \frac{n}{1} \) for \( n \) in the Rule for Reciprocals and using the fact that \( n \cdot 1 = n \), the equation becomes

\[
\frac{1}{n} \cdot n = \frac{1}{n} \cdot \frac{n}{1} = \frac{n \cdot n}{n} = \frac{n}{n} = 1
\]

**EXPLORATION 3**

What fraction can be multiplied by \( \frac{2}{3} \) to get 1? In other words, what times \( \frac{2}{3} \) equals 1? Explain your answer.

Remember that the **reciprocal** of a number is the number that, when multiplied by the original number, equals 1. What is the reciprocal of \( \frac{3}{4} \)? Verify that the product of \( \frac{3}{4} \) and its reciprocal is 1.

Make a conjecture about the reciprocal of any fraction \( \frac{a}{b} \).

For example, the reciprocal of \( \frac{x}{y} \) is \( \frac{y}{x} \). This makes sense because \( \frac{x}{y} \cdot \frac{y}{x} = \frac{xy}{yx} = \frac{xy}{xy} = 1 \).

You found that the product of \( \frac{2}{3} \) and \( \frac{3}{2} \) equals 1, and the fractions are reciprocals of each other. Notice that \( \frac{3}{2} \) is a fraction larger than 1. In general, if a positive number is less than 1, then its reciprocal is greater than 1.

**RULE 9.3: RECIPROCAL OF A FRACTION**

In general, the **multiplicative inverse** or **reciprocal** of \( \frac{x}{y} \) is the fraction \( \frac{y}{x} \) since

\[
\frac{x}{y} \cdot \frac{y}{x} = \frac{xy}{yx} = \frac{xy}{xy} = 1.
\]
EXAMPLE 2

Lisa has 24 books in her library, one third of which are hardback books.

a. How many of her library books are hardback?
b. How many of her books are not hardback?

SOLUTION

a. Take $\frac{1}{3}$ of 24: $24 \cdot \frac{1}{3} = \frac{24}{3} \cdot \frac{1}{3} = \frac{24}{3} = 8$ books.
b. Take $\frac{2}{3}$ of 24: $24 \cdot \frac{2}{3} = \frac{24}{1} \cdot \frac{2}{3} = \frac{48}{3} = 16$ books.

Or, you can subtract 8 from 24 to get 16 books.

PROBLEM 4

Compute each of the following, and simplify as needed.

a. $12 \cdot \frac{1}{4}$
b. $12 \cdot \frac{1}{3}$
c. $\frac{4}{5} \cdot \frac{1}{3}$
d. $\frac{3}{7} \cdot 35$

EXAMPLE 3

Compute the product $\frac{21}{32} \cdot \frac{16}{35}$.

SOLUTION

Using the Rule for Multiplying Fractions,

$$\frac{21}{32} \cdot \frac{16}{35} = \frac{21 \cdot 16}{32 \cdot 35} = \frac{336}{1120}$$
The answer certainly does not look simplified. To simplify this fraction, before you multiply the numerators and denominators, write the numerator and denominator in their prime factorization form. Completing the factoring process,

\[
21 \cdot 16 = (3 \cdot 7) \cdot (2 \cdot 2 \cdot 2) = 2^4 \cdot 3 \cdot 7 \\
32 \cdot 35 = (2 \cdot 2 \cdot 2 \cdot 2) \cdot (5 \cdot 7) = 2^5 \cdot 5 \cdot 7
\]

The answer can be simplified by looking for an equivalent fraction as follows:

\[
\frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 7} = \frac{2^4 \cdot 3 \cdot 7}{2^5 \cdot 5 \cdot 7} = \frac{3}{2 \cdot 5} = \frac{3}{10}
\]

In the next section, we will explore division of fractions more carefully.

**PROBLEM 5**

Compute each of the following products and simplify as needed:

a. \( \frac{5}{3} \cdot \frac{3}{5} \)

b. \( \frac{12}{17} \cdot \frac{5}{24} \)

d. \( \frac{24}{25} \cdot \frac{15}{36} \)

c. \( \frac{36}{49} \cdot \frac{13}{15} \)

Consider the problem of computing the product \( 2 \frac{1}{3} \cdot 3 \frac{1}{2} \). Recall from Section 8.2 that \( 2 \frac{1}{3} \) is also the sum \( 2 + \frac{1}{3} \) and that \( 3 \frac{1}{2} \) is \( 3 + \frac{1}{2} \). To multiply two mixed numbers, convert the product of mixed numbers \( 2 \frac{1}{3} \cdot 3 \frac{1}{2} \) to the product of sums:

\[
(2 + \frac{1}{3}) (3 + \frac{1}{2}) = 2 \cdot 3 + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} = 6 + 1 + 1 + \frac{1}{6} = 8 \frac{1}{6}
\]

Below is a visual model of this product using the area model.
The area model illustrates the use of the distributive property that was introduced previously in Section 4.2. One advantage of this method is that you can estimate that the product will be greater than 2 · 3 and less than 3 · 4. That is, the product will be between 6 and 12. Explain why this is true.

Remember in Section 8.5 Example 2, you can add two mixed numbers by first rewriting the fractions as improper fractions. Similarly, you can multiply two mixed numbers by first rewriting them as improper fractions. For example, the product \(2\frac{1}{3} \cdot 3\frac{1}{2}\) could be computed as

\[
2\frac{1}{3} \cdot 3\frac{1}{2} = \frac{7}{3} \cdot \frac{7}{2} = \frac{49}{6}.
\]

Check to see if the answer is in simplest form. The answer can be left as an improper fraction or converted back to a mixed number, whichever is more useful for the original problem.

**PROBLEM 6**

Find the products of each of the following problems. Write your answer as an improper fraction in simplest form and as a simplified mixed number.

- a. \(4\frac{1}{2} \cdot 3\frac{2}{3}\)
- b. \(3\frac{2}{5} \cdot 1\frac{1}{3}\)
- c. \(2\frac{3}{4} \cdot 4\frac{2}{3}\)
- d. \(3\frac{4}{5} \cdot 7\frac{1}{3}\)
EXERCISES

1. Compute the following products and simplify if needed:
   a. \( \frac{2}{3} \cdot \frac{2}{3} \)  
   b. \( \frac{1}{2} \cdot \frac{3}{5} \)  
   c. \( \frac{2}{5} \cdot \frac{1}{4} \)  
   d. \( \frac{2}{7} \cdot \frac{5}{6} \)  
   e. \( \frac{5}{8} \cdot \frac{1}{3} \)

2. Find the reciprocal of the following:
   a. \( \frac{3}{5} \)  
   b. \( \frac{21}{16} \)  
   c. \( \frac{28}{7} \)  
   d. \( \frac{91}{118} \)  
   e. \( \frac{x}{y} \)  
   f. \( \frac{3a}{5b} \)

3. Compute the following products and simplify if needed:
   a. \( \frac{5}{12} \cdot \frac{3}{10} \)  
   b. \( \frac{7}{5} \cdot \frac{5}{7} \)  
   c. \( \frac{6}{1} \cdot \frac{12}{1} \)  
   d. \( \frac{1}{6} \cdot 12 \)  
   e. \( \frac{10}{21} \cdot \frac{18}{25} \)  
   f. \( \frac{1}{20} \cdot \frac{1}{75} \)  
   g. \( \frac{25}{48} \cdot \frac{27}{100} \)  
   h. \( \frac{19}{21} \cdot \frac{7}{38} \)  
   i. \( 27 \cdot \frac{2}{3} \)  
   j. \( \frac{8}{45} \cdot \frac{9}{40} \)

4. Beth lives \( \frac{4}{5} \) of a mile from her school. Walking to school, \( \frac{1}{3} \) of the trip is downhill. How long is the trip downhill?

5. Three-eighths of Ms. Chen’s class have a birthday in the summer. If her class has 24 students, how many of them have birthdays in the summer?

6. Mr. Alexander earned $345.50 commission this week. He earned \( \frac{2}{5} \) of his commission on Tuesday. Which equation can be used to determine the amount of commission, \( m \), Mr. Alexander earned on Tuesday? Select the best choice and explain your answer.
   a. \( 345.50 \div \frac{2}{5} \)  
   b. \( 345.50 - \frac{2}{5} \)
c. $345.50 \times \frac{2}{5}$

d. $345.50 \div \frac{2}{5}$

7. Mr. Rodriguez has a large farm that is in the shape of a rectangle. It is $\frac{3}{4}$ of a mile long and $\frac{2}{5}$ of a mile wide. What is his farm’s area?

8. Which rule can be used to find the value of the $n$th term in the sequence below, where $n$ represents the position of the term? Explain your answer.

<table>
<thead>
<tr>
<th>Position</th>
<th>Value of Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>5</td>
<td>17</td>
</tr>
<tr>
<td>$n$</td>
<td></td>
</tr>
</tbody>
</table>

a. $5n$  

b. $3n + 1$  

c. $2n+1$  

d. $3n+2$

9. Jack has $\frac{3}{8}$ of a pan of brownies left over from a party. He wants to give what is left equally to his 3 younger sisters. How much will each sister receive?

10. In a school survey, $\frac{2}{5}$ of the students preferred juice, $\frac{1}{3}$ preferred soft drinks, and the rest preferred water.

a. What fraction of the student body preferred water?

b. Of those students who preferred soft drinks, $\frac{3}{4}$ of them preferred sugar-free to regular soft drinks. What fraction of the student body prefers sugar-free soft drinks?

c. There are 300 students in the school. How many students prefer each kind of beverage?

11. Mr. Johnson took a survey of his class. He found that $\frac{3}{5}$ of his students have dogs and that $\frac{1}{3}$ of these students have a Chihuahua. If there are 30 students in his class, how many have a Chihuahua?

12. Ramon is making a bulletin board that is $2\frac{2}{3}$ meters wide and $4\frac{2}{3}$ meter long. He plans to cover the board with cloth. How much cloth does he need?
13. In 1960, a large ranch in south Texas had a population of about 1200 deer. By 1995, the deer population on the ranch was $1 \frac{4}{5}$ times as large as in 1960. What was the deer population on the ranch in 1995?

14. Compute each of the following products. Assume that each variable is a non-zero number. Simplify the answer if needed.

   a. $\frac{x}{y} \cdot \frac{y}{x}$
   b. $\frac{3}{xy} \cdot \frac{y}{6x}$
   c. $\frac{4x}{y^2} \cdot \frac{y}{10x^2}$
   d. $\frac{5ac}{b^2} \cdot \frac{bc}{7a^2}$
   e. $\frac{3b^3c^2}{4a^2} \cdot \frac{b^3c^2}{3a^2}$
   f. $\frac{2yz}{5x^2} \cdot \frac{15x}{42}$
   g. $\frac{6ax}{b^2y} \cdot \frac{by^3}{10ax^2}$
   h. $\frac{4bx}{3ay} \cdot \frac{3ay}{4bx}$
   i. $\frac{4x}{y} \cdot \frac{2x^2}{5y^2}$

15. **Ingenuity:**

   At a wedding party, the cook made a number of equal size cakes. Each person is served $\frac{1}{16}$ of a piece of a cake. The server counts that there are $5 \frac{1}{4}$ cakes left. How many more servings can she cut?

16. **Investigation:**

   Compute each of these products:
   
   a. $(1 + x)(1 - x)$
   b. $(1 + x + x^2)(1 - x)$
   c. $(1 + x + x^2 + x^3)(1 - x)$
   d. Guess what the following product is: $(1 + x + x^2 + x^3 + \ldots + x^{14} + x^{15})(1 - x)$
SECTION 9.2  DIVISION OF FRACTIONS

EXPLORATION 1

Rene has 6 pounds of jellybeans. She plans to make little party bags containing $\frac{3}{8}$ pound of jellybeans. How many party bags can she make?

PROBLEM 1

Compute the following quotients:

What is $2 \div \frac{1}{3}$, that is, $2 \div \frac{1}{3}$?

What is $3 \div \frac{1}{5}$, that is, $3 \div \frac{1}{5}$?

What is $3 \div \frac{3}{4}$, that is, $3 \div \frac{3}{4}$?

EXPLORATION 2

Madison has two-thirds of a pan of brownies and shares it evenly among her five friends. How much does each friend receive? Using the area model, each friend gets $\frac{2}{15}$ of Madison’s brownie pan or

$$\frac{1}{5} \cdot \frac{2}{3} = \frac{2}{3} \cdot \frac{1}{5} = \frac{2}{15}.$$ 

PROBLEM 2

Maria has $\frac{3}{4}$ pizza in the refrigerator. She wants to share this equally with 3 friends. Use an area model to determine how much of a pizza each person will get.
In Exploration 2, dividing by 5 produces the same result as multiplying by \(\frac{1}{5}\). You can verify that \(\frac{2}{3} \div 5 = \frac{2}{15}\) by checking to see that
\[
5 \cdot \frac{2}{15} = \frac{10}{15} = \frac{2}{3}.
\]

Use the area model to check that this relationship between division by a whole and multiplication by a fraction works for each of the following:

a. \(\frac{1}{4} \div 2\)  
b. \(\frac{3}{5} \div 6\)  
c. \(\frac{2}{5} \div 4\)  
d. \(\frac{5}{6} \div 3\)

You've discovered that dividing by \(n\) is the same as multiplying by \(\frac{1}{n}\). The following sequence of calculations shows the connection between division and multiplication with fractions:

\[
\begin{align*}
1 \div 5 &= 1 \cdot \frac{1}{5} = \frac{1}{5} & 4 \div 10 &= 4 \cdot \frac{1}{10} = \frac{4}{10} \\
2 \div 3 &= 2 \cdot \frac{1}{3} = \frac{2}{3} & 5 \div 5 &= 5 \cdot \frac{1}{5} = \frac{5}{5} = 1 \\
3 \div 8 &= 3 \cdot \frac{1}{8} = \frac{3}{8}
\end{align*}
\]

In general, a fraction \(\frac{m}{n}\) is another way to write the division problem \(m \div n\). Also, as above, \(\frac{m}{n} = m \cdot \frac{1}{n}\).

Look at a similar problem but with fractional quantities: How many \(\frac{1}{4}\)-pound bags does it take to pack \(\frac{3}{4}\) pounds of sand? In other words, what is \(\frac{3}{4} \div \frac{1}{4}\)?
Using a repeated subtraction model, make 3 equal parts. With the first \(\frac{3}{4}\)-pound bag, \(\frac{3}{4} - \frac{1}{4} = \frac{1}{2}\) pounds are left. The second \(\frac{1}{4}\)-pound leaves \(\frac{1}{2} - \frac{1}{4} = \frac{1}{4}\), so the third \(\frac{1}{4}\)-pound bag leaves no sand.

Writing this as a division problem, \(\frac{3}{4} ÷ \frac{1}{4} = 3\). At first, it might be surprising that when dividing two fractions, the answer is an integer, especially when the integer is large compared to the fractions. What does 3 represent in this case?

Another way to think about this problem uses the missing factor method. What number times \(\frac{1}{4}\) equals \(\frac{3}{4}\)? Starting at 0 on the number line, 3 jumps of length \(\frac{1}{4}\) equals \(\frac{3}{4}\). So, \(\frac{3}{4} ÷ \frac{1}{4} = 3\).

Notice in the earlier example with bags of sand, the quantity of sand exceeded the bag size. A \(\frac{3}{4}\)-pound bag was being separated into smaller \(\frac{1}{4}\)-pound bags. The number of bags was \(\frac{3}{4} ÷ \frac{1}{4} = 3\) bags. What if the initial quantity is less than the bag size, like having \(\frac{8}{1}\) pound of sand and a bag that holds \(\frac{1}{4}\) of a pound? What is \(\frac{8}{1} ÷ \frac{1}{4}\)?

It is impossible to use a “repeated-subtraction” model, because there is no way to fill even one \(\frac{1}{4}\)-pound bag with only \(\frac{1}{8}\) pound of sand. You can see that with the \(\frac{1}{8}\) pound, only \(\frac{1}{2}\) of the bag is filled. Therefore, \(\frac{1}{8} ÷ \frac{1}{4} = \frac{1}{2}\).
Using the relationship between fractions and division, that \( m \div n \) is the same as \( \frac{m}{n} \), rewrite \( \frac{1}{8} \div \frac{1}{4} \) as a big fraction, that is \( \frac{1}{8} \div \frac{1}{4} = \frac{1}{8} \cdot \frac{4}{1} \). Now this looks pretty complicated, but luckily it can be simplified.

The first problem with the big fraction is that the denominator is a fraction. Recall that simplifying a fraction requires rewriting the fraction as an equivalent fraction. But instead of factoring the numerator and denominator, create an equivalent fraction by multiplying both the numerator and denominator by a number that will convert the denominator to a very friendly product. To get a friendly product, multiply the denominator by its own reciprocal. What happens to the denominator when it is multiplied by its reciprocal? To produce an equivalent fraction, the numerator must also be multiplied by the reciprocal of the denominator.

\[
\frac{1}{8} \div \frac{1}{4} = \frac{1}{8} \cdot \frac{4}{1} = \frac{4}{8} = \frac{1}{2}
\]

In general, when the denominator of a fraction is a fraction, multiplying both the numerator and denominator by the reciprocal of the denominator produces a simpler fraction. Another way to simplify complicated fractions uses the pattern that \( m \div n = \frac{m}{n} = m \left( \frac{1}{n} \right) \). Because of this, dividing by \( n \) is the same as multiplying by the reciprocal of \( n \). Using this pattern, dividing by \( \left( \frac{1}{n} \right) \) is the same as multiplying by the reciprocal of \( \frac{1}{n} \), which is \( n \).

Using this pattern, rewrite \( \frac{1}{8} \div \frac{1}{4} \) as \( \frac{1}{8} \cdot \frac{4}{1} \), because the reciprocal of \( \frac{1}{4} \) is \( \frac{4}{1} \). Then multiply to find the answer: \( \frac{1}{8} \cdot \frac{4}{1} = \frac{4}{8} = \frac{1}{2} \).

**PROBLEM 3**

Compute the following division of fractions using the stacking method from above.

a. \( \frac{1}{2} \div \frac{1}{3} \)

b. \( \frac{1}{3} \div \frac{1}{2} \)
**PROBLEM 4**

Christina’s bird feeder holds \( \frac{1}{6} \) of a cup of birdseed. Christina is filling the bird feeder with a scoop that holds \( \frac{2}{3} \) of a cup. How many scoops of birdseed will Christina put into the feeder? Use the numerical technique from above. Write your answer in simplest form.

**PROBLEM 5**

Compute the following quotients.

a. \( \frac{7}{10} \div \frac{2}{5} \)  
b. \( \frac{2}{5} \div \frac{7}{10} \)  
c. \( \frac{7}{8} \div \frac{1}{6} \)  
d. \( \frac{3}{11} \div \frac{6}{33} \)

**RULE 9.4: RECIPROCAL OF AN INTEGER**

If \( n \) is a positive integer, then the **multiplicative inverse** or **reciprocal** of \( n \) is the unit fraction \( \frac{1}{n} \). The product \( n \cdot \frac{1}{n} \) is

\[
n \cdot \frac{1}{n} = 1
\]

Note that, by the **commutative property of multiplication**:

\[
n \cdot \frac{1}{n} = \frac{1}{n} \cdot n = 1
\]

To summarize, when dividing by a fraction, or simplifying a fraction whose denominator is a fraction, use one of the two following techniques:

**Method 1**: Write the division problem as a fraction and multiply the numerator and denominator of this fraction by the reciprocal of the denominator. Notice that multiplying both the numerator and denominator of a fraction by the same number, does not change the value of the fraction, because \( \frac{X}{X} = 1 \), so the fraction is multiplied by 1. This results in an equivalent fraction with denominator 1:

\[
\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c} = \frac{ad}{bc}
\]
Section 9.2 Division of Fractions

Method 2: Or, because division is equivalent to multiplication by the reciprocal, rewrite the division as multiplication:

\[ \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} \]

Either approach will find the quotient. The major point is division transforms into multiplication, not magically, but from a well-motivated reason using a deep understanding of fractions and how they work.

Remember that in \( m \div n \), \( n \) cannot be zero.

EXAMPLE 1

A 3 \( \frac{1}{4} \) ft. long party sub is being cut into pieces that are \( \frac{1}{2} \) ft. pieces. How many servings can be cut?

SOLUTION

Let \( x = \) the number of \( \frac{1}{2} \) ft. pieces in 3 \( \frac{1}{4} \) ft. of sub. \( \frac{1}{2} \cdot x = 3 \frac{1}{4} \). Solving for \( x \), we say

\[
x = 3 \frac{1}{4} \div \frac{1}{2} = \frac{13}{4} \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{26}{4} = \frac{26}{4} = 6 \frac{1}{2}
\]

We can also use the multiplicative inverse property along with other properties previously studied to solve equations.

EXAMPLE 2

Solve the equation \( 3(x+2) = 7 \) algebraically, indicating the properties you use at each step.

SOLUTION

We simplify the equation by writing:

\[ 3(x + 2) = 7 \]
Chapter 9  Multiplying and Dividing Fractions

3x + 6 = 7  
by the distributive property

(3x + 6) – 6 = 7 – 6  
by the subtraction property

3x + (6 – 6) = 7 – 6  
by the associative property

3x + 0 = 7 – 6  
by the additive inverse property

3x = 7 – 6  
by the additive identity property

3x = 1  
subtraction identity

$\frac{1}{3} \cdot (3x) = \frac{1}{3} \cdot 1$  
multiplication property of equality  
(just like addition and subtraction)

$(\frac{1}{3} \cdot 3) \cdot x = \frac{1}{3} \cdot 1$  
associative property of multiplication

$(\frac{1}{3} \cdot 3) \cdot x = (3 \cdot \frac{1}{3}) \cdot x$  
by commutative property of multiplication

1 \cdot x = \frac{1}{3} \cdot 1  
Multiplicative inverse property

x = \frac{1}{3}  
Multiplicative identity property of 1.

The key idea is that if you have an expression like 3x, then when you multiply by $\frac{1}{3}$, you will get x back again, using associativity of multiplication, commutativity of multiplication, the multiplicative inverse, and multiplicative identity properties.

EXERCISES

1. Use a visual model to compute the following quotients. Check your work by using the product of the quotient and the divisor.

   a. 1 ÷ $\frac{2}{3}$  
   b. $\frac{7}{5}$ ÷ $\frac{1}{5}$

   c. 6 ÷ $\frac{3}{2}$  
   d. $\frac{3}{2}$ ÷ 6

   e. $\frac{1}{2}$ ÷ $\frac{1}{3}$  
   f. $\frac{1}{3}$ ÷ $\frac{1}{2}$

   g. $2 \div \frac{2}{5}$  
   h. $\frac{2}{5} ÷ 2$

   i. $\frac{1}{2} ÷ \frac{1}{5}$  
   j. $\frac{1}{5} ÷ \frac{1}{2}$
2. Compute the following quotients by using the process developed in Method 1. Check by using Method 2. Simplify if needed.

\[
\begin{array}{ccccccc}
\text{a.} & \frac{5}{3} & \text{c.} & \frac{1}{6} & \text{e.} & \frac{3}{4} & \text{g.} & \frac{5}{9} \\
\text{b.} & \frac{5}{7} & \text{d.} & \frac{5}{12} & \text{f.} & \frac{1}{20} & \text{h.} & \frac{1}{75} \\
\text{i.} & \frac{7}{15} & \text{j.} & \frac{27}{16} \\
\end{array}
\]

3. Sylvia has 21 meters of cloth on a roll. She needs a piece of material \(\frac{3}{5}\) of a meter long for each decoration she is making. How many decorations can she make with the cloth she has?

4. A duck walks at a rate of \(\frac{3}{4}\) mph. What fraction of an hour does it take the duck to walk \(\frac{1}{8}\) of a mile?

5. Henry walks at a speed of \(3\frac{3}{2}\) miles per hour. How long will it take him to walk 12 miles?

6. Linda biked 9 miles in \(\frac{3}{4}\) of an hour. What was her average speed?

7. A recipe calls for \(\frac{2}{5}\) cup of sugar and you have \(\frac{1}{2}\) cup of sugar. What part of the recipe can you make?

8. Robert has \(\frac{4}{5}\) of a meter of electrical cord. He needs pieces \(\frac{3}{25}\) meters long to make adapters for computers. How many adapters can he make with the \(\frac{4}{5}\) meter of cord? How much cord will be left over?

9. James can build a wooden box in \(\frac{3}{5}\) of an hour. How many boxes can he build in 9 hours?

10. Shelly needs a costume that requires \(\frac{3}{4}\) of a yard of material. The seamstress has \(\frac{1}{2}\) yard of material. How many costume(s) can she make?

11. Solve each of the following equations for \(x\). Describe how you solved a below:

\[
\begin{align*}
\text{a.} & \quad 3 \cdot x = \frac{3}{2} & \quad \text{b.} & \quad \frac{1}{4} \cdot x = \frac{3}{4} & \quad \text{c.} & \quad \frac{2}{3} \cdot x = \frac{1}{6} & \quad \text{d.} & \quad 5 \cdot x = \frac{3}{4} \\
\end{align*}
\]

12. Solve each of the following equations for \(x\).

\[
\begin{align*}
\text{a.} & \quad 3 \cdot x = \frac{3}{2} & \quad \text{c.} & \quad \frac{1}{4} \cdot x = \frac{3}{4} & \quad \text{e.} & \quad \frac{2}{3} \cdot x = \frac{1}{6} \\
\text{b.} & \quad 5 \cdot x = \frac{3}{4} & \quad \text{d.} & \quad \frac{x}{4} = \frac{3}{4} & \quad \text{f.} & \quad \frac{2}{3} \cdot x = \frac{1}{6} \\
\end{align*}
\]
13. Compute each of the following quotients. Assume that each variable is a non-zero number. Simplify the answer if needed.

   a. \( \frac{15}{28a} \div \frac{10}{7a} \)
   b. \( \frac{12x}{yz} \div \frac{z}{10xy} \)
   c. \( \frac{24bc}{a} \div \frac{36c}{25ab} \)
   d. \( \frac{24xy}{z} \div \frac{xy}{10z} \)
   e. \( \frac{15ac}{b} \div \frac{c}{ab} \)
   f. \( \frac{y}{x} \div \frac{y}{1} \)

14. Solve the following equations for \( x \). Show your work and indicate what properties you are using at each step. Check your answers.

   a. \( 2x + 4 = 12 \)
   b. \( \frac{1}{5}(x - 2) = 20 \)
   c. \( 3(x + 4) = 2x - 3 \)
   d. \(-2x = 20 \)
   e. \(-2(x - 3) = 15 \)

15. Compute the following quotients.

   a. \( \frac{4}{5} \div \frac{1}{2} \)
   b. \( \frac{4}{5} \div \frac{1}{2} \)
   c. \( \frac{1}{2} \div \frac{4}{5} \)
   d. \( \frac{1}{2} \div \frac{4}{5} \)
   e. \( \frac{1}{4} \div \frac{2}{3} \)
   f. \( \frac{1}{4} \div \frac{2}{3} \)

16. **Ingenuity:**

   Two-thirds of Ms. Adams’ fifth-grade students are girls. To make the number of boys and girls equal, 4 girls go to the other fifth-grade class, and 4 boys come from that class into Ms. Adams’ class. Now one-half of her students are boys. How many students are in Ms. Adams’ class?
17. **Investigation:**

Consider the following number line.

![Number Line Diagram](image)

a. What point best represents the product of the fractions $C$ and $D$? Why?

b. What point best represents the quotient the quotient $D \div C$? Explain.

c. Is the quotient $B \div C$ greater than or less than $D \div C$? Explain.

d. In which interval would you most likely find $C \div D$: between 0 and $B$, between $B$ and $C$, between $C$ and $D$, between $D$ and 1, or greater than 1? Explain why you think so.
SECTION 9.3 FRACTION, DECIMAL, & PERCENT EQUIVALENTS

In the past, you have probably referred to one-half of a dollar as $0.50, or 50 cents. One half is a fraction that is equal to 0.50, a decimal. We say that $\frac{1}{2}$ is equivalent to 0.50. In this section, we will review how a fraction can be represented as a decimal number and how some decimals can be represented as fractions.

If you buy four apples for a dollar, how much does each apple cost? In Chapter 6, we found that each apple costs $1 \div 4 = $0.25, or 0.25 dollars. We can also say each apple costs a quarter, or $\frac{1}{4}$ of a dollar. So, $\frac{1}{4}$ and 0.25 are equal, or equivalent. Does the fraction $\frac{5}{1}$ have a decimal form? If we buy 5 bananas for $1, we know that each banana costs $1 \div 5 = $0.20, or 20 cents. In other words, each banana costs $\frac{1}{5}$ dollar because it takes 5 ($0.20) to make a whole dollar. So, $\frac{1}{5} = 1 \div 5 = 0.20$, or 20 hundredths. But the decimal 0.20, or twenty hundredths, has the same name as the fraction $\frac{20}{100}$. Does this mean the fraction $\frac{1}{5}$ is equal to $\frac{20}{100}$? These are equivalent fractions, so the decimal 0.20 and the fractions $\frac{1}{5}$ and $\frac{20}{100}$ are equal.

<table>
<thead>
<tr>
<th>PROPERTY 9.1: FRACTIONS AND DIVISION</th>
</tr>
</thead>
<tbody>
<tr>
<td>For any number $m$ and nonzero number $n$ the fraction $\frac{m}{n}$ is equivalent to the quotient $\frac{n}{m}$.</td>
</tr>
</tbody>
</table>

Now we ask, “What decimal is equivalent to $\frac{1}{3}$?” We could also ask, “What is $\frac{1}{3}$ of a dollar?” We can use our new rule to see that $\frac{1}{3}$ is equivalent to the quotient $1 \div 3$. In Section 6.4, we discovered that this quotient is $1 \div 3 = 0.3333\ldots = 0.\overline{3}$. Thus, $\frac{1}{3}$ of a dollar is $0.333\overline{3}$, and we cannot practically divide $1$ into 3 equal parts with our present set of coins. There are other fractions that equal repeating decimals:

\[
\begin{align*}
2 \div 3 &= 0.6666\ldots = 0.\overline{6} = 0.6 \\
1 \div 6 &= 0.1666\ldots = 0.1\overline{6} = 0.16\overline{6}
\end{align*}
\]

Indeed, we learned in the Repeating Decimal Game from Section 6.4 that there are many fractions that have repeating decimals.
EXPLORATION

Convert the following fractions to decimal form. You may verify your answer by dividing with a calculator, if necessary.

\[
\begin{align*}
a. & \quad \frac{1}{4} & \quad b. & \quad \frac{1}{5} & \quad c. & \quad \frac{1}{8} & \quad d. & \quad \frac{1}{3} \\
& \quad 0.25 & \quad & \quad 0.2 & \quad & \quad 0.125 & \quad & \quad 0.3333(\overline{3})
\end{align*}
\]

In converting a decimal to a fraction, we take advantage of the fact that we use the base ten system to write each decimal number. For example, the number 0.3 is called three-tenths and so is equivalent to \(\frac{3}{10}\). The number 0.35 is read as 35 hundredths and so is the same as \(\frac{35}{100}\).

Some fractions can be converted to an equivalent fraction with a denominator that is a power of 10, such as 10, 100, or 1000. In this case, the new equivalent fraction can be written as a decimal quite easily. For example, \(\frac{3}{25}\) is equivalent to \(\frac{3 \cdot 4}{25 \cdot 4} = \frac{12}{100} = 0.12\). We used the factor 4 because 4 times 25 gives us the product 100. Compare this to the process of dividing 3 by 25:

\[
\begin{align*}
25 & \overline{)3.00} \\
& \underline{25} \\
& \quad 50 \\
& \quad 48 \\
& \quad 2
\end{align*}
\]

PROBLEM 1

Convert the following fractions to decimal form by using equivalent fractions:

\[
\begin{align*}
a. & \quad \frac{4}{5} & \quad b. & \quad \frac{11}{20} & \quad c. & \quad \frac{49}{50} & \quad d. & \quad \frac{3}{8}
\end{align*}
\]

You have learned that fractions such as \(\frac{3}{4}\) can be written as the equivalent fraction \(\frac{75}{100}\). This equivalent fraction can also be represented by the decimal 0.75. In some instances, this number can then be converted to 75 percent, 75%. The word percent means “in a hundred” in Latin.
Chapter 9  Multiplying and Dividing Fractions

It is useful to convert some fractions into decimal form by finding the equivalent fraction with the denominator of 100. In converting decimals to percents, you can multiply the decimal by 100 to get the percent. For instance \((0.75)(100) = 75\%\). We illustrate this using the chart below. Test your skills by completing it.

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Fraction in Hundredths</th>
<th>Decimal</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{3}{4})</td>
<td>(\frac{75}{100})</td>
<td>0.75</td>
<td>((0.75)(100) = 75%)</td>
</tr>
<tr>
<td>(\frac{12}{25})</td>
<td>(\frac{48}{100})</td>
<td>0.48</td>
<td>((0.48)(100) = 48%)</td>
</tr>
<tr>
<td>(\frac{4}{5})</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\frac{9}{15})</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\frac{4}{32})</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\frac{7}{10})</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\frac{1}{20})</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\frac{5}{8})</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Similarly, you can reverse the pattern of converting percents to decimals by dividing the percent by 100. For example, 75\% is equivalent to \(75 \div 100 = \frac{75}{100} = 0.75\). Even if the percent includes a decimal part, simply divide by 100 to get its decimal equivalent. For example, 6.5\% is equivalent to \(6.5 \div 100 = 0.065\).

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12.5%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.25%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(24 \frac{1}{7})%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
While the process used in the table is useful for many fractions, it only works for "friendly fractions." A friendly fraction is one that can easily be converted to a fraction whose denominator is a power of ten.

**EXAMPLE 1**

How do you convert a fraction like \( \frac{5}{16} \) into a decimal and a percent?

**SOLUTION**

When a fraction is not "friendly," you can use the division process to convert it to a decimal. For example, \( \frac{5}{16} \) is equivalent to \( 5 \div 16 = 0.3125 = 31.25\% \).

The trickiest conversion from a fraction to a percent involves fractions with repeating decimals. For example, \( \frac{1}{3} = 0.333\ldots \). To convert from a decimal to a percent, multiply \((0.333\ldots)(100) = 33.333\ldots\%\), which can also be written as the mixed fraction \( 33 \frac{1}{3}\% \). The idea of mixed fractions was discussed in more detail in Section 8.6.

**EXAMPLE 2**

In a random survey to find people's favorite car color, 60% of the people surveyed liked red, 10% liked blue, and 30% liked black.

a. Represent this data with a pie graph.

b. If 12 people in the survey liked red, how many people total were surveyed? Solve numerically, and also by using your pie graph.

c. Based on this survey, in a group of 1000 people, about how many might be expected to like red?

**SOLUTION**

a. [Pie chart showing percentages]
Chapter 9  Multiplying and Dividing Fractions

b. Let $T$ is the total number of people. Setting up a proportion of red to total, $\frac{12}{T} = \frac{60}{100}$.

Multiplying both sides of the equation by $100T$

$$\frac{12}{T} \cdot 100T = \frac{60}{100} \cdot 100T$$

$$12 \cdot 100 = 60T$$

$$1200 = 60T$$

$$T = 20$$

c. In 1000 people, approximately 60% of the total should prefer red.  
60% of 1000 = 0.60 \cdot 1000 = 600.

PROBLEM 2

Convert each of the following fractions to a decimal and a percent:

a. $\frac{4}{25}$

b. $\frac{9}{32}$

c. $\frac{5}{6}$

EXAMPLE 3

A class of 25 students has 12 girls. What percent of the class are girls?

SOLUTION

Notice that this problem does not give the percent. You are, in fact, asked to find the percent of girls given that of the 25 students, 12 are girls.

If 12 of the 25 students are girls, the fraction of girls to the total number of students is $\frac{12}{25}$. We can convert this fraction to a decimal and then to a percent. Or, we can find an equivalent fraction for $\frac{12}{25}$ that has a denominator of 100. The fraction $\frac{48}{100}$ is equivalent to $\frac{12}{25}$ and to the decimal 0.48. It is good practice to check your work by checking that the product of the percent of the class that is girls times the class total, $(0.48)(25)$, is equal to 12 = the number of girls in class.
Section 9.3 Fraction, Decimal & Percent Equivalents

PROBLEM 3
Amy was shooting hoops in her backyard. She made 9 of 15 baskets. What percent of her shots did she make?

PROBLEM 4
In a small bag of 32 pieces of mixed candy, there are 4 pieces of lemon candy. What percent is lemon? 12.5%

LINEAR MODEL FOR FRACTIONS ACTIVITY
On a number line, each integer corresponds to a point. Recall that there are many other points between each pair of integers on the number line, and each of these points also corresponds to a number. We will now fill out a table based on a number line. We will locate points and label them with fractions (above) and decimals (below).

Materials: You will need a long strip of paper like a sentence strip or an 18-inch piece of adding machine paper and 10x10 grid paper.
1. Locate and label the point representing \(\frac{1}{2} = 0.5\). Color each of these strips the same color.
2. Do the same for \(\frac{1}{3}\) and \(\frac{1}{4}\) in groups. Check that the whole class is on target.
3. Fill in the remaining strips.
4. Compare the number line with a typical foot ruler or yardstick.
5. Use your new number line to estimate the decimal and percent form of the following fractions:
   a. \(\frac{7}{20}\)  b. \(\frac{5}{12}\)  c. \(\frac{5}{16}\)  d. \(\frac{2}{25}\)
6. Use your 10x10 grid paper to find the decimal and percent form for:
   a. \(\frac{1}{20}\)  b. \(\frac{1}{12}\)  c. \(\frac{1}{16}\)  d. \(\frac{1}{25}\)
7. Use the number line chart to determine which fraction is greater, $\frac{2}{5}$ or $\frac{3}{7}$.

Given two fractions, how can you determine which of them is greater? We can now locate fractions on the number line. The fraction that is to the right of the other is the greater. What problems might arise from this method? For one, its accuracy depends on the quality of the comparative number lines. It becomes harder as the fractions get closer to the same value.

Use the master number line that you constructed or the Fraction Chart to decide which fraction is greater, $\frac{2}{3}$ or $\frac{3}{4}$. Explain your answer. Is there another way to explain which is greater?

**COMPARING AND ORDERING FRACTIONS: THREE METHODS**

**I. Linear Model:**

Use the number line from 0 to 1 to answer the following. Verify your answer by comparing the decimal form for each pair of fractions.

A. Which fraction is greater: $\frac{3}{5}$ or $\frac{4}{9}$? Can you tell how much greater using only the number line?

B. Which fraction is greater, $\frac{3}{4}$ or $\frac{4}{5}$?

C. Use the number line to order the following fractions from least to greatest. Justify your choices. $\frac{2}{3}$, $\frac{3}{5}$, $\frac{5}{8}$, $\frac{7}{10}$, $\frac{5}{9}$.

D. Write the following fractions in order from greatest to least:

$\frac{3}{2}$, $\frac{5}{3}$, $\frac{8}{5}$, $\frac{7}{10}$, $\frac{5}{9}$.

**II. Area Model:**

Use the area model to compare the following pairs of fractions. Use vertical cuts for one fraction and horizontal cuts for the second fraction.

A. Which fraction is greater, $\frac{1}{4}$ or $\frac{1}{3}$? How much greater?
B. Which fraction is greater, $\frac{2}{3}$ or $\frac{3}{4}$? How much greater?
C. Which fraction is greater, $\frac{4}{5}$ or $\frac{3}{7}$? How much greater?

III. Conversion to Decimal

Use the method of converting fractions to their decimal equivalents to place each group of fractions in order.

A. $\frac{3}{4}, \frac{7}{10}, \frac{2}{3}$
B. $\frac{3}{8}, \frac{4}{9}, \frac{1}{2}$
C. $\frac{2}{5}, \frac{3}{7}, \frac{5}{12}$
D. $\frac{8}{13}, \frac{3}{5}, \frac{5}{8}$

Comparing fractions can be useful in real-life problems as well.

EXAMPLE 4

Sara has a 16-ounce cup that has ten ounces of water in it. Mary has a 12-ounce cup that has eight ounces of water in it.

a. They look about equal. Are they? Which cup is fuller?

b. Sara and Mary each drink three ounces of water from their glasses. Which cup is fuller now?

SOLUTION 4

a. Calculate the percentage of water in Sara’s cup. The ratio is 10 ounces water: 16 ounces total, or $\frac{10}{16}$, so Sara’s cup is $\frac{5}{8}$-full. Converting to a percentage, Sara’s glass is 62.5% full.

The ratio of water to the capacity for Mary’s cup is 8 ounces water: 12 ounces total, or $\frac{8}{12}$, so Mary’s cup is $\frac{2}{3}$ full. Converting to a percentage, Mary’s cup is 66.6% full.
b. When Sara drinks three ounces, there will be only 7 ounces of water remaining. Her glass is now $\frac{7}{16}$ or 43.75% full.

When Mary drinks three ounces of water, her glass is $\frac{5}{12}$ or 41.67% full.

Mary’s cup is not as full as Sara’s because she drank $\frac{3}{12} = \frac{1}{4}$ or 25% of her cup’s capacity.

Sara only drank $\frac{3}{16}$ or 18.75% of her cup’s capacity. So three ounces makes a bigger difference to Mary’s amount of water left than to Sara’s.

Thinking of the situation above in terms of fractions, originally Sara’s water to cup ratio versus Mary’s was $\frac{10}{16} < \frac{8}{12}$.

Subtracting 3 from each denominator, yielded the fractions $\frac{7}{16}$ and $\frac{5}{12}$, but $\frac{7}{16} > \frac{5}{12}$.

PROBLEM 5

Finding percents from picture models.

All 750 students at Miller Middle School in San Marcos were asked whether they played a musical instrument and whether they played on a sports team at the school. The Venn Diagram shows the results of the survey.

a. What percent of the students played a musical instrument?

b. What percent of the students played on a sports team?

c. What percent of the students played a musical instrument and played on a sports team?

d. What percent did neither?
EXERCISES

1. Convert each fraction to its equivalent decimal and order them least to greatest.

   a. \( \frac{1}{5} \)  
   b. \( \frac{7}{20} \)  
   c. \( \frac{1}{3} \)  
   d. \( \frac{12}{16} \)

   \[ \begin{array}{l}
   \frac{1}{10} \\
   \frac{1}{20} \\
   \frac{1}{40} \\
   \frac{1}{80}
   \end{array} \]

   \[ \begin{array}{l}
   \frac{11}{25} \\
   \frac{35}{100} \\
   \frac{44}{100} \\
   \frac{1}{48}
   \end{array} \]

   \[ \begin{array}{l}
   \frac{1}{6} \\
   \frac{1}{12} \\
   \frac{1}{24} \\
   \frac{1}{48}
   \end{array} \]

   \[ \begin{array}{l}
   \frac{27}{36} \\
   \frac{15}{20} \\
   \frac{16}{20}
   \end{array} \]

2. Convert each percent to a decimal and then to a fraction in simplest form.

   a. 30%  
   b. 25%  
   c. 40%  
   d. 2%  
   e. 85%  
   f. 0.1%  
   g. 2.4%  
   h. 24%  
   i. 0.6%  
   j. 20.6%  
   k. 2.08%

3. Complete the table below by converting each unit fraction to an equivalent decimal. Use calculators as necessary.

   \[
   \begin{array}{llll}
   \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\
   \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \\
   \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} \\
   \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} \\
   \frac{1}{18} & \frac{1}{19} & \frac{1}{20} & \frac{1}{21}
   \end{array}
   \]

4. Use the number line, paper strip, and 10x10 grid paper to find the following in decimal and percent form:

   (a) \( \frac{3}{10} \)  
   (b) \( \frac{3}{25} \)  
   (c) \( \frac{1}{8} \)
5. For each fraction in simplest form, write an equivalent fraction with a denominator of 100, or in hundredths. Then convert each fraction to its decimal form.

<table>
<thead>
<tr>
<th>Simplest Form</th>
<th>Fractions in Hundredths or Thousandths</th>
<th>Decimal Form</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>3/5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7/10</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9/20</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13/40</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7/25</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6. For each of the following decimals, find a fraction that is equivalent.
   a. 0.4    b. 0.15   c. 0.375
   0.9       0.125   0.98
   0.04      0.12     0.075
   0.05      0.65     0.48

7. Ms. Johnson’s class of 20 students has 9 boys.
   a. The boys make up what percent of the class?
   b. Draw a pie graphs representing the students in the school, showing the percentage of boys and percentage of girls based in the class.
   c. The percentage of boys is the same in Mr. Johnson’s class as in the whole school, and there are 360 boys in the school. How many students are in the school? How many girls are in the school?

8. In Mr. Henry’s math class, 25 students out of 29 passed their test. In Ms. Wiley’s class, 21 students out of 25 passed their test. Which class had a higher passing percentage?

9. In a survey of 24 children, 21 preferred chocolate to vanilla ice cream. What percent of those surveyed preferred chocolate ice cream?

10. A biologist collected 40 turtles of a certain species at the beach. Fifteen of them had spots on their shells.
Section 9.3 Fraction, Decimal & Percent Equivalents

a. What percent of the turtles had spots on their shells?
b. If a random sample of turtles of this species had 25 spotted turtles, about how many turtles were in the sample?

11. For each fraction in the answers to Exercise 5, find the simplest equivalent fraction.

12. In a survey, 745 students were asked what to choose their favorite subject between Math, English, PE, and music. The results were the following:

<table>
<thead>
<tr>
<th>Class</th>
<th>Boys</th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>Math</td>
<td>105</td>
<td>164</td>
</tr>
<tr>
<td>English</td>
<td>125</td>
<td>114</td>
</tr>
<tr>
<td>PE</td>
<td>54</td>
<td>65</td>
</tr>
<tr>
<td>Music</td>
<td>85</td>
<td>103</td>
</tr>
<tr>
<td>Total</td>
<td>369</td>
<td>446</td>
</tr>
</tbody>
</table>

a. Make a pie graph of the results for the boys.
b. Make a pie graph of the results for the girls.
c. Make a pie graph of both boys and girls.
d. Use the pie graphs to find the percentage of boys, the percentage of girls, and the percentage of both who preferred each subject. List your results in a table.
e. Is the percentage of both boys and girls the sum of the other percentages? Why or why not?
f. Use the pie graph to find the percentage of girls who preferred either math or music.
13. For each of the following decimals, find a fraction that is equivalent. Check your answer with a calculator, if necessary.
   a. \(0.333\ldots\)  \(0.0333\ldots\)  \(0.0033\ldots\)  
   b. \(0.111\ldots\)  \(0.444\ldots\)  \(0.0444\ldots\)  
   c. \(0.166\ldots\)  \(0.0166\ldots\)  \(0.00166\ldots\)  

14. Use the number line to discover and represent three fractions that are greater than \(\frac{1}{4}\) and less than \(\frac{1}{2}\). Explain why each fraction is between \(\frac{1}{4}\) and \(\frac{1}{2}\).

15. Which fraction is greater, \(\frac{4}{5}\) or \(\frac{6}{7}\)? By how much? Explain your reasoning.

16. Your Uncle Jim ate \(\frac{1}{4}\) of a cherry pie. Your Aunt Peggy ate \(\frac{1}{6}\) of the same pie. How much pie was eaten? How much pie is left over? Which fraction model did you use to answer these questions?

17. Why is it important to know how to simplify fractions?

18. Determine whether a decimal or fraction representation is more appropriate in the following situation. Explain your answer.
   a. Renee orders a meal at a restaurant. What best represents the cost of the meal, \$14.25 or \$14\frac{1}{4}\?  
   b. Billy orders a hamburger at a fast food restaurant. What best represents the weight of the hamburger, \(\frac{1}{3}\) lb or 0.333\ldots lb?  
   c. A large pizza is divided into 12 slices. Danny and Marie eat 9 slices from a large supreme pizza. What best represents the portion of the pizza eaten, 0.75 or \(\frac{9}{12}\)?
SECTION 9.4 FRACTIONS AND ALTERNATIVES

In examining quantities like “half of a page,” “a tenth of a block,” or “25% of the class,” it is easy to see that fractional parts are represented in more than one way. For example, from the previous sections in Chapters 8 and 9, we wrote “half of a page” numerically as \( \frac{1}{2} \) of a page. Similarly, “a tenth of a block” can be written as 0.1 of a block. In other words, we use fractions, decimals, and percents to indicate parts of a whole.

We could have written \( \frac{1}{2} \) of a page as 0.5 of a page, or 50% of a page. Express 0.1 of a block in fractional terms. How is 0.1 expressed as a percent? Is \( \frac{1}{10} \) or 0.10 better? When is 25% better than its decimal equivalent? When is it better than its equivalent fraction?

There are times when one fractional representation is easier to use than another. In such situations, being able to convert one form to another is very useful.

EXPLORATION 1

Order the numbers below from least to greatest and identify which of the numbers are equal to each other. Explain your reasoning.

\[
\frac{1}{3}, \quad 30\%, \quad 0.33, \quad 0.3, \quad \frac{3}{10}, \quad \frac{1}{3}\%
\]

PROBLEM 1

In Terry’s class there are 36 students, and 25% of the class are absent due to a terrible flu epidemic. How many students are absent?
Chapter 9  Multiplying and Dividing Fractions

The problem leads to the question, “What is 25% of 36?” Earlier you learned that 25% is equivalent to the decimal 0.25 and to the fraction \( \frac{1}{4} \). So, we can rephrase the question in terms of fractions: “What is \( \frac{1}{4} \) of 36?” Why are these the same?

You have two options for finding the answer. One option is to use what you learned about fractions in Section 9.3 to multiply. The other is to use the decimal equivalent and multiply. Which do you prefer?

EXPLORATION 2

Roger needs 75% of a 160-centimeter rope. Decide which of the following equivalent representations would be the easiest for him to use: 75%, 0.75, or \( \frac{3}{4} \). What if he has 200 centimeters and wants to use \( \frac{1}{5} \) of it? Decide which of the following equivalent representations would be easiest for him to use: 20%, 0.20, or \( \frac{1}{5} \). If he wanted \( \frac{1}{6} \) of the 200-centimeter rope, which method, fraction or decimal, would be better? In the problems above, what determines which method is better?

Next, revisit equations using an understanding of fractional and decimal multiplication and division. Notice how useful one form of a fractional representation might be over another.

In Chapter 4, you learned how to solve equations like \( 4x = 7 \). Remember, to solve this equation, divide both sides of the equation by 4.
Because dividing by 4 is equivalent to multiplying by its multiplicative inverse $\frac{1}{4}$, the solution to the equation $4x = 7$ is

$$4x \div 4 = 7 \div 4$$

or

$$4x \cdot \frac{1}{4} = 7 \cdot \frac{1}{4}$$

$$\frac{4x}{4} = \frac{7}{4}$$

$$x = \frac{7}{4} = 1.75$$

To check the answer, find that $4 \cdot \frac{7}{4} = 7$, and $4 \cdot (1.75) = 7$.

**PROBLEM 2**

Solve each of the following pairs of equivalent equations. Choose the most convenient form of the equation and method for solving each equation.

a. $\frac{1}{4}x = 3$ and $0.25x = 3$

b. $\frac{1}{6}x = 4$ and $(0.166\ldots)x = 4$

c. $\frac{2}{5}x = 7$ and $0.8x = 7$

**EXAMPLE 1**

There are 18 green marbles in Julia’s bag. Two-thirds of the marbles in her bag are green. How many marbles does she have in the bag? Justify each step.
SOLUTION

Let \( x \) be the number of marbles in the bag. Then, \( \frac{2}{3} \cdot x \) is the number of green marbles in the bag, so \( \frac{2}{3} \cdot x = 18 \). Now solve the equation:

\[
\frac{2}{3} \cdot x = 18
\]

\[
\frac{3}{2} \cdot \frac{2}{3} \cdot x = \frac{3}{2} \cdot 18
\]

\[
x = 27
\]

Therefore, Julia’s bag contains \( x = 27 \) marbles.

EXAMPLE 2

Amy has some pens in her desk. Nathan starts out with twice as many pens as Amy. Nathan then gives \( \frac{1}{3} \) of his pens to his sister, Lisa. Lisa received 18 pens from Nathan. How many pens does Amy have?

SOLUTION

Let \( A \) be the number of pens Amy has. How many pens does Nathan have? He has \( 2A \) pens. How many pens does Nathan give to Lisa? He gives \( \frac{1}{3} \) of \( 2A \) pens to Lisa, that is, he gives \( \frac{1}{3} \cdot 2A = \frac{2A}{3} \) pens to Lisa. Lisa has 18 pens, so \( \frac{2A}{3} = 18 \). Does this equation look familiar? It has the same form as the one above. So, just as above, the solution is Amy has \( A = 27 \) pens.

As you have seen, there are many equivalent ways to write a product involving fractions. For example, the following expressions are equivalent:

\[
\frac{3x}{4} = 3 \cdot \frac{x}{4} = \frac{3}{4} \cdot x = 3x \cdot \frac{1}{4} = 0.75x
\]

In Sections 8.1, you discovered that \( \frac{a}{b} = a \cdot \frac{1}{b} = \frac{1}{b} \cdot a \). Extend this equivalence to see that

\[
\frac{ax}{b} = ax \cdot \frac{1}{b} = \frac{1}{b} \cdot ax = a \cdot \frac{x}{b} = x \cdot \frac{a}{b}.
\]
If you substitute numbers for the variables, are the expressions still equivalent?

For the following equivalent equations, which form leads to the simplest way to compute a solution? Compute the solution to these equations as many ways as possible and compare the different approaches.

\[
\frac{3x}{8} = 12 \quad \text{or} \quad \frac{3}{8}x = 12 \quad \text{or} \quad 0.375x = 12
\]

**EXERCISES**

1. Jeffrey’s school has 780 students. Find the number of students in each group below. Determine which form is the most convenient to use.
   a. \( \frac{1}{3} \) of the students live within 2 miles of the school.
   b. \( \frac{7}{20} \) of the students bring a sack lunch to school.
   c. \( \frac{1}{12} \) of the students prefer salad to pizza for lunch.
   d. 75% of the students prefer ice cream to pie.
   e. 2 out of every 3 students take the bus to school.

2. Convert each fraction or decimal into a percent:
   a. \( \frac{6}{1} \)
   b. \( \frac{20}{11} \)
   c. \( \frac{5}{8} \)
   d. \( \frac{12}{1} \)
   e. 0.274
   f. 0.028
   g. 0.005

3. Solve each of the following equations in as many ways as you can. Compare the different methods you used.
   a. \( \frac{3x}{4} = 9 \)
   b. \( \frac{3x}{4} = 2 \)
   c. \( \frac{3x}{4} = 1 \)
   
   d. \( \frac{2x}{5} = 6 \)
   e. \( \frac{2x}{5} = 3 \)
   f. \( \frac{2x}{5} = 1 \)

   g. \( \frac{3}{10}x = 6 \)
   h. \( \frac{3}{10}x = 4 \)
   i. \( 0.3x = 2 \)

   j. \( \frac{3x}{20} = 9 \)
   k. \( \frac{3}{20}x = 4 \)
   l. \( 0.15x = 2 \)
4. It takes $\frac{1}{4}$ cup of uncooked rice for a meal for one person. Let $R$ be the amount of rice it takes to feed the guests at a banquet.
   a. What is a formula for $R$?
   b. How much uncooked rice will it take to prepare a meal for 20 people? 30 people?
   c. How many people can you serve with 30 cups of uncooked rice?

5. A farmer grew peach trees and expected a yield of $\frac{1}{2}$ bushel of peaches per tree.
   a. How many trees does she need to pick to fill an order of 75 bushels of peaches for a market?
   b. The actual yield later that year was on average $\frac{2}{3}$ bushel per tree. How many trees did she end up needing to fill the order in part a?

6. It takes Mary $2 \frac{1}{3}$ days to build a table. How many tables can she make in 20 days? 21 days?

7. A store sells pencils for $0.40 each. The cost of pencils is given by the formula $C = 0.4x = \frac{2}{5}x$. Use the formula to answer the following questions:
   a. What is the cost of 12 pencils?
   b. How many pencils can you buy with $8?
   c. How many pencils can you buy with $15?

8. A wholesale company sells pens for $1.60 each. The cost of pens is given by the formula $C = 1.6x = \frac{8}{5}x$. Use the formulas to answer the following questions:
   a. What is the cost of 9 pens?
   b. How many pens can you buy with $200?
   c. How many pens can you buy with $350?

9. The tax rate in Pottsboro is 8%.
   a. Write a formula for the tax paid for buying $x$ dollars of goods.
   b. What is the tax on a $54.50 purchase?
   c. How much was spent if the tax was $6.64?
   d. What is the amount of tax paid if the total cost was $75.06?
10. Frida’s Fruit Punch is made from water and syrup. Twelve percent of every cup is syrup.
   a. Write a formula for the number of cups of syrup needed to make $x$ cups of fruit punch.
   b. How many cups of syrup are in 16 cups of fruit punch?
   c. How many cups of fruit punch can be made from 2.6 cups of syrup?

11. Solve each of the following equations:
   a. $\frac{2x}{3x} = \frac{3}{4}$
   b. $\frac{x}{3} = \frac{2}{5}
   c. \frac{7x}{12} = \frac{3}{16}$
   d. $\frac{2x}{5} - \frac{3}{7} = \frac{5}{7}$
   e. $\frac{3x}{10} + \frac{5}{2} = 3$
   f. $\frac{7x}{12} - \frac{3}{20} = \frac{11}{18}$

12. Ingenuity:
    Review the definition of factorial in the ingenuity in Section 4.3. Find $\frac{104!}{102!}$.

13. Investigation:
    Meg has a bag of marbles containing 5 red marbles and 5 green marbles.
    a. What fraction of the marbles are red?
    b. Meg buys 4 more identical bags of marbles. What fraction of the marbles are red?
    c. How many red marbles will she have when she has 12 such bags of marbles?
REVIEW PROBLEMS

1. Compute the following. Write answers in simplest form.
   a. \(5 \cdot \frac{2}{3}\)
   b. \(\frac{2}{3} \cdot \frac{3}{4}\)
   c. \(\frac{4}{5} \cdot \frac{5}{6}\)
   d. \(\frac{10}{3} \cdot \frac{2}{5}\)
   e. \(3 \cdot \frac{5}{6}\)
   f. \(\frac{9}{14} \cdot \frac{7}{12}\)
   g. \(\frac{10}{27} \cdot \frac{9}{25}\)
   h. \(\frac{36}{21} \cdot \frac{35}{48}\)

2. Iris baked a dozen cupcakes. James ate \(\frac{1}{4}\) of the cupcakes. How many cupcakes did James eat?

3. Daisy would like to use \(\frac{2}{3}\) of her 42 turquoise jewels to make a necklace. How many jewels will she use?

4. A poll showed that \(\frac{3}{4}\) of those surveyed prefer pizza to hamburgers. If there were 56 people surveyed, how many of them prefer pizza?

5. A tank was \(\frac{2}{3}\) full of water. If you drain \(\frac{2}{5}\) of the water presently in the tank, how much water will be left in the tank?

6. Pablo is covering the face of his math book that measures \(6\frac{4}{1}\) inches wide and \(10\frac{1}{8}\) inches tall. How many total square inches will Pablo be covering?

7. Compute the following. Write answers in simplest form.
   a. \(2 \div \frac{1}{4}\)
   b. \(\frac{3}{4} \div \frac{1}{4}\)
   c. \(\frac{5}{8} \div \frac{1}{2}\)
   d. \(\frac{3}{2} \div \frac{1}{3}\)
   e. \(\frac{2}{3} \div \frac{2}{5}\)

8. Daisy is making some jewelry for her sister. It takes her \(5\frac{2}{5}\) hours to make 3 pieces of jewelry. How long does it take Daisy to make 1 pieces of jewelry?

9. Nila is stringing together some yarn to make the hair on her paper doll. She would like each string to measure \(4\frac{3}{4}\) inches long. She is starting with a string that is 30 inches long. How many yarn hair strings can she get from this 30 inch string?

10. Sandra is running a marathon of 26 miles. If she wants to drink a bottle of water every \(3\frac{1}{2}\) miles, how many bottles does she need to prepare for the entire race?
11. Using the number line below, give three fractions that are greater than $\frac{1}{2}$ and less than $\frac{4}{5}$. Plot each on the number line to show how they are ordered.

![Number line](image)

12. Convert the following fractions to an equivalent decimal.
   a. $\frac{2}{5}$
   b. $\frac{12}{25}$
   c. $\frac{6}{12}$
   d. $\frac{3}{4}$
   e. $\frac{7}{10}$

13. Convert the following decimals to an equivalent fraction.
   a. 0.25
   b. 0.3
   c. 0.07
   d. 0.60
   e. 0.125

14. Convert each fraction to a percent.
   a. $\frac{1}{5}$
   b. $\frac{3}{4}$
   c. $\frac{6}{10}$
   d. $\frac{3}{8}$
   e. $\frac{7}{14}$

15. Convert each decimal to a percent.
   a. 0.22
   b. 0.1
   c. 0.75
   d. 0.125
   e. 0.05

16. Convert each decimal to a fraction then a percent.
   a. 0.24
   b. 0.35
   c. 0.3
   d. 0.375
   e. 0.09

17. Using the number line below, compare and order the following: 0.75, $\frac{1}{3}$, 20%, 0.07, 50%, 0.25, $\frac{2}{5}$.
Chapter 9 Operations with Fractions

CHALLENGE PROBLEMS

Section 9.1:
How many fractions (in simplest form) between $\frac{1}{4}$ and $\frac{1}{2}$ have a denominator less than 10?

Section 9.2:
How many pairs of positive integers (a,b) with $a \leq b$ satisfy $\frac{1}{a} + \frac{1}{b} = \frac{1}{6}$?

Section 9.3:
Compute the denominator, after simplifying, of $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{19}{20}$.

Section 9.4:
Simplify $1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}}}$.

Section 9.5:
If $1 + \frac{2}{5} + 4 \cdot \frac{5}{6} + 7 \cdot \frac{8}{9} + \cdots + n \cdot \frac{n + 1}{n + 2} < 100$, what is the largest possible value of n?
Fractions are often used to compare quantities. For example, Miller Junior High School has 400 students, and 280 of them live within 3 miles of the campus. Simplifying, \( \frac{280}{400} = \frac{7}{10} \). Notice that both units are “students” and they simplify. How can you interpret the meaning of the simplified fraction?

In Miller Junior High School, 7 out of every 10 students live within 3 miles of the school. Converting the fraction \( \frac{7}{10} \) into a percent, 70% of the students live within 3 miles. The fractional form of this comparison is called a ratio. A ratio is a division comparison of two quantities with or without the same units.

Ratios can also be written in the form of first one quantity, then a colon, followed by a second quantity. Ratios can be written using the word “to” in place of the colon and in fraction form. In this example, the unit of measure is the number of students.

Because there are 280 students who live within 3 miles, write

\[ 280 \text{ students} : \quad \text{______} \]

Compared to these 280 students within 3 miles, there are 400 total students. So the ratio of students who live within 3 miles to total number of students is

\[ 280 \text{ students who live within 3 miles} : 400 \text{ total students} \]

Ignoring for a moment the units and using only the numbers, write the ratio as \( 280:400 \). Just as with fractions, we can simplify this to \( 7:10 \), that is, \( \frac{280}{400} = \frac{7}{10} \). Always remember what kinds of things are being compared. In this problem, what does the ratio \( 7:3 \) describe?
**Chapter 10  Rates, Ratios, and Proportions**

**EXPLORATION 1**

Suppose a class has 12 boys and 18 girls. How many other ratios can you discover in this situation? Write the following comparisons as ratios:

a. the number of boys to the number of girls \( \frac{12}{18} \), \( \frac{2}{3} \)

b. the number of girls to the number of students \( \frac{3}{5} \), \( \frac{18}{30} \)

c. the number of students to the number of girls \( \frac{18}{3} \), \( \frac{30}{5} \)

d. the number of boys to the number of students \( \frac{12}{30} \), \( \frac{2}{5} \)

e. the number of students to the number of boys \( \frac{30}{12} \), \( \frac{5}{2} \)

In computing the ratios above, notice that some of the ratios are between a part and the whole, such as girls to students, and some of them are between two parts, such as boys to girls.

Rates are special ratios that compare different units. Suppose, you earn 48 dollars for doing 6 hours of yard work and mowing the lawn. You know that \( 48 \div 6 = 8 \) indicates how much money you earned per hour. Using fractions, this calculation looks like \( \frac{48}{6} = 8 \). However, it is usually helpful to write this problem using the units that describe each quantity. So the calculation becomes

\[
\frac{48 \text{ dollars}}{6 \text{ hours}} = \frac{8 \text{ dollars}}{1 \text{ hour}} = 8 \text{ dollars per hour}.
\]

<table>
<thead>
<tr>
<th>Hours</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dollar</td>
<td>8</td>
<td>16</td>
<td>24</td>
<td>32</td>
<td>40</td>
<td>48</td>
</tr>
</tbody>
</table>

You read “\( 8 \text{ dollars per hour} \)” as “8 dollars per hour.” The answer explains exactly how many dollars you earned each hour. This quantity is an example of a rate. A rate is defined as a division comparison between two quantities, usually with two different units, like dollars and hours. The simplified fractional answer in the example is called a unit rate because it represents a number or quantity per 1 unit, or hour in this case. The units may be written in fractional form, like \( \frac{\text{dollars}}{\text{hour}} \), \( \frac{\text{miles}}{\text{hour}} \) or \( \frac{\text{gallons}}{\text{mile}} \).
Section 10.1 Rates and Ratios

The word *rate* is usually used to distinguish an even more specific kind of ratio: one involving change, such as \(\frac{\text{dollars}}{\text{hour}}\), \(\frac{\text{miles}}{\text{hour}}\), and \(\frac{\text{miles}}{\text{gallon}}\). From Exploration 1, suppose you wish to compare the genders in a class with 12 boys and 18 girls. Comparing the number of girls to boys in this class gives the ratio \(\frac{18\text{girls}}{12\text{boys}}\). Simplifying this fraction, the ratio of girls to boys in the class is \(\frac{3}{2}\). Although it is mathematically correct to say the rate of girls per boy in the class is \(\frac{3}{2}\), we rarely use the word rate in that comparison because the units are different and the ratio does not involve change.

EXPLORATION 2

Juan drove 150 miles in 3 hours and used 5 gallons of gasoline. Make as many rates using these quantities and their units as possible. Explain what each unit fraction means.

EXAMPLE 1

Sandra’s bakery uses 12 cups of flour to make 3 cakes when she bakes. How many cups of flour will she use when she bakes 7 cakes for a customer?

SOLUTION

The unit rate for the amount of flour per cake is given as \(\frac{4\text{ cups}}{\text{1 cake}} = 4\text{ cups/cake}\).

<table>
<thead>
<tr>
<th>cups</th>
<th>cakes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

To find how much flour is used to bake 7 cakes, multiply the unit rate by the number of cakes:

\[
4\text{ cups/cake} \cdot (7\text{ cakes}) = 28\text{ cups/cake} \cdot \text{cake} = 28\text{ cups}
\]
Chapter 10 Rates, Ratios, and Proportions

Notice that the units of “cake” simplify to one and the answer is in cups. This is similar to computing the product \( \left( \frac{7}{4} \right) \cdot 4 = 7 \). This way of keeping track of the units is very useful in application problems, especially in science.

What happens when we compare two quantities with the same unit?

PROBLEM 1
Karla rode her bike for \( 2 \frac{1}{2} \) hours and traveled 20 miles. What was her average rate, or speed? Approximately how far did she travel in the first hour and a half?

EXPLORATION 3
Sally works as a computer consultant for the Lennox Company and earns $612 for working 36 hours. If she charges the same amount per hour, how much will she earn working 5 hours? 10 hours? 15 hours? Use unit rate to solve and show your work in table form.

PROBLEM 2
Karen and Karla both jog everyday. Karen jogs an average of 1,800 meters in 40 minutes and Karla jogs an average of 1,500 meters in 30 minutes.

a. Who jogs faster?
b. On average, how far does Karen jog in 12 minutes?
c. Maintaining her pace, how long does it take Karla to jog 600 meters?

Problem 2 is an example of the rate formula that assumes that something is traveling at a constant rate. The distance traveled is equal to the constant rate multiplied by the time traveled: \( d = r \cdot t \) or \( d = rt \). The constant rate is often written as the unit rate. Typical units for \( r \) are:

- miles per hour \( \left( \frac{\text{mi}}{\text{hr}} \right) \) = miles traveled in one hour
- feet per second \( \left( \frac{\text{ft}}{\text{sec}} \right) \) = feet traveled in one second
- meters per minute \( \left( \frac{\text{m}}{\text{min}} \right) \) = meters traveled in one minute
PROBLEM 3

Sandy went canoeing for 3 hours and traveled 12 miles down river. How fast was she traveling? At the same rate, how far will she travel in 5 hours?

Some typical equivalents with the corresponding conversion rates equal to 1 are

\[3 \text{ feet} = 1 \text{ yard}\]
\[
\frac{3 \text{ ft}}{1 \text{ yd}} = 1 = \frac{1 \text{ yd}}{3 \text{ ft}}
\]

\[9 \text{ sq ft} = 1 \text{ sq yd}\]
\[
\frac{9 \text{ sq ft}}{1 \text{ sq yd}} = 1 = \frac{1 \text{ sq yd}}{9 \text{ sq ft}}
\]

\[60 \text{ minutes} = 1 \text{ hour}\]
\[
\frac{60 \text{ min}}{1 \text{ hr}} = 1 = \frac{1 \text{ hr}}{60 \text{ min}}
\]

\[8 \text{ pints} = 1 \text{ gallon}\]
\[
\frac{8 \text{ pints}}{1 \text{ gal}} = 1 = \frac{1 \text{ gal}}{8 \text{ pints}}
\]

\[10 \text{ millimeters} = 1 \text{ centimeter}\]
\[
\frac{10 \text{ mm}}{1 \text{ cm}} = 1 = \frac{1 \text{ cm}}{10 \text{ mm}}
\]

Because the conversion rate is equal to 1, multiplying by one of the rates maintains equivalence. For example,

EXAMPLE 2

a. Convert 4 miles to yards

\[
4 \text{ miles} \left( \frac{1760 \text{ yd}}{1 \text{ mile}} \right) = 7040 \text{ yards}
\]

b. Convert 3 1/2 hours to minutes

\[
3 \frac{1}{2} \text{ hours} = \left( 3 \frac{1}{2} \text{ hr} \right) \left( \frac{60 \text{ min}}{1 \text{ hr}} \right) = \frac{7}{2} \cdot 60 \text{ min} = 7 \cdot \frac{1}{2} \cdot 60 \text{ min} = 7 \cdot 30 \text{ min} = 210 \text{ minutes.}
\]
EXAMPLE 3

Susan has $5\frac{1}{3}$ yards of cloth. How many feet of cloth is this? How many inches of cloth does she have?

SOLUTION

Convert $5\frac{1}{3}$ yards to $\frac{16}{3}$ yards. Then multiply $\frac{16}{3}$ yards by the conversion factor $\frac{1}{3}$ or $\frac{3}{1}$ yd to convert to feet:

$$\frac{16}{3} \text{ yds} \cdot \frac{3 \text{ ft}}{1 \text{ yd}} = 16 \text{ ft}.$$

Now convert feet into inches, using the conversion unit $\frac{12 \text{ in}}{1 \text{ ft}}$:

$$16 \text{ ft} \cdot \frac{12 \text{ in}}{1 \text{ ft}} = 192 \text{ in}.$$

EXERCISES

1. Gladys has 3 shirts, 5 pairs of pants and 2 hats in a suitcase.
   a. What is the ratio of shirts to pants?
   b. What is the ratio of pants to shirts?
   c. What is the ratio of pants to hats?
   d. What is the ratio of hats to pants?
   e. What is the ratio of shirts to hats?
   f. What is the ratio of hats to shirts?

2. Mr. Morton decides to give candy to only 2 out of every 5 students in his class.
   a. What is the ratio of students who receive candy to those who don’t?
   b. If the class has 30 students, how many will receive candy?
3. The table below hangs on the wall of a bakery. The baker claims that each offering yields the same price per donut. Is she correct? Explain how you decided.

<table>
<thead>
<tr>
<th>Donuts</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$2.00</td>
</tr>
<tr>
<td>8</td>
<td>$3.20</td>
</tr>
<tr>
<td>12</td>
<td>$5.00</td>
</tr>
<tr>
<td>20</td>
<td>$8.00</td>
</tr>
</tbody>
</table>

4. A bag contains red, blue and green marbles. Twenty percent of the marbles are red, thirty percent are blue and fifty percent are green. Draw a picture using area to represent the portion of each color of marble. Does the picture help to determine the ratios?
   a. What is the ratio of blue marbles to the total number of marbles?
   b. What is the ratio of green marbles to the total number of marbles?
   c. What is the ratio of blue marbles to red marbles?
   d. What is the ratio of red marbles to green marbles?
   e. What is the ratio of green marbles to blue marbles?
   f. Let \( T \) be the total number of marbles. Write an expression using \( T \) that describes the number of red marbles. Then, write an expression using \( T \) that describes the number of green marbles, and then of blue marbles.

5. Steve rides his bike 42 miles in 3 hours. How far will he ride at a constant rate for 1 hour? How far does he travel in \( 2 \frac{1}{4} \) hours? 14 miles; 31.5 miles

6. Ron went to the grocery store to purchase cereal. He found 15 ounces for $4.05, 8 ounces for $1.89 and 20 ounces for $5.60. What is the best price for Ron’s cereal? What is the unit cost for each container of cereal?
7. The entering freshman class of Texas Technical College has 1,200 students. Of those, 782 are from Texas and 102 are international students. There are 525 science majors, 413 liberal arts majors, and 150 music majors. There are 630 females. Determine whether the statements in parts a–f are true or false, and explain your reasoning.

a. More than 3 out of 4 students are science or liberal arts majors.
b. More than 1 out of every 11 students is an international student.
c. The ratio of music majors to non-music majors is 1 : 7.
d. The ratio of science majors to non-science majors is 5 : 6.
e. The ratio of male to female freshman is 19 : 20.
f. The ratio of Texan to non-Texan is less than 1.87 to 1.
g. Four out of five students usually graduate in 4 years. How many will that be from this class?
h. Three out of five students who enter as freshman usually go to graduate school. At that rate, how many of the entering class of 1,200 students will go to graduate school?

8. Use dimensional analysis to convert the following:

a. \(2 \frac{1}{2}\) gallons = _____ pints
b. 1.3 km = _______ meters
c. 2.5 km = _______ cm
d. 48.5 cm = _______ m

9. Which of the following can be used to find \(y\), the number of yards in 7 miles? Explain your answer.

a. 5280÷3
b. 7(5280)
c. 1760(7)
d. 7(3)(5280)

10. Manuel is driving his car at 60 mph.

a. How many feet does he travel in one hour?
b. How many feet does he travel in one second?
11. In baseball, a batting average is computed by dividing the number of hits by the number of times at bat, and then multiply by 1,000. Tony has a 326 batting average. How many hits would he have in a typical baseball season of 500 times at bat?

12. Jasmine is building birdhouses. It takes her 4 hours to build 7 houses. Which of the following is an equivalent rate? Explain your answer.
   a. 10 hours to build 16 houses
   b. 28 hours to build 49 houses
   c. 18 hours to build 35 houses
   d. 5 hours to build 8 houses

13. Bobbie and Marie play soccer for the same team. The ratio of Bobbie’s goals scored to the rest of the team, including Marie, is 3 : 7. The ratio of Marie’s goals scored to the rest of the team, including Bobbie, is 4 : 5. What is the ratio of goals scored by Bobbie and Marie to the rest of the team?

14. **Ingenuity: Geometric Series**
   It is a curious fact that in arithmetic sometimes sums of fractions go on forever and are still relatively small. Find the value of the following infinite sum. It may help to visualize the sum geometrically.
   \[
   \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots
   \]

15. **Investigation:**
   Jeff and Shawn are going for a half-mile swim, then a 5-mile run. Shawn swims at a rate of 1.5 mph and runs at a rate of 8 mph. Jeff swims at rate of 2 mph and runs at a rate of 10 mph. Jeff gives Shawn a 10-minute head start.
   a. Who finished the swim first? How much longer did it take the second swimmer to finish? How far ahead was the first person?
   b. Who finished the run first? How long did the winner wait for the other to finish? How far ahead did the winner finish?
16. Fineas and Monica go to the library. Fineas goes to the math books section and chooses every third book. Monica goes to the physics section and checks out every seventh book. If the library has the same number of math and physics textbooks, what is the estimated ratio of the number of books Fineas checks out to the number of books Monica checks out?
SECTION 10.2 RATES OF CHANGE AND LINEAR FUNCTIONS

Let us continue to look at comparing quantities while also looking at changes that occur in those quantities. For example, if a bag contains 3 black marbles and 4 white ones, the ratio of black to white marbles is 3:4. Pictorially, a 3:4 ratio could look like this:

Notice that for every three black marbles, there are four white ones. Another bag has double the number of black and white marbles. Draw a picture like the one above to represent the ratio of marbles in this bag. Are there still four white marbles for every three black ones? What is the overall ratio of black to white marbles? What is the relationship between the two ratios? Such ratios are equivalent because they form equivalent fractions.

EXPLORATION 1

A restaurant makes and sells a famous dish that contains rice and beans. The ratio of rice to beans in its secret recipe is 1:2, and the ratio of beans to rice is 2:1.

<table>
<thead>
<tr>
<th>Cups of beans</th>
<th>Cups of rice</th>
<th>( \frac{\text{rice}}{\text{beans}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>y</td>
<td>( y = )</td>
</tr>
</tbody>
</table>

Step 1: Complete the table of possible amounts of rice and beans that the chef uses to make the dish.
Step 2: Use the numbers in the table as coordinates of points. Make a graph using the data points. For each point, the number of cups of beans is the $x$-coordinate and the number of cups of rice is the $y$-coordinate. Describe the graph.

Step 3: For each of the rows in the table, what is the ratio of rice to beans? How is the second column of amounts of rice changing?

Step 4: Could you start with a smaller amount of beans? Pick two possible smaller amounts of beans and find the corresponding amounts of rice.

Step 5: The graph is a set of points on a straight line. Define $R(x)$ as the number of cups of rice needed for $x$ cups of beans in the recipe. Write a rule for this linear function $R$. What is the ratio of rice to beans?

Step 6: Use the rule for $R$ to compute the amount of rice needed for $3\frac{1}{2}$ cups of beans. Use the graph to confirm that your answer makes sense.

This exploration is an example where two quantities change but the relationships or rates of change between them remain the same. The quantities have a constant rate of change and result in graphs that are lines. There are other non-linear ways in which two quantities can vary. You will study these interesting relationships in algebra.

Another setting for rates of change is in movement, like walking, biking or riding in a car. In each of these instances some distance is traveled over a time interval. For example, you might walk 20 feet in 10 seconds, bike 10 miles in 30 minutes, or travel 60 miles in a car in 1 hour. Notice in each example that there are two quantities: one is a length unit and the other a time unit. Terms like feet per second, miles per minute or miles per hour all refer to rates, which were discussed in a previous section. Walking 20 feet in 10 seconds can be restated as a certain number of feet walked in 1 second, which is a unit rate as we discussed in 10.1. What is the walking unit rate? the biking unit rate?

Information about distance and time is useful in finding rates of change.
EXAMPLE 1

Priscilla takes a long walk on the weekend, walking at a steady rate of 4 miles per hour. Using the formula \( d = rt \), if \( r = 4 \) notice that the distance \( d \) depends on the amount of time \( t \) Priscilla walks. This process is like the frog-jumping model on the number line from Section 4.1.

From the picture, make the following table:

<table>
<thead>
<tr>
<th>Time ( t )</th>
<th>Distance ( d )</th>
<th>Rate ( \frac{d}{t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>( \frac{4}{1} = 4 )</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>( \frac{8}{2} = 4 )</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>( \frac{12}{3} = 4 )</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>( \frac{16}{4} = 4 )</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>( \frac{20}{5} = 4 )</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
<td>( \frac{24}{6} = 4 )</td>
</tr>
</tbody>
</table>

Now represent the situation as a graph
Priscilla walks four miles each hour. Looking at the table, it is easy to see that after each hour, Juan’s distance has changed by 4 miles. As in Section 5.5, the constant change in the second coordinates $d$ is called a constant rate of change for the distance function $d$. The third column $r$ gives us the same constant. From Section 4.5, this number is the constant of proportionality. For linear functions in the form $y = mx$, the two constant properties are the same and are both equal to $m$. However, a general linear function $y = mx + b$ when $b$ is not 0 has a constant rate of change but does not have a constant of proportionality. Explain why.

**PROBLEM 1**

In the graph below, $d$ represents the distance traveled by a bicycle rider and $t$ represents the number of hours ridden. What is the rate in miles per hour? Use this rate to write a formula for $d$ in terms of $t$. 
EXAMPLE 2

Jane bought 4 drinks for $9.

a) What is the cost of 1 drink? What is the rate of cost for drink?

b) Fill in the following table.

<table>
<thead>
<tr>
<th>x</th>
<th>y = C(x)</th>
<th>y ÷ x</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c) Does the cost function have a constant rate of change? If so, what is it? Where is it in the equation?

What do you notice about the third column?

SOLUTION

We form the rate of \( \frac{\$9}{4 \text{ drinks}} \), so the unit rate is $2.25 per drink. Use the unit rate of $2.25 per drink to find a formula for the function \( C \) where the cost \( C(x) \) is the cost for \( x \) drinks. Use the function \( C(x) = \frac{9}{4}x \) to find that Jane spent \( 2.25(7) = \$15.75 \).

PROBLEM 2

Make up table (with at least 5 points) and write the formula for a linear function that has a constant rate of change of 2.5.
PROBLEM 3

Find the constant rate of change for the linear functions represented by their graphs on the coordinate system below:

\[ y = 3x \]
\[ y = -2x \]

PROBLEM 4

In the graph below, \( y \) represents the number of teaspoons of honey used to make \( x \) number of cupcakes. What is the ratio of teaspoons of honey to number of cupcakes? Use this ratio to write a formula for \( y \) in terms of \( x \).
EXERCISES

1. Write an equation and make a table with five points for a linear function with both a constant rate of change and proportionality of 6.

2. Write an equation and make a table with five points for a linear function with both a constant rate of change and proportionality of -3.

3. For each of the following linear functions, determine both the constant rate of change and the constant of proportionality, if it exists. If there is no constant of proportionality, explain why it doesn’t exist.
   a. The function given by \( y = 7x \)
   b. The function given by \( y = -5x \)
   c. The function given by \( y = 3x + 2 \)
   d. The function given by \( y = \frac{x}{4} = \frac{1}{4} \cdot x \)
   e. The function given by \( y = \frac{3x}{2} \)

4. The table below is from a linear function. What is the constant rate of change for this function? What is the constant of proportionality? What is the ratio of \( y \) to \( x \)? What is the linear function?

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( \frac{y}{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-4</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-8</td>
<td></td>
</tr>
</tbody>
</table>

5. The table below is a record of the number miles a school bus has traveled during a band trip. What is the constant rate of change for the function? What is the formula for the distance \( d \) in miles as a function of time \( t \) in hours? What is the ratio of distance to time?

<table>
<thead>
<tr>
<th>Hours</th>
<th>Distance in miles</th>
<th>( \frac{d}{t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>64</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>128</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>192</td>
<td></td>
</tr>
</tbody>
</table>
6. The picture below represents the distance a jackrabbit travels. Each hop covers the same distance. If the scale is in feet, what is the constant rate of feet-per-hop? Write a formula for the distance as a function of the number of hops.

![Diagram of a jackrabbit with hops and distance markers]

7. For each of the linear functions given by graphs below, determine the constant rate of change and the constant of proportionality.

![Graphs of linear functions: y = 2x, y = -2x, y = x/2, y = 2x]

8. A coffee shop sells hot chocolate for $2.75 per cup and 8 cups of regular coffee for $28.
   a. How much does it cost to buy 13 cups of hot chocolate?
   b. How much do 5 cups of regular coffee cost?
   c. How much does each cup of coffee cost?
   d. Make a table for each of the cost functions.
   e. Calculate formulas for the costs of hot chocolate and for coffee.
f. Graph each of the cost functions on the same coordinate system.
g. Determine the constant of proportionality for each of these linear functions.

9. In the graph below, \( y \) represents the cups of sugar and \( x \) represents the number of kilograms of cake batter made at a large baker. What is the ratio of sugar to cake batter? Use this ratio to write a formula for \( y \) in terms of \( x \).


11. Molly’s Binding binds writing assignments for students. The company charges $0.70 for binding one writing assignment and, in addition, $0.05 per page.
   a. How much does it cost to bind an assignment with 6 pages? 15 pages?
   b. Make a table for this cost function with at least 6 inputs/outputs.
   c. Write a rule for the cost function \( C \) where \( C(x) \) is the cost of binding a writing assignment with \( x \) pages.
   d. Draw a graph of this cost function.
   e. What is the constant rate of change? Does the cost function have a constant of proportionality? Explain.

12. Zack’s bicycle wheel can travel about 8.5 feet per revolution. Which statement is best supported by this information? Explain your answer.
   a. The wheel can travel about 415 feet in 50 revolutions.
   b. The wheel can travel about 72 feet in 8 revolutions.
10.2 Rates of Change and Linear Functions

c. The wheel can travel about 297.5 feet in 35 revolutions.

d. The wheel can travel about 300 feet in 36 revolutions.

13. Ron went to the grocery store to purchase cereal. He found 15 ounces of brand A for $4.05 and 20 ounces of brand B for $5.60. Which brand of cereal has the best price for? What is the unit cost of each container of cereal? Make a table and determine the unit rate.

14. In a factory, one machine makes flashlights at a rate of 140 flashlights per hour, and another machine makes the same flashlights at a rate of 175 flashlights per hour. Which of the following equations can be used to find \( t \), the total number of flashlights both machines will make in 6 hours? Explain your answer.
   a. \( (140 + 175) ÷ 6 \)
   b. \( (175)(6) - (140)(6) \)
   c. \( (140 + 175)(8) \)
   d. \( 140(8) + 175 \)

15. Sandra bought 6 T-Shirts for $39.00.
   a. How much does one T-Shirt cost?
   b. How much would 4 T-Shirts cost?
   c. What is the formula for \( C(x) \), the cost of \( x \) T-Shirts?
   d. What is the constant of proportionality for this cost function?

16. Mike bought 8 fruit drinks for $11.20. How much would 20 drinks cost?

17. Scientists developed a new variety of flower. They found that 40% or \( \frac{2}{5} \) of the seeds produced plants with purple flowers and the rest produced plants with white flowers. Let \( P(x) \) be the number of kilograms of seeds that will produce purple flowers from a bag with \( x \) kilograms of the new variety's seeds. How many kilograms of purple seeds will there be in a 5kg bag, 8 kg bag, and 1 kg bag of seed?
   a. Write a rule for \( P(x) \), where \( P \) is the purple flower seeds.
   b. Fill in the table below.
Chapter 10 Rates, Ratios, and Proportions

<table>
<thead>
<tr>
<th>x</th>
<th>y = P(x)</th>
<th>y ÷ x</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c. Does P have a constant of proportionality?

18. A truck driver travels at an average speed of 62 miles per hour. Which expression can be used to find d, the distance the truck driver will travel in 217 hours? Explain your answer.

a. 62(217)
b. 62÷217
c. 217÷62
d. 217+62

19. Tommy’s Taters brand of potato chips has 8 grams of fat, 16 grams of carbohydrates, and 2 grams of protein per serving.

a. Write a function $C(x)$ for the number of grams of carbohydrates if $x$ grams of fat are eaten. What is $C(7)$?
b. Write a function $P(x)$ for the number of grams of protein if $x$ grams of fat are eaten. What is $P(12)$?
20. The Slugger Company has enough employees to devote 1680 work-hours a day making baseball bats. It takes 6 work-hours to make a wooden baseball bat, and it takes 14 work-hours to make an aluminum bat.
   a. How many aluminum bats can the company make a day if it only makes aluminum bats?
   b. How many wooden bats can it make if it makes no aluminum bats?
   c. How many aluminum bats can it make per wooden bat?

21. A cement company makes three grades of cement. The company makes three times as much low-grade as high-grade cement, and it makes twice as much medium-grade as high. The company produces on average $x$ tons of cement per day. How much of each grade of cement does it make each day? Express each amount as a fraction of $x$.

22. **Ingenuity:**

   Here is a problem students in Baghdad solved over 2000 years ago.

   Two students sat down to eat. One had five loaves of bread and the other, three. Just as they were about to begin, a third student came in and asked to eat with them, promising to pay eight cents for her part of the meal. If they ate the same amount and consumed all the bread, how should the two original students divide the eight cents?

23. **Investigation:**

   People in San Marcos want 1360 DVDs if they are given away for free. However, if the cost of DVDs increases, the number of people who want them decreases at a constant rate as the cost increases. If the cost is increased to $80, no one in San Marcos wants any DVDs.
   a. For every dollar the price of DVDs increases, by how much does the amount of DVDs wanted change?
   b. Write a function $D(x)$ expressing the number of DVDs people in San Marcos want if a DVD costs $x$ dollars.
24. **Investigation:**

In the same dish, the recipe calls for 2 teaspoons of oregano for each 3 teaspoons of salt. Repeat the first four steps from Exploration 1 for these two ingredients. Include a graph where the $x$-coordinate is the amount of salt and the $y$-coordinate is the amount of oregano.

Compare your graph and table to those from Exploration 1. What changes and how? What remains the same? Describe in words what the rate of change for each graph means.

25. **Investigation:**

Make a table and draw the graph using points with a $2:1$ ratio between the first and second coordinates. Do the same for a graph with the ratio $1:2$. What do you notice about these graphs? Write an equation for each of these functions. What do the graphs tell you about the relationship between the ratios $2:1$ and $1:2$?
SECTION 10.3 PROPORTIONS

When you look at a map of Texas, you know that the actual state is much larger than the map. For example, one inch can represent 50 miles, according to a scale designation on the map legend. That means that the ratio of the map distance to the actual distance is 1 inch to 50 miles. This ratio is written $1:50$ or $\frac{1}{50}$.

Using this information, what actual distance does 2 inches represent? This time, writing the information as a ratio of actual distance to the map distance, the fraction is $\frac{x}{2}$, where $x$ is the actual distance in miles, represented by 2 inches on the map. Using the scale of 50 miles to 1 inch from the map, combine the two ratios in the equation $\frac{x}{2 \text{ inches}} = \frac{50 \text{ miles}}{1 \text{ inch}}$. Solve the equation for $x$. We will explore a way to solve equations like this later in this section.

In a proportion, each side of the equation is a ratio. Sometimes, a proportion can compare two different types of the same units, like inches to inches and miles to miles, as long as both fractions are equivalent: $\frac{x \text{ miles}}{2 \text{ inches}} = \frac{50 \text{ miles}}{1 \text{ inch}}$ or $\frac{2 \text{ in}}{1 \text{ in}} = \frac{x \text{ mi}}{50 \text{ mi}}$.

Sometimes, the first ratio in the proportion compares different units. If the first ratio compares inches to miles, then the second ratio also compares inches to miles: $\frac{1 \text{ in}}{50 \text{ mi}} = \frac{2 \text{ in}}{x \text{ mi}}$. If the first ratio compares miles to inches, then the second ratio also compares miles to inches: $\frac{50 \text{ mi}}{1 \text{ in}} = \frac{x \text{ mi}}{2 \text{ in}}$. It does not matter which quantity, miles or inches, is in the numerator. The important thing is that the fractions are equivalent.

**DEFINITION 10.1: PROPORTION**

A proportion is an equation of ratios in the form $\frac{a}{b} = \frac{c}{d}$, where $b$ and $d$ are not equal to zero.

**EXAMPLE 1**

Four water bottles costs $6. How much will it cost to buy 10 of the same water bottles?
SOLUTION

One way to set up the ratio is to use number of water bottles to cost.

We use the ratio \( \frac{4 \text{ bottles}}{6 \text{ dollars}} \) and then equate this to \( \frac{10 \text{ bottles}}{x \text{ dollars}} \), where \( x \) represents the cost of 10 water bottles.

The proportion we have is:
\[
\frac{4 \text{ bottles}}{6 \text{ dollars}} = \frac{10 \text{ bottles}}{x \text{ dollars}}
\]

Note that both sides of the equation have the same units: bottles per dollar. Thus, we can drop the units.

\[
\frac{x \cdot \left( \frac{4}{6} \right)}{1} = \frac{x \cdot \left( \frac{10}{x} \right)}{1}
\]

Multiply both sides by \( \frac{x}{1} \).

\[
\frac{4x}{6} = \frac{10x}{x}
\]

\( \frac{x}{x} \) equals 1, so we are left with 10 on the right side of the equation.

\[
\frac{4x}{6} = 10
\]

Multiply both sides by 6.

\[
6 \left( \frac{4x}{6} \right) = 10 \cdot 6 \Rightarrow \frac{6 \cdot 10}{6} \text{ equals 1, so we are left with 4x on the left side of the equation.}
\]

\[
4x = 60
\]

Divide both sides of the equation by 4.

\[
\frac{4x}{4} = \frac{60}{4}
\]

Again, since \( \frac{4}{4} \) equals 1, the left side of the equation is \( x \).

\[
x = 15
\]

The situation is \( x = 15 \).

Alternate method: Or we can use an alternate ratio of cost to water bottles

\[
\frac{6 \text{ dollars}}{4 \text{ bottles}}\]

The proportion then looks like:

\[
\frac{6 \text{ dollars}}{4 \text{ bottles}} = \frac{x \text{ dollars}}{10 \text{ bottles}}
\]

Note that both sides of the equation have the same units: dollars per bottle. Thus, we can drop the units.

\[
\frac{6}{4} = \frac{x}{10}
\]
Section 10.3 Proportions

\[ \frac{10}{1} \left( \frac{6}{4} \right) = \frac{10}{x} \]

equals 1, so the right side of the equation is

\[ \frac{60}{4} = x \]

simplifying the left side of the equation

\[ 15 = x \]

Notice the result is the same regardless of the ratio set up. In fact, there are other proportions that can be used as you can see in Example 2. Which of the two choices required fewer steps to solve? Why do you think this happened?

**EXAMPLE 2**

Set up the following problem using a proportion.

Three bags of chips cost $2.79. How much do 7 bags of chips cost?

**SOLUTION**

In the chips problem, any one of these proportions will correctly determine x, the cost of 7 bags of chips. The possible proportions are indicated:

\[ \frac{\text{bags of chips}}{\text{cost}} = \frac{3}{2.79} = \frac{x}{7} \]

\[ \frac{\text{cost}}{\text{bags of chips}} = \frac{3}{7} = \frac{7}{x} \]

\[ \frac{\text{cost}}{\text{bags of chips}} = \frac{x}{7} = \frac{2.79}{3} \]

\[ \frac{\text{cost}}{\text{bags of chips}} = \frac{x}{2.79} = \frac{7}{3} \]

Use one of the proportions to solve for x. Verify that the cost for 7 bags of chips is the same when each proportion is solved. Which of the possible proportions above involved a unit rate?
Chapter 10 Rates, Ratios, and Proportions

Setting up the correct proportion is often the hardest part of solving a proportion problem. Some proportions are easier to solve than others because of the way they are set up. Which of the proportions above was easiest to solve and why?

EXAMPLE 3

Marla estimates her party guests will consume an average of a pint of punch each, so she will need 28 pints of punch. She has a family recipe that makes one gallon of punch. How many gallons of punch does she need for the party?

SOLUTION

Recall there are four quarts in a gallon and two pints in a quart. So there are eight pints in a quart. To calculate the number of gallons of punch needed, set up a proportion

\[
\frac{g \text{ gallons}}{28 \text{ pints}} = \frac{1 \text{ gallon}}{8 \text{ pints}}
\]

So, \( g \text{ gallons} = \left( \frac{1 \text{ gallon}}{8 \text{ pints}} \right) (28 \text{ pints}) = \left( \frac{28}{8} \right) \text{gallons} = 3.5 \text{ gallons} \).

Marla needs to make 3.5 gallons of punch.

ALTERNATE SOLUTION

Another way to approach this problem is to use the unit rate of conversion, which in this problem is \( \frac{1 \text{ gallon}}{8 \text{ pints}} \). To convert 28 pints into gallons, multiply the quantity of punch in pints times the unit rate in gallons per pint:

\[
(28 \text{ pints}) \left( \frac{1 \text{ gallon}}{8 \text{ pints}} \right) = \left( \frac{28}{8} \right) \text{gallons} = 3.5 \text{ gallons}
\]

Notice that in each solution the pint units canceled through multiplication leaving the gallon units.
EXAMPLE 4

A colony of leafcutter ants cuts up 4 leaves in 7 minutes. How many leaves does the colony cut in an hour?

SOLUTION

Tabular Method:

Construct a table to record the time and the number of leaves cut.

<table>
<thead>
<tr>
<th>Time in minutes</th>
<th>Number of leaves cut</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>14</td>
<td>8</td>
</tr>
<tr>
<td>21</td>
<td>12</td>
</tr>
<tr>
<td>28</td>
<td>16</td>
</tr>
<tr>
<td>35</td>
<td>20</td>
</tr>
<tr>
<td>42</td>
<td>24</td>
</tr>
<tr>
<td>49</td>
<td>28</td>
</tr>
<tr>
<td>56</td>
<td>32</td>
</tr>
<tr>
<td>63</td>
<td>36</td>
</tr>
</tbody>
</table>

Notice that the problem asked how many leaves the colony cut in 1 hour or 60 minutes. The table shows that the number of leaves must be between 32 and 36 leaves. This gives a good estimate for the solution, but there is a way to get an exact answer.

Unit Rate Method:

Set up a proportion that compares the ratio of leaves to minutes. Note that 1 hour must be converted into 60 minutes.
Because the ants cut 4 leaves in 7 minutes, using division, the ants must cut \( \frac{4}{7} \) of a leaf in 1 minute. This is the unit rate or the number of leaves cut per minute. If the ants keep cutting at this rate, they will cut 60 times this number of leaves in 60 minutes. Call the number of leaves cut in 60 minutes \( x \). Then 

\[
x = \frac{4 \text{ leaves}}{7 \text{ min}} \cdot 60 \text{ min} = \frac{240}{7} \text{ leaves} = 34 \frac{2}{7} \text{ leaves}
\]

**Proportion Method:**

Set up a proportion by comparing amounts for the two different times.

The ants cut 4 leaves in 7 minutes. How many leaves \( x \) will the ants cut in 60 minutes?

\[
\frac{x \text{ leaves}}{60 \text{ min}} = \frac{4 \text{ leaves}}{7 \text{ min}}
\]

To solve, multiply both sides of the equation by the denominator 60.

\[
x = \frac{4}{7} \cdot 60 = \frac{240}{7} = 34 \frac{2}{7} \text{ leaves}.
\]

This proportion method involves the rate of change in the form of leaves cut per unit time or minute. This is a rate of change like miles per hour or mph.

**EXAMPLE 5**

Leo runs \( \frac{1}{4} \) of a mile in 3 minutes. How many miles will he run in 45 minutes, assuming he continues to run at the same rate?
SOLUTION

To see a visual way of organizing the problem, set up a table. Make a table with time and distance as headers.

<table>
<thead>
<tr>
<th>Time (in minutes)</th>
<th>Distance (in miles)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>(\frac{1}{4})</td>
</tr>
<tr>
<td>6</td>
<td>(\frac{2}{4} = \frac{1}{2})</td>
</tr>
<tr>
<td>9</td>
<td>(\frac{3}{4})</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
</tr>
<tr>
<td>15</td>
<td>1 (\frac{1}{4})</td>
</tr>
<tr>
<td>18</td>
<td>1 (\frac{1}{2})</td>
</tr>
<tr>
<td>21</td>
<td>1 (\frac{3}{4})</td>
</tr>
<tr>
<td>24</td>
<td>2</td>
</tr>
<tr>
<td>27</td>
<td>2 (\frac{1}{4})</td>
</tr>
<tr>
<td>30</td>
<td>2 (\frac{1}{2})</td>
</tr>
<tr>
<td>33</td>
<td>2 (\frac{3}{4})</td>
</tr>
<tr>
<td>36</td>
<td>3</td>
</tr>
<tr>
<td>39</td>
<td>3 (\frac{1}{4})</td>
</tr>
<tr>
<td>42</td>
<td>3 (\frac{1}{2})</td>
</tr>
<tr>
<td>45</td>
<td>3 (\frac{3}{4})</td>
</tr>
</tbody>
</table>

You can also use proportions to determine the distance Leo runs in 45 minutes. Let x represent the distance Leo runs in 45 minutes. You can use the proportion \(\frac{x \text{ miles}}{45 \text{ minutes}} = \frac{1}{4} \text{ miles} \). Multiply both sides of the equation by 45.

The resulting equation is: \(x = \frac{45}{3} \cdot \frac{1}{4} = \frac{1}{4} \cdot \frac{1}{3} \cdot 45 = \frac{45}{12} = \frac{3}{12} \cdot 3 = \frac{3}{4} \) miles.

Can you use the Unit Rate Method on this problem?

EXAMPLE 6

Lucinda is studying prairie dog populations in Colorado. She captures and tags 15 prairie dogs and then releases them back into the wild. Two weeks later she captures 35 prairie dogs and discovers 3 are tagged. What is the approximate population of prairie dogs in the region?
SOLUTION

Setting \( x \) equal to the approximate total population of prairie dogs in the area,

\[
\frac{x}{15 \text{ tagged prairie dogs}} = \frac{35 \text{ total prairie dogs}}{3 \text{ tagged prairie dogs}}
\]

To solve, multiply each side of the equation by the denominator 15.

\[
x = \frac{35}{3} \cdot 15 = \frac{525}{3} = 175 \text{ prairie dogs}
\]

EXPLORATION 1

Materials: You will need a map of any region that contains a legend with the distance scale and a ruler or tape measure in the same unit system as the map.

Step 1: Find the legend in the map and write a ratio that relates the map measure to the actual measure.

Step 2: Use a measuring instrument to measure the straight-line distance between two major cities on the map.

Step 3: Determine the actual straight-line distance between the cities using proportions.

Step 4: Repeat Steps 2 and 3 with two other cities.

What are the actual straight-line distances between the cities that you chose?
EXPLORATION 2

From Math Explorer, December 1999, vol. 2.3

A globe is approximately 30 cm in diameter. Using that measurement, calculate the scaled size for the measurements indicated in the table.

<table>
<thead>
<tr>
<th>Object</th>
<th>Actual size (miles)</th>
<th>Scaled size (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth diameter</td>
<td>8,000</td>
<td>30</td>
</tr>
<tr>
<td>Top of the atmosphere</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>Space shuttle orbit height</td>
<td>200</td>
<td></td>
</tr>
<tr>
<td>Height of satellite</td>
<td>18,000</td>
<td></td>
</tr>
<tr>
<td>Moon diameter</td>
<td>2,100</td>
<td></td>
</tr>
<tr>
<td>Distance from Earth to Moon</td>
<td>240,000</td>
<td></td>
</tr>
</tbody>
</table>

EXERCISES

1. Alvin the squirrel picks as many acorns as he can hold, then brings them home. For every seven acorns he picks, he loses two but manages to get the remaining five home. After one hour Alvin has fifteen acorns stored in his home. How many acorns has he dropped?

2. Farmer Al sells peaches at $9 a dozen at the farmer’s market. At that price, what is the cost of 8 peaches? At that rate how much does each peach cost?

3. A large restaurant makes a soup that uses twelve pounds of carrots for each fifteen pounds of potatoes. What is the proportion of potatoes to carrots in the soup? What is the rate of pounds of carrots per pound of potatoes?

4. A toy manufacturer makes colored marbles and packages them. Each package contains red, blue, and green marbles, the ratio of red to green is 3:5, and the ratio of blue to green is 8:5. If the package contains 42 red marbles:
   a. How many green marbles does each package contain?
   b. How many blue marbles does each package contain?
5. It takes 120 minutes to wash 30 vehicles at a car wash. At this rate, how many minutes does it take to wash 6 vehicles? Select the best choice and explain your answer.
   a. 30 minutes
   b. 24 minutes
   c. 28 minutes
   d. 6 minutes

6. Gabby can assemble 10 music books in 8 minutes. At this rate, how many music books can she assemble in 3 hours? Select the best choice and explain your answer.
   a. 24
   b. 4
   c. 225
   d. 300

7. Use proportions to convert the original units to the new units of measure:
   a. Convert 11 quarts to gallons.
   b. Convert 8 feet to yards.
   c. Convert 8 yards to feet.
   d. Convert 150 seconds to minutes.
   e. Using the fact that 440 yards is equivalent to ¼ mile, find the number of yards are in a mile. How can you compute the same answer mentally?
   f. Using the answer in e, compute the number of feet in a mile. Explain how to compute the same answer mentally.
   g. Convert 84 hours to days.

8. Use each given unit rate to convert the following quantities to the new unit of measure:
   a. \( \frac{12 \text{ inches}}{1 \text{ foot}} \); Convert 4 feet into inches.
   b. \( \frac{12 \text{ inches}}{1 \text{ foot}} \); Convert 2 \( \frac{1}{4} \) feet into inches.
   c. \( \frac{1 \text{ foot}}{12 \text{ inches}} \); Convert 30 inches into feet.
Section 10.3 Proportions

d. \( \frac{1 \text{ foot}}{12 \text{ inches}} \); Convert 15 inches into feet.
e. \( \frac{24 \text{ hours}}{1 \text{ day}} \); Convert 4 days into hours.
f. \( \frac{24 \text{ hours}}{1 \text{ day}} \); Convert 1 \( \frac{1}{2} \) days into hours.
g. \( \frac{1 \text{ day}}{24 \text{ hours}} \); Convert 8 hours into days.

9. Solve the following equations.
   a. \( \frac{3}{4} = \frac{x}{12} \)
   b. \( \frac{5}{9} = \frac{x}{36} \)
   c. \( \frac{25}{x} = \frac{5}{2} \)
   d. \( \frac{8}{5} = \frac{36}{x} \)

10. If a museum requires at least one chaperone for every six students, what is the minimum number of chaperones required for a field trip with 80 students?

11. If six shepherds can watch 180 sheep, how many sheep can four shepherds probably watch?

12. Jane has 15 kites. She says \( \frac{2}{3} \) of them are box kites. How many box kites does she have?

13. Bobby has 3 green marbles. He says that \( \frac{1}{4} \) of his marbles are green. How many marbles does he have?

14. A class has 18 students and \( \frac{1}{3} \) of them are boys. How many boys are there in the class? Explain your thinking on this problem.

15. The mileage for Dr. Thompson’s new car is \( 34 \frac{\text{miles}}{\text{gallon}} \). That is the average mileage per gallon in France, Denmark and Italy, where the cost of gas is over $4.00 a gallon. When returning from a research expedition in the desert, her car runs out of gas seven miles from the nearest gas station. She plans to walk to the gas station, bring back just enough fuel to drive her car to the gas station, and refill there.
   a. How much gas does she need to buy?
   b. What is the least amount of money she needs to buy the necessary gas?

16. Heather is studying the fish population for Canyon Lake. She catches 30 fish and tags them. After a month, she catches another 50 and discovers that 3 of them are tagged. What is the approximate population of fish in the lake?

17. Ernie averages 3 hits for every 10 times he goes to bat during the season. If he gets 136 hits over the whole season, about how many times did he bat?
18. In baseball, the Earned Run Average, or ERA, is determined by the average number of earned runs a pitcher allows during a 9-inning game over a period of time. Roy has allowed 28 earned runs in 84 innings.
   a. What is Roy’s ERA?
   b. Generally, how many earned runs does Roy allow in 120 innings?
   c. Roy’s teammate, Roger, has an ERA of 2.40. On average, how many earned runs does Roger allow in 210 innings?

19. Marcy has a bag of red and blue jacks. The ratio of red jacks to the total number of jacks is 4 : 9. If she has 45 blue jacks, how many jacks does she have in all?

20. A recipe for 48 cookies requires $1 \frac{3}{4}$ cups of sugar. How much sugar is needed to make 168 cookies?

21. Mrs. Jackson bought 12 pounds of potatoes for $5.16. Which of the following represents the same price per pound? Select the best choice and explain your answer.
   a. 8 pounds of potatoes for $3.28
   b. 10 pounds of potatoes for $4.00
   c. 15 pounds of potatoes for $6.45
   d. 9 pounds of potatoes $4.05

22. George is running a marathon, a 26.2-mile. Researchers have determined that to run in a marathon, it is generally safe to eat up to one power bar for every ten pounds a person weighs. George weighs 160 pounds. According to these guidelines, over the whole race, approximately what part of a power bar would he eat per mile?

23. Ingenuity:
   Reagan dropped a rock from the top of a high cliff. The speed of the rock was recorded at random times while the rock was falling. The first 5 speeds recorded, in feet per second, were 0, 48, 80, 96 and 144.
   If the rate of change in speed is constant, is there enough information to find how fast the speed is changing? If there is, what is the rate of change? If not, what information do you need?
24. **Investigation:**

Four numbers $a$, $b$, $c$ and $d$ satisfy the relationships $\frac{a}{b} = \frac{2}{3}$, $\frac{b}{c} = \frac{7}{2}$ and $\frac{c}{d} = \frac{3}{8}$. Find $\frac{a + b + c}{d}$. 
In this chapter, we have been comparing quantities using ratios and proportions. It is also possible to make comparisons using percents. Ratios can be used to compare a part to the whole or to compare one part to another part. A percent is a ratio of a part to the whole. For example, in a class of 18 girls and 12 boys, the ratio of girls to total students is $18 : 30$, or equivalently, \( \frac{18 \text{ girls}}{30 \text{ students}} \). This ratio or rate is equivalent to the fraction \( \frac{3}{5} \), the decimal 0.60, and represents 60%. The whole is the original amount, called its base. In this case, the base is the class size 30 students. Likewise, the rate \( \frac{12 \text{ boys}}{30 \text{ students}} \) is equivalent to the fraction \( \frac{2}{5} \), and the decimal 0.4, and represents 40%. In each case, we are comparing a part of the class to the whole class. We say that 60% of the class is female, which is equivalent to the statement that \( \frac{3}{5} \) of the class is female. From Chapter 9, we know that to compute \( \frac{3}{5} \) of 30 is a multiplication problem:

\[
\left( \frac{3}{5} \right)(30) = 18.
\]

We can compute the number of girls by using the decimal form of \( \frac{3}{5} \):

\[
(0.60)(30) = 18.
\]

We say that “60% of the class is female,” and that 60% of 30 is 18.

**PROBLEM 1**

Joe and Moe went to the pizza arcade. Their parents bought 75 tickets for them to spend. Joe used 30 tickets while his brother used the rest.

a. What percent of the tickets did Joe use?

b. What percent of the tickets did Moe use?

c. How could you solve for these problems using proportions?
PERCENT INCREASE AND PERCENT DECREASE

Percents can also be used to express the change between two quantities. **Percent change** is the ratio of the amount of change compared to the original quantity.

\[
\text{Percent change} = \frac{\text{amount of change}}{\text{original amount}}
\]

**EXAMPLE 1**

A store receives a shipment of summer shirts. Each shirt costs the store $80 wholesale. They sell the shirts for $104. What was the percent markup, or **percent increase**?

**SOLUTION**

Amount of change = new amount – original amount = $104 – $80 = $24

Original amount = $80

Percent change (increase) = \(\frac{24}{80} = 0.3 = 30\%\)

**EXAMPLE 2**

At the end of summer, a store manager decides to mark down the price of a dress from $120 to $84. What was the markdown, or **percent decrease**?

**SOLUTION**

Amount of change = $120 – $84 = $36

Original amount = $120

Percent change (decrease) = \(\frac{36}{120} = 0.3 = 30\%\)
PROBLEM 2

Molly Drilling Company employed 640 people. When the oil boom began, their workforce grew to 672 people. What was the percent change in the number of employees? Was this a percent increase or decrease?

PROBLEM 3

Briley Machine Shop builds oil field drill bits. They had 45 bits in stock. Three weeks later, their stock contained 40 drill bits. What was the percent change? Was this a percent increase or decrease?

Sometimes in a problem, we are given the percent that the part is of the whole and the amount of the whole. We can use the percent to compute the amount of the part.

EXAMPLE 3

Suppose you know that a class of 30 students conducted a survey about pets and 80% of the students in the class had a dog. How do you calculate the number of students in this class that have a dog?

SOLUTION

We shall solve this problem two ways.

Method 1: The first method uses proportions. First, we define a variable \( x \).

Let \( x = \) the number of students in class that have a dog.

The ratio of students with dogs to total students can be written two ways:

\[
\frac{80}{100} = \frac{4}{5} \quad \text{and} \quad \frac{x}{30}.
\]

Thus, we can set up the proportion:

\[
\frac{4}{5} = \frac{x}{30}
\]
Multiplying both sides by 30, we get
\[ \frac{4}{5}(30) = x \]. Thus, \( x = 24 \).

**Method 2:** Another way to think of this question is to ask: “What is 80\% of 30?” If we used the fractional equivalent of 80\%, \( \frac{80}{100} = \frac{4}{5} \), then the question becomes: “What is \( \frac{4}{5} \) of 30?” From Chapter 9, we know this means that we use multiplication to compute the number of students who have a dog.

\[ \frac{4}{5}(30) = 24 = \text{number of students with a dog} \]

Thus, computing the percent of a number involves multiplication of the percent (in its decimal form) by the quantity that answers the question: “80\% of what?” The answer is 80\% of 30, which is equivalent to the product \((.80)(30) = 24\). Note that 30 students is the base of 80\%. We always follow the computation of a percent times its base with a description that explains what the product represents. This written description is very helpful to remember what the product means.

**PROBLEM 4**

If 40\% of a group of 35 students participate in athletics, how many of these 35 students participate in athletics? Solve this using both methods.

**EXAMPLE 4**

A region averages 28 inches of rain each year. Last year, there was a 30\% increase above the average rain. How much rain did the region receive?

**SOLUTION**

**Method 1:** We let \( x = \) rainfall last year. The normal rainfall is 100\% and the increase in rainfall last year was 30\%, the total rainfall last year was 130\%. Thus, the ratio of \( \frac{130\%}{100\%} \) is the ratio of the amount of rainfall last year compared to the average rainfall. Thus, we can write the proportion:

\[ \frac{130}{100} = \frac{x}{28} \]

Rewriting this equation, we get \((1.3) = \frac{x}{28}\). Solving this equation, we get

\[ (1.3)(28) = x \].

Thus, \( x = 36.4 \) inches.
Method 2: We compute 30% of 28 to get \((0.30)(28) = 8.4\) = amount of increase of rain.

So, the total rainfall in this region last year was \(28 + 8.4 = 36.4\) inches of rain. Notice the importance of writing a description of what \((0.30)(28) = 8.4\) represents.

PROBLEM 5
Jack and Jill ate at the Hill Restaurant and their bill was $22.00. If they include a 15% tip for the waiter, how much is their payment?

Another example of percent increase is computing sales tax to include in the overall price of an item.

EXAMPLE 5
Vera buys a CD for $12.00. The sales tax rate is 8%. How much tax does she pay? What is her final bill?

SOLUTION
The tax paid is 8% of $12.00, or \((12.00)(0.08) = 0.96\). So her final bill is

\[
12.00 + 0.96 = 12.96.
\]

You could also compute the final bill directly as \((12.00)(1.08) = 12.96\). Explain why.

EXPLORATION 1
A store receives a shipment of summer shirts. Each shirt costs the store $100 wholesale. The shirts are marked up 30%.

a. What is the selling price for each shirt?

b. At the end of the summer, the store puts the shirts on sale at a 30% discount from the selling price. What is the discounted price for each shirt? Does the store make a profit, suffer a loss, or break even on the discounted shirts?
To compute the selling price in the Exploration, the store must mark up the price to the selling price. This requires that you take 30% of $100 to compute the amount of markup and then add this to the wholesale price to get the selling price. Other examples of this kind of percentage increase can be seen in adding a tip to a bill at a restaurant or in computing the cost of an item at a store with the tax included. The discounted price in part b of the Exploration is an example of a percentage decrease. You multiply the percent of discount 30% by the selling price and then subtract this amount from the selling price to find the discounted price.

EXAMPLE 6

Suppose 80% of the students in a class have backpacks and we know that 20 students have backpacks. How many students are in the class?

SOLUTION

In each problem involving a percent, we have identified the “base” of the percent and then multiplied the percent times this base to get a useful quantity. What is the base of 80% in this problem? We need to compute 80% of the number of students in the whole class. But this is the number that we want to compute. We let \( x \) represent the number of students in the whole class:

\[
\text{Let } x = \text{the number of students in the whole class.}
\]

**Method 1:** The number \( x \) represents 100% of the class. The ratio of students with backpacks to the whole class is \( \frac{20}{x} \). This is equivalent to the ratio \( \frac{80\%}{100\%} \). Thus, we set up the equation:

\[
\frac{80}{100} = \frac{20}{x} \quad \text{or equivalently } \quad \frac{4}{5} = \frac{20}{x}.
\]

Solving this equation, we multiply each side by \( 5x \) to get

\[
4x = 100. \quad \text{Thus, } x = 25.
\]

**Method 2:** The base of 80% is the number of students in the whole class, \( x \). Therefore, we compute 80% of \( x \) and write a description of what this product means:

\[
(0.80)(x) = 0.8x = \text{the number of students with backpacks}.
\]
We know that 20 students have backpacks so we can make an equation: \(0.8x = 20\).

In solving this equation, we divide both sides by 0.8 to get:

\[
\frac{0.8x}{0.8} = \frac{20}{0.8}.
\]

Thus, \(x = 25\). We have calculated that the whole class must have 25 students and that 80% of these, 20 students, have backpacks.

**PROBLEM 6**

a. Store A has a $100 skirt on sale for 20% off. At the end of the summer, it offers an additional 40% off then reduced price. What is the final sale price?

b. Suppose Store B has a $100 skirt on sale for 40% off. At the end of the summer, it offers an additional 20% off the first sale price. What is the final sale price for this skirt?

c. Compare the answers in parts a and b.

**PROBLEM 7**

A store manager marks a cell phone 40% above the wholesale cost to the selling price of $56. What was the original wholesale price of the cell phone?

**EXPLORATION 2: Sales, Rebates, and Coupons**

Stores are eager to promote new business and offer various ways to encourage new customers to shop with them. Some of these ways include special sales, coupons, and rebates. In a sale, the store reduces the selling price, usually by some percentage discount. A coupon could either provide a percentage discount or a fixed amount off. A rebate is money that the customer receives, typically from the manufacturer, after making the purchase. Generally, rebates require the customer’s further action.

Stores are eager to get your business and offer ways to encourage you to shop with them. An item may have a certain price tag but if it is on sale, a newspaper has a special coupon, or there is a rebate offer, what is the actual cost to you the customer?
For example, here are three stores with their various promotions:

- Pascal Shop offers 15% off on all the items in its store.
- Euclidmart has a 5% store coupon in the newspaper that can be used on all clearance where you can take 10% off the tag price.
- Fibonacci’s Department Store is giving a rebate of $15 on certain items.

**PROBLEM 8**

Suppose Ruth is considering buying four items available at all three stores for $15, $45, $100, and $200.

Use the table below to compare the actual dollar amount that Ruth will pay at each of the stores with their promotions. Assume that the coupon can be used with any purchase from Euclid Mart, and that each of the items at Fibonacci’s has a rebate.

<table>
<thead>
<tr>
<th>Cost of Item</th>
<th>$15</th>
<th>$45</th>
<th>$100</th>
<th>$200</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pascal’s Shop 15% off Sale</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Euclidmart 5% off in addition 10% off</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fibonacci’s Department Store $15 rebate</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Analyze each of the monetary incentives—sales, rebates, and coupons. Discuss which offer is the best for each item and why. Should Ruth shop at just one store for all of her items? Generally, how much does the cost of the item affect your choice of store?

What is the advantage of a sale or coupon to a rebate? Why might some stores offer rebates and not sales?
EXAMPLE 7

Henry’s Clothing store is running a sale of 60% off of the list price. Marvin’s Clothing store has already reduced the price by 25%, and has a new sale offering 40% off of the marked price. If the original price of a shirt is $75 at both stores,

a. How much is the final discount at each store?
b. Which store has the lower price, or are the prices be the same? Explain

SOLUTION

a. At Henry’s, the discount is 60% of $75, (0.60)75 = $45.
   At Marvin’s, the original discount is 25% of 75, (0.25)75 = $18.75. So the original sale price is 75 – 18.75 = $56.25. Marvin’s second discount is 40% from the sale’s price, so the additional discount is (0.40)56.25 = $22.50. The total discount taken is 18.75 + 22.50 = $41.25.

b. Taking 60% off the original price leads to a final sales price of $30 at Henry’s.
   Taking a discount of 25%, followed by an additional discount of 40% leads to a final sales price of $33.75. It is better to take 60% off of the original price—the shirt will be cheaper at Henry’s.

Now explore and compare different types of promotions that a store might run.

PROBLEM 9

Fermat Market offers a rebate of $50 for each of your purchases while Euclidmart has 10% off sale for each item purchased. Use the table below to determine what is the actual price you will pay for each item with the indicated price tag.

<table>
<thead>
<tr>
<th>Item’s price tag</th>
<th>Fermat Market</th>
<th>Euclidmart</th>
</tr>
</thead>
<tbody>
<tr>
<td>$100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$200</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$400</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$500</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$600</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. Discuss which store provides a better savings. Explain why. What is your strategy for where you would shop and why?
2. Both stores have an item that is listed at the price of $480. If the stores provide the incentives mentioned above, which store would you go in order to get the most in savings? Explain why.

3. Both stores have an item that is listed at the price of $563. If the stores provide the incentives mentioned above, which store would you go in order to get the most in savings? Explain why.

EXERCISES

1. A pet store owner had 45 parrots in her store. She received 9 more parrots in a shipment. What was the percent of increase in the number of parrots? 20%

2. A shirt was selling regularly for $40. The store manager put up a sign offering a discount of $5. What was the percent of discount? 12.5%

3. The wholesale price of a refrigerator is $480. If the store marked the selling price as $590, what was the percent of markup? 22.9%

4. Kendall has a collection of pets. She has 4 dogs and 9 fish. What percent of animals are dogs? What percent are fish? 30.8%; 69.2%

5. A department store has a dress that is originally priced at $200. A sale offers a discount of 20% off the original price.
   a. How much is the discount in dollars?
   b. What is the sale price?
   c. What percentage of the original price does the sale price represent?

6. A food store offers a $5 rebate on any purchase over $10. It also gives a 25% rebate off the sales price for any purchase. Unfortunately, you may use only one of the offers. How much is each rebate for
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7. Connie’s grocery bill totals $125.24. However, she has three coupons, one for $1.50, one for $2.00, and one for a free $12 turkey for a $19.50 purchase she made.

a. What is the total of all her coupons?
b. How much will her grocery bill be when she subtracts the coupons?
c. What percent discount equals the coupons’ savings?

8. A printer costs a store $180 wholesale. The store’s retail markup is 28%. What is the retail price for this printer?

   a. 10% of 70  
   b. 10% of 64  
   c. 10% of 263  
   d. 10% of 25.4  
   e. 50% of 24  
   f. 75% of 24  
   g. 20% of 55  
   h. 80% of 55  
   i. 12.5% of 16  
   j. 37.5% of 16  
   k. $33 \frac{1}{3}$% of 9  
   l. $66 \frac{2}{3}$% of 9  
   m. 79% of 100  
   n. 16% of 50  
   o. 21% of 200  
   p. 8% of 150  
   q. 40% of what is 20  
   r. 20% of what is 3  
   s. 50% of $x$ is 42. What is $x$?
   t. 35% of $A$ is 28. What is $A$?

10. A bag of red and blue marbles contains 40% red marbles.
   a. If there are 200 marbles total, how many of them are red?
   b. If there are 260 marbles in all, how many of them are red?
   c. If there are 64 red marbles, how many marbles are in the bag?

11. Amy plays on the school basketball team. She shot a total of 70 free throws this season and made 90% of them. How many free throws did she make? Draw a picture that represents the quantities in this problem.

12. A lab has an alcohol solution (alcohol mixed with water) of 100 gallons. This solution contains 30% alcohol. How much of the solution, in gallons, is pure alcohol? How much of the solution, in gallons, is pure water? Draw a picture of the solution, showing the amount of alcohol and water present.
13. The cafeteria has 30 gallons of milk that contains 2% fat. How much fat does the 30 gallons of milk contain?

14. A family ate out after a soccer game. Their bill was $45.60. If they included a 15% tip, what was their final bill? How much tip did they give the waitress?

15. Amanda bought a new blouse for $36.50. If the sales tax is 8%, how much was her final bill? How much tax did she pay?

16. The tax rate in Gainesville is 8%.
   a. What is the tax on a $27.25 purchase?
   b. If the tax on an item was $6.64, how much did the item cost?
   c. What is the amount of tax paid if the total cost is $153.90?
   d. How much does the item cost if the total cost is $85.05?

17. Juan lost 40% of his marbles. If he lost 30 marbles, how many marbles did he have originally? Use both methods.

18. Bill lost 12 pounds on his diet. If he lost 8% of his weight during this diet, what was his original weight?

19. An mp3 player is on sale for $40 after a 20% discount. What was the original price? What was the amount of the discount? Show how a picture can help solve this problem.

20. The discount during a summer clearance sale is 20% and the sale price is $50. What was the original price?

21. A computer costs a store $360 wholesale and is sold for $486 retail. What is the percent markup?

22. Sunshine Department Store offers a rebate of $20 for one purchase over $25. Joe’s Department Store has a 25% off sale for any one item. Sam’s Department Store has a $10 coupon for each item purchased over $50. Use the table below to determine what is the actual price paid for each item with the indicated price tag. Assume the cost of sending in the paperwork for a rebate is $1.
Chapter 10  Rates, Ratios, and Proportions

<table>
<thead>
<tr>
<th></th>
<th>Sunshine</th>
<th>Joe's</th>
<th>Sam's</th>
</tr>
</thead>
<tbody>
<tr>
<td>$25</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$50</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$100</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$200</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$400</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

How do these different promotions compare? Explain how to spend the least amount of money.

23. Mr. Robinson ate supper at a locally owned restaurant. His meal cost $14.95 and he included a 20% tip. How much was his final bill?

24. Ms. Puente bought a new TV for $500. How much sales tax did she pay if the sales tax in her town is 9%? What was her final bill? What would her final bill have been if the sales tax was $8 \frac{1}{4}\%$?

25. A pair of jeans was marked-up 45% to a selling price of $92.80. What was the original wholesale cost of this pair of jeans?

26. A clothing store marks a dress up 10% right before a 10% discount sale. Does the dress cost the same on sale as it did before the markup? Explain.

27. Jan finds a $60 dress on sale for 10% off. She offers to buy the dress if the store manager gives her another 10% off the sale price. What would be her new sale price?

28. Allison bought some new dolls to add to her collection. She increased her collection by 60% and has 96 dolls now. How many dolls did she have before she bought the new dolls?

29. Jameer scored 36% more points than his average basketball score this past game. He finished the game with 34 points. What was Jameer’s average before the game?

30. Toby sold 70% of his baseball cards and was left with 36 cards. How many cards did he have in his collection before the sale?

31. Janice lost 35% of her marbles. She has 52 marbles now. How many marbles did she originally?
32. Kirsten is a real estate agent and earns 4% commission for each property she sells. If she sells a house for a client for $180,000, how much money will she earn?

33. Ian put $300 into a savings account at his bank. If he leaves the money in the bank for an entire year he will earn 2% interest. How much money will he have in the account in total at the end of one year?

34. **Ingenuity:**
   The Guerrero family has a 48-gallon rain barrel that contains 24 gallons of water and a 5-gallon water jug that contains 3 gallons of water.
   a. Which container has more water? Which container is fuller?
   b. If we drain a gallon of water from each, does this change the second answer to the previous problem? Explain your reasoning.
   c. How many gallons of water should the Guerreros have in the 5-gallon jug to make it as full as the 24 gallons in the 48 gallon barrel?
   d. How many more gallons of rain do the Guerreros need to catch in the barrel in order to for it to be as full as the jug is?

35. **Investigation:**
   Ms. Campos, the science teacher, mixed 3 gallons of a 20% percent alcohol solution with 6 gallons of a 40% alcohol solution.
   a. Draw a picture of the mixtures showing the alcohol amount and the water amount.
   b. Make a table or a diagram that indicates the number of gallons of the different solutions, the number of gallons of the alcohol, and the corresponding percentage concentrations. Notice that there are three solutions: the 20% solution, the 40% solution and the resulting solution.
   c. How many gallons of alcohol are in the resulting solution?
   d. How many gallons of solution are there total?
   e. What fraction of the resulting solution consists of alcohol?
   f. What percent alcohol is the resulting solution?
SECTON 10.5  SCALING

So far in this chapter you examined rates and ratios, two of the most important ways to relate two quantities. For example, relating the number of miles to number of hours is called a rate of miles per hour. Other examples of rates and ratios included feet per yard, miles per hour, and even the number of Texas students to the number of international students. In this section, you will explore relationships that occur in geometry by observing relationships involving rectangles.

EXPLORATION 1: SORTING RECTANGLES

1. On a printed copy of the shaded rectangles below, cut and sort these rectangles in any way that seems natural. Discuss how you decided to sort them.
2. What are the attributes of a rectangle? Which did you use to sort? Explain.

EXPLORATION 2: ATTENDING TO SHAPES AND SIZES

Now consider the following rectangles from a printed copy:

1. Which rectangles have exactly the same shape as rectangle $Z$? Explain your answer.

2. Copy and complete the table. What do you notice? What do rectangles B, C, E, and G have in common?

<table>
<thead>
<tr>
<th>Rectangle</th>
<th>Length</th>
<th>Width</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$A$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
You probably discovered that the dimensions of rectangles B, C, E, and G are multiples of the dimensions of rectangle Z.

<table>
<thead>
<tr>
<th>Rectangle</th>
<th>Length</th>
<th>Width</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>$3 \cdot 2$</td>
<td>$2 \cdot 2$</td>
</tr>
<tr>
<td>C</td>
<td>$3 \cdot 3$</td>
<td>$2 \cdot 3$</td>
</tr>
<tr>
<td>E</td>
<td>$3 \cdot 4$</td>
<td>$2 \cdot 4$</td>
</tr>
<tr>
<td>G</td>
<td>$3 \cdot 5$</td>
<td>$2 \cdot 5$</td>
</tr>
</tbody>
</table>

The bold numbers for each rectangle are called the \textit{scale factors}, or the \textit{constant rate of proportionality}, of the dimensions of the original rectangle Z.

4. For each rectangle, how can you find the scale factor that relates it to Z?

5. Do rectangles A, D, and F have scale factors that relate to Z? Explain.

\textbf{DEFINITION 10.2: SCALE FACTOR}

If rectangle A has dimensions $b$ and $h$, two positive numbers, and rectangle B has dimensions $kb$ and $kh$, where $k$ is also a positive number, then $k$ is the \textit{scale factor} from A to B. It is also called the \textit{constant rate of proportionality}.

\textbf{PROBLEM 1}

Suppose we start with the rectangle $P$ below. What is the scale factor from $P$ to $R$? From $R$ to $T$? From $T$ to $T$? From $R$ to $P$? From $P$ to $T$? From $T$ to $P$?
EXPLORATION 3

1. Make a 4 by 8 rectangle on blank grid paper and label it \( R \). Create 5 more rectangles that have the same shape but different size from \( R \) and label them \( A, B, C, D \) and \( E \). What is the scale factor of each new rectangle in relation to the rectangle \( R \)?

2. Copy the table below and complete the data.

<table>
<thead>
<tr>
<th>Rectangle</th>
<th>Length</th>
<th>Width</th>
<th>Scale Factor</th>
<th>Perimeter</th>
<th>Area</th>
</tr>
</thead>
<tbody>
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<td>32</td>
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<td>( F )</td>
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</table>

Did you create a rectangle that is smaller than \( R \)? If not, make one now and label it \( F \).

3. What is the scale factor from \( R \) to \( F \)? Make a conjecture about scale factors less than 1.
4. Make a rectangle the same shape as $R$ that is 6 units long. Label it $G$. What is the scale factor? from $R$ to $G$?

5. Make a new rectangle using a scale factor of $\frac{5}{2}$. Label it $H$. What are the dimensions of this new rectangle? How can you check to see if the scale factor is $\frac{5}{2}$?

6. Make a rectangle with the same shape as $R$ that is 3 units long. Label it $J$. What is the scale factor for this rectangle?

EXAMPLE 1

Sue is making a dessert from a recipe that will serve 3 people. The recipe uses 5 tablespoons of chocolate, and 2 cups of milk.

a. Sue wants to serve six people. How much milk will she need?

b. Sue decides to invite four more people. How much milk will she need now? What is the scale factor between her final recipe and the original?

c. On another occasion, Sue uses 7 tablespoons of chocolate. How much milk should she use this time?

SOLUTION

a. When Sue doubles the people, the scale factor is 2. So the amount of milk will be $2 \cdot 2 \text{ cups} = 4 \text{ cups of milk}$.

b. What is the scale factor between 4 people and 10 people? Call the scale factor $x$.

Set $4 \cdot x = 10$. Dividing both sides of the equation by 4, $x = \frac{10}{4} = 2.5$.

Using the scale factor 2.5, the recipe will use $2.5 \cdot 2 = 5 \text{ cups of milk}$.

c. The recipe uses a ratio of 5 tablespoons chocolate for every 2 cups of milk. What is the scale factor when Sue increases the chocolate to 7 tablespoons? Call this scale factor $x$. Set $5 \cdot x = 7$, so $x = \frac{7}{5}$. The amount of milk will be $2 \cdot \frac{7}{5} = \frac{14}{5} = 2\frac{4}{5} \text{ cups of milk}$.

Alternatively, let $C =$ the number of cups of milk needed with 7 tablespoons of chocolate. Then
$5:2 = 7:C$

$$\frac{5}{2} = \frac{7}{C}$$

Multiplying both sides of the equation by $2C$

$$\left(\frac{5}{2}\right) \cdot 2C = \left(\frac{7}{C}\right) \cdot 2C$$

$$5C = 14$$

$$C = \frac{14}{5}$$

**PROBLEM 2**

Make a 5 units x 9 units rectangle on grid paper and label it Rectangle M.

A. Construct a new rectangle using a scale factor of 0.6. Label the new dimensions and name it Rectangle N.

B. Construct another rectangle using a scale factor of 1.5. Label the new dimensions and name it Rectangle P.

**PROBLEM 3**

Suppose Rectangle A is a 6 by 9 rectangle. Draw a Rectangle B so that the scale factor from A to B is $\frac{1}{3}$.

A. What is the scale factor from B to A?

B. Draw a Rectangle C so that the scale factor from B to C is 2.

C. What is the scale factor from A to C?

D. What is the scale factor from C to A?

E. What do you notice about the scale factors from A to B and B to A? From A to C and C to A?
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PROBLEM 4

A. Draw a rectangle that is 3 inches by 4 inches. In a map with a scale factor of 1 inch = 2 miles, how big an area does the rectangle represent?

B. What are the dimensions of a rectangle that represents an area 5 miles wide and 8 miles long on the map? What is the area of the scaled rectangle? What is the area represented by the map?

EXERCISES

1. Three rectangles have the following dimensions:

   - Rectangle A: 6 cm by 8 cm
   - Rectangle B: 9 cm by 12 cm
   - Rectangle C: 18 cm by 24 cm

   Determine the scale factor, if there is one, from
   a. Rectangle A to Rectangle B.
   b. Rectangle B to Rectangle C.
   c. Rectangle A to Rectangle C.

2. In each problem below, apply the specified scale factor to a rectangle. What is the percent increase or decrease in the length of the new rectangle?
   a. scale factor of 1.8
   b. scale factor of 0.8
   c. scale factor of 2.5
   d. scale factor of 3

3. Draw a 6-by-15 rectangle and label it A, a 4-by-10 rectangle and label it B, and a 8-by-20 rectangle and label it C.
   a. What is the scale factor from A to B?
   b. What is the scale factor from B to A?
   c. What is the scale factor from A to C?
   d. What is the scale factor from C to A?
   e. What is the scale factor from C to B?
   f. What is the scale factor from B to C?
4. Draw a 4-by-6 rectangle and call it \( A \). Increase the length and the width of \( A \) by \( \frac{1}{2} \) to form rectangle \( B \). What is the scale factor from \( A \) to \( B \)? What is the scale factor from \( B \) to \( A \)?

5. Increase the length and the width of rectangle \( A \) by \( \frac{2}{5} \) to form rectangle \( B \). What is the scale factor from \( A \) to \( B \)? (Call the dimensions of \( A \) length \( L \) and width \( W \).)

6. Decrease the length and the width of 15 \( \times \) 10 rectangle \( A \) by \( \frac{2}{5} \) to form rectangle \( B \). What is the scale factor from \( A \) to \( B \)? What is the scale factor from \( B \) to \( A \)?

7. Make a 2-by-3 rectangle and call it \( R \). Make a new rectangle using a scale factor of 3, and label it \( S \). What is the perimeter of \( S \)? the area of \( S \)? How do the perimeter and area of \( S \) compare to the perimeter and area of \( R \)?

8. For rectangle \( S \) with length \( L \) and width \( W \), use a scale factor of 2 to make a larger rectangle \( T \) (sketch a picture for this problem.)
   a. Compute and compare the perimeters of both rectangles. What do you notice?
   b. Compute and compare the areas of both rectangles. What do you notice?
   c. Compare the perimeter of rectangle \( S \) to \( T \). Determine the percent increase in the perimeter.
   d. Compare the area of rectangle \( S \) to \( T \). Determine the percent increase in the area.

9. On a coordinate plane, plot the points \( A \) (1, 0), \( B \) (4, 0), \( C \) (5, 5), \( D \) (0, 3) and \( E \) (0, 1).
   a. Draw line segments connecting these points in alphabetical order.
   b. Double each coordinate of the given points \( A \), \( B \), \( C \), \( D \), and \( E \). Plot these points and draw line segments connecting the points as you did in part a.
   c. Find half of each coordinate of the given points \( A \), \( B \), \( C \), \( D \), and \( E \). Plot these points and draw line segments connecting the points as you did in part a.
   d. What do you notice about the figures in parts a, b, and c?
10. Nama is making a dog house for her dog Cloud using scale drawings. If the dimensions on the pattern are 1 1/2 inches by 3 inches, what will the final dimensions of the dog house be using a scale factor of 12?

11. **Ingenuity:**

What is the value of the product

\[
(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{4^2})\cdots(1 - \frac{1}{101^2})?
\]

Look at other products of this form. Do you see a pattern?

12. **Investigation:**

Measure the dimensions of an interesting room in your house. Using a scale of 1 inch: 2 feet, draw the floor plan. Include doorways, closets, windows, and furniture.

a. Find the scale factor that makes the area of the room 4 times the original area. What are the dimensions of the bigger room?

b. Find the scale factor that makes the perimeter 4 times the original perimeter. What are the dimensions of the bigger room?
Amy's teacher asked her to draw a floor plan of her house. The boundary is a rectangle 100 feet long and 32 feet wide. In her drawing, she cannot use the actual dimensions of the house, because her paper is not 100 feet long. She decides to use a scale of one inch in her drawing to represent 8 feet in the house. After Amy has completed the drawing, filling in the walls of all the rooms in her house, how can her teacher find the dimensions of each room of Amy's house?

Scaling is the process of taking a figure, like the floor plan of a house, and using it to make a new figure, for example the real floor plan. In Section 10.4, you learned how to use a scale factor to make a smaller and a larger rectangular figure with the same shape.

EXPLORATION 1

a. Draw rectangle A. Apply the scale factor to each new rectangle. Draw and label appropriately. Use the new rectangle to complete the tables.
b. Using your information, make a prediction for the Perimeter and Area for a new rectangle F with a scale factor of 6.

c. What is the relationship between the scale factor and the perimeters of the rectangles? What is the relationship between the scale factor and areas of the scaled rectangles?

When an \(n\)-by-\(m\) rectangle is scaled in each direction using the scale factor \(k\), the old perimeter and the new perimeter are

\[
P_{\text{old}} = 2n + 2m, \quad \text{and} \quad P_{\text{new}} = 2(k \cdot n) + 2(k \cdot m) = k \cdot (2n + 2m) = k \cdot P_{\text{old}}.
\]

**The new perimeter is the old perimeter multiplied by \(k\).**

The old area and the new area are

\[
A_{\text{old}} = n \cdot m, \quad \text{and} \quad A_{\text{new}} = (k \cdot n)(k \cdot m) = k^2(n \cdot m) = k^2 \cdot A_{\text{old}}.
\]

**The new area is equal to the old area multiplied by \(k^2\).** The area has been scaled by the same number in each direction. It is also possible to scale by different numbers in each direction, but the resulting figure will not look like an enlarged or reduced copy of the original shape.
EXPLORATION 2

What do you notice about the triangles below?

![Triangle ABC and Triangle DEF](image)

a. Measure the angles and the side lengths of triangle $ABC$ and triangle $DEF$.

b. Compare the following angle pairs, $\angle A$ to $\angle D$, $\angle B$ to $\angle E$, and $\angle C$ to $\angle F$. What do you notice?

c. Compare the side lengths $AB$ to $DE$, $BC$ to $EF$, and $AC$ to $DF$. What do you notice?

d. Compute the ratio of each pair of side lengths from part c. What pattern do you notice in these ratios? Do you see a scale factor from triangle $ABC$ to triangle $DEF$? Do you see a scale factor from triangle $DEF$ to triangle $ABC$?

e. Graph the two triangles on a coordinate plane and discuss your findings. How do the area and perimeter compare? Explain using algebraic notation.

The smallest angles in triangles $ABC$ and $DEF$ had the same measure, so they represent corresponding angles of both triangles. Similarly, the shortest sides in triangles $ABC$ and $DEF$ had the same ratio as the longest sides in triangles $ABC$ and $DEF$. That characteristic makes them corresponding sides. Look again at the picture. It’s easiest to recognize corresponding sides and angles visually.
Two triangles are similar when their corresponding angles have equal measures and their corresponding sides have the same ratio. In the picture, \( \angle A \) corresponds to \( \angle D \), \( \angle B \) corresponds to \( \angle E \), and \( \angle C \) corresponds to \( \angle F \). From this information, we can also determine that \( \overline{AB} \) corresponds to \( \overline{DE} \), \( \overline{AC} \) corresponds to \( \overline{DF} \), and \( \overline{BC} \) corresponds to \( \overline{EF} \). Choosing corresponding sides is essential in working with similar triangles.

**Theorem 10.1: Triangle Similarity Theorem**

If two triangles have corresponding angles of the same measure, then the ratios of their corresponding sides are the same. Conversely, if two triangles have corresponding sides with the same ratio, then the triangles' corresponding angles have equal measure.

Triangle \( S \) has sides measuring 2, 3 and 4 units, and triangle \( T \) is triangle \( S \) scaled by a factor of 3. Triangle \( T \) has corresponding sides that measure \( 3\times2 = 6 \), \( 3\times3 = 9 \) and \( 3\times4 = 12 \) units. Although the lengths of the sides are all different, the ratios of corresponding sides are the same, because \( \frac{6}{2} = \frac{9}{3} = \frac{12}{4} = 3 \). The ratio of the corresponding sides is the scale factor, 3.

What is the effect on the angles of the change from triangle \( S \) to \( T \)?

In summary, for similar triangles, one of the triangles can be created from the other by stretching or compressing the sides by the same scale factor, without changing the measures of the angles. The triangles have the same shape, but different sizes.
Because the scaling relationship holds for all similar triangles, when you find the ratio of any two sides of a triangle, this will be the same ratio for the corresponding sides in any similar triangle. Generalizing the example above, call the lengths of the sides of the first triangle \(a, b\) and \(c\). With a scale factor of \(k\), the similar triangle will have sides that measure \(ka = A\), \(kb = B\) and \(kc = C\), respectively.

In the picture below, the original triangle has been scaled by a factor \(k = 2\): Each side is twice as long.

With two similar shapes, order is important. Corresponding sides must be labeled in the same order. For instance, \(\triangle rst\) is similar to \(\triangle RST\) and \(\angle str\) has the same measure as \(\angle STR\). (By convention, \(\triangle rst\) is not said to be similar to \(\triangle STR\), because there is no fixed scale factor such that \(rs\) scales to \(ST\), \(st\) scales to \(TR\), and \(tr\) scales to \(RS\).) The ratios between corresponding sides in similar triangles are equivalent.

\[
\frac{a}{b} = \frac{ka}{kb}, \quad \frac{a}{c} = \frac{ka}{kc} \quad \text{and} \quad \frac{b}{c} = \frac{kb}{kc}
\]

In general, we can have similarity in any type of polygon.

**THEOREM 10.2: POLYGON SIMILARITY THEOREM**

Two polygons are similar when their corresponding angles have equal measure and their corresponding sides have the same ratio.

**EXAMPLE 1**

In the afternoon, a tree casts a shadow of 28 feet. If a 5 foot post casts a 4 foot shadow, what is the height of the tree? Draw a picture.
From the picture above, notice that the shadows of the figures create 2 similar triangles, due to the position of the sun and the right angles between the figures and the ground. Thus, if we find the scale factor from figure A to figure B, we can compute the height of the tree. To find the scale factor, we can compare the lengths of the shadows. So, \(4k = 28\), or \(k = 7\). Since the heights of the figures are corresponding sides, then the height of the tree must be \(5 \times 7 = 35\) ft.

However, we can use our knowledge of ratios between corresponding sides in similar triangles to set up the following proportion to solve this problem in a different way:

\[
\frac{5}{4} = \frac{x}{28}.
\]

Therefore, \(x = \left(\frac{5}{4}\right)(28) = 35\). So, the height of the tree must be 35 ft.

**EXERCISES**

1. Suppose that \(\triangle ABC\) is similar to \(\triangle A'B'C'\), and the side of length 10 in triangle \(ABC\) has length 50 in \(A'B'C'\). What is the scale factor?

2. Suppose that the sides of \(\triangle ABC\) are of length 4, 5, and 6. Suppose that \(\triangle A'B'C'\) is similar and that the side corresponding to the side of length 4 has length 12. What are the lengths of the other two sides of \(\triangle A'B'C'\)?
3. Which statement best describes the change in the perimeter of a triangle if all its side lengths are multiplied by 3? Select the best choice and explain your answer.
   a. The new perimeter will be 9 times as large as the perimeter of the original triangle.
   b. The new perimeter will be 12 times as large as the perimeter of the original triangle.
   c. The new perimeter will be 3 times as large as the perimeter of the original triangle.
   d. The new perimeter will be 6 times as large as the perimeter of the original triangle.

4. Triangle ABC has sides of lengths 10, 20, and 25.
   a. Triangle DEF has sides of lengths 20, 40, and 50. Are these two triangles similar? Explain.
   b. Triangle DEF has sides of lengths 50, 20, and 40. Are these two triangles similar? Is there a different ordering of the angles that would make the triangles similar? If so, what ordering would make them similar?

5. On a grid, draw a right triangle with sides of length 4 and 6.
   a. Draw two similar triangles with scale factors of 2 and 1.5. Label and measure all of the corresponding angles and compare them.

6. Use a grid to make copies of each of the triangles below. For each triangle, draw a similar triangle that uses a different scale factor.
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7. Find all pairs of similar rectangles in the drawing below and give the scale factor for each pair.

8. On grid paper, draw a triangle \( M \) with base 4 and height 3. Then draw a similar triangle \( N \) with base 10.
   a. Determine the scale factor \( k \) from \( M \) to \( N \).
   b. Compute the area \( A_M \) of triangle \( M \) and the area \( A_N \) of triangle \( N \).
   c. What is the ratio \( \frac{A_N}{A_M} \)?

9. Draw a triangle \( S \) with base 6 in., another side with length 4 in. and height 2 in. Then draw a similar triangle \( T \) with base 3 in.
   a. Determine the scale factor \( k \) from \( S \) to \( T \).
   b. Compute the area \( A_S \) of triangle \( S \) and the area \( A_T \) of triangle \( T \).
   c. What is the ratio \( \frac{A_T}{A_S} \)?
10. Find all pairs of similar triangles in the drawing below and give the scale factor for each pair.

11. A farmer builds a chicken coop with a front that has the shape of an isosceles triangle. The base of the triangle is 12 meters long. In the plans that she used to build the barn, the base was 40 cm and the lengths of the other two sides were 25 cm.

a. What is the length of each equal side of the coop?

b. What is the height of the coop?

12. One night a woman 5 feet tall stood 10 feet from a light post. She measured the length of her shadow and found that her shadow was 12 feet long. What was the height of the light post?

13. A triangle originally has area 20 cm². Each of its sides is increased by a scale factor of 3. What is the area of the new triangle?

14. Two triangles are similar, and one triangle’s sides are twice as long as the other triangle’s. What is the relation between the areas of the triangles? What is the relation between their perimeters?
15. If a 10 foot pole casts a 6 foot shadow, find the length of the shadows of each of the following:
   a. 5 foot boy
   b. 3 foot child

16. A flagpole casts a shadow of 96 feet and a 4 foot child casts a shadow of 5 feet.
   a. How tall is the flagpole?
   b. How long of a shadow will a 6 foot person cast at this time?

17. In downtown New York, Jack looks up from a park bench to see the top of a flagpole and behind it is the top of a building. The pole is 80 feet tall and Jack is 30 feet from the pole. If Jack is sitting 240 feet from the base of the building, how tall is the building?

18. Ingenuity:
   In right triangle $\triangle ABC$, draw a perpendicular line from right angle $C$ to hypotenuse $AB$. The perpendicular line intersects side $AB$ at point $D$. Show that $\triangle ACD$ is similar to $\triangle ABC$ and that both are similar to $\triangle CBD$.

19. Investigation:
   There is a tall pole with a straight wire from point $A$ on the ground to the top of the pole. The bottom of the pole is 20 feet from the wire. Sam, who is 5 feet, 6 inches tall, stands a yard away from point $A$ and the top of his head touches the wire. How high is the pole?
REVIEW PROBLEMS

1. The Grinnell Mountain Goats and the Texas State Bobcats are playing a water polo match. The Mountain Goats score an average of 2 times every 5 minutes and the Bobcats scored an average of 1 point every 4 minutes. What is the score of the game if they played a 60 minute game? What if they played an 80 minute game? Does the length of time played change who won the game?

2. Quicko’s makes copies for $.10 a page. They also allow people to pay $10 and become Quicko’s Preferred Customers and pay half a cent for each copy. Write a function for the cost per copy of members and non members. When is it worth it to buy a membership?

3. Mr. Boney’s class has 3 boys for every 2 girls.
   a. Suppose there are 10 girls in Mr. Boney’s class. How many boys are there in the class? How many students total?
   b. Suppose Mr. Boney has 30 students in his class. How many boys are there in the class?

4. The Quadlings are an alien race from the planet Quadloo. If 8 Quadlings have 32 legs, how many legs does each Quadling have?

5. Jeff’s local grocery store normally sells grapes for $.03 per grape. He uses these grapes to make his delicious grape jam. If it takes 200 grapes to make 3 jars of jam, how much is cost to make 5 jars of jam?

6. If it takes 12 waiters to serve 192 customers, how many customers can 5 waiters serve?

7. At the Texas State Summer Math Camp, each study group is made up of 4 students and 1 counselor.
   a. How many counselors are there if there are 56 campers total?
   b. How many students are there if there are 12 counselors?
8. RJ wants to run a neighborhood lemonade stand. It costs $10 dollars for the startup costs of the stand itself, the pitchers, etc. It costs $0.50 to make each large glass of lemonade. How much does it cost to make 20 glasses of lemonade? How many glasses can he make for $50? Write a function that describes the amount of money RJ makes based on the number of glasses of lemonade that he sells.

9. Heather is working in Denali National Park for 63 days (9 weeks) in the summer. If she averages sighting 15 Pine Grosbeaks every 5 days, how many will she have seen over her entire time there?

10. Kate is working at a bookstore. She sells twice as many mystery books as science fiction books and she sells 10 mystery books a day. How many science fiction books does she sell in a week? Do this without figuring out the number of mystery books she sells.

11. A bird watcher sees 3 hawks every 8 miles she drives across Texas. How many hawks does she expect to see if she drives 100 miles?

12. Peter walks 3 miles in 2 $\frac{1}{4}$ hours. If he walks all three miles at the same pace, how long will it take him to walk two miles?

13. A rancher increases his herd from 84 to 106 cattle. What is the percent of increase of the herd?

14. A dress was discounted from $96 to $84. What was the rate (percent) of discount?

15. Sketch a rectangle A with dimensions 4 by 10 units.
   a. Sketch a rectangle B with a scale factor of 3.5 from A to B. Label the dimensions of rectangle B.
   b. Sketch a rectangle C with a scale factor of $\frac{1}{4}$ from A to C. Label the dimensions of rectangle C.

16. Let X be a 12 by 20 rectangle. Determine which of the following rectangles are similar to rectangle X and, if it is, what is the scale factor from X to it?
   a. Rectangle W is a 3 by 4 rectangle
   b. Rectangle Y is a 3 by 5 rectangle
c. Rectangle Z is an 18 by 30 rectangle

17. Jack and Jill ate at a cafe and the bill (including tax) was $13.50. If they leave a 20% tip, how much was their final bill?

18. Juan and Sherry ate at a cafe and their total bill (with tip) was $19.25. If they included a 10% tip, what was their original bill?

19. A store marks up the wholesale price of a shirt by 35% to the selling price of $75.60. What was the wholesale price of the shirt?

20. A CD is marked down 30% to the sale price of $8.82. What was the original price of the CD?

21. Every evening for dinner, Leopold the rabbit eats $1 \frac{1}{2}$ ounces of leafy greens, such as lettuce, and 1 ounce of carrot.
   a. By weight, what percentage of Leopold’s dinner consists of carrot?
   b. How many ounces of vegetables does Leopold eat in one week, given that he only eats vegetables at dinner?
   c. Leopold weighs 5 pounds. How long will it take him to eat his body weight in vegetables?

22. Triangle $ABC$ is similar to triangle $DEF$. What is the measure of $\angle E$?

23. At 10:00 a tree casts a 20-foot shadow. At the same time, a 6-foot man casts a 4-foot shadow. Write a proportion to find the height of the tree.

24. Katrina was enlarging a photo of her mom for her mom’s birthday. She enlarged the photo 150% of its original size on a copy machine. If the photo is $3 \frac{1}{2}$ inches by 5 inches, what are the dimensions of the enlarged photo?
25. Which rectangle is similar to rectangle \( RSTU \)?

![Rectangle Diagram]

26. Find the value of \( x \) in each pair of similar figures.

a. \[
\begin{align*}
\text{6 cm.} & \quad \text{12 cm.} & \quad \text{12 cm.} \\
\text{3 cm.} & \quad \text{12 cm.} \\
\end{align*}
\]

b. \[
\begin{align*}
\text{2 ft.} & \quad \text{7 ft.} & \quad \text{5 ft.} \\
\text{x ft.} & \quad \text{x ft.} \\
\end{align*}
\]

c. \[
\begin{align*}
\text{15 in.} & \quad \text{24 in.} & \quad \text{x in.} \\
\text{15 in.} & \quad \text{8 in.} \\
\end{align*}
\]
Section 10.1:
Carl and Bob can demolish a building in 6 days, Anne and Bob can do it in 3, Anne and Carl in 5. How many days does it take all of them working together if Carl gets injured at the end of the first day and can’t come back?

Section 10.2:
An estimate for converting temperatures from Celsius to Fahrenheit is to double and add 30. Thus 0°C is approximately 30°F (actually 32), and 100°C is approximately 230°F (actually 212). For what temperature in °C is the estimate correct?

Section 10.3:
If 2 authors can write 3 pages in 4 days, how many days does it take 5 authors to write 6 pages?

Section 10.5:
In triangle ABC the length of \( AB = 15 \) cm. Point D is a point on side AB so that \( AD = 10 \) cm. What is the ratio of the area of triangle ADC to the area of triangle BDC?
Chapter 10 Rates, Ratios, and Proportions
SECTION 11.1 MEASURING ANGLES

How would you answer the question, “What is an angle?”

You have probably seen angles in many places in everyday life. Can you name a few of these places? In this section, you will learn what angles are, how to construct angles, and how to measure the size of an angle.

Let’s begin by constructing an angle. To do this, first draw two rays from a common point $P$. A ray is part of a straight line that has a starting point and continues forever in only one direction. In the figure above, there are two rays, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Both of these rays begin at point $P$ and pass through the points $Q$ and $R$ respectively.

To name the ray, list the starting point first and then any other point on the ray. There are many different ways to name it, $\overrightarrow{PQ}$, or ray $PQ$. Picking another point on this ray, for instance $S$, a point between $P$ and $Q$, gives the ray $\overrightarrow{PS}$, which is the same ray as $\overrightarrow{PQ}$. However, the ray $\overrightarrow{SP}$ is a different ray because it begins at point $S$. In fact, $\overrightarrow{SQ}$ is a third ray that is different from all of the rays mentioned so far.
Now we can answer our original question: “What is an angle?”

**DEFINITION 11.1: ANGLE**

An angle is formed when two rays share a common vertex.

The common endpoint $P$ on both rays is called the vertex of the angle. In the diagram, the rays $PQ$ and $PR$ form an angle called angle $QPR$ or $\angle QPR$. The symbol “$\angle$” is the math symbol for the word angle. To name an angle, you can do the following three steps:

1. Write the name of one of the non-vertex points on one of the rays.
2. Write the name of the vertex.
3. Write the name of the other non-vertex point.

There can be many ways to name the same angle because there are many choices of points on the two rays in steps 1 and 3. You could also label this angle $\angle RPQ$. The order in which the points are written does not matter as long as the middle point identifies the vertex of the angle.

**EXAMPLE 1**

In angle $\angle XYZ$, identify the rays and identify the vertex.

**SOLUTION**

$XY$ and $YZ$ are the rays, and $Y$ is the vertex of the angle. We can use different letters to label different angles, like $\angle QPR$ or $\angle XYZ$.

Sometimes a single letter is used to name an angle. Sometimes $\angle QPR$ is called angle $P$, or $\angle P$, when there is no confusion. Rather, the letter $P$ is the name for the angle, so $\angle QPR = \angle P$. For instance, in the triangle on the left, $\angle QPR$ is the same as $\angle P$. In the triangle on the right, however, $\angle QPR$ is the largest angle, but $\angle P$ could be one of three angles.
Section 11.1 Measuring Angles

Once you understand the definition of an angle, the next step is to measure the size of the angle. Angles are measured as part of a circle. A circle is divided into 360 equal parts. Each part is one degree. A full circle contains 360 degrees, written $360^\circ$.

The instrument used to measure angles is called a protractor. When you use a protractor, the units commonly used to measure the size of an angle are degrees as described above.

Protractors have degree markings along the outside of the curved edge. To measure an angle, place the vertex at the center of the semicircle so that one ray passes through $0^\circ$ or $180^\circ$ and the other ray passes through a mark on the curved edge. If necessary, extend the other ray so that it falls on a mark along the curved edge. The degree at this mark is the measure of the angle, or its supplement, which we will define later in this section.

If two rays with a common endpoint form a straight line, the angle they form has a measure of 180 degrees, or $180^\circ$. This is called a straight angle.
EXPLORATION

1. Estimate the measure of each angle below and then use a protractor to check your answers.

   a. 

   b. 

   c. 

   d. 

   e. 

2. Consider the rays and points above. Write each angle in two ways.

3. Divide a straight angle into two parts. Describe how you constructed it to another student or your teacher. Measure each angle using your protractor.

4. Use your protractor to draw rays making the following angles: 35°, 80°, and 100°.
How did you construct the angles in question 4? Here is one approach to construct an angle with a given measure.

1. Draw an initial ray and label it $PR$. The initial ray is usually, but not necessarily, horizontal.
2. Place the center of the semicircle of the protractor on top of the point $P$, with the ray passing through $0^\circ$.
3. Find the place along the curved edge of the protractor that corresponds to the degree measure you are constructing, and mark it with a new point $Q$.
4. Draw a line connecting the point $P$, which is the vertex of the angle, to the new point $Q$ with a straight edge to obtain an angle with a given measure.
5. What is the measure of the angle $QPR$ below?

PROBLEM 1

Construct an angle starting at $0^\circ$ with measure $45^\circ$.

PROBLEM 2

Construct an angle starting at $20^\circ$ with measure $45^\circ$. 
Mathematicians use the notation $m(\angle QPR)$ to mean the measure of angle $QPR$. Notice that $\angle QPR$ and $\angle RPS$ divide the $\angle QPS$ into two parts. The measure of each angle is a number, and we can add these numbers together to get the equations below:

$$m(\angle QPR) + m(\angle RPS) = m(\angle QPS)$$

$QS$ is a line, so $m(\angle QPS) = 180^\circ$. Then we have

$$m(\angle QPR) + m(\angle RPS) = 180^\circ.$$

**DEFINITION 11.2: SUPPLEMENTARY**

Two angles are **supplementary** if the sum of their measures totals $180^\circ$.

In the example above, $\angle QPR$ and $\angle RPS$ are supplementary angles. This means $\angle RPS$ is the supplement of $\angle QPR$, and $\angle QPR$ is the supplement of $\angle RPS$.

Now divide a straight angle in half. Each angle formed is called a **right angle** and measures $90^\circ$ because $\frac{1}{2}$ of $180^\circ$ is $90^\circ$. We label the right angle with a $\perp$ to indicate that it is $90^\circ$. When two lines meet and form a right angle, they are **perpendicular** to each other.
When two rays meet to form a right angle, they are perpendicular rays.

Next, divide the right angle $PRQ$ above into two parts:

Because $\angle PRS$ and $\angle SRQ$ divide $\angle PRQ$,

$$m(\angle PRS) + m(\angle SRQ) = m(\angle PRQ) \text{ and } m(\angle PRS) + m(\angle SRQ) = 90^\circ.$$
DEFINITION 11.3: COMPLEMENTARY

Two angles are **complementary** if the sum of their measures totals 90°.

In the example above, ∠PRS and ∠SRQ are complementary angles. This means ∠SRQ is the complement of ∠PRS, and ∠PRS is the complement of ∠SRQ.

Angles that have a measure between 0° and 90° are called **acute angles**. Angles that have a measure greater than 90° but less than 180° are called **obtuse angles**. Using a protractor, construct and label a straight angle, a right angle, an acute angle, and an obtuse angle.

EXERCISES

1. Use the drawing to answer the following questions:

   ![Diagram](image)

   a. Name 2 acute angles.
   b. Name 2 obtuse angles.
   c. Name all right angles.
   d. Find all supplementary angle pairs.
   e. Find all complementary angle pairs.

2. If ∠QPR has measure 40°, what is the measure of the angle that is supplementary to ∠QPR? Construct the two angles and measure them using your protractor. Do your answers agree?
3. A certain angle measures 75°. What is the measure of its complement? Use your protractor to draw angles with these measures.

4. Draw an angle that measures 20°. What is the measure of the complementary angle to the original angle?

5. An angle and its supplement have equal measures.
   a. Find their measures.
   b. Draw these angles.

6. Two angles are supplementary, and one angle has a measure that is 10° more than the other angle. What is the measure of each of the angles? (Hint: Call the measure of the original angle $x$; then write an equation involving $x$ and solve the equation.)

7. Two angles are complementary, and one of the angles has a measure that is twice as large as the measure of the other angle. What is the measure of each of the angles? (Hint: Call the measure of your original angle $x$. Write an equation involving $x$ and solve the equation.)

8. An angle’s measure is twice its supplement’s. What is the measure of the two angles?

9. How many acute angles can you find with measures between 80° and 90°?

10. If $L_1$ and $L_2$ are parallel, which of the following angles are supplementary to angle A?
   a. $\angle 1$
   b. $\angle 2$
   c. $\angle 3$
   d. $\angle 4$
11. **Ingenuity:**

The pitch of the roof is a ratio used in place of the angle that the roof makes with a horizontal ray. A common pitch on a roof is given by a right triangle with horizontal side 12 and vertical side 6. This roof has pitch $\frac{6}{12}$. In fact, $\frac{6}{12}$ gives a measure for how much the roof is sloped.

a. Draw a scale model of a roof with this pitch, and measure the angle of the roof.

b. If the vertical side is 7, will the roof be steeper or less steep? What is the angle of a roof with a $\frac{7}{12}$ pitch?

c. Draw angles A, B, and C that correspond to roofs with pitches $\frac{3}{12}$, $\frac{4}{12}$, and $\frac{5}{12}$. Measure each angle.

d. Which pitch corresponds to the largest angle? Which pitch corresponds to the smallest angle? What do you notice about the angles as the pitch ratio increases?

12. **Investigation:**

Do you know historically why there are 360° in a full circle? Who first used the number 360? Why didn’t they choose 300 or 400 or 500 degrees to make a circle?
SECTION 11.2 PARALLEL LINES AND ANGLES IN A TRIANGLE

In the previous section you measured angles and worked with complementary and supplementary angles. These ideas are very useful for studying triangles and other shapes in geometry.

EXPLORATION 1

Draw two straight lines that intersect at a point $P$. Label several other points on both lines, and use these to name all four of the angles formed by the two lines. Measure the angles using your protractor. What do you notice about the angles? What do you notice about the sums of the angles?

Repeat the Exploration using a different pair of intersecting lines. What do you notice about the measures of the angles? Write a sentence or equation that describes what you have discovered about the angles made by intersecting lines.

Consider the four labeled angles formed by the two intersecting lines:

If two of these angles share a common ray between them, then the angles are said to be adjacent. Note that two angles might share a common ray but not be adjacent if the ray is not between the two angles. In the figure above, $\angle 1$ and $\angle 2$ are adjacent, and similarly, $\angle 1$ and $\angle 4$ are also adjacent. What other pairs of adjacent angles can you find?

If the two adjacent angles have their non-adjacent rays lying on a straight line, then the two adjacent angles will have measures that add up to $180^\circ$. So, $m(\angle 1) + m(\angle 2) = 180^\circ$. Do you see why this is true?
From the same figure, consider a pair of angles that are not adjacent. These angles form a pair of **vertical angles**.

In the figure, \( \angle 1 \) and \( \angle 3 \) are vertical angles, and \( \angle 2 \) and \( \angle 4 \) are vertical angles. This is described more precisely in the following definition:

**DEFINITION 11.4: VERTICAL ANGLES**

If two straight lines intersect at a point, then each line is divided into two rays. The angles formed by using the opposite rays from each line segment are called **vertical angles**.

Note: Opposite rays are two rays with a common endpoint that form a straight line.

When you measure each pair of vertical angles, do you always get the same answer? It seems to be the case that any two vertical angles will always have the same measure. Let’s explore this in the example that follows.

**EXAMPLE 1**

For each pair of supplementary angles, write down a corresponding equation that expresses the relationship between the two supplementary angles.

**SOLUTION**

From the picture, \( \angle 1 \) and \( \angle 2 \) are supplementary because their sum totals \( 180^\circ \), and they lie on line \( m \). Also \( \angle 1 \) and \( \angle 4 \) are supplementary because together they make a \( 180^\circ \) angle, and they lie on line \( n \). So,

\[
m(\angle 1) + m(\angle 2) = 180^\circ, \quad \text{and} \quad m(\angle 1) + m(\angle 4) = 180^\circ.
\]

From these two equations, you can substitute for \( 180^\circ \) and obtain
Section 11.2 Angles in a Triangle

\[ m(\angle 1) + m(\angle 2) = m(\angle 1) + m(\angle 4). \]

Subtracting \( m(\angle 1) \) from both sides of this equation, you get that

\[ m(\angle 2) = m(\angle 4). \]

This is a proof of your observation that vertical angles always have the same measure! This is a famous theorem, called the Vertical Angle Theorem:

**THEOREM 11.1: VERTICAL ANGLE THEOREM**

If two lines intersect at a point \( P \), then the vertical angles will have the same measure.

You have begun the process of exploring problems in geometry. The fun part is seeing why observations that you make always hold true. Let’s think carefully about what seems like a simple concept, the idea of “parallel” lines. The question is how you decide whether two lines are actually parallel. In fact, what does it mean to say that they are parallel in the first place?

One way to describe parallel lines is to say that two lines in a plane are parallel if they never intersect, even if they are extended forever in both directions. The question is how to decide whether the lines will have this property or not.

This is a problem studied by the ancient Greeks. One approach they took was to begin by adding in a third line called a transversal. A transversal is a line that intersects the pair of lines that you begin with. For instance, in the drawing, line \( N \) transverses, or goes across, lines \( L \) and \( M \). The transversal cuts the two lines \( L \) and \( M \). What they observed was the following:

1. Begin with two original lines.
2. Cut these with a transversal.
3. Form a pair of corresponding angles using part of the transversal as one ray and the part of the lines on the same side of the transversal as the other ray. (See picture.) Note that $\angle 1$ and $\angle 2$ are corresponding angles, and $\angle 3$ and $\angle 4$ are also corresponding angles. So, corresponding angles are pairs of angles on the same side of the transversal.

4. Observe that $m(\angle 1) = m(\angle 2)$, and $m(\angle 3) = m(\angle 4)$.

The **Corresponding Angle Postulate** of Euclid says the following:

<table>
<thead>
<tr>
<th>THEOREM 11.2: CORRESPONDING ANGLE POSTULATE</th>
</tr>
</thead>
<tbody>
<tr>
<td>If two parallel lines are cut by a transversal, then the corresponding angles have the same measure, and if the two lines are cut by a transversal so that the corresponding angles have the same measure, then the two lines are parallel.</td>
</tr>
</tbody>
</table>

What this says is that two lines will be parallel precisely when the corresponding angles from a transversal are equal!

**EXAMPLE 2**

Lines $M$ and $N$ are parallel, and lines $O$ and $P$ are transversals. What are the measures of angles $A$ and $B$?

![Diagram of lines M and N with transversals O and P, showing angles A, B, C, D, E, and F with measures 135°, 70°, and others.]
Section 11.2 Angles in a Triangle

SOLUTION

Using the corresponding angle postulate, we know that \( m(\angle D) + 70^\circ = 135^\circ \)
because line O is a transversal of lines M and N. Therefore, the \( m(\angle D) = 135^\circ - 70^\circ \)
which equals 65°. Since angles D and B are vertical angles, and the \( m(\angle D) = 65^\circ \),
then the \( m(\angle B) = 65^\circ \).

Using the definition of supplementary angles, \( m(\angle C) + 135^\circ = 180^\circ \). Therefore,
the \( m(\angle C) = 180^\circ - 135^\circ \), which equals 45°.

Since line O is a transversal of lines M and N, then angles E and C are
 corresponding. Since the \( m(\angle C) = 45^\circ \), then the \( m(\angle E) = 45^\circ \).
Using the corresponding angle postulate, we know that \( m(\angle D) + m(\angle E) = m(\angle A) \)
because line P is a transversal of lines M and N. Since the \( m(\angle D) = 65^\circ \) and the
\( m(\angle E) = 45^\circ \), then the \( m(\angle A) = 65^\circ + 45^\circ = 110^\circ \).

EXAMPLE 3

In a triangle ABC, \( m(\angle C) = 108^\circ \) and the measure of \( \angle A \) is twice the measure of
\( \angle B \). What are the measures of angles \( \angle A \) and \( \angle B \)?

SOLUTION

Call the measure of \( \angle A = x \) and the measure of \( \angle B = 2x \). By the Triangle Sum
Theorem,

\[
m(\angle A) + m(\angle B) + m(\angle C) = 180^\circ
\]
\[
x + 2x + 108 = 180^\circ
\]
\[
3x + 108^\circ = 180^\circ
\]
\[
3x = 180^\circ - 108^\circ = 72^\circ
\]
\[
3x = 72^\circ
\]
\[
x = \frac{72^\circ}{3} = 24^\circ, \text{ so } A = 24^\circ \text{ and } B = 48^\circ.
\]
EXPLORATION 2

Draw a large triangle on a sheet of paper using a straight edge. Color or label the three angles of the triangle with different colors. Carefully cut out the triangle. Next, cut the triangle into 3 triangular pieces, each including one angle from the triangle. Put the angles together. What is the sum of the three angles of the triangle? Compare your result with others.

In each case, the sum of the measures of the angles in the triangles appears to be $180^\circ$. This leads to a conjecture: “The sum of the measures of the angles in any triangle is $180^\circ$.” This is a conjecture because it is a statement we think might be true based on our observations, but we have not yet proved it is always true. Is there a way to give a convincing argument or proof that the sum of the measures of the angles in any triangle is $180^\circ$? To answer that, we investigate further with the next explorations.

EXPLORATION 3

Divide the class into groups. In each group, make a small triangle and several copies of it on lined paper using a straight edge. Be as exact in your work as possible. Use these copies to tessellate the paper, or plane. A tessellation, or tiling of the plane with some shape, is a way of covering the plane with that shape with no gaps. This tessellation can be used to show that the sum of the measures of the angles of any triangle adds up to $180^\circ$.

To do this, begin by putting your triangles together to make a series of equal four-sided figures whose opposite sides are parallel. Use these to cover the plane. Label one of the triangles $ABC$. Place it in the middle of a sheet of paper and, using a straight edge, draw lines parallel to the three sides. Then draw three sets of equally-spaced parallel lines like the example.
This forms tessellated tiles in which each tiling piece is a triangle congruent to triangle $ABC$. **Congruent** means that all the tiling pieces, or triangles, have exactly the same size and shape. Label each of the angles in the picture $A$, $B$, or $C$. One way to see that an angle with measure $A$ appears at different places is to use the Corresponding Angle Postulate and the Vertical Angle Theorem. It is now easy to see something quite remarkable: the measures of angles $A$, $B$, and $C$ sum to a straight angle. Explain why.

We can now state the **Triangle Sum Theorem**: 

<table>
<thead>
<tr>
<th>POSTULATE 11.2: TRIANGLE SUM THEOREM</th>
</tr>
</thead>
<tbody>
<tr>
<td>The sum of the measures of the angles in any triangle equals $180^\circ$.</td>
</tr>
</tbody>
</table>

The tessellation is a sketch of the proof that the sum of the measures of the angles in a triangle always adds up to $180^\circ$. In geometry, you will learn how to prove many interesting properties of geometric shapes using only the basic ideas above.

Much of the geometry that you study in middle school comes from the studies developed by Euclid several thousand years ago. He based the study of geometry on the foundations of axioms, postulates, and theorems. Part of the excitement of mathematics involves seeing new relationships based on simple ideas, like corresponding angles.

**EXERCISES**

1. Answer parts a, b, and c based on the figure below.
a. Write an equation that describes the relationship between \( \angle 1 \) and \( \angle 2 \) and between \( \angle 3 \) and \( \angle 4 \)?
b. If \( m(\angle 1) = 105^\circ \), what are the measures of the other angles?
c. Write equations that express the measures of \( \angle 2 \), \( \angle 3 \), and \( \angle 4 \) if the measure of \( \angle 1 = x \).

2. Given that lines \( M \) and \( N \) are parallel, answer parts a, b, and c based on the figure below.

\[ \text{M} \quad \text{L} \quad \text{N} \]

a. What term describes line \( L \), which crosses line \( M \) and line \( N \)?
b. Write an equation to describe the relationship between \( \angle 1 \) and \( \angle 2 \)?
c. If you are only given that \( m(\angle 1) = m(\angle 2) \), what can you conclude about line \( M \) and line \( N \)?

Find the measure of the angle missing in each of the triangles in Exercises 3 – 6.

3. 

4. 

5. 

6. 

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7. A triangle has two angles whose measures are 45° and 60°. Find the measure of the third angle.

8. A right triangle is a triangle where one of the angles has measure 90°. If a right triangle has another angle whose measure is 45°, find the measure of the third angle.

9. All of the angles in $\triangle ABC$ have the same measure. What is the measure of each of the angles?

10. A 90° angle is trisected, or divided into three equal angles. What is the measure of each of these angles?

11. A triangle has three angles with measures $x$, $2x$, and $3x$. Find the measure of each of these angles.

12. Lines J and K are parallel. Lines L and M are transversals of lines J and K.

```
\hspace{1cm}
\begin{array}{c}
\hspace{1cm}
\end{array}
```

a. Calculate the measure of angle A.
b. Calculate the measure of angle B.

13. Lines R and Q are parallel. Lines S and T are transversals of lines R and Q.

```
\hspace{1cm}
```
a. Calculate the measure of angle A.
b. Calculate the measure of angle B.
c. Calculate the measure of angle C.

14. **Investigation:**

The word polygon comes from the Greek words *poly*, meaning “many,” and *gon*, meaning “angles.” A polygon is made by joining a number of line segments to make a closed shape. Each polygon can have many sides. The simplest polygon is a triangle. A quadrilateral is a polygon with “quad” or 4 sides.

a. Show that the sum of the angles of every quadrilateral is the same. What is this sum?
b. Show that the sum of the angles of every pentagon is the same. What is the sum?
c. Find the sum of the angles in an *n*-gon, a polygon with *n* sides.
SECTION 11.3 TWO-DIMENSIONAL FIGURES

The word polygon comes from the Greek words poly, meaning “many”, and gon, meaning “angles.” A polygon is made by joining a finite number of line segments to make a closed shape. Each polygon can have many sides. The simplest polygon is a triangle. A quadrilateral is a polygon with “quad” or four sides.

**DEFINITION 11.5: POLYGON**

A polygon is a simple, closed, plane figure formed by 3 or more line segments.

In a regular polygon, the line segments are equal, and the interior angles are congruent. An irregular polygon is a polygon that is not a regular polygon.

Look at the table below:

<table>
<thead>
<tr>
<th># of sides</th>
<th>Name</th>
<th>Regular Polygon</th>
<th>Irregular Polygon</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Triangle</td>
<td><img src="image" alt="Triangle" /></td>
<td><img src="image" alt="Irregular Triangle" /></td>
</tr>
<tr>
<td>4</td>
<td>Quadrilateral</td>
<td><img src="image" alt="Quadrilateral" /></td>
<td><img src="image" alt="Irregular Quadrilateral" /></td>
</tr>
<tr>
<td>5</td>
<td>Pentagon</td>
<td><img src="image" alt="Pentagon" /></td>
<td><img src="image" alt="Irregular Pentagon" /></td>
</tr>
<tr>
<td>6</td>
<td>Hexagon</td>
<td><img src="image" alt="Hexagon" /></td>
<td><img src="image" alt="Irregular Hexagon" /></td>
</tr>
<tr>
<td>7</td>
<td>Heptagon</td>
<td><img src="image" alt="Heptagon" /></td>
<td><img src="image" alt="Irregular Heptagon" /></td>
</tr>
<tr>
<td>8</td>
<td>Octagon</td>
<td><img src="image" alt="Octagon" /></td>
<td><img src="image" alt="Irregular Octagon" /></td>
</tr>
<tr>
<td>9</td>
<td>Nonagon</td>
<td><img src="image" alt="Nonagon" /></td>
<td><img src="image" alt="Irregular Nonagon" /></td>
</tr>
<tr>
<td>10</td>
<td>Decagon</td>
<td><img src="image" alt="Decagon" /></td>
<td><img src="image" alt="Irregular Decagon" /></td>
</tr>
<tr>
<td>11</td>
<td>Undecagon</td>
<td><img src="image" alt="Undecagon" /></td>
<td><img src="image" alt="Irregular Undecagon" /></td>
</tr>
<tr>
<td>12</td>
<td>Dodecagon</td>
<td><img src="image" alt="Dodecagon" /></td>
<td><img src="image" alt="Irregular Dodecagon" /></td>
</tr>
</tbody>
</table>
A polygon with 21 sides is called a 21-gon. In this section, we will focus on 3 and 4 sided polygons. We will study different attributes of these figures, such as perimeter and area.

What is the definition of perimeter? What is the perimeter of a $3 \times 4$ rectangle? In general, if a rectangle has length $L$ and width $W$, what is the perimeter of the rectangle?

Just as you found the perimeter of a rectangle by adding the lengths of the four sides, you can find the perimeter of any polygon.

**DEFINITION 11.6: PERIMETER**

The perimeter of a polygon is the sum of the lengths of all of its sides.

If the units of the length and width are inches, then the perimeter is measured in inches, but the area is measured in square inches. Each square inch corresponds to a unit square.

Next, look at other shapes.

Triangles can be classified by different properties - their size, shapes, and angles. In Exploration 1, you may discover different properties of triangles from making measurements of some of these properties.

**EXPLORATION 1**

a. Draw a line segment that is five units long. Now draw two other line segments to complete a triangle. Repeat this process several times. What do you notice about the sum of the lengths of the other two sides relative to the length of the original side?
b. Make a triangle with all three sides of equal length. In groups, reflect on what you notice about each of these triangles. Measure and then make a rule about the angles of a triangle with all three sides of equal length.

c. Make a triangle with two of the sides of equal length. In groups, reflect on what you notice about each of these triangles. Measure and then make a rule about the angles of a triangle having two sides of equal length.

d. Make up three different triangles whose sides are different lengths. Measure the angles opposite each of the sides. What do you notice about the measures of the angles opposite the larger sides?

A triangle in which all three sides have the same length is called an **equilateral triangle**. This term comes from the Greek *equi*, meaning “the same,” and Latin *latus*, meaning “side.” You have probably discovered in the Exploration that all the angles of an equilateral triangle have the same measure.

![Triangle](image)

**DEFINITION 11.7: EQUILATERAL TRIANGLE**

An **equilateral triangle** is a triangle with three congruent sides. An equilateral triangle also has three congruent angles and is also called an **equiangular triangle**.

From previous years, you may recall another type of triangle with two equal sides.

![Triangle](image)
DEFINITION 11.8: ISOSCELES TRIANGLE
A triangle with at least two sides of equal length is called an isosceles triangle.

EXPLORATION 2
Draw two different isosceles triangles with two equal sides of length 1 inch. Measure each of the angles.

PROBLEM 1
Draw two isosceles triangles with two equal sides of length 2 inches but different lengths for the third sides. Measure each of the angles.

In each of the problems above, did you notice that the angles opposite the equal sides are also equal? This is actually one of the properties of all isosceles triangles. In an isosceles triangle, the angles opposite the equal sides are always equal. Conversely, if two of the angles in a triangle are equal, then the sides opposite these equal angles will be equal, and the triangle will be isosceles. These are properties that you will learn when you study geometry. Do you see why they might be true?

It is possible that all three sides of a triangle have different lengths. We call this type of triangle a scalene triangle. In a scalene triangle, the three angles will all have different measures.

DEFINITION 11.9: SCALENE TRIANGLE
A triangle with all three sides of different lengths is called a scalene triangle.
Categorizing the triangles by angles rather than sides, one kind of triangle is a **right triangle**. A right triangle is a triangle with a **right angle**, an angle whose measure is 90°.

The longest side of a right triangle is called the **hypotenuse**. The right angle is **opposite** the hypotenuse. The two shorter sides are called the **legs** of the right triangle.

You will learn a special theorem that relates the lengths of the legs of a right triangle to the length of the hypotenuse. This theorem, the Pythagorean Theorem, enables you to find the length of any side of a right triangle if you are given the lengths of the other two sides.

In addition to right triangles, there are other ways to classify triangles by their angles. If all three angles of a triangle are acute, or less than 90°, the triangle is called an **acute** triangle. If one of the angles is larger than 90°, the triangle is called an **obtuse** triangle. Is it possible for a triangle to have two angles larger than 90°? Explain.

**CLASSIFYING TRIANGLES ACTIVITY:**

1. Can an equilateral triangle be a right triangle? Justify your answer.
2. Is it possible for a scalene triangle to be an acute triangle? Justify your answer.
3. Can you draw a triangle that is both obtuse and isosceles? Justify your answer.
4. Is it possible for an obtuse triangle to also be equilateral? Justify your answer.
QUADRILATERALS
A four-sided polygon is called a quadrilateral.

CLASSIFYING QUADRILATERALS ACTIVITY
1. Is every rectangle a parallelogram? Justify your answer.
2. Is a trapezoid a parallelogram? Justify your answer.
3. Is every square a rectangle? Justify your answer.
AREA OF POLYGONS

We will now discuss the area of a parallelogram.

**DEFINITION 11.10: PARALLELOGRAM**

A parallelogram is a four-sided figure with opposite parallel sides.

![Parallelogram Diagram]

**EXPLORATION 3**

Draw four parallelograms using grid paper. For this Exploration, make sure the longest side is on one of the grid lines.

a. Measure the length of each of the sides and the measure of each angle. What do you observe?

b. Find the area of one of the parallelograms by cutting the parallelogram apart, as illustrated below, and reassembling it to make a rectangle.

![Cut and Reassemble Diagram]

Label one of the horizontal parallel sides of the parallelogram the base, with length $b$. To find the height, draw a line segment between the two bases, perpendicular to each base. The height, $h$, is the length of the perpendicular distance between the two bases. Notice that the height is not the same as the length of either of the two non-horizontal sides. What is the formula for the area of the new figure?
When reassembled, the parallelogram creates a rectangle with length, or base, $b$, and width, or height, $h$. That means the formula for the area $A$ of a parallelogram is:

$A = b \cdot h$ or $A = bh$.

However, what happens if you have a long, skinny parallelogram, ABCD? In this case, in order to find the height, you will have to extend the base at the bottom. The height is the length of a perpendicular from the top to the bottom. Does our formula still work? In order to investigate this situation, we enclose our original parallelogram in a bigger rectangle AEDF as shown.

**EXAMPLE 1**

![Diagram of parallelogram and rectangle]
For shorthand, we label the area of \(ABCD\) as \([ABCD]\) and the area of triangle \(ADF\) as \([ADF]\). If we have a figure with vertices \(PQR\), we will put square brackets around these vertices to indicate the area of the figure.

a. What is the width of the rectangle \(AECF\), i.e. what is the length of side \(AE\)?

b. What is the length of the rectangle \(AECF\), i.e. what is the height of the rectangle?

c. What is the total area of the rectangle \(AECF\), \([AECF]\)?

d. What is the sum of the areas of the two triangles \(ADF\) and \(BEC\)?

e. What is the area of parallelogram \(ABCD\)?

**SOLUTION**

a. The width of the rectangle is \(AB + BE = b + x\).

b. The height of the rectangle is \(h\).

c. The total area of rectangle \(AECF\) is \([AECF] = (b + x) \cdot h = bh + xh\) (by the distributive property)

d. The two right triangles \(ADF\) and \(BEC\) can be put together to form a rectangle. This rectangle has area \(hx\). Each triangle has area \([ADF] = [BEC] = \frac{1}{2}(hx)\)

e. The sum of areas of the two triangles and the area of the parallelogram equals the total area of the rectangle.

\[
[ADF] + [BEC] + [ABCD] = [AECF]
\]

\[
\left(\frac{1}{2}\right) \cdot hx + \frac{1}{2}(hx) + [ABCD] = bh + hx
\]

\[
hx + [ABCD] = bh + hx
\]

\([ABCD] = bh\), exactly as before!

Rectangles are special parallelograms with four right angles. In this case, the height of the parallelogram is nothing more than the width of the rectangle, which produces the original formula for the area of a rectangle.
EXPLORATION 4

Using the grid below, find the area of each triangle. Calculate the area of the triangles below. How does the area of each rectangle relate to the area of the triangle inside it?
a. How did you compute the areas of each triangle?
b. What patterns did you notice? Explain.
c. Using the triangles above, make a copy of each triangle and paste it together with the original triangle. What shape do you get? Use this process to find a rule for the area of these triangles.

In the exploration above, you were able to put together two triangles of any shape to form a parallelogram. Another way of saying this is that you can decompose a parallelogram into 2 congruent triangles.
Section 11.3 Two-Dimensional Figures

So \[ ABDC = [ABC] + [BCD] = 2 \cdot [ABC] \]
\[ \frac{1}{2} [ABDC] = [ABC] \]

So, what is the area of a triangle, and is there a formula to compute this area? You have seen that taking any triangle, copying it exactly, and putting the two triangles together creates a parallelogram. Use the formula for the area of the parallelogram and take one-half of it to compute the area of the triangle. Each of the triangles will have area that equals \( \frac{1}{2} \) the area of the parallelogram. Be careful in identifying the base and the height of the triangle. The base, \( b \), must be a side of the triangle, and the height, or altitude, \( h \), must be perpendicular to the base, or an extension of the base, and be drawn from the vertex opposite the base. With those restrictions, the formula for the area \( A \) of a triangle is:

**FORMULA 11.2: AREA OF A TRIANGLE**

\[
A = \frac{1}{2} b \cdot h \quad \text{or} \quad A = \frac{1}{2} b h \quad \text{or} \quad A = \frac{bh}{2}.
\]

**EXPLORATION 5**

For each of the trapezoids below, make two copies on a grid paper. Cut them out and put them together to form a parallelogram. Use a strategy similar to that developed for triangles to compute the area of each trapezoid.
Chapter 11  Geometry

The parallel sides of the trapezoids are referred to as the bases of the trapezoid and can be labeled base 1 and base 2, with base lengths $b_1$ and $b_2$. The height, $h$, of the trapezoid is the length of a line segment between the two bases that is perpendicular to the bases. As with the parallelogram, notice that the height of the trapezoid is not usually the same length as either side of the trapezoid.

We can follow a similar strategy used to determine the area of triangles to compute the area of trapezoids. Starting with trapezoid A, label one of the horizontal, parallel sides of the trapezoid $b_1$, for base 1, and the other $b_2$, for base 2. Label the corresponding sides of the copy of trapezoid A with identical labels. Flipping the copy over, the two trapezoids can be combined to form a parallelogram with a new base of length $b_1 + b_2$.

The area for the created parallelogram from two trapezoids put together is $A = b \cdot h$, where $b = b_1 + b_2$, or $A = (b_1 + b_2) \cdot h$. Therefore, the area of the one trapezoid is half of the area of the parallelogram. We have the formula.
FORMULA 11.3: AREA OF A TRAPEZOID

The area $A$ of a trapezoid is given by

$$A = \frac{1}{2}(b_1 + b_2)h$$

where $b_1$ and $b_2$ are the length of the parallel sides, and $h$ is the height.

EXAMPLE 2

Find the area of the following polygons, whose sides are the rational numbers as indicated.

**a.**  

![Diagram of a trapezoid with sides 3 1/5 cm and 5 1/3 cm]

**b.**  

![Diagram of a trapezoid with sides 6 1/2 in and 8 1/2 in]

**c.**  

![Diagram of a rectangle with sides 7 1/2 m and 10 1/6 m]

**d.**  

![Diagram of a trapezoid with sides 6 ft and 8 1/2 ft]

**SOLUTION**

**a.**  

$$A = \frac{bh}{2} = \frac{(5 \frac{1}{3})(3 \frac{1}{5})}{2} = \frac{(16\frac{3}{5})}{2} \cdot \frac{1}{2} = \frac{(16 \frac{3}{5})(8 \frac{1}{5})}{2} = \frac{128}{15} \text{ sq. cm.}$$

**b.**  

$$A = (8 \frac{2}{3})(6 \frac{1}{2}) = \left(\frac{26}{3}\right)\left(\frac{13}{2}\right) = \frac{13}{3} \cdot \frac{13}{1} = \frac{169}{3} = 56 \frac{1}{3} \text{ sq. in.}$$

**c.**  

$$A = bh = lw = \left(7 \frac{3}{5}\right)(10 \frac{1}{2}) = \left(\frac{38}{5}\right)(\frac{21}{2}) = \left(\frac{19}{5}\right)(21) = \frac{399}{5} = 79 \frac{4}{5} \text{ sq. m.}$$

**d.**  

$$A = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}(6 + 8 \frac{5}{7}) \cdot 4 \frac{1}{2} = \frac{1}{2}(14 \frac{5}{7}) \cdot (4 \frac{1}{2}) = \frac{1}{2} \left(\frac{103}{7}\right) \left(\frac{9}{2}\right)$$
EXAMPLE 3

What is the sum of the measures of the angles inside of each of the following polygons?

a. 

b. 

c. 

PROBLEM 2

What is the sum of the measures of the angles for each of the following polygons? Write the name of each polygon. For example, a pentagon or a quadrilateral.

a. 

b. 

c. 

d. 

e. 

PROBLEM 3

<table>
<thead>
<tr>
<th>Polygon</th>
<th>Number of Sides</th>
<th>Sum of the measures of all interior angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quadrilateral</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pentagon</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hexagon</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Septagon</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Octagon</td>
<td></td>
<td></td>
</tr>
<tr>
<td>n-gon</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
EXERCISES

1. Find the area and perimeter of the following parallelograms:

   a. \[ \text{Length} = 18 \text{ in}, \text{Width} = 10 \text{ in}, \text{Height} = 8 \text{ in} \]

   b. \[ \text{Length} = 21 \text{ in}, \text{Width} = 13 \text{ in}, \text{Height} = 13 \text{ in} \]

   c. \[ \text{Diagonal} = 5 \text{ in}, \text{Height} = 4 \text{ in} \]

2. Calculate the areas of the triangles below using the area formula for a triangle:

   \[ \text{Base} = 5 \text{ in}, \text{Height} = 5 \text{ in} \]

3. A rectangular house has a porch on the rear of the house as shown. Find the area of the house and the porch.
4. Find the area of the following trapezoids in several ways.

a. Find the area by decomposing the trapezoid into triangles and rectangles, and finding the area of each.

b. Find the area by putting two trapezoids together to form a parallelogram, and then rearranging the shapes to form a rectangle.

c. Find the area by using the formula for the area of a trapezoid to check your answers.

5. Ms. Garcia wants to buy new carpet for her living room. What information about the room will she need when she talks to the salesman about the cost of the carpet?

6. A gardener wants to put a fence around his garden. The garden is twice as long as it is wide. If he uses 210 feet of fencing, what is the area of the garden?

7. Compute the sum of the measures of the angles for each of the following polygons.
8. Write equations and solve to find the area and perimeter of the following triangles.

![Triangle A](image1)

- a. 16 ft, 11.8 ft, 6.2 ft, 8.6 ft

![Triangle B](image2)

- b. 12.2 mm, 14.9 mm, 8.5 mm

![Triangle C](image3)

- c. 14.7 m, 4.8 m, 13.9 m, 14.7 m

9. Write an equation that represents the area and perimeter of the following figures. Use this equation to find the area and perimeter.

![Figure D](image4)

- 3.5 in., 5.33 in., 15.2 in., 8.5 in., 8.75 in., 4.25 in., 4.25 in.
10. For each triangle above, decide whether the triangle is a right triangle, an acute triangle, or an obtuse triangle. Explain your decision.

11. Is it possible to have a triangles with sides of lengths:
   a. 10, 11, 12
   b. 8, 10, 15
   c. 10, 20, 50
   d. 5, 20, 30
   e. 1, 2, 3
   f. 2, 4, 8

12. Draw a triangle with sides of lengths a=6, b=8, c=9. Label the angles opposite these sides as angles A, B, C. Which of these angles has the largest measure? The smallest measure?

13. Draw 3 different shaped quadrilaterals. Measure the interior angles of each, and find the sum of these angles. Make a conjecture about what the sum is of the interior angles of a quadrilateral.

14. Use a protractor and ruler to draw an isosceles triangle with two sides 4 inches long and having an area that is
   a. less than 1 square inch.
   b. more than 2 square inches.

15. A rectangular house has a porch on the rear of the house as shown. What is the area of the house and porch combined? Note, this picture is not drawn to scale.
16. A room has the following floor plan and dimensions.

a. Find the area of the room.

b. If the approximate perimeter is 43 feet and the walls in the room are 8 feet high, what is the total area of the walls?

c. It takes one gallon of paint to cover 200 square feet of wall. How many gallons will it take to paint the room?

17. Rhonda baked a cake shaped like a sailboat for her nephew’s birthday, as shown below.

a. What is the area of the top surface of the cake?

b. What is the perimeter of the top surface of the cake?
18. On a coordinate plane, plot the points \( A (0, 0), B (5, 0) \) and \( C (4, 3) \).
   a. Draw line segments connecting these points in alphabetical order.
   b. Double each coordinate of the points given in part a. Plot these points and draw line segments connecting the points as you did in part a.
   c. On a coordinate plane, plot the points \( D (2, 1), E (7, 1) \) and \( F (6, 4) \). Draw line segments connecting these points in alphabetical order.
   d. Double each coordinate of the points given in part c. Plot these points and draw line segments connecting the points as you did in part c.
   e. What do you notice about the figures in parts a and c? Parts a and b? Parts c and d? Parts b and d? Do you see a scale factor between any of these figures? Explain.

19. Explain how to find the area of a triangle, parallelogram, and trapezoid by decomposing and rearranging into simpler shapes. Use this method to check your answers on exercises 8 and 9.

20. **Ingenuity:**

   Draw five different triangles with the same area and the same base.

21. **Investigation:**

   Two parallel lines are \( DE \) and \( AB \) cut by two transversals. Discuss everything you notice about the triangles \( CDE \) and \( CBA \).
The previous section involved the area and perimeter of rectangles, parallelograms and triangles. To find the perimeter, find the lengths of each of the sides and add them together. This is certainly possible if you know the lengths of all three sides, but what if you only know the lengths of two of the sides in the triangle? Is it possible to find the length of the third side?

Start exploring this question with right triangles. Look at each of the right triangles below and measure the lengths of the sides.

You already know that the hypotenuse is the longest side and that the sum of the lengths of the legs is more than the length of the hypotenuse. Is there another way to find the lengths of the sides of a triangle without using a ruler?

Sometimes it is difficult to see a pattern with only three examples. In the following diagram the squares attached to the triangle have been drawn. Draw a square off of each side of the right triangle as shown below so that the base of the right triangle is one side of the square. Remember to look for a pattern involving the lengths of the sides of right triangles.
EXPLORATION 1

Your teacher will give you copies of the following right triangles to measure.

Copy and fill out the table below to record the lengths of the sides and the areas of the attached squares.

<table>
<thead>
<tr>
<th>Length of Vertical Leg</th>
<th>Length of Horizontal Leg</th>
<th>Length of Hypotenuse</th>
<th>Area of Square 1</th>
<th>Area of Square 2</th>
<th>Area of Square 3</th>
</tr>
</thead>
<tbody>
<tr>
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</tbody>
</table>
Looking at the triangle below, what is the area of each square?

The area of the square with side length $a$ is $a \cdot a = a^2$. The areas of the other two squares are $b^2$ and $c^2$.

You might have noticed that the area of the square attached to the hypotenuse is equal to sum of the areas of the other two squares. This is usually written $a^2 + b^2 = c^2$, where $c$ is the length of the hypotenuse, and $a$ and $b$ are the lengths of the legs. Check that this formula works for each of the right triangles in the exploration. The formula is called the **Pythagorean formula** or the **Pythagorean Theorem**.

<table>
<thead>
<tr>
<th>THEOREM 11.3: PYTHAGOREAN THEOREM</th>
</tr>
</thead>
<tbody>
<tr>
<td>If $a$ and $b$ are the lengths of the legs of a right triangle and $c$ is the length of the hypotenuse, then</td>
</tr>
<tr>
<td>$a^2 + b^2 = c^2$</td>
</tr>
</tbody>
</table>

You discovered the pattern by looking at examples and computing areas. But how do you know that it is always true for any right triangle with legs of length $a$ and $b$ and hypotenuse of length $c$? There are many different proofs of this ancient
One of the most beautiful proofs can be seen from the following two pictures. In each picture there are four copies of the original right triangle, with side lengths $a$, $b$, $c$. Each of the pictures represents a square with side length $a + b$. On the left, there are four triangles and one square with side length $c$. On the right, there are the same four triangles and two squares, with side lengths $a$ and $b$. Copy and label all the sides in the two pictures to check the statements. Why does the picture prove that the area of the large square is the sum of the areas of the two smaller squares? The result is the Pythagorean Theorem: $a^2 + b^2 = c^2$.

**EXPLORATION 2**

On the floor, table or board, draw a square with sides 1 meter long. Next, draw a diagonal line from one corner to the opposite corner. Your challenge is to find the length of the diagonal.

First, estimate the length of the diagonal.

Next, measure the length of the diagonal. How precise an estimate is possible using a meter or yardstick?

Now, use the Pythagorean formula to find the length, $c$, of the diagonal. What is the value of $c$?

Use the Pythagorean formula, $1^2 + 1^2 = 2 = c^2$. This means that $c^2$ or “$c$ squared” is 2. So $c$ is a number that when multiplied by itself equals 2. The square root notation $\sqrt{n}$ denotes the positive number whose square is $n$. The symbol $\sqrt{ }$ is called the **square root symbol**. So $\sqrt{25} = 5$ because 5 is a number whose
Section 11.4 The Pythagorean Theorem

The square is 25. The square is a function of \( x \): \( f(x) = x^2 \). The square root is called the inverse function of the square because taking the square root of the square of a number gives the number. For instance, \( 4^2 = 16 \) and \( \sqrt{16} = 4 \), so \( \sqrt{4^2} = 4 \).

EXERCISES

1. Using a calculator and the same technique as in Exploration 2, estimate the exact values for the following square roots to four decimal places. Then use the “square root” button on the calculator to check your answer.
   a. \( \sqrt{3} \)  b. \( \sqrt{4} \)  c. \( \sqrt{10} \)  d. \( \sqrt{42} \)
   e. \( \sqrt{9} \)  f. \( \sqrt{169} \)  g. \( \sqrt{49} \)  h. \( \sqrt{10000} \)

2. As you know, \( \sqrt{25} = 5 \). Are there other numbers whose square is 25? What are all the numbers whose square is 16?

3. A right triangle has legs 3 inches and 4 inches long. Find the length of the hypotenuse and the area of the triangle. Confirm your answer using a lined paper and a ruler.

4. A right triangle has hypotenuse 10 inches long, and one leg 6 inches long. Find the length of the other leg. What are the area and perimeter of the triangle?

5. Call the lengths of the sides of a right triangle \( a \) and \( b \), and the length of the hypotenuse \( c \). By the Pythagorean Theorem, \( a^2 + b^2 = c^2 \). Find \( c \) using square root notation.

6. Draw a right triangle with legs of length 5 cm and 12 cm. Measure the length of the hypotenuse, then calculate the exact length of the hypotenuse.

7. Draw a right triangle with legs of length 7 cm and 9 cm. Measure the hypotenuse to estimate its length. Then use the Pythagorean Theorem to find the exact length of the hypotenuse, as well as a better decimal approximation.

8. The grade of a road is the change in elevation times 100 divided by the horizontal distance traveled. Sam traveled 1.25 miles along a local road and the change in elevation was 395 feet. What is the grade of the road?
9. Mike and Larry are carrying a square 8 ft by 8 ft piece of glass through a 7 ft by 4 ft doorway. Will the glass fit through the doorway? Explain.

10. In rectangle $ABCD$, the diagonal is $\sqrt{320}$ cm and the length $AB$ is twice as long as the width $BC$. Find the dimensions of the rectangle.

11. A 10-foot ladder is leaning against a wall. The bottom of the ladder is six feet from the wall. Where will the top of the ladder touch the wall?

11. **Ingenuity:**

Suppose $\triangle ABC$ is a right triangle $m(\angle ACB) = 90^\circ$ and with base $\overline{AB}$ as pictured. Draw altitude $\overline{CP}$. Draw a line through $C$ that is parallel to the base $\overline{AB}$. Find the area of $\triangle ABC$.

```
   C
  /|
 / |
A---B
```

a. Show that $\angle CAB$ has the same measure as $\angle BCP$.
b. Show that $\triangle ABC$ has the same angles as $\triangle PBC$.
c. Show that $\triangle ABC$ has the same angles as $\triangle APC$. 

SECTION 11.5 CIRCLES

Everyone has seen circles of various sizes, but what is the definition of a circle? How do you draw a circle? Try to describe a circle to someone without using the word "circle."

EXPLORATION 1

How do you draw a circle? Once you have drawn a circle, write directions someone could use to draw a circle. Then state your definition for a circle.

In general, one way to draw a circle is by marking a point $P$, called the center of the circle. Then take a length of string, $r$ units long, place one end of the string at point $P$ and attach a pencil to the other end. Stretch the string to its full length and draw the circle with the pencil. Each point on the circle is $r$ units from $P$. The circle is made up of all points that are distance $r$ from $P$. The fixed distance $r$ from the center $P$ to the edge of the circle is called the radius of the circle. A straight line connecting two points on a circle is called a chord. A straight line connecting two points on the circle and passing through the center $P$ is called a diameter. The length of the diameter is equal to the length of 2 radii. Radii is the plural of radius. The distance around the circle is called the circumference and is like the perimeter of a polygon.
EXPLORATION 2

Use a variety of different size circles. Using a piece of string, carefully measure the radius and circumference. Place the circle on grid paper and estimate the area, then complete the table below:

<table>
<thead>
<tr>
<th>Circle</th>
<th>Radius</th>
<th>Diameter</th>
<th>Circumference</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Looking at the table, what patterns do you notice?

Do you notice a relationship between the radius and the diameter? Using the variable $d$ to denote the length of the diameter, express the diameter in terms of the radius $r$. 
What is the relationship between the circumference of a circle and its diameter? Compute the ratio of the circumference to its diameter for the five circles. What do you notice about the ratios? The ratio you computed approximates the exact ratio of the circle’s circumference to its diameter, the number \( \pi \), written as the Greek letter \( \pi \). This ratio is the same regardless of the size of the circle.

Look at a circle with diameter 1 unit. Remember, a unit can be any length you choose. Take out a string or tape ruler and measure the circumference. The circumference has a length of \( \pi \) units. The number \( \pi \) is approximately equal to the fraction \( \frac{22}{7} \) or the decimal 3.14. These two approximations are not exactly equal to \( \pi \). However, the two approximate values are very close to the actual value of \( \pi \), which begins 3.1415926. . . .

**Definition 11.11: Pi**

The ratio of the circumference to the diameter of any circle, represented either by the symbol \( \pi \), or the approximation \( \frac{22}{7} \) or 3.1415926.

What happens to the circumference when the radius doubles? What pattern do you notice when you measure the scaled circumference and compare it to the original circumference? Just as with a square, when scaled by a factor of 2, the perimeter, or circumference \( C \), doubles. The ratio \( \frac{C}{d} \) of the circumference to the diameter remains the same, \( \pi \).

In summary, call \( C \) the circumference of a circle, \( d \) the diameter, and \( r \) the radius. Then,

\[
\begin{align*}
d &= 2 \cdot r \\
C &= \pi \cdot d \\
&= 2 \pi r
\end{align*}
\]
EXPLORATION 3

What is the area $A$ of a circle whose radius is 1? Draw a circle with radius 1 and circumference $2\pi$ and cut it in half. Then cut each half into many small pie slices:

Take the slices from one half of the circle and lay the points of the slices along a line:

Do the same with the bottom half of the circle, filling in the spaces:

The shape looks a little like a rectangle. The more slices, the closer the shape is to a rectangle. If this cutting process continued infinitely, the area of the circle with radius 1 would approximate the area of the rectangle with length $\pi$ and width 1. The area $A = \pi \cdot 1 = \pi$ square units.
What happens to the area of the circle when its radius is a number $r$? One way to visualize this is to create slices in the circle with radius $r$, like the process with radius 1.

Cut the circle into two equal semicircles as you did in the unit circle and fit one semicircle into the other semicircle.

What is the length of this rectangular shape? What is its width? What is the area of the rectangle?

In this rectangle the length is half the circumference $2\pi r$, or $\pi r$, and the width is $r$. The area of the rectangle is length times width or $\pi r \cdot r$ or $\pi r^2$. Any area is measured in square units. So if $r$ is measured in inches, $r \cdot r$, or $r^2$, is measured in square inches. To summarize:

**FORMULA 11.4: AREA OF A CIRCLE**

The area of a circle with radius $r$ is $A = \pi r^2$ square units.
EXAMPLE 1

A circle has radius 4 inches.

a. Find the exact circumference of the circle.
b. Approximate the circumference to the nearest tenth of an inch.
c. Find the exact area of the circle.
d. Approximate the area to the nearest hundredth of an inch.

SOLUTION

Apply the above formulas.

a. The exact circumference needs to be written as $8\pi$ inches because $\pi$ is a non repeating decimal number.
b. 25.1 inches
c. $16\pi$ square inches because $\pi$ is a non repeating decimal.
d. 50.24 square inches

The circumference is $C = 2\pi \cdot 4$ inches = $8\pi$ inches.
The area is $A = \pi \cdot (4$ inches$)^2 = 16\pi$ square inches = $16\pi$ in$^2$.

EXAMPLE 2

A circle has the circumference of 37.68 ft. What is the circle’s diameter? Find the radius and its area.

In mathematics, a product that includes a constant, or number, times a variable is written with the constant first, like $2x$. In the product of a constant and a variable, the constant 2 in this example is called the coefficient of the product $2x$. However, even though $\pi$ is a constant, not a variable, the product of $\pi$ and a constant like 16 is usually written $16\pi$. 

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EXERCISES

1. Find a circle large enough to measure easily in your home or school. Use a measuring tape or string to mark the length of the circumference of each circular object. Use the tape or string to measure the diameter of the circle. Calculate the approximate ratio of circumference to diameter? Explain why this result makes sense.

2. Find the area and circumference of the circles below. Use \( \pi, 3.14, \) or \( \frac{22}{7} \).

   a. \( d=4 \text{ mm} \)
   b. \( r=7 \text{ yds} \)

   c. \( d=15 \text{ miles} \)
   d. \( r=9 \text{ units} \)

3. Use the EXPLORATION 3 in this section to explain to someone at home or someone in class how to discover the formula for the area of a circle. In particular, explain how the number \( \pi \) is used from the beginning of the activity.

4. The diameter of a circle is 10 centimeters. What is the circumference of the circle? What is the area of the circle?

5. A circle has a radius of 20 miles. Find its circumference. Find its area.

6. The circumference of a circle is 100 inches. To the nearest tenth of an inch, what is the circle’s diameter? Its radius? Use \( \pi = 3.14 \).

7. The radius of a circle is 10 inches. What is the circle’s exact diameter? Its exact area? Its approximate area, to the nearest inch?
8. A track is shaped like a rectangle with semi-circles at each end. The track is 100 yards long, and 50 yards wide. What is its perimeter to the nearest yard? Its area to the nearest square yard?

9. A circle has area 20 square units. Each dimension is scaled by a factor of 3 to make a new circle. What is the area of the new circle?

10. A circle with radius $r$ lies inside a square with each side $2r$ long.

11. A circle with radius $r$ is contained in a larger circle with radius $2r$ touching at the bottom.

What is the area outside the smaller circle and inside the larger circle?
12. Angelique wants to water her front lawn with sprinklers that cover a circular area. Would a giant sprinkler cover more area or would 4 smaller sprinklers cover more area? Explain.

13. In the figure below, five circles are nested inside a larger circle. Consider the region in the upper right of the large circle between the top circle, the circle on the right, the center circle, and the circumference of the large circle. Is the area of this region greater than, less than, or equal to the area of one of the smaller circles? First make a guess, and then use geometry to check your answer.

14. Barry is making a stained glass window for a client. The window has an arch on the top, like the picture below. How many square feet of glass does he need to make the window? Use $\pi$ as 3.14, and round your answer to the nearest tenth.
15. **Ingenuity:**
A *sector* of a circle is the part of the interior of the circle between two radii, like a slice of pie. A circle has radius 4 inches, and two radii make a sector with a 60° angle. Find the exact area of the sector these radii enclose.

![Sector of a circle](image)

16. **Investigation:**
   a. Draw a circle with radius $r$ and center $O$. Pick two points $A$ and $B$ on the circle that do not lie on a common diameter.
   b. Draw the diameter that goes through point $A$. It intersects the circle at point $A$ and another point, $A'$. The diameter that goes through point $B$ likewise intersects the circle at another point $B'$. In what way are triangles $ABO$ and $A'B'O$ related? Why?
SECTION 11.6 THREE-DIMENSIONAL SHAPES

In the previous sections, you studied shapes in two dimensions: triangles, squares, rectangles, parallelograms, trapezoids, and circles. In this section, you will learn about three-dimensional shapes. Some of these shapes appear as familiar objects like beach balls, blocks, paper towel rolls or cardboard boxes. In this section, you will learn some mathematical terminology and ways to measure volume.

A basic kind of three-dimensional figure is called a polyhedron. This word comes from the Greek words *poly*, meaning "many," and *hedra* meaning "faces." So a polyhedron is a three-dimensional figure with many faces. Each face of a polyhedron is a polygon. The vertices of the polygons are the vertices of the polyhedron. The edges are the borders of the faces that are also the line segments that join the vertices.

A box shape is an example of the most common type of polyhedron called a prism. In a prism, two of the faces, called bases, are parallel and congruent. Prisms are named by their bases. In the case of a box, the polyhedron is a rectangular prism, because the bases are rectangles. The lateral surfaces are the faces of a geometric figure, excluding the bases. In a prism, the lateral surfaces are always rectangles.
PROBLEM 1

How many faces does a rectangular prism have? How many are bases? How many are lateral surfaces? How many vertices? How many edges? How does this change for a triangular prism?

A cube is a regular rectangular prism. A cube is regular because each of its faces has equal sides and angles. All the cube’s faces are squares.

EXAMPLE 1

Identify the 5 prisms below. Determine the number of vertices, faces, and edges for each.

### SOLUTION

<table>
<thead>
<tr>
<th>Prism</th>
<th>Name</th>
<th>Vertices</th>
<th>Faces</th>
<th>Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Triangular</td>
<td>6</td>
<td>5</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>Prism</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Rectangular</td>
<td>8</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>Prism</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A **pyramid** is a 3-dimensional figure with one polygonal base that connects to a point called the **apex**. The lateral surfaces formed when the base is connected to the apex are all triangles.

The pyramids are named for their bases.

Let's consider another three-dimensional figure called the **cylinder**. Like a prism, it has two congruent and parallel bases. It cannot be classified as a prism because the bases are not polygons but are circles.

A **circular cone** is a 3-dimensional figure with one circular base that connects to a point called the apex.

The simplest three-dimensional shape to measure is a **cube**, two parallel congruent square bases connected by four perpendicular congruent squares.
A cube one unit long, one unit wide and one unit high has a volume of one cubic unit. Recall that the area of a two-dimensional figure is measured by the number of unit squares needed to cover it. The volume of a three-dimensional shape is measured by the number of unit cubes needed to fill it. For example, if each side of a cube is 1 foot long, the volume of the cube is 1 cubic foot, written 1 cu. ft. or 1 ft³.

**EXPLORATION 1**

How many inch cubes (also called cubic inches) are there in a cube that is 2 inches long on each side? How many cubic inches are there in a cube that is 3 inches long on each side?

**Materials:** You will need approximately 30 cubes of the same size for this activity.

In groups, make a cubic box that is two units long on each side. Fill this box using one unit cubes. How many one unit cubes does it take to fill the box that is 2 units long on each side? A box that is 3 units long on each side?

Are you surprised that the two unit cube has volume 8 cubic units, while the three unit cube has volume 27 cubic units? In general, if a cube has side length \( s \) units, what is the volume? You should get the formula:

<table>
<thead>
<tr>
<th>FORMULA 11.5: VOLUME OF A CUBE</th>
</tr>
</thead>
<tbody>
<tr>
<td>The volume of a cube with each side of length ( s ) units is ( s^3 ) cubic units. ( V = s^3 ) or ( V = B \cdot h )</td>
</tr>
<tr>
<td>where ( B ) is the area of the base of a 3-dimensional figure.</td>
</tr>
</tbody>
</table>
EXPLORATION 2

How many cubic inches are there in one cubic foot?

In order to think about this problem, let’s begin by reviewing how to change units in computing areas. A square that is one foot long on each side has an area of one square foot. Thinking in terms of smaller units, each side of the square foot is 12 inches long. Using this ratio of feet to inches,

\[ 1 \text{ square foot} = (1 \text{ foot}) (1 \text{ foot}) = (12 \text{ inches}) (12 \text{ inches}) = 144 \text{ square inches}. \]

Using the same pattern,

\[ 1 \text{ cubic foot} = (1 \text{ foot}) (1 \text{ foot}) (1 \text{ foot}) = (12 \text{ inches}) (12 \text{ inches}) (12 \text{ inches}) = 1728 \text{ cubic inches}. \]

Another way to compute the volume of this cube is to use the formula you just learned. Since one foot = 12 inches, each side of the cube is 12 inches long. So the volume of your cube is \(12^3 = (12)(12)(12) = 1728 \text{ cubic inches}.\)

In three-dimensions, conversions to smaller units make volumes seem much larger even though the shape and size have not changed at all!

EXAMPLE 2

Draw a rectangular prism with edges that are 2, 3 and 4 units long.

a. Find its volume.

b. Now scale the prism using a scale factor of 2, then a scale factor of 3. What are the new dimensions with each scale factor? What is the new volume in each case?

SOLUTION

The first step is to draw the two base rectangles. For example, make the bases \(2 \times 3\) rectangles. Place these rectangles 4 units apart to make the height of the box.
a. How many unit cubes does it take to fill the box? Using the two-dimensional pattern of cutting a $2 \times 3$ rectangle into 6 unit squares, cut the box into $6 \cdot 4 = 24$ unit cubes. Notice that each layer has the same number of cubes: There are 6 cubes in the first layer, 6 cubes in the second layer, 6 in the third layer and 6 in the fourth layer.

b. When you scale with a scale factor of 2, the new edges will be 4, 6, and 8 units. The volume will then be:

$$4 \cdot 6 \cdot 8 = (2 \cdot 3) \cdot (2 \cdot 3) \cdot (2 \cdot 4) = 8 \cdot 24 = 192 \text{ cubic units}$$

When you scale with a scale factor of 3, the new edges will be 6, 9, and 12 units. The volume will then be

$$6 \cdot 9 \cdot 12 = (3 \cdot 3)(3 \cdot 3)(3 \cdot 4) = 27 \cdot 24 = 648 \text{ cubic units}$$

Do you see a relation between the scale factor and the new volume?

In each case, to find the volume multiply the area of the base by the height to get the volume.

In general the volume of a prism is equal to the area of the base times the height. This formula is often written as $V = B \cdot h = Bh$. The variable $B$ is the area of the base. This general formula is true for any prism, regardless of the shape of the base, whether it is a rectangle, a triangle, a hexagon or any polygon.

### FORMULA 11.6: VOLUME OF A PRISM

The volume of a prism is the area of the base of the three-dimensional figure times the height of the prism.

$$V = B \cdot h$$

where $B$ is the area of the base of the 3-dimensional figure.
EXAMPLE 3

Using the triangular prism below, which has a height of 7 units, a triangle base of 4 units and a height of 5 units, determine the volume of the prism.

![Triangular Prism Diagram]

SOLUTION

Begin with the area of the base, since the base is a triangle. Use the formula

\[ A = \frac{1}{2} \cdot b \cdot h \]

where \( h \) is 5, the height of the triangular base.

Area of the Triangle:

\[ A = \frac{1}{2} \cdot (4)(5) = 10 \text{ units}^2 \]

After you find the area of the triangle, which in the volume formula is \( B \), you need to “stack” the area to determine the volume of the prism. Do this by multiplying the area of the triangle by the height of the prism.

So, \( V = B \cdot h \)

\[ V = 10 \text{ units}^2 \cdot (7 \text{ units}) = 70 \text{ units}^3 \]

PROBLEM 2

Write equations to represent the volume, and use these equations to calculate the volume of the following prisms:

a. 

![Triangular Prism Diagram]

b. 

![Cube Diagram]
c.  
\[ \text{Volume of the prism} = \text{base area} \times \text{height} \]
\[ = 2 \times 4 \times 8 = 64 \text{ cubic inches} \]

d.  
\[ \text{Volume of the prism} = \text{base area} \times \text{height} \]
\[ = 1.5 \times 6 \times 7 = 63 \text{ cubic feet} \]

e.  
\[ \text{Volume of the prism} = \text{base area} \times \text{height} \]
\[ = 2.2 \times 7.4 \times 1.6 \approx 35.9 \text{ cubic inches} \]

Remember, the volume in cubic inches of three-dimensional shapes is the number of one-inch cubes it takes to fill the shape exactly. Because some shapes cannot be easily filled with one-inch cubes, the volume might be a fraction or a decimal part of a cube. As in the case of prisms, you can examine volumes and arrive at formulas that will make the computation much easier than counting blocks every time.

**EXPLORATION 3**

Find a hollow cylinder, like a paper towel roll or an empty can. Measure the volume inside the cylinder.

If possible, fill the inside space with something such as a non-drying clay. Remove the play dough and form a rectangular prism. Measure its dimensions and determine its volume. Measure and record the dimensions of the cylinder using a ruler. Use the radius of the circular base and the height of the cylinder to compute the volume of the cylinder. Use several measurements to get an average. Review the definition and approximate value of \(\pi\).

Compare the two computed volumes with the play dough or with the ruler to the cylinder’s volume.

Adapting the prism volume formula, the volume of a cylinder is the area of the base times the height. Because the base is a circle, its area is \(\pi r^2\). Therefore, the formula for the volume of a cylinder is \(V = \pi r^2 h\). 

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EXAMPLE 4

Determine the volume of the cylinder below.

\[ \text{SOLUTION} \]

In order to calculate the volume of a cylinder, we can use the formula \( V = B \cdot h \) (area of the base of the figure times the height of the figure). Since the base is a circle, the formula to determine the area is \( \pi r^2 \).

\[ A = \pi r^2 = 3.14 \cdot 7 \cdot 7 \]
\[ A = 153.86 \text{ square inches} \]

The next step is to multiply the area of the base times the height of the cylinder.

\[ V = B \cdot h \]
\[ V = 153.86 \cdot 10 \]
\[ V = 1538.6 \text{ square inches} \]

Imagine a pyramid and a prism like the ones below with congruent bases and the same height.
How do you suppose the volume of the pyramid is related to the volume of the prism? Do you think it would be \( \frac{1}{2} \) as much? \( \frac{1}{4} \) as much? If we were to have two paper models like the ones in the picture above, we could calculate the relationship using rice or sand to fill the prism and see how many times the amount of material would fill the pyramid. In fact, the volume of a prism is equal to 3 times the volume of a pyramid with congruent bases and the same heights. Alternately, the volume of the pyramid is \( \frac{1}{3} \) the volume of the prism.

The formula for the volume of a pyramid is often written as \( V = \frac{1}{3}Bh \), where \( B \) is the area of the base, and \( h \) is the height of the pyramid. As with prisms, this formula is true for all pyramids, regardless of the shape of their base, whether it is a rectangle, a triangle or any polygon.

**FORMULA 11.7: VOLUME OF A PYRAMID**

The volume of a pyramid is equal to \( \frac{1}{3} \) the area of the base, \( B \), times the height of the pyramid.

\[
V = \frac{1}{3}Bh
\]

**PROBLEM 3**

Calculate the volume of a triangular pyramid, in which the base is identical to the triangular base in Example 3, with a height of 7 units.

**EXERCISES**

1. What is the volume of a rectangular prism with edges 2 ft, 3 ft and 5 ft?

2. Sketch a rectangular prism with dimensions 2 by 6 by 8 in three different ways by switching the dimensions for each sketch.
   
   a. Using each prism, inscribe a rectangular pyramid inside the prism with the same base and height as the original prism. Compare the shapes of the three pyramids. Can you see that each pyramid has a different shape?
   
   b. Compute the volume for each of these inscribed pyramids.

3. What is the volume of a cube with side lengths of 4 inches? Now scale using a scale factor 5.
   
   a. What is the new volume?
b. How does the volume of the original prism change if you scale using a scale factor of 10?
c. Do you notice a pattern between the scale factor and the volume?

4. Make a scaled sketch of a cylinder whose base has radius 2 inches and whose height is 5 inches, and label the dimensions. What is the volume of the cylinder?

5. A cylinder has bases with radius 4 inches and height 10 inches.

a. What is its volume?
b. What happens to the volume if both the radius and height are doubled?
c. What happens to the volume if only the radius is doubled?
d. What happens to the volume if only the height is doubled?

6. Draw a triangular prism whose base is a right triangle with sides 6 cm, 8 cm and 10 cm and whose height is 12 cm. Find the volume of the triangular prism.

7. What is the volume of a cylinder that has base radius 0.6 in and height 6 in?

8. The height of a triangular prism is 20 cm. The triangular base has height 12 cm, and the length of the base of the triangle is 5 cm. What is the volume of the prism?

9. A rectangular prism has volume 234 in$^3$ and edges with lengths 4 in, 5 in and $x$ in. What is the value of $x$?

10. A solid two-inch cube weighs 10 pounds and is worth $144. How much is a solid three-inch cube made of the same material worth?

11. A cylindrical tank 12 feet in diameter and 15 feet high fills with water at the rate of 1.5 cubic feet per second. How long does it take to fill the tank?

12. Calculate the volumes of a pyramid and a prism, in which the base is 3"x 4" rectangle, and the height is 6".
13. The volume of a rectangular pyramid that is 8 inches high is 96 cubic inches. A rectangular prism has the same base and height as the rectangular pyramid. What are some possible dimensions for the base of both the rectangular prism and pyramid?

14. The base of a triangular prism is a right triangle with dimensions of 6, 8, and 10 units. The height of the prism is 5 units.
   a. What is the volume of the prism?
   b. Sketch three possible triangular pyramids that have the same base triangle and the same height as the original triangular prism.
   c. Calculate the volume of these three pyramids. What do you notice about your answers?

15. Ingenuity:

   A rectangular prism has a square base. Then one pair of opposite sides of the base of the prism are each increased by two inches. The other pair of opposite sides of the base are decreased by two inches.

   a. What is the change in volume if the height of the prism is 10 inches and the original base had side length 4 inches?
   b. What is the change in volume if the height of the prism is 7 inches and the original base had side length 5 inches?
   c. What is the change in volume if the height of the prism is \( h \) inches and the original base had side length \( s \) inches?

16. Investigation:

   a. What is the radius of a cylinder that has height 1 in and volume \( \pi \) in\(^3\)?
   b. What is the radius of a cylinder that has height \( \pi \) in and volume 1 in\(^3\)?
   c. What is the radius of a cylinder that has height 1 in and volume 1 in\(^3\)?
SECTION 11.7 SURFACE AREA AND NETS

In the previous section, you explored finding the volume of three-dimensional shapes. Finding volume involves measuring the space inside shapes. You can measure the volume of a cereal box. But there are other interesting aspects of the cereal box to explore. For example, how much cardboard is needed to make a cereal box, ignoring the flaps? To find out, flatten the box and measure the area of the cardboard.

The surface area of a three-dimensional figure is the area needed to form its exterior. One way to see surface area of a three-dimensional figure is to cut the figure along its edges and “flatten” its exterior on a surface to make a two-dimensional shape. The flattened exteriors are called nets. The shapes that make up the nets are often rectangles, triangles and circles. The net of a cereal box might look like this:

Multiple nets can be drawn for the same geometric solid depending on which edges are cut. The figures below are two additional nets of the same prism that is deconstructed above.

EXAMPLE 1

Find the surface area of the net of a rectangular $2 \times 3 \times 4$ prism.

\[ h = 2 \]
\[ w = 3 \]
\[ l = 4 \]

SOLUTION

To find the surface area, construct a net from your figure and place it on a flat surface. What does it look like? Compare various nets formed from the same prism. For one possible net, the top and bottom, or the bases, each have area $3 \times 4$. Two of the opposite sides have area $2 \times 3$, and the other two sides have area $2 \times 4$. One way to cut this apart is shown below:

The total surface area is $6 + 6 + 12 + 12 + 8 + 8 = 52$ square units. Can you identify from where each term in this sum came? Remember, a net or surface area is measured in square units, like all two-dimensional areas.
Is there a more efficient way to find the surface area of a rectangular prism? The prism below has dimensions \( l \times w \times h \), where \( l \) = length, \( w \) = width, and \( h \) = height of the rectangular prism.

What are the areas of the two congruent bases and opposite sides? Each base has area \( B = l \times w = l \cdot w \). Two of the parallel sides have equal area \( h \times l \). The other two parallel sides have equal area \( h \times w \). Notice that all four sides have equal height \( h \). The sum of the areas of each of the four sides, also called the **Lateral Surface Area**, is

\[
(l \cdot h) + (w \cdot h) + (l \cdot h) + (w \cdot h) = h(l + w + l + w) = (2l + 2w)h.
\]

The common factor \( h \) is the height of the prism, and the sum \( (2l + 2w) \) is the perimeter, \( P \), of the base rectangle. The sum of the areas of each surface, which includes all lateral surfaces and the bases, is known as the **Total Surface Area**.

From the figure below, you can see that \( S = 2lh + 2wh + 2lw \), where \( S \) is the total surface area of the rectangular prism.

Since \( S = 2lh + 2wh + 2lw = (2l + 2w)h + 2lw \), this observation leads to a formula for the total surface area of a rectangular prism, \( S = 2B + Ph \), where \( S \) is the total surface area of the rectangular prism, \( B \) is the area of the base, \( P \) is the perimeter of the base rectangle and \( h \) is the height of the prism.
FORMULA 11.7: SURFACE AREA OF A RECTANGULAR PRISM

The surface area of a rectangular prism, $S$, is given by the formula

$$S = 2B + Ph$$

where $B$ is the area of the base, $P$ is the perimeter of the base rectangle, and $h$ is the height of the prism.

PROBLEM 1

Use a similar process to find the surface area of a cube. Sketch a net to explain how you found the formula.

FORMULA 11.8: SURFACE AREA OF A CUBE

The surface area, $S$, of a cube is given by the formula

$$S = 6 \cdot s^2 = 6B$$

where $s$ is the length of a side, and $B$ is the area of the base.

PROBLEM 2

a. There are many ways to draw a net for a cube. Draw as many nets as you can that form a cube. What would be an example of one that would not work? Explain why.

b. Draw a net for a triangular prism with side lengths of 3, 5, and 6 and height 10. Compute the lateral surface area of this prism.
EXPLORATION 2

Find the surface area of the cylinder by examining the net for the cylinder.

The net for a cylinder has a middle section that is a rectangle representing the lateral surface. This rectangle has width \( h \) equal to the cylinder’s height, and length \( l \) equal to the circle’s circumference: \( l = 2\pi r \). The area of the lateral surface of a cylinder is the product \( 2\pi rh \). The two bases have a combined area of \( 2\pi r^2 \).

**FORMULA 11.9: SURFACE AREA OF A CYLINDER**

The total surface area, \( S \), of a cylinder is the sum of the areas of the bases and the lateral surface,

\[
S = 2\pi r^2 + 2\pi rh,
\]

where \( r \) is the radius of the base and \( h \) is the height of the cylinder.

PROBLEM 3

Draw a net for a cylinder with a radius of 4 cm and a height of 6 cm. Using \( \pi = 3.14 \), calculate the surface area of the cylinder.
EXPLORATION 3

Below are two nets, a net of a cube and a net of a square pyramid. Using the formula for the total surface area of a prism, which is $S = 2B + Ph$, can you create a formula to calculate the surface area of a pyramid?

a. What is different about the two nets?

b. What is similar about the two nets?

c. What should be done to the net of the cube to make it resemble the net of a pyramid?

d. What should be done to the formula for the surface area of a cube for it to resemble the changes in question c?
In a pyramid the surface area is equal to the area of the base plus the areas of each triangular lateral side. It is important when determining the areas for the triangular sides to distinguish between the height of the pyramid $h$, and the height of the individual sides, also known as the slant height $l$.

In the picture below, the height of the pyramid is 15 cm, the slant height is equal to 17 cm, and the dimensions of the base are 16 cm x 16 cm.

The area of the base is $B = 16 \text{ cm} \times 16 \text{ cm} = 256 \text{ cm}^2$. The surface area of each lateral side $S = \frac{1}{2} 16 \text{ cm} \times 17 \text{ cm} = 136 \text{ cm}^2$. The total surface area is equal to the sum of the area of the base and each lateral side, $S = 256 \text{ cm}^2 + 4 \times (136 \text{ cm}^2) = 800 \text{ cm}^2$. This can be written as $S = B + \frac{1}{2} Ph$, since the combined surface area of all the lateral sides is equal to $\frac{1}{2}$ the perimeter times the slant height.

**FORMULA 11.10: SURFACE AREA OF A PYRAMID**

The total surface area, $S$, of a square pyramid is the sum of the area of the base and the area of the lateral surfaces,

$$S = B + \frac{1}{2} Ph,$$

where $B = x^2$ is the area of the $x$-by-$x$ square base, $P = 4x$ is the perimeter of the base and $h$ is the slant height of each of the triangles that form the sides of the pyramid. Note: the slant height of each side is not the length of the edge. It is the perpendicular distance from the apex of the pyramid to the base of each side.
PROBLEM 4
The picture below is the net of a rectangular pyramid.

![Net of a Rectangular Pyramid]

a. Calculate the lateral area using the net that is labeled below.
b. Calculate the volume of this pyramid.
c. Write a general formula for the lateral area of a rectangular pyramid

PROBLEM 5
Johnny has a toy in the shape of a tetrahedron, which is a triangular pyramid of which each side is an equilateral triangle with sides of length 10cm. He measures the height of each triangular side (they are all congruent) to be approximately 8.7 centimeters.

a. Sketch a net for this triangular pyramid.
b. Compute the approximate lateral surface area of this pyramid.
c. Compute the approximate total surface area of this pyramid.

PROBLEM 6
Now scale the figure using a scale factor of 2. What is the new surface area?

EXPLORATION 4: DIFFERING VIEWS
Work with a partner. Each of you should take 12 unifix cubes, unit base-ten blocks or some kind of cubes and construct a three-dimensional shape. Each face must fully touch another face or touch nothing. Declare one side the front. Now change places with your partner and draw on grid paper the front, left, right and top view of your partner’s three-dimensional shape. After each of you has finished drawing your two-dimensional views of your partner’s shape, discuss your results with your partner.
PROBLEM 7

Consider the following two-dimensional views of a three-dimensional solid. Create the three-dimensional figure that corresponds to the three views. Is there only one such figure? Could there be more?

Top View:  
Front View:  
Side View:

PROBLEM 8

Consider the irregularly shaped object: A rectangle 10 feet by 15 feet with a right triangle with its hypotenuse on one side, and a semicircle on the opposite side. The height of the object is 10 feet.

a. What is the area of the base of the object?

b. What is the lateral area?

c. What is the volume of the object?

EXERCISES

1. Sketch the following nets:
   a. triangular prism
   b. triangular pyramid
   c. Explain the differences in the nets from a and b.

2. Name the three-dimensional figure represented by the following net.
3. Use the following three-dimensional solid to answer a – c.

![Diagram of a solid]

a. Draw the top view of this solid.
b. Draw the front view of this solid.
c. Draw the side view of this solid.

4. Draw a scaled net of a rectangular prism with edges of lengths 4 meters, 5 meters, and 7 meters. What is the surface area of the prism?

5. Draw a net of the rectangular prism below. What is the prism’s surface area? Shade the lateral area of the net, excluding the bases. What is the lateral area?

![Diagram of a rectangular prism]

6. Draw a net for the triangular prism and compute the lateral area. Compute the total surface area of this prism.

![Diagram of a triangular prism]

7. A triangular prism has sides of length a, b, and c and a height of h. Write a formula for the lateral area for this prism.
8. Which of the following three-dimensional solids correspond to the following top, front, and side views?

Top

Front

Side

a.  

b.  

c.  

d.  

   a. What is cylinder’s surface area?
   b. Scale the cylinder by multiplying each dimension by 3. What is the new surface area?
10. A prism 5 inches high has parallel square bases 2 inches long.
   a. What is its volume? What is its surface area?
   b. Scale the cylinder by multiplying each dimension by 5. What is the new surface area? What is the new volume? What pattern do you notice between the scale factor of the cylinder and the surface area? The volume?

11. The length of each side of a cube decreases by 20%. What is the percent decrease in surface area of the cube?

12. The surface area of a cube is 864 sq. cm. What is the length of the each side of the cube?

13. What is the surface area of a rectangular prism with side lengths $\frac{3}{5}$ ft, $\frac{7}{4}$ ft and $\frac{1}{3}$ ft?

14. Determine the lateral surface area and the total surface area of the rectangular pyramid given by the net below.

15. A triangular pyramid has an equilateral triangle as its base with side length 6 cm. Each lateral side is an isosceles triangle with slant height of 9 cm.
   a. Sketch a net for this triangular pyramid.
   b. Compute the lateral surface area of this pyramid

16. What is the surface area of a cylinder with height 0.35 mm and base radius 0.6 mm?
17. A cylindrical soup can has a radius of 3.8 cm and a height of 6 cm. The soup company needs to determine how much paper is required to label each can. Find the lateral surface area of the soup can.

18. William is painting a wall with a cylindrical paint roller. The diameter of the base is 2 inches and the distance between the bases is 8 inches.
   a. What is the lateral surface area of the paint roller?
   b. What is the area William will paint in 5 revolutions?

19. A glass company is designing a cylindrical fish tank for an aquarium-themed restaurant. The fish tank needs to be open at the top. The restaurant wants the radius of the tank to be 12 inches and the height to be 5 feet. What is the surface area of the aquarium in terms of \( \pi \)?

20. A rectangular swimming pool is 20 feet wide and 50 feet long. There is a semi-circular extension at the end of the pool, with a 20-foot diameter. The pool is 10 feet deep.
   a. What is the area of the base of the object?
   b. What is the lateral area?
   c. What is the volume of the object?

21. What is the ratio between the surface area of a 24 in \( \times \) 24 in \( \times \) 24 in cube in square inches and the surface area in square feet?

22. Ingenuity:
   An 8 \( \times \) 8 \( \times \) 8 cube is made of 512 unit cubes glued together. If the large cube is dipped in paint, how many of the unit cubes are painted?

23. Investigation:
   The following diagram shows the base of a trapezoidal prism:

   ![Trapezoidal Prism Diagram]

   Draw the prism and its net. The prism has surface area 1998 cm\(^2\). What is the measure of its height?
REVIEW PROBLEMS

1. In the following figure, lines $\overline{AB}$ and $\overline{CD}$ are parallel.

Identify each of the following:

a. Right angles
b. Acute angles
c. Obtuse angles
d. Supplementary angle pairs
e. Complementary angle pairs
Review Problems

2. In each of the figures below, fill in the missing angle measures with the information you are given.

a. 

![Diagram with angles](image)

b. Lines $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel.

![Diagram with angles](image)

c. Lines $\overrightarrow{EF}$ and $\overrightarrow{GH}$ are parallel

![Diagram with angles](image)

3. Figure ABCD is a rectangle. Fill in the missing angle measures with the information you are given.

![Diagram with angles](image)
4. What is the area and circumference of the circle pictured?

   a. \( r = 13 \)
   
   b. \( r = 1.6 \)

   c. \( d = 7 \)

5. Parallelogram ABCD has area 10. Triangle ABC has the same base and same height as the parallelogram. What is the area of triangle ABC? Can you figure this out in multiple ways?

6. Find the sum of the angles in each polygon.

   a. 
   
   b. 
   
   c. 
7. Find the area of each of the following figures:
   a. 
   ![Diagram of a trapezoid](image)
   b. 
   ![Diagram of a parallelogram](image)

8. What is the area and perimeter of triangle DEF?

   ![Diagram of a triangle](image)

9. Describe the lines of symmetry in this figure.

   ![Diagram of a hexagram](image)
10. Find the length of the missing sides of each of the following right triangles:

a. 
\[
\begin{array}{c}
6 \\
8
\end{array}
\]

b. 
\[
\begin{array}{c}
4 \\
5
\end{array}
\]

c. 
\[
\begin{array}{c}
8 \\
5
\end{array}
\]

11. The fire department has a cylindrical holding tank of water that measures 8 feet deep and has a diameter of 12 feet. How much water can the tank hold?

12. The measurements of a cereal box are 12 inches by 2 inches by 7 inches. Determine the volume and the surface area of the cereal box.

13. Determine the volume and surface area of the given figure.

14. Determine the volume of the following cylinder.
15. Draw a net of the rectangular prism below.
   a. What is the prism's surface area?
   b. What is its volume?
   c. What would be the volume and surface area if a new rectangular prism is formed by doubling the dimensions of the old rectangular prism?

![Rectangular Prism Net](image)

16. We have the a right triangular prism as pictured.
   a. What is the length of the third side of the triangular base?
   b. What is its volume and its surface area?

![Triangular Prism](image)
Chapter 11 Geometry

CHALLENGE PROBLEMS

Section 11.1:
For how many values of x are the angles x° and (x^2)^° complementary?

Section 11.2:
A polygon is called convex if every line segment joining any two vertices lies inside the polygon or forms one of its sides. At most how many right angles can a convex polygon have?

Section 11.3:
A square garden with sides of length 3m is surrounded by a fence. There is a metal fence post in each corner and a wooden fence post every meter along the sides. What is the largest area of a triangle whose vertices are all at wooden fence posts?

Section 11.4:
Points A and B lie on a circle with center O and a radius of 6 in. If OA and OB are perpendicular and chord AB is drawn, what is the area enclosed by chord AB and the shorter arc AB?
SECTION 12.1 MEASURES OF CENTRAL TENDENCY

Sets are useful for grouping interesting and related numbers. One such set is the heights of all of the people in your class. In order to use these sets, we need to analyze the numbers, or data, in context. The first step in data analysis, the process of making sense of a set, is collecting data. In data analysis, the idea of a data set is slightly different from that of a set. Unlike regular sets, data sets can have repetition of elements, and the order or arrangement matters.

EXPLORATION 1

Measure each person in your class in inches and record their name, age in months, and height in inches in a table like the one below. The numbers below were taken from another class, so your own class will have different results. Try to find ways to summarize the information in the table so that you can share your results with a friend without showing her the whole table. Would your strategy still work if there were 100 people in the survey? 1000 people?
The entire collection of numbers is called the data and each individual piece of information is called a data point. Data is plural for datum.

A major goal of data analysis is to find a simple measure of the data, called a measure of central tendency, that summarizes or represents, in a general way, the majority of the data. There are three common measures of central tendency: the mean, median, and mode. The mean, median and mode are different ways to identify the center of the data. We are also interested in how spread out our data is. The range, the difference between the largest and smallest values of the data, provides a simple measure of how much the data varies.

A dot (line) plot orders data and displays frequency. Each data point is represented by a dot on the line. For example, given the following data set (2, 8, 4, 8, 6, 5, 7, 9, 3, 7, 5), the corresponding dot plot is shown below.

<table>
<thead>
<tr>
<th>Name</th>
<th>Height (in)</th>
<th>Age (months)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sophia</td>
<td>52</td>
<td>113</td>
</tr>
<tr>
<td>Rhonda</td>
<td>51</td>
<td>112</td>
</tr>
<tr>
<td>Edna</td>
<td>57</td>
<td>112</td>
</tr>
<tr>
<td>Danette</td>
<td>61</td>
<td>115</td>
</tr>
<tr>
<td>Hesam</td>
<td>55</td>
<td>117</td>
</tr>
<tr>
<td>Eloi</td>
<td>62</td>
<td>110</td>
</tr>
<tr>
<td>Vanessa</td>
<td>58</td>
<td>113</td>
</tr>
<tr>
<td>Michelle</td>
<td>60</td>
<td>108</td>
</tr>
<tr>
<td>Mari</td>
<td>58</td>
<td>125</td>
</tr>
<tr>
<td>Calvin</td>
<td>56</td>
<td>129</td>
</tr>
<tr>
<td>Moises</td>
<td>57</td>
<td>124</td>
</tr>
<tr>
<td>Amanda</td>
<td>57</td>
<td>120</td>
</tr>
<tr>
<td>Hannah</td>
<td>55</td>
<td>131</td>
</tr>
<tr>
<td>Tricia</td>
<td>55</td>
<td>129</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Name</th>
<th>Height (in)</th>
<th>Age (months)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kristen</td>
<td>57</td>
<td>130</td>
</tr>
<tr>
<td>Max</td>
<td>52</td>
<td>135</td>
</tr>
<tr>
<td>Jim</td>
<td>50</td>
<td>142</td>
</tr>
<tr>
<td>Karen</td>
<td>57</td>
<td>136</td>
</tr>
<tr>
<td>Diane</td>
<td>49</td>
<td>138</td>
</tr>
<tr>
<td>Tiankai</td>
<td>58</td>
<td>138</td>
</tr>
<tr>
<td>Oscar</td>
<td>51</td>
<td>137</td>
</tr>
<tr>
<td>Jenny</td>
<td>60</td>
<td>138</td>
</tr>
<tr>
<td>Bence</td>
<td>59</td>
<td>142</td>
</tr>
<tr>
<td>Pat</td>
<td>53</td>
<td>134</td>
</tr>
<tr>
<td>Teri</td>
<td>59</td>
<td>135</td>
</tr>
<tr>
<td>Sally</td>
<td>57</td>
<td>139</td>
</tr>
<tr>
<td>Will</td>
<td>57</td>
<td>140</td>
</tr>
</tbody>
</table>
EXAMPLE 1

Use the dot plot above to answer the following questions.

a. What percent of the data has a value of 5 or less?
b. What percent of the data has a value of 7 or more?
c. What is the proportion of the data with a value of 7 or less to data with a value of 8 or more?
d. What is the proportion of data that is 5 or less to data that is 7 or more?

The mean, also called the arithmetic mean or average, is the sum of all the data values divided by the number of data points. For a visual example, suppose we have five containers, each containing a certain number of blocks:

These data can be grouped into a data set: {7, 3, 5, 7, 3}. There are 25 blocks total. The mean number of blocks in a container is the number of blocks each container has if these 25 blocks are distributed evenly among the 5 containers: \( \frac{25}{5} = 5 \).
The **median** is the value of the middle data point when the values are arranged in increasing order. If the data set has an even number of data points, the median is the average of the two middle values. To find the median value for the container example, order the data, with the smallest number of blocks first and the largest number last:

![Image of containers with blocks]

The median is the number of blocks in the middle, or third container with respect to the sorted ordering. The median is a helpful measure of central tendency because half of the values are less than or equal to the median and the other half of the values are greater than or equal to it.

**Frequency** is the number of times a data point appears in a data set. For example, if there are 4 people in the class who are 56 inches tall, then the frequency of the height 56 inches in the class is 4. The **mode** is the value or element that occurs the most often or with the highest frequency in the data set. In Exploration 1 the mode in our data is 57 because it appears 7 times in the data. What is the mode in your class’ data? A set of data can have more than one mode. For the containers of blocks example, the modes are 3 and 7 because both appear twice.

**PROBLEM 1**

Use the data table from Exploration 1 for the following questions.

a. Construct a dot (line) plot of students’ heights.

b. What qualitative characteristics do you notice about the dot plot, such as the shape, the centers, or the spread? What other statements can you infer about the class from the data?

c. Determine the mean, median, mode and range for the data set.

Age and height measurements for another class in the same school were collected and are shown in the following table.
Section 12.1 Measures of Central Tendency

<table>
<thead>
<tr>
<th>Name</th>
<th>Height (in)</th>
<th>Age (months)</th>
<th>Name</th>
<th>Height (in)</th>
<th>Age (months)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alexandra</td>
<td>63</td>
<td>140</td>
<td>Britanny</td>
<td>64</td>
<td>145</td>
</tr>
<tr>
<td>Ian</td>
<td>66</td>
<td>138</td>
<td>Shazil</td>
<td>69</td>
<td>148</td>
</tr>
<tr>
<td>Jongwook</td>
<td>62</td>
<td>136</td>
<td>Stanley</td>
<td>68</td>
<td>138</td>
</tr>
<tr>
<td>Keith</td>
<td>58</td>
<td>58</td>
<td>Eric</td>
<td>65</td>
<td></td>
</tr>
<tr>
<td>Alison</td>
<td>57</td>
<td>145</td>
<td>Gregory</td>
<td>58</td>
<td>141</td>
</tr>
<tr>
<td>Ming</td>
<td>54</td>
<td>126</td>
<td>Maurice</td>
<td>59</td>
<td>150</td>
</tr>
<tr>
<td>Izzy</td>
<td>64</td>
<td>169</td>
<td>Jazmine</td>
<td>57</td>
<td>137</td>
</tr>
<tr>
<td>Aishu</td>
<td>59</td>
<td>126</td>
<td>Maria</td>
<td>61</td>
<td>145</td>
</tr>
<tr>
<td>Nannette</td>
<td>60</td>
<td>154</td>
<td>Jeffrey</td>
<td>56</td>
<td>129</td>
</tr>
<tr>
<td>Patty</td>
<td>61</td>
<td>127</td>
<td>Amy</td>
<td>56</td>
<td>138</td>
</tr>
<tr>
<td>Michael</td>
<td>63</td>
<td>136</td>
<td>Carlos</td>
<td>57</td>
<td>137</td>
</tr>
<tr>
<td>Lisa</td>
<td>68</td>
<td>141</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

d. Create a dot plot of the student’s heights for this class. What statements can be made about the data from this class based on the dot plot?
e. Compare the dot plots for the two classes. Compare their means, medians and range. Do you notice any similarities or differences between the two classes?
f. What percent of the students are five feet tall or shorter?
g. What percent of the students are at least 5 feet 3 inches tall?
h. What is the proportion of data that is 7 or less to data points that is 8 or more?
i. What is the proportion of data that is 5 or less to data that is 7 or more?

EXAMPLE 2

Find the mean, median, mode, and range of the following data set.

\{2, 8, 4, 8, 8, 6\}
SOLUTION

The mean is found by adding the values together and dividing by the number of values. The sum of the values is $2 + 8 + 4 + 8 + 8 + 6 = 36$. The number of values in the set is 6. The mean is $\frac{36}{6} = 6$.

Putting the data set into order from smallest to greatest value results in \{2, 4, 6, 8, 8, 8\}. Because there are an even number of values in the set, the median is the average of the two middle values. The median is the average of 6 and 8, $\frac{6 + 8}{2} = \frac{14}{2} = 7$.

The most commonly occurring value in the data set is 8, so 8 is the mode.

The range is the difference between the highest and lowest value. The range is $8 - 2 = 6$.

PROBLEM 2

In the following data set, what is the mean? the median? the mode? the range?

\{4, 9, 12, 5, 9, 14, 11, 15, 5, 6, 7, 5\}

The mean depends on all the numbers in the data, but the median only depends on the value of the data point in the middle position. That does not, however, suggest that the mean is a better measure of central tendency than the median.

PROBLEM 3

Find the mean and median of the following six weeks test grades:

\{95, 30, 98, 93, 100\}.

Compare the value of each as a measure of the data.

PROBLEM 4

Two high school soccer teams have a scheduled game. The rosters from each school are reflected in the chart
Section 12.1 Measures of Central Tendency

<table>
<thead>
<tr>
<th>School</th>
<th>Freshmen</th>
<th>Sophomore</th>
<th>Juniors</th>
<th>Seniors</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>B</td>
<td>5</td>
<td>7</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

a. Construct dot plots for both of these teams.

b. Compare these dot plots to determine if one team is more experienced than the other team.

EXPLORATION 2

Using the data from Exploration 1, compute the mean and the median of the heights of the class. Then, imagine that a giant who is 400 inches tall joins the class. Compute the new mean and find the new median. How has each changed?

If the data is skewed, or uneven, a median value is a more accurate picture of the representative value than the mean is. Exploration 2 had a very tall giant join the class. The mean was affected by this outlier, a term used to refer to a value that is drastically different from most of the data values. The median, however, was not affected. The mean is usually more influenced by extreme values than the median.

Let us review the ways in which we summarized data in this section.

If we have a set of \( n \) values, then we can find the following measures:

- Find the mean by adding the values and dividing by \( n \).
- Find the median by ordering the values and finding the value that is in the middle, if \( n \) is odd, or taking the average of the middle values, if \( n \) is even.
- The mode is the most frequent value that occurs. There could be two or more such values.
- The range is the difference between the largest and the smallest values in the set.
EXERCISES

1. Using the example class from Exploration 1,
   a. Find the mean age, in years, to the nearest tenth.
   b. Now look at the ages of students in your own class. Find the mean, median and mode of the ages, in years, to the nearest tenth.
   c. How are the two means different? What are the factors that cause the difference?
   d. To the nearest inch, is the median height of students in your class different from the mean height? If so, why do you think they are different?

2. Separate the data from your class into categories by age and find the mean height for each age. Does the mean height increase with age? Explain the results of your analysis.

3. An English teacher assigned the same writing assignment to two different classes. In class x, the teacher only gave written instructions about the assignment. In class y, the teacher had the class discuss the assignment in small groups as well handing out written instructions. The assignment was graded on a scale of A, B, C, D or F. The results are in the chart below:

<table>
<thead>
<tr>
<th>Class</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>2</td>
<td>4</td>
<td>9</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Y</td>
<td>7</td>
<td>8</td>
<td>7</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

   a. Construct dot plots for both of these classes.
   b. Compare these dot plots to determine which class performed better than the other. Why did class y have better results on the assignment?

4. Find the mean, median, mode and range of the following data sets:
   a. {8, 6, 10, 14, 10, 9, 8, 3, 8, 16, 0, 8, 6, 2, 15, 2, 12, 8, 16, 5}
   b. {74, 66, 66, 66, 64, 66, 71, 66, 71, 66, 74, 64, 66, 73, 71, 60, 71, 65, 63, 74}
   c. Construct a dot plot for the data in part b.
   d. Use the dot plot to find the percent of the data points between 60 and 69.
   e. What is the ratio of data between 60 and 69 to those between 70 and 79?
5. Use the following results of a math test as data to create a dot plot. Determine the range, mean, mode, and median for the math test.
{100, 92, 79, 65, 86, 80, 78, 63, 91, 83, 91, 87, 79, 86, 85, 92, 75, 76, 95, 78, 68, 67, 73, 76, 71, 86, 89, 85, 91, 96, 83, 77, 93}

a. Use the dot plot to find the percent of test scores 90 or above?

b. What is the ratio of test scores above 90 to those below 90?

6. Two brothers live 300 miles from each other. Tom lives near the coast and his brother Mike lives inland. They each keep a record of the number of inches of rainfall each month for a year. The results are:

Tom:  {3, 4, 6, 2, 4, 4, 5, 3, 4, 6, 5, 4}

Mike:  {1, 2, 0, 1, 0, 1, 8, 9, 1, 3, 2, 4}

a. Make a dot plot for each of these sets of data.

b. Determine the mean, median, mode, and range for each set of data. Compare the measures and make some generalizations about Tom's weather versus Mike's.

c. Compare the shape of each dot plot. What does each shape indicate about the annual rainfall in the two areas?

7. A class has 17 students and the total height of all the students is 935 inches. What is the mean height of the class? What is the median height?

8. Rhonda joins a class that has 17 students. The class mean height was 58 inches. Rhonda is 65 inches tall. What is the new mean height for the class with Rhonda as an additional student? Give your answer to the nearest hundredth of an inch.

9. A class has six students with a mean height of 55 inches. The class mean height changes to 56 inches after Hannah, a new student, joins the class. How tall is Hannah?

10. A 14-person class with an average height of 54 inches merges with a 12-person class with an average height of 50 inches. What is the average height of the combined class to the nearest hundredth of an inch?
11. Which measure of central tendency is most helpful in representing the following?

a. Your grade in math.
b. The winner of the race for mayor.

12. Choose the appropriate measure of central tendency or range to describe the data in the table. Justify your reasoning.

<table>
<thead>
<tr>
<th>School</th>
<th>Number of Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jones Middle School</td>
<td>32</td>
</tr>
<tr>
<td>Lampasas Middle School</td>
<td>36</td>
</tr>
<tr>
<td>Falls Middle School</td>
<td>28</td>
</tr>
<tr>
<td>Miller Middle School</td>
<td>37</td>
</tr>
<tr>
<td>Fossum Middle School</td>
<td>51</td>
</tr>
</tbody>
</table>

13. The heights of various buildings in the city are listed. Which measure of central tendency or range would make the heights appear tallest? 168 ft., 186 ft., 221 ft., 73 ft., 152 ft., 186 ft., 199 ft.

14. The January mean daily temperatures for Castolon, TX and Galveston, TX are approximately the same. However, their ranges are quite different. The temperature data, in °F, from NOAA are

<table>
<thead>
<tr>
<th>City</th>
<th>Maximum</th>
<th>Minimum</th>
<th>Mean</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Galveston</td>
<td>61.9</td>
<td>49.7</td>
<td>55.8</td>
<td>12.2</td>
</tr>
<tr>
<td>Castolon</td>
<td>67.7</td>
<td>33.6</td>
<td>50.7</td>
<td>34.1</td>
</tr>
</tbody>
</table>

Even though Galveston and Castolon have about the same daily mean temperature for January, would you consider packing different clothes for the two places? Which measure of central tendency influenced your decision? Why?
15. On the right are estimated national median heights in inches for 9- through 14-year-olds in 2000, according to the National Center for Health Statistics (NCHS). Based on this data, what is your estimate for the median height for 15-year-olds? Do you think the median heights for 24-year-olds and 25-year-olds are that much different? Explain.  

<table>
<thead>
<tr>
<th>Age group</th>
<th>Height (in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>9-year-olds</td>
<td>52.5</td>
</tr>
<tr>
<td>10-year-olds</td>
<td>54.5</td>
</tr>
<tr>
<td>11-year-olds</td>
<td>56.5</td>
</tr>
<tr>
<td>12-year-olds</td>
<td>59.0</td>
</tr>
<tr>
<td>13-year-olds</td>
<td>62.0</td>
</tr>
<tr>
<td>14-year-olds</td>
<td>65.0</td>
</tr>
</tbody>
</table>

16. Ingenuity:  

It takes 1263 digits to number all the pages of a book. How many pages are there in the book?

17 Investigation:  

Supply an example that applies to your hometown where the median is a more appropriate measure than mean.
SECTION 12.2  GRAPHING DATA

When collecting data, it is often useful to draw a picture or graph to represent the data that has been collected. A graph of the data gives a quick, easy way to see what the data represents.

EXAMPLE 1

Ms. Garcia’s class has twenty students. Each student was asked which color they like best. The survey shows that 10 students prefer red, 5 students prefer green, and 5 students prefer blue. What are the best ways to represent or display this information?

SOLUTION

One way to display the data is to make a special kind of graph called a bar graph. To construct a bar graph, draw an $x$- and $y$-axis, subdivide the horizontal or $x$-axis into three equally-spaced intervals and label the intervals with the categories Red, Green, and Blue. Then label the vertical or $y$-axis with points from 0 to 12. For each color, draw a vertical bar equally separated from the other bars. The height of each bar represents the number of people who liked a particular color best. The bar graph should look like this:

![Bar Graph]

Favorite Color

Explain why the vertical axis has a number scale from 0 to 12. Why is there no number scale on the horizontal axis? Explain whether the order of the colors is important.
Another way to represent this data is by using percents. Because there are 20 students in all, \( \frac{10}{20} \) of the class likes the color red. Convert \( \frac{10}{20} \) to the decimal 0.50 and then to the percent 50%. Similarly, \( \frac{5}{25} = 0.25 = 25\% \) of the class likes green.

**PROBLEM 1**

Calculate the percents of the other two colors. Use these percents as data on the vertical axis to build another bar graph. Label the axes and draw the graph.

![Bar Graph](image)

**Favorite Color**

a. Calculate the percents of green and blue. What do they represent in the situation?

b. What proportion of the students like red?

c. What is the proportion of students who like green to those who like red?

Although the shape of the bar graph is the same, the percentage graph gives a picture of the relative quantity of the class’ preference for each color, not the number of students directly. Instead, the bar graph shows the relative number or percentage of students immediately.

Another way to represent the percentage data is to use a **circle or a pie graph** of the data. Use your protractor to draw a circular outline for the circle graph. You have already computed the percentage of each color. The proportion of the circle graph with a given color corresponds to the percentage of students who prefer that color. The larger the sector of the circle graph, the greater the percentage of people who liked the color.
Because 50% of the students prefer the color red, the question is what angle corresponds to 50% of the circle? To calculate 50% of 360°, multiply \((0.50 \times 360) = 180°\). Construct an angle that measures 180° and color that part of the circle red and label it. You can also calculate 50% of 360° by using a proportion as follows:

\[
50\% = \frac{50}{100} = \frac{x}{360}
\]

\[
\frac{1}{2} = \frac{x}{360}
\]

\[
(360) \frac{1}{2} = \frac{x}{360} \quad (360)
\]

\[
\frac{360}{2} = \frac{x}{1}
\]

\[
180 = x
\]

Next, calculate 25% of 360 and construct a section of the circle with this angle and color it green. Label this region as well. Finish making the circle graph by computing the angles and then coloring and labeling the last region. The completed circle graph on the next page clearly represents which color students like the best and makes the result of the survey visually obvious. Why does a circle graph make it easy to catch computation errors in converting from percents to degrees?
PROBLEM 2

Mr. Ruiz asked his 12 students which color they liked best as well. In his class, he found that 6 students prefer green, 4 students prefer red, 1 student prefers blue, and 1 student prefers purple.

a. Make a bar graph to represent the data from Mr. Ruiz’s class.

b. Make a circle graph to represent the same data.

c. What differences do you notice between the bar graph and circle graph?

EXPLORATION 1

The rainfall record for a region over an 8-year period from 1990 to 1997 is listed to the right.

<table>
<thead>
<tr>
<th>Year</th>
<th>Rainfall</th>
</tr>
</thead>
<tbody>
<tr>
<td>1990</td>
<td>30 inches</td>
</tr>
<tr>
<td>1991</td>
<td>32 inches</td>
</tr>
<tr>
<td>1992</td>
<td>24 inches</td>
</tr>
<tr>
<td>1993</td>
<td>18 inches</td>
</tr>
<tr>
<td>1994</td>
<td>28 inches</td>
</tr>
<tr>
<td>1995</td>
<td>36 inches</td>
</tr>
<tr>
<td>1996</td>
<td>42 inches</td>
</tr>
<tr>
<td>1997</td>
<td>31 inches</td>
</tr>
</tbody>
</table>

a. Plot the data on a coordinate plane. Label the horizontal axis to represent time in years, and the vertical axis to represent inches of rainfall. To convert the set of points to a line graph, connect the points sequentially with straight lines. Line graphs are typically used when the first data deal with change in time.

EXPLORATION 2

Another way to represent the data is in a histogram. A histogram is a special kind of bar graph that shows the frequency of data between set intervals. The chart below shows the results of Mr. Red’s students’ final exam grades.

<table>
<thead>
<tr>
<th>Student</th>
<th>Grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>Joe</td>
<td>78</td>
</tr>
<tr>
<td>Bryan</td>
<td>97</td>
</tr>
<tr>
<td>Cathy</td>
<td>98</td>
</tr>
<tr>
<td>Kendall</td>
<td>85</td>
</tr>
<tr>
<td>Amy</td>
<td>68</td>
</tr>
</tbody>
</table>
A. Complete the following table with the grades organized into the proper range.

<table>
<thead>
<tr>
<th>Range</th>
<th>Number of Grades (Frequency)</th>
</tr>
</thead>
<tbody>
<tr>
<td>91 – 100</td>
<td>7</td>
</tr>
<tr>
<td>81 – 90</td>
<td>3</td>
</tr>
<tr>
<td>71 – 80</td>
<td>4</td>
</tr>
<tr>
<td>61 – 70</td>
<td>1</td>
</tr>
</tbody>
</table>

The histogram organizes the data in the following chart.

B. Why did we choose the range 91 - 100 instead of 90 - 100 in the table?

C. When you look at the histogram, what do you notice about the grade distribution?

D. From the histogram, what score(s) do most of the students make?
Organizing data is an important part in the process of analyzing data. Venn Diagrams are a way to represent data that have different or similar characteristics.

Venn Diagrams commonly use circles to represent sets of items from the data. The circles are usually within a large rectangle that represents all the data gathered.

It may have two or more circles that may or may not overlap, depending on the relationship among the sets. The sets should have a description and the Venn Diagram may contain numbers in the regions of the diagram that tell us how many elements are in those specific regions of the circles.

**EXPLORATION 3**

If the three sets in the example below represent students who have a brother(s), who have a sister(s), who have a pet(s), then explore what some of the overlapped parts of the circle can represent. Be very careful with how you word your descriptions. Within the regions in the circle, we can write the description as well as place numbers to indicate how many elements are in that region of the circle. Remember that a person can have a brother and also have a sister.
In a survey of 100 students at Fossum Middle School, the following results were obtained. Use the Venn Diagram above and write the description and the numbers in the appropriate regions.

a. 15 students have a brother, sister and a pet
b. 45 students have a brother and a sister
c. 25 students have a brother and a pet
d. 17 students have a sister and a pet
e. 59 students have a sister
f. 35 students have a pet
g. 71 students have a brother
h. Are there any students in your survey that have no brother, sister nor pet? Explain your reasoning.

EXAMPLE 2

The Venn Diagram below shows the number of students at Miller Junior High School who have a favorite sports team, a favorite movie, and a favorite singer. The total number of students surveyed was 200.

Using the information from the above Venn Diagram, answer the following questions and explain your reasoning.
Section 12.2 Graphing Data

a. How many students have only a favorite singer?
b. How many students have a favorite singer?
c. How many students have a favorite sports team?
d. How many students have all three favorites?
e. How many students have no favorites?
f. What percent of students have no favorites?

SOLUTION

The three circles represent the sets of students that have a favorite singer, $S$; favorite sports team, $T$; and favorite movie, $M$. The numbers represent students that have the respective favorites.

The number of students that have only a favorite singer is 50. However, if we want the number of students that have a favorite singer, then we must add up all the numbers that are in the various regions that make up the circle for $S$. For example, a person may have a favorite singer and also have a favorite movie. Some of them may have a favorite singer and have a favorite movie and a favorite sports team. The total number of students that have a favorite singer is then $50 + 15 + 10 + 4 = 79$.

Similar to the argument about the students who have a favorite singer, the number of students with a favorite sports team is equal to 70. This is the result of adding $36 + 20 + 10 + 4$.

The number of students that have all three favorites is 10. Notice this is the intersection of the three circles.

The number of students that have no favorites is 8. These are the students not situated in any of the circular regions.
PROBLEM 3

The results of a random survey of 2000 people about their preference for the color of a car are shown in the circle graph below. Answer the following questions:

a. How many people surveyed preferred a blue car?
b. How many people surveyed preferred a black car?
c. What is the ratio of the number of people who preferred a blue car to the number of people who preferred a black car?
d. What is the ratio of the number of people who preferred a red car to the number of people who preferred a green car?
e. What is the ratio of the number of people who preferred a black car to the number of people who preferred a green car?
f. If you were an automotive executive, what would you decide to do, based on the data above?
EXAMPLE 3

Twelve students received the following scores on a math test:
97, 75, 73, 75, 83, 85, 54, 98, 97, 65, 75, and 83.

a. Graph this data with a box and whisker plot.
b. What percentage of the class has scores that appear in the box?
c. What is the interquartile range of the data?
d. What is the range of the data?

SOLUTION

A box and whisker plot divides the data into 4 parts, each containing \( \frac{1}{4} \) of the data points. First, order the data to find the points that divide the data into fourths. We will need to graph five numbers:

1. The minimum — the left-hand end of the left whisker.
2. The maximum — the right-hand end of the right whisker.
3. The median, which is the same as the second quartile, is Q2, the middle of the box.
4. The left side of the box is the first quartile Q1
5. The right side of the box is the third quartile Q3.

The first step is to arrange the data in order, from smallest to largest:
54, 65, 73, 75, 75, 75, 83, 83, 85, 97, 97, 98

1, 2: The minimum is 54, and the maximum is 98.

3. The median is the middle number if there are an odd number of data points, and the average of the two numbers in the middle, if there are an even number of data points. Because there are an even number of data points, the average of the two numbers in the middle is \( \frac{75 + 83}{2} = 79 \). This is the median and the second quartile Q2.

4. The first quartile Q1 is the median of the numbers to the left of the median: 54, 65, 73, 75, 75, 75. Again there are an even number of numbers so the median is \( \frac{73 + 75}{2} = 74 \). The first quartile Q1.
5. The third quartile Q3 is the median of the numbers to the right of the median: 83, 83, 85, 97, 97, 98. So the third quartile Q3 is \( \frac{85 + 97}{2} = 91 \).

Using these 5 numbers, we now make our box and whiskers plot:

```
```

b. 50% of the data is in the box.

c. The Interquartile Range (IQR) is the difference between the third quartile and the first quartile, IQR = Q3 – Q1 = 91 – 74 = 17. Notice that 50% of the data is in the IQR. The IQR gives the spread of the middle half of the data.

d. The range of the data is the difference between the maximum value and the minimum value. For our problem, the range = 98 – 54 = 44.

**EXAMPLE 4**

a. Now graph the same data with a stem and leaf plot.

b. What is the advantage of the stem and leaf plot over the box and whisker in demonstrating how the numbers are distributed?
c. Make a histogram that shows the number of students with scores in the 50’s, 60’s, 70’s, 80’s, and 90’s.

d. Compare the stem and leaf plot to the histogram.

**SOLUTION**

a. When we make a stem and leaf plot, the stems are the tens digit

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
</tr>
</tbody>
</table>

The leaves are the one’s digits:

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>5 4</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>3 5 5 5 5</td>
</tr>
<tr>
<td>8</td>
<td>8 3 3 5</td>
</tr>
<tr>
<td>9</td>
<td>9 7 7 8</td>
</tr>
</tbody>
</table>

b. The score that appears most often, the mode, is easy to see.

c. A histogram, shows the same data and shape in a vertical direction.

| The horizontal axis labels are the scores 50, 60, 70, 80 90 that show the range of the data. |
| The vertical axis is the number of scores in each range |

| 50’s | 1 |
| 60’s | 1 |
| 70’s | 4 |
| 80’s | 3 |
| 90’s | 3 |

d. The number of scores in the histogram is the number of leaves on each stem. Both show the distribution of scores.
EXAMPLE 5

The range of salaries in San Marcos compared with Austin is summarized in the box and whiskers plot below:

San Marcos:

\[\begin{align*}
\text{min} & \quad \text{Q1} \quad \text{Q2} \quad \text{Q3} \quad \text{max} \\
7,500 & \quad 18,000 \quad 21,750 \quad 50,250 \quad 64,500 \\
28,000 & \quad 28,750 \quad 36,000 \quad 78,750 \quad 107,250 \\
18,000 & \quad 72,000 \quad 121,500 \quad 135,750 \\
72,000 & \quad 135,750 \\
\end{align*}\]

Austin:

\[\begin{align*}
\text{min} & \quad \text{Q1} \quad \text{Q2} \quad \text{Q3} \quad \text{max} \\
9,000 & \quad 20,000 \quad 35,000 \quad 35,000 \\
50,100 & \quad 60,100 \quad 91,200 \quad 214,500 \\
91,200 & \quad 173,400 \quad 214,500 \quad 337,800 \\
91,200 & \quad 378,900 \\
20,000 & \quad 378,900 \\
\end{align*}\]

a. From the box and whisker plot, which city has the higher median salary?
b. Compare the distribution of salaries in San Marcos and Austin.
c. Which city has the highest salaries?

SOLUTION

a. Austin has a higher median salary--$35,000 compared to $28,000 in San Marcos.
b. San Marcos salaries are a little lower. Also, there is a wider range of salaries in Austin.
c. Austin also has the highest salaries.
PROBLEM 4

Consider the numbers from 1–25. Let E equal the set of all the even numbers from 1–25 and list them. Let F equal the set of all multiples of 3 from 1–25 and list them. Let G equal the set of all prime numbers from 1–25 and list them.

1) Construct a Venn Diagram with the appropriate elements listed in the regions of the Venn Diagram. Describe, in words, each region.
2) Construct a different Venn Diagram and write in each appropriate region the number of elements in the region.

When might a Venn Diagram be a good choice for representing data? How is this better than a bar graph? A circle graph?

The following table summarizes the type of graph chosen depending on the information provided.

<table>
<thead>
<tr>
<th>Graph Type</th>
<th>When To Use</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bar Graph</td>
<td>To compare groups of data</td>
</tr>
<tr>
<td>Circle (pie) Graph</td>
<td>To compare parts to the whole</td>
</tr>
<tr>
<td>Histogram</td>
<td>To show frequency within intervals</td>
</tr>
<tr>
<td>Venn Diagram</td>
<td>To show different or similar characteristics</td>
</tr>
<tr>
<td>Line Graph</td>
<td>To show change over time</td>
</tr>
<tr>
<td>Dot (line) Plot</td>
<td>Orders data and displays frequency</td>
</tr>
</tbody>
</table>

PROBLEM 5

Choose the appropriate type of graph for each situation.

a. Favorite colors of students.

b. The number of students who prefer pepperoni pizza to all types of pizza.

c. Students were polled according to their favorite subjects in school. They were given the choice of math, science, English, and social studies. How many chose math and science?

d. Jose's height over the past year.
EXAMPLE 6

The scores on a test are given in the Stem and Leaf graph below:

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
</tr>
</tbody>
</table>

a. How many students took the test?
b. What score occurred the most often?
c. What is the median score?
d. What is the highest score?
e. What is the range of the scores?

SOLUTION

a. 24, the total number of leaves
b. 64 occurred most often.
c. The median is the average of 75 and 76, 75.5.
d. The highest score is 98.
e. The range of scores is from 52 to 98, so the range is 98 - 52 = 46.

EXERCISES

1. a. Describe what the data in the circle graph shows about the favorite time of the year of students in the class.
b. What information do you need about the class to give more specific information about the class’ preference for time of year?
c. The approximate percent of students who favor winter.
d. The approximate percent of students who favor summer.
e. The approximate percent of students who favor fall.
Section 12.2 Graphing Data

f. The ratio of students who favor summer to those who favor fall.
g. The ratio of students who favor fall to those who favor winter.
h. The ratio of students who favor summer to those who favor spring.

2. Poll your class. Ask all the students what their favorite month is. Make a bar graph and a circle graph for data gathered from a survey of favorite months.
   a. The percent that favor the summer months.
   b. The percent that favor November or December.
   c. The ratio between those who favor summer months to those who favor non-summer months.
   d. The ratio between those who favor June to those who favor September.

3. Poll your class for size-of-family data. Make a bar graph and a circle graph using this data.

4. A poll is taken as to what is each person’s favorite season. The results are

<table>
<thead>
<tr>
<th>Season</th>
<th>Number of People</th>
</tr>
</thead>
<tbody>
<tr>
<td>Winter</td>
<td>10</td>
</tr>
<tr>
<td>Spring</td>
<td>35</td>
</tr>
<tr>
<td>Summer</td>
<td>10</td>
</tr>
<tr>
<td>Fall</td>
<td>40</td>
</tr>
</tbody>
</table>

   a. Make a bar graph using the numbers in each group.
   b. Make an equivalent bar graph showing the percent of those polled favoring each season.
   c. Make a circle graph using the numbers in each group
   d. Make an equivalent circle graph showing the percent of those polled favoring each season.

5. Biologists found the estimated population of rabbits in the Palo Duro Canyon during the 8-year period from 1990 to 1997. Make a bar graph to represent the population estimates to the right.

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1990</td>
<td>12,500</td>
</tr>
<tr>
<td>1991</td>
<td>13,000</td>
</tr>
<tr>
<td>1992</td>
<td>11,000</td>
</tr>
<tr>
<td>1993</td>
<td>7,500</td>
</tr>
<tr>
<td>1994</td>
<td>8,000</td>
</tr>
<tr>
<td>1995</td>
<td>11,500</td>
</tr>
<tr>
<td>1996</td>
<td>15,000</td>
</tr>
<tr>
<td>1997</td>
<td>14,000</td>
</tr>
</tbody>
</table>
6. Make a line graph to represent the data from Exercise 5.

7. Connie was working on important data when her computer crashed and she lost all the numbers. Luckily, she had printed a bar graph of her data earlier.

![Car Sales for August](image)

Based on the data from the bar graph:

a. What percent of cars sold cost less than $20,000?

b. What percent of cars sold cost more than $17,000?

c. What is the proportion of cars sold that cost less than $19,000 to the number that cost more than $20,000?

d. How many cars were sold?

e. What is the total sales amount for the month of August?

f. If Connie gets a 1% commission on the sales amount, how much did Connie earn from her commission in the month of August?

8. Make a circle graph for the birthday data to the right. How do you represent months with no birthdays?

<table>
<thead>
<tr>
<th>Month</th>
<th>Birthdays</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>12</td>
</tr>
<tr>
<td>February</td>
<td>13</td>
</tr>
<tr>
<td>March</td>
<td>0</td>
</tr>
<tr>
<td>April</td>
<td>13</td>
</tr>
<tr>
<td>May</td>
<td>15</td>
</tr>
<tr>
<td>June</td>
<td>22</td>
</tr>
<tr>
<td>July</td>
<td>24</td>
</tr>
<tr>
<td>August</td>
<td>25</td>
</tr>
<tr>
<td>September</td>
<td>12</td>
</tr>
<tr>
<td>October</td>
<td>13</td>
</tr>
<tr>
<td>November</td>
<td>0</td>
</tr>
<tr>
<td>December</td>
<td>23</td>
</tr>
</tbody>
</table>
9. Make a histogram based on the birthday data to the right. Group the months by quarters of the year. Which quarter of the year has the most birthdays?

10. The box and whisker plot below shows the distribution of ages of workers at Max’s Department Store.

   a. From this box and whisker plot, what is the median age?
   b. What is the youngest employee’s age?
   c. What is the oldest employee’s age?
   d. Do you think this department store caters to young or older customers? Explain.
   e. What percentage of the employees is younger than 35?
   f. What percentage of the employees is between 21 and 35 years old?

11. Rachel and Liz are preparing to bake cookies, but they want to bake the most popular kind. They polled their friends to find out what kind of cookie is the most preferred. The results are shown in the circle graph below.

   Cookie Choices

   Snickerdoodle 12%
   Oatmeal Raisin 15%
   Peanut Butter 12%
   Sugar 38%
   Chocolate Chip 23%
a. If 25 friends are polled, how many prefer Snickerdoodle?

b. What angle measure corresponds to sugar cookies?

c. They decide to poll the entire seventh grade. 57 said they like sugar cookies. How many students are in the seventh grade?

d. What percent of students preferred Sugar or Chocolate Chip cookies?

12. A survey was conducted that asked for the salaries of ten people:

$25,700, $32,000, $14,000, $115,000, $39,000, $36,000, $74,000, $18,000, $12,000, and $38,000

a. Graph this data with a box and whisker plot.

b. What percentage of the respondents has salaries that appear in the box between Q2 and Q3?

c. Graph this data with a stem and leaf plot.

d. From the stem and leaf plot, which salaries occur most often?

e. Make a histogram that shows the salaries in $10,000 increments.

f. Compare the stem and leaf plot to the histogram

13. The coaches from junior high schools compared the total basket scored by their teams during the preceding season. The results are shown in the graph below. The combined total baskets scored by all teams was 1,340.

Total Baskets Scored

![Bar Graph showing Total Baskets Scored by Teams]
14. In a circle graph for hair color from another class, the angle for red hair is 45°.
   a. What percent of the students have red hair?
   b. How many of the 24 class members have red hair?

15. A circle chart cannot be made in all cases where data is presented in percentages. All four seventh grade teachers in a school asked their students which color they liked best. The table shows the results for what percentage in each class liked green best. Make a bar graph. Explain why you couldn’t make a circle chart for the data in the table.

<table>
<thead>
<tr>
<th>Class</th>
<th>Percent that prefer green</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ms. Garcia</td>
<td>32%</td>
</tr>
<tr>
<td>Mr. Ruiz</td>
<td>50%</td>
</tr>
<tr>
<td>Ms. Serviere</td>
<td>25%</td>
</tr>
<tr>
<td>Ms. Voigt</td>
<td>20%</td>
</tr>
</tbody>
</table>

16. Which type of graph would be most appropriate to display the grade distribution in Dr. Warshauer’s Algebra class? Why?

17. Which type of graph would be most appropriate to display the change in temperature over the month of February? Why?

18. All the students in a class measured their heights. The results of this survey are given in the stem and leaf graph below, with the heights in inches. The stem is the tens digit, and the leaf is the units digit.

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3 4 4 4 5 6 6 9</td>
</tr>
<tr>
<td>6</td>
<td>1 1 1 2 2 2 3 4 5 8 8 9</td>
</tr>
<tr>
<td>7</td>
<td>0 0 0 1 2 5</td>
</tr>
</tbody>
</table>
a. How many students were surveyed?
b. What height occurred the most often?
c. What is the median height of the class?
d. What is the highest?
e. What is the range of heights?

19. **Ingenuity:**
Each time a question is given with three possible answers, it is possible to get a different result. For example, if the possible responses to the survey question are A, B and C, one possibility with 5 people is 2 A’s, 1 B and 2 C’s. How many different results are possible if there is 1 person? 2 people? 3 people?

20. **Investigation:**
Conduct a survey of 20 people on a question of your choice that has three possible answers. Collect your data into a table, and display the results with a bar graph, a line graph, if possible, and a circle graph. Which representation seems most appropriate for your survey? Why
SECTION 12.3  PROBABILITY

The study of probability allows us to make educated guesses about what might happen in the future based on past experience, to determine how likely different outcomes are. This knowledge can help us make the informed choices.

You have heard people say, “There is a 50-50 chance of getting heads on a coin flip,” or “There is a 80% chance of rain today.” What do statements like these mean? How can we determine the chance that some event will happen?

To study this question, we first need to define or review a few important terms. An experiment is a repeatable action with a set of outcomes. For example, the experiment of flipping a coin has two possible outcomes, either heads or tails. The set of all possible outcomes of an experiment is called the sample space. There are different ways to display the probable outcomes, some of which are creating a tree diagram, a list, or a table.

In studying an experiment, the question is, “What are all the possible outcomes?” If you flip a coin once, you will observe only one of the two possible outcomes, heads or tails. The key to finding any probability is to determine the likelihood of each possible outcome.

It is often helpful to use abbreviations or draw diagrams to represent the outcomes of the experiment. For example, when flipping a coin, we often write H for the outcome of getting a head and T for the outcome of getting a tail. The sample space is the set \{H, T\}; there are only two possible outcomes.

The sample space \{H, T\} represents a simple coin-flipping experiment. Each event arises from a single toss. Other examples of simple experiments are listed below:

a. Roll a number cube or die once. The possible outcomes are the sample space \{1, 2, 3, 4, 5, 6\}.

b. Randomly pick a marble from a bag containing a blue, a red, a green, and a purple marble. The sample space for this simple experiment is \{B, R, G, P\}.
Another way to represent a sample space for these simple experiments is to draw a tree diagram. For the sample of flipping a coin, the tree diagram might be drawn like this:

```
  H
 / \
T   
```

**EXAMPLE 1**

Now consider a more complicated experiment, flipping a coin twice. What is the set of all possible outcomes? How many outcomes show no tails?

**SOLUTION**

The order of outcomes is important. The outcome of getting a head and then a tail, denoted by HT, is a different outcome from getting a tail and then a head, denoted by TH. This experiment has the sample space \{HH, HT, TH, TT\}. Notice that there are four possible outcomes. When the experiment involves flipping a coin twice, \{H\} is an impossible outcome. The simple event \{HH\} is the subset containing the outcome that both flips show heads and is the only outcome that shows no tails. The following chart lists all the possible outcomes of this experiment.

<table>
<thead>
<tr>
<th>1st flip</th>
<th>2nd flip</th>
<th>Outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>H</td>
<td>HH</td>
</tr>
<tr>
<td>H</td>
<td>T</td>
<td>HT</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>TT</td>
</tr>
<tr>
<td>T</td>
<td>H</td>
<td>TH</td>
</tr>
</tbody>
</table>

We can also summarize all the possible outcomes with a Tree Diagram, as shown below.

```
  H
 / \   2nd flip
H   T   HH
 |   |   /  \  
|   |   H   T
 |   |   /   \  
|   |   H   T
 |   |   /   \  
|   |   T   T
 |   |   /   \  
|   |   T   T
```
In both of the experiments so far, each outcome in the sample spaces has the same chance of occurring as any other outcome. Each outcome is said to be equally likely.

**DEFINITION 12.1: EVENT, SIMPLE EVENT AND COMPOUND EVENT**

An event is any subset of the sample space. A simple event is a subset of the sample space containing only one possible outcome of an experiment. A compound event is a subset of the sample space containing two or more outcomes.

The words simple and compound are used to describe both events and experiments. The main thing to remember is that a simple event has just one outcome while a compound event has more than one outcome listed. A simple experiment has just one action, such as pick a card, roll a number cube, or flip a coin. A compound experiment may roll two number cubes or flip a coin more than once.

Call $E$ the event of “getting at least one head” in the two-flip experiment. The outcomes HH, HT, and TH satisfy the criteria for being in $E$, so $E = \{\text{HH, HT, TH}\}$. Because $E$ contains three possible outcomes, it is a compound event.

In order to study a complicated event $E$, a technique that is often useful is to find the possible outcomes that are not in $E$. We call this set the complement of $E$, or $E^c$. (TT), the event that there are no heads, is the complement of the event of at least one head, $[\text{TH, HT, HH}]$.

Therefore, the probability of showing no heads is $\frac{1}{4}$, because 1 of the 4 equally likely outcomes show no heads. The probability of showing at least one head is $\frac{3}{4}$.

**DEFINITION 12.2: PROBABILITY**

In an experiment in which each outcome is equally likely, the probability $P(A)$ of an event $A$ is $\frac{m}{n}$, where $m$ is the number of outcomes in the subset $A$ and $n$ is the total number of outcomes in the sample space $S$.

$$P(A) = \frac{\# \text{ of outcomes}}{\text{total number of possible outcomes}}$$
Notice that the probability of an event from an experiment is always a number between 0 and 1. Explain why this is true, using the coin-flipping example if necessary. Explain why \( P(S) = 1 \), where \( S \) is the sample space for an experiment.

Explain why \( P(E^c) = 1 - P(E) \).

**EXAMPLE 2**

Consider the experiment of rolling one number cube.

a. What is the probability of rolling a four?

b. What is the probability of not rolling a four?

**SOLUTION**

Identify the sample space, \( S = \{1, 2, 3, 4, 5, 6\} \), for the experiment of rolling one number cube. Recall, a sample space is the set of all the possible outcomes of an experiment.

a. Let \( A = \) the event of rolling a 4 in the experiment. Written as a subset of the sample space we have \( A = \{\text{rolling a 4}\} \) or just \( \{4\} \). \( A \) is a simple event. The probability of \( A \) is the fraction with the numerator equal to 1, the number of outcomes in \( A \) and the denominator equal to 6, the number of total outcomes in the same space \( S \). Therefore, \( P(A) = \frac{1}{6} \).

b. Let \( B = \) the event of not rolling a 4 in this experiment. Written as a subset, we have \( B = \{1, 2, 3, 5, 6\} \). Notice that \( B \) is the complement of event \( A \) because the outcomes in \( B \) are “not 4.” \( B \) is a compound event. The probability of \( \{\text{not getting a 4}\} = P(B) = \frac{5}{6} \). Another way to think about the probability of “not an event” is 1 minus probability of the event. A notation for this is:

\[
P(B) = P(A^c) = 1 - \frac{1}{6} = \frac{5}{6}
\]

When we consider an experiment of rolling one number cube, we do not actually roll a number cube. Instead, we think about what could possibly happen if we rolled a number cube. This is called theoretical probability. If we actually rolled the number cube, that would be experimental (empirical) probability.
To see the difference, go back to the simple experiment of tossing a coin. The theoretical probability of getting heads is $\frac{1}{2}$. Now, gather some empirical data by performing the experiment many times: Toss a coin 50 times and record the heads and tails. What was the result of your empirical experiment? Was the ratio of heads to total tosses $\frac{1}{2}$? Was it close to $\frac{1}{2}$? On average, as the number of tosses, or repetitions of the experiment, increases, the experimental probability gets closer to the theoretical probability.

**PROBLEM 1**

a. Draw a tree diagram for the simple experiments of rolling the number cube from part a.

b. Draw a tree diagram for the simple experiment of picking a marble out of the bag from part b.

**EXPLORATION 1**

Many middle school students prefer red black tennis shoes, but some student prefer neither. How might the percent of middle school students who prefer red tennis shoes be estimated? Explain.

**EXPLORATION 2**

Bilateral Junior School has 1000 students, and each student prefers either the color red or blue. Each student in this unusual school also prefers either chocolate or strawberry ice cream. Use a survey to calculate the percent of your class that favors colors red or blue. Repeat for the percent favoring chocolate or strawberry ice cream. Then find the following percentages:

a. the number of students who favor red and chocolate

b. the number who favor red and strawberry

c. the number who favor blue and chocolate

d. the number who favor blue and strawberry

Use the work above to predict

e. the number of students in the school who favor red and chocolate.

f. the number in the school who favor red and strawberry.

g. the number in the school who favor blue and chocolate.

h. the number in the school who favor blue and strawberry.
Chapter 12  Data Analysis

i. $P(\text{favors red and chocolate})$

j. $P(\text{favors red and strawberry})$

k. $P(\text{favors blue and chocolate})$

l. $P(\text{favors blue and strawberry})$

PROBLEM 2

Suppose there is a bag containing 10 marbles: 5 green, 3 blue and 2 red.

a. What is the theoretical probability of randomly drawing a green marble from the bag?

b. What is the probability of drawing a red marble from your original bag of 10 marbles?

c. Imagine you conducted an experiment 100 times. Each time you would draw a marble from the bag, record the color, and then replace it in the bag. How many times do you anticipate you would draw each color?

PROBLEM 3

Consider an experiment in which we flip a coin.

a. If we flip the coin 3 times, what are the possible outcomes? Draw a tree diagram and make a list for the sample space.

b. Conduct an experiment of flipping a coin twenty times. How many outcomes are heads? How does this compare to the probability of getting heads from part a?

c. Conduct the same experiment flipping the coin fifty times. How do the experimental and theoretical probabilities compare?

The above example demonstrates the law of large numbers, which states as more experiments are conducted, experimental probability will approach theoretical probability.

EXPLORATION 3

There are three bags containing red and blue marbles, as shown below.
Section 12.3 Probability

a. Each bag is shaken. If you were to close your eyes, reach into a bag, and remove one marble, which bag would give you the best chance of picking a blue marble? Explain your answer.

b. Which bag gives you the best chance of picking a red marble?

c. How can you change Bag 2 to have the same chance of getting a blue marble as Bag 1? Explain how you reached this conclusion.

d. How can you change Bag 2 to have the same chance of getting a blue marble as Bag 1 if Bag 2 must contain 60 total marbles?

e. Consider only Bags 1 and 2. Make a new bag of marbles so that this bag has a greater chance of getting a blue marble than Bag 1, but less of a chance of getting a blue marble than Bag 2. Explain how you arrived at the number of blue and red marbles for your new bag.

EXAMPLE 3

As an experiment, a class bought five bags of the same brand of candy from five different stores, opened them and randomly selected 1000 pieces. There were 198 red, 305 green, and 497 yellow candy. Next they opened another bag and randomly selected 150 pieces of candy. Approximately how many pieces of each color might the class expect to be in this sample?

SOLUTION

The percent of each candy in this sample might be as follows:

\[
\frac{198}{1000} \text{ is approximately } \frac{200}{1000} = .20 = 20\% \\
\frac{305}{1000} \text{ is approximately } \frac{300}{1000} = .30 = 30\% \\
\frac{497}{1000} \text{ is approximately } \frac{500}{1000} = .50 = 50\%
\]
So the class expected about 20% of the last sample to be red, or \((0.20)(150) = 30\) pieces, about 30% of the last sample to be green, or \((.30)(150) = 45\) pieces and about 50% of the last sample to be yellow, or \((.50)(150) = 75\) pieces.

**PROBLEM 4**

In a survey of middle school students, 180 have only a dog, 60 have neither a dog nor cat, 160 have both a cat and a dog, and 200 have only a cat.

a. What is the probability that a student selected at random will have a cat?

b. What is the probability that a student selected at random will have a dog?

c. What is the probability that a student selected at random will have both a cat and a dog?

d. What is the probability that a student selected at random will have neither a cat nor a dog?

e. What is the probability that a student selected at random will have only a cat?

f. In a random sampling of 30 students, about how many might have a cat?

There are many applications of probability in society. Every day the newspaper and other news outlets use surveys and polls to predict the outcome of an upcoming election or to discover public opinion about issues like whether to build a new highway or to allow drivers to text while driving. Professional statisticians use a random sample of between several hundred to a few thousand people and ask their opinion. For example, the size of the random sample is 300 people. From the sample, 180 favor candidate A, 90 favor candidate B, and 30 are undecided. From this data pollsters conclude that on election day approximately \(\frac{180}{300} = 60\%\) of the voters to vote for candidate A, or more if she attracts some undecided voters. So, if about 10,000 people vote in the election, unless something happens, about 60% of the voters will cast their ballot for candidate A.

This is a likely outcome if the random sample accurately reflects the people who actually vote on the day of the election, if the undecided vote is not significant or something happens to change the predicted outcome. For instance, the voters who want vote for candidate A might be affected by bad weather, a flu epidemic or bad press.
EXAMPLE 4

If we roll two number cubes, determine the sample space using a list (table).

SOLUTION

What is a possible outcome? You might roll a 3 and a 4. As with tossing a coin twice, the order matters. Rolling a 3 and a 4 is different from rolling a 4 and a 3. To see this more easily, think about rolling one red number cube and one green number cube. Rolling a red 4 and a green 3 is a different outcome from rolling a red 3 and a green 4.

To list all the outcomes of this experiment, abbreviate the outcome red 3 and green 4 as (3, 4). The sample space for the two-number cube experiment can be listed in several ways. One convenient method is to make a table like the one below.

<table>
<thead>
<tr>
<th>(R, G)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1, 1)</td>
<td>(2, 1)</td>
<td>(3, 1)</td>
<td>(4, 1)</td>
<td>(5, 1)</td>
<td>(6, 1)</td>
</tr>
<tr>
<td>2</td>
<td>(1, 2)</td>
<td>(2, 2)</td>
<td>(3, 2)</td>
<td>(4, 2)</td>
<td>(5, 2)</td>
<td>(6, 2)</td>
</tr>
<tr>
<td>3</td>
<td>(1, 3)</td>
<td>(2, 3)</td>
<td>(3, 3)</td>
<td>(4, 3)</td>
<td>(5, 3)</td>
<td>(6, 3)</td>
</tr>
<tr>
<td>4</td>
<td>(1, 4)</td>
<td>(2, 4)</td>
<td>(3, 4)</td>
<td>(4, 4)</td>
<td>(5, 4)</td>
<td>(6, 4)</td>
</tr>
<tr>
<td>5</td>
<td>(1, 5)</td>
<td>(2, 5)</td>
<td>(3, 5)</td>
<td>(4, 5)</td>
<td>(5, 5)</td>
<td>(6, 5)</td>
</tr>
<tr>
<td>6</td>
<td>(1, 6)</td>
<td>(2, 6)</td>
<td>(3, 6)</td>
<td>(4, 6)</td>
<td>(5, 6)</td>
<td>(6, 6)</td>
</tr>
</tbody>
</table>

How do we represent the above solution as a tree diagram?

PROBLEM 5

Consider again the experiment of rolling two number cubes. What is the probability of each of the following events?

a. \( A = \{ \text{getting at least one 6} \} \)

b. \( B = \{ \text{getting a double (both number cubes have the same number)} \} \)

c. \( C = \{ \text{the sum of the two number cubes is 7} \} \)
What is the difference between flipping a coin two times and flipping two coins simultaneously? What is the difference between rolling a number cube two times and rolling two number cubes? Run the thought experiments for both the coins and the number cubes to see the difference or similarities.

Two groups are surveyed about their preference for sodas or diet sodas. A random sample of 500 people between 45 and 55 years of age was selected. Another group of 400 people, between the ages of 18 and 28, was randomly selected. The results of the survey are in the table below.

<table>
<thead>
<tr>
<th>Group</th>
<th>Regular Soda</th>
<th>Diet Soda</th>
</tr>
</thead>
<tbody>
<tr>
<td>A: 18-28</td>
<td>284</td>
<td>116</td>
</tr>
<tr>
<td>B: 45-55</td>
<td>188</td>
<td>312</td>
</tr>
</tbody>
</table>

The percent of the group A sample that favor diet soda is 29%. The percent of the group B sample that favor diet soda is 63.4%. Explain how the percentages were reached. From these random samples, why can you infer that someone in the age range of 45 to 55 is twice as likely to prefer diet soda?

**SOLUTION**

The percent of the group A sample who favor diet soda is $116/400 = 29\%$. The percent of group B sample who favor diet soda is $312/500 = 63.4\%$. From these random samples, you can infer that someone 45 to 55 years old is twice as likely to prefer diet soda. Because the data was derived from a random sample of a relatively large group, it is safe to say that the probability that someone between the ages of 18 and 28 prefers diet soda is approximately 29%, and the probability that someone between the ages of 45 and 55 prefers diet soda is approximately 63.4%.

**EXERCISES**

1. A citrus farmer has 2,000 orange trees. He suspects that he might have a fruit fly problem. Devise a plan that would enable the farmer to decide whether or not his orchard has fruit flies and what percent of his orange trees might be infested.

2. For each of the following simple experiments, write a sample space and draw a tree diagram to represent the sample space:
3. Write a sample space $S$ for the experiment of rolling a number cube.

a. Let $B = \{4\}$ be the simple event of getting a 4. What is $P(B)$?

b. Write the subset of the complement of $B$, $B^\circ$. Is $B^\circ$ a simple event or a compound event? Explain how you found the complement of $B$.

c. What is $P(B^\circ)$? TE: $P(B^\circ) = 5/6 = 1 - 1/6 = 1 - P(B)$.

d. Call $A$ the event of rolling a number that is at least 5. Write $A$ as a subset of $S$. Is event $A$ simple or compound? What is $P(A)$? Does your answer make sense?

e. Call $D$ the event of getting an even number. Write $D$ as a subset of $S$. What is $P(D)$?

f. What is the complement of $D$, $D^\circ$? Explain how to find $D^\circ$. What is $P(D^\circ)$?

4. A random group of sixty students from a Fastfood Junior School were asked the following questions: Do you prefer your French fries with or without catsup? Do you prefer regular of diet soda? The following tree diagram shows the outcome in this sample:

```
   Catsup (35)
     /       \
    /         \
No Catsup (25)  Regular Soda (20)
                /       \
           /         \
     Diet Soda (15)  Regular Soda (15)
                    /       \
               /         \
         Diet Soda (10)
```

A. Estimate the following percentages:

a. students who favor catsup and regular soda

b. those who favor catsup and diet soda

c. those who favor no catsup and regular soda
d. those who favor no catsup and diet soda

B. Fastfood Junior School has 900 students. Predict the number of students who:

a. favor catsup and regular soda.
b. favor catsup and diet soda.
c. who favor no catsup and regular soda.
d. favor no catsup and diet soda.

C. Compute the experimental probabilities for each of the following events:

a. \( P(\text{favors catsup and regular soda}) \)
b. \( P(\text{favors catsup and diet soda}) \)
c. \( P(\text{favors no catsup and regular soda}) \)
d. \( P(\text{favors no catsup and diet soda}) \)

5. In a large lake, a scientist randomly catches forty small mouth bass using a net, tags each fish, and returns her catch. A week later, she returns to the lake and randomly catches 100 small mouth bass. She finds that ten of these fish have her tags. How many small mouth bass might be in this lake?

6. Marta has a large bag of marbles that are red, blue or green. She draws out 20 marbles and gets 7 red, 9 blue and 4 green marbles. She knows there are exactly 500 marbles in the bag, predict how many of each color marble there might be in the bag.

7. Two large containers are filled with red and white balls. In container \( A \), the proportion of red to white balls is 3:2. In container \( B \), the proportion of red to white balls is 5:4. If a ball is randomly selected from each container, which ball is more likely to be white, the ones for container \( A \) or \( B \)?

8. Consider the experiment of rolling 1 number cube.

a. What is the probability of \( A = \{\text{rolling a number that is at least 5}\} \)?
b. List the outcomes that involve rolling an even number. What is the probability of this event?
c. Perform an experiment by rolling the number cube 50 times and record the outcomes in a table. Using these results, compute the experimental probability of \( \{\text{getting a 1}\} \) or \( P(1) \).
d. Compute the following experimental probabilities: \( P(2) \), \( P(3) \), \( P(4) \), \( P(5) \) and \( P(6) \).

e. Compare these experimental probabilities to the theoretical probabilities. Now average the individual experimental probabilities from everyone in class. What do you notice?

9. Perform the experiment of first flipping a coin and then rolling a number cube. What is the sample space of this experiment? How many outcomes does it have?

10. Suppose we spin the spinner and roll the number cube, as shown below.

What is the sample space? How did you organize your sample space? How many outcomes are there? What is \( P(\text{getting an A and a 1}) \)? What is \( P(\text{getting an A or a 1}) \)?

11. Consider the experiment of flipping four coins simultaneously. What is the sample space? What is the probability of getting 4 heads?

12. Krystal bought a bag of 24 jellybeans at the store. The bag has 8 red jellybeans, 6 yellow jellybeans, 6 green jellybeans, and 4 white jellybeans.

   a. What is \( P(\text{Green}) \)?

   b. What is \( P(\text{Red}) \)?

   c. What is \( P(\text{Yellow}) \)?

   d. Compute the sum \( P(\text{Green}) + P(\text{Red}) + P(\text{Yellow}) + P(\text{White}) \). Explain why the answer to this sum makes sense.
13. Consider the experiment of flipping a coin four times.
   a. How many outcomes get no heads?
   b. How many outcomes get exactly 1 head?
   c. How many outcomes get exactly 2 heads?
   d. How many outcomes get exactly 3 heads?
   e. How many outcomes get exactly 4 heads?
   f. How many outcomes get at least 1 head?

14. Roll two number cubes, one red and one green.
   a. What is \( P(E) \) for the event \( E \) that at least one number cube rolled a 3?
   b. What is \( P(F) \) for the event \( F \) that neither of the number cubes rolled a number greater than 3?
   c. What is \( P(G) \) for the event \( G \) that the numbers on the number cubes add to 7?

15. The city of Carthage has a population of about 500,000. It has about 240,000 registered voters. In an upcoming bond election, voters will be asked whether they support a new water treatment plant. A poll is taken using 500 voters chosen at random asking whether they support the new plant. The poll results are 220 in favor, 200 against, and 80 undecided. What can you say about the chances of the bond issue passing?

16. A factory has 850 employees. Of these, 550 work the day shift and 300 work the night shift. The day shift has 240 males and the night shift has 120 females. If an employee is selected at random, what is the probability that the employee is
   a. a female who works the night shift.
   b. a male who works the night shift.
   c. a male who works the day shift.
   d. a female who works the day shift.

17. Think about rolling two standard five-sided number cubes: a green number cube and a red number cube.
a. What is the set of all possible outcomes from rolling the two number cubes?

b. What is the set of all possible outcomes where the sum of the pips is less than 7?

c. What is the set of all possible outcomes where the outcome is a double?

d. What is the intersection of the sets from parts b and c?

e. What is the set of all possible outcomes where the product of the pips is odd?

18. Two groups were surveyed about their preference, milk chocolate or dark chocolate. A random sample of 250 people between the ages of 15 and 20 was selected from group A. Another group of 280 people between the ages of 35 and 40 was randomly selected. The results were

<table>
<thead>
<tr>
<th>Group</th>
<th>Milk Chocolate</th>
<th>Dark Chocolate</th>
</tr>
</thead>
<tbody>
<tr>
<td>A: 15-20 age</td>
<td>194</td>
<td>56</td>
</tr>
<tr>
<td>B: 35-40 age</td>
<td>92</td>
<td>188</td>
</tr>
</tbody>
</table>

a. What percent prefer dark chocolate in group A?

b. What percent prefer dark chocolate in group B?

c. What can you infer from these surveys about the differences in the two populations?

d. What two criteria in the survey allow anyone to infer results from the data?

e. Estimate the probability of that someone between the age of 15-20 might prefer milk chocolate to dark chocolate.

f. Estimate the probability of that someone between the age of 35-40 might prefer milk chocolate to dark chocolate.

19. Tommy bought a bowl of ice cream with two scoops. For each of the scoops, he can choose chocolate, strawberry or vanilla. List the different two-scoop choices Tommy can make.

20. Consider flipping a coin four times. Which outcome is more likely, getting HHHH or HTHT?
21. Harry has 5 coins in his pocket: 3 nickels and 2 dimes.
   a. He pulls a coin from his pocket. Then, without replacing the first coin, he pulls a second coin from his pocket. What is the sample space for this experiment? (Hint: Consider labeling the nickels n1, n2, and n3, and the dimes d1 and d2.)
   b. What is the probability he pulled exactly 15 cents from his pocket?
   c. Now, he pulls a coin from his pocket, replaces it and then pulls a second coin from his pocket. What is the sample space for this experiment?
   d. What is the probability he pulled exactly 15 cents from his pocket in the second experiment?

22. Ingenuity:
   A group of 6 students place their names in a hat.
   a. Consider the experiment of drawing a name from the hat. How many outcomes are there for this experiment?
   b. Consider another experiment of drawing two names from the hat, where the order in which the names are drawn matters. How many possible outcomes are there?
   c. Now consider the same experiment of drawing two names from the hat, but this time simultaneously, so that order does not matter. How many possible outcomes are there for this experiment?

23. Investigation:
   Consider the experiment of drawing one card randomly from a standard 52-card deck.
   a. What is the probability of getting a heart? a club?
   b. What is the probability of getting a heart or a club?
   c. What is the probability of getting a heart and a club?
   d. What is the probability of getting a face card: a jack, queen or king?
   e. What is the probability of getting a heart and a face card?
SECTION 12.4 INDEPENDENT EVENTS

One of the major goals of mathematics is to find simple underlying ideas to explain how and why things work. To do this, mathematicians analyze problems by breaking them into simpler steps.

EXAMPLE 1

Suppose you have a hat and a box. The hat contains the numbers 1, 2, 3, and 4, and the box contains the letters A, B and C. Imagine the following experiment: Without looking, reach into the hat and pull out one number, and then reach into the box and without looking, pull out one letter. What is a possible outcome? How can you represent all the possible outcomes? How many possible outcomes are there?

SOLUTION

This experiment can be broken down into two simpler experiments:
Step 1: Draw a number from the hat. This has sample space $S_1 = \{1, 2, 3, 4\}$. There are 4 possible outcomes.
Step 2: Draw a letter from the box. This has sample space $S_2 = \{A, B, C\}$ with 3 possible outcomes.

For the combined experiment, list the outcomes as ordered pairs, like $(2, B)$ and $(3, A)$, or shorten the notation to $2B$ and $3A$. Always try to write the sample space with some sort of order, if possible, so you do not miss a possible outcome. In this case, the sample space is

$$S = \{1A, 1B, 1C, 2A, 2B, 2C, 3A, 3B, 3C, 4A, 4B, 4C\}.$$

The outcomes can also be listed in a table:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1A</td>
<td>1B</td>
<td>1C</td>
</tr>
<tr>
<td>2</td>
<td>2A</td>
<td>2B</td>
<td>2C</td>
</tr>
<tr>
<td>3</td>
<td>3A</td>
<td>3B</td>
<td>3C</td>
</tr>
<tr>
<td>4</td>
<td>4A</td>
<td>4B</td>
<td>4C</td>
</tr>
</tbody>
</table>
What is the size of the sample space? How does the arrangement in the table help to count the number of outcomes?

A tree diagram is another method to visually represent the sample space:

The 4 branches that come from the left node or root of the tree represent the first experiment, choosing from \{1, 2, 3, 4\}. Each branch splits into three smaller branches, representing the second experiment, choosing from \{A, B, C\}. There are 12 branches on the right. To obtain the 12 outcomes in the combined experiment, follow the branches from left to right, reading the labels to obtain the 12 outcomes in the combined experiment.

The action of drawing a number from the hat and drawing a letter from the box are said to be independent, meaning that the probabilities of one of these actions does not depend on the other. Because they are independent, the two actions can occur either in succession or simultaneously, it doesn’t matter which. Using the table model, the 4 rows represent the 4 choices in the hat. The choices correspond to the first 4 branches in the tree model. Each row has 3 columns that represent the 3 choices in the box. The rows correspond to the second level of branching. For each of the first 4 choices, there are 3 second choices. So to count the number of members in the array, or the number in the sample space, compute the sum of 4 rows with 3 outcomes in each row: $3 + 3 + 3 + 3$. This is the same as the area model for multiplication. The total number of outcomes is $4 \cdot 3$. 
This process is a formal rule in counting:

**THEOREM 12.1: THE RULE OF PRODUCT**

If one action can be performed in $m$ ways and a second independent action can be performed in $n$ ways, then there are $m \cdot n$ possible ways to perform both actions.

Next, let’s use our counting principles to compute the probabilities. For the experiment above, consider the following two events:

Let event $E$ be the event that the number drawn from the hat is a 2.

Let event $F$ be the event that the letter drawn from the box is an A.

These events are independent since the probability of either experiment does not depend on the other. The probability of $E$ is $\frac{1}{4}$. You can compute this in two ways. If you think of $E$ as the experiment of drawing a number from a hat, then there are 4 equally likely outcomes. So, $P(E) = \frac{1}{4}$. However, if you think of the combined experiment, then the sample space $S$ has 12 possible outcomes, and event $E = \{2A, 2B, 2C\}$. So, $P(E) = \frac{3}{12} = \frac{1}{4}$.

Similarly, the probability of $F$ is $\frac{1}{3}$, since the sample space contains 3 equally likely outcomes. Alternatively, $F = \{1A, 2A, 3A, 4A\}$. So, $P(F) = \frac{4}{12} = \frac{1}{3}$.

We can now extend this problem and ask:

**EXAMPLE 2**

What is the probability that both $E$ and $F$ will occur?

**SOLUTION**

Looking at our tree diagram, the only way that both $E$ and $F$ will occur is if the number drawn is a 2 and the letter drawn is an A. So, the probability of $E$ and $F$ is $\frac{1}{12}$.
Another way to think about this is entirely with probabilities. If $E$ occurs $\frac{1}{4}$ of the time, and an independent event $F$ occurs $\frac{1}{3}$ of the time, then both $E$ and $F$ will occur $\frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$ of the time.

We can summarize this rule, which extends the rule of product, as:

**THEOREM 12.2: THE RULE OF PRODUCT OF PROBABILITIES**

If two events, $E$ and $F$, are independent, then the probability that both will occur is the product of their probabilities, namely

$$P(E \text{ and } F) = P(E) \cdot P(F)$$

This leads to the idea of “conditional probability.” The question is what do we mean when we talk about the “probability of event $E$, if event $F$ occurs?” To explain this, consider the following example:

**EXAMPLE 3**

Consider the experiment of rolling a number cube. Let $E$ be the event that the outcome is a 1 or 2. Let $F$ be the event that the outcome is odd.

1. What is the probability of event $E$?
2. What is the probability of event $F$?
3. What is the probability of event $E$, if event $F$ occurs, written $P(E|F) = P(E \text{ given that } F \text{ occurred})$?

**SOLUTION**

1. The sample space of this experiment is $S = \{1, 2, 3, 4, 5, 6\}$. Event $E = \{1, 2\}$. So, $P(E) = \frac{2}{6} = \frac{1}{3}$.
2. Event $F = \{1, 3, 5\}$. The probability of $F$ is $\frac{3}{6} = \frac{1}{2}$.
3. If you are given that the event $F$ occurs, then the new sample space is $F$. The question is, how likely is it that $E$ occurs if we know that the outcome is odd? The only way that $E$ could occur would be if the outcome is a 1. So,
Section 12.4 Independent Events

\[ P(E \mid F) = \frac{\text{# of outcomes in both } E \text{ and } F}{\text{# of outcomes in } F} \]

i.e. \( P(E \mid F) = \frac{1}{3} \). Since this is the same as the probability of event \( E \), \( E \) and \( F \) are independent events. By this, we mean that \( P(E) \) is the same, whether or not event \( F \) occurs. We can write the set of objects in both \( E \) and \( F \) as \( E \cap F \). So,

\[ P(E \mid F) = \frac{\text{# of elements in } E \cap F}{\text{# of elements in } F} = \frac{P(E \text{ and } F)}{P(F)}. \]

EXAMPLE 4

Suppose, with the same hat of numbers and box of letters, we change the experiment to the following: Assume cards with numbers 1-4 and cards with letters A-C are same shape and size. Put all of the cards in some box. Select one card. How many possible outcomes are there for this experiment?

SOLUTION

This is the same as choosing one item from the combined sample space \( \{1, 2, 3, 4, A, B, C\} \), which is the set \( S_1 \) or \( S_2 \). The combined sample space has seven elements, or possibilities.

The main difference between the earlier situation and this one lies in the change of one word. In the first example we chose from the hat and the box, while in the second we chose from the hat or the box. This illustrates the importance in mathematics of reading words carefully, especially words like “and” and “or.” The following rule captures the number of ways to perform one action “or” another:

**THEOREM 12.3: THE RULE OF SUM**

If one action can be performed in \( m \) ways and a second action can be performed in \( n \) ways, then there are \( (m + n) \) ways to perform one action or the other, but not both. This assumes that the two actions have no elements in common.
The rule of sum and the rule of product provide a powerful way to examine experiments made up of several actions. By carefully using the rule of sum and the rule of product, it can often be far easier to compute the number of possible outcomes in such a compound experiment.

Note: Two events that have no elements in common are said to be “mutually exclusive.” Using Example 4, where the sample space is \{1, 2, 3, 4, A, B, C\}, let E be the event of getting a number and F be the event of getting a letter.

Then \( P(E) = \frac{4}{7} \) and \( P(F) = \frac{3}{7} \). \( P(E \cup F) = P(E) + P(F) = \frac{4}{7} + \frac{3}{7} = \frac{7}{7} = 1. \)

**THEOREM 12.4: THE RULE OF MUTUALLY EXCLUSIVE EVENTS**

For mutually exclusive events, \( E \) and \( F \), the probability of \( E \) or \( F \), written \( P(E \text{ or } F) \), occurring is given by the rule of sum

\[
P(E \text{ or } F) = P(E) + P(F).
\]

This follows since the number of ways that \( E \) or \( F \) can occur is the number of outcomes for event \( E \) plus the number of outcomes for event \( F \), by the rule of sum.

**PROBLEM 1**

For the experiment in Example 1, compute the probability of each of the following events:

a. The event of drawing an even number and an A.

b. The event of drawing neither a 1 nor an A.

Use the definition of complement to answer c and d.

c. The event of drawing a 1 or an A.

d. The event of drawing an odd number or a B.

e. Why can’t we use the rule of mutually exclusive events to answer c and d?
PROBLEM 2

We describe two experiments that each involve a two-step process. Determine whether each experiment is independent or not. Why?

**Experiment 1**: Sally opens a drawer with pairs of socks. There are 4 pairs of green, 3 pairs of purple, 2 pairs of red, and 1 pair of black socks. Without looking, she pulls out one pair of socks. If she doesn't like the color, she puts it back and pulls out another pair of socks. But Sally is not careful, so each time she puts the socks back, they get all mixed up.

a. What is the probability that the first two pairs of socks that Sally selects are both green?

b. Suppose the first pair of socks was green; what is the probability that the second pair will be green also?

**Experiment 2**: Rhonda wants to pack two pairs of socks for a trip. In her drawer, there are 4 pairs of yellow, 3 pairs of blue, 2 pairs of pink, and 1 pair of white socks. Without looking, she pulls out one pair of socks, puts it in her suitcase, and then selects another pair.

c. What is the probability of Rhonda pulling out two pairs of socks that are both yellow?

d. Suppose the first pair of socks was yellow; what is the probability that the second pair will be yellow also?

e. Compare Sally's and Rhonda's two-step processes. Does the outcome of the second step (the color of the second pair of socks selected) depend on the outcome of the first step? Explain.
EXERCISES

1. Tim and Diana plan to spend a day at the beach. Tim wants to take a walk, play volleyball or collect shells. Diana wants to bike, swim or surf.
   a. If they each choose one of their own activities, what are the possible outcomes for the day? Draw and label a tree diagram representing all the possible outcomes.
   b. If Tim and Diana decide to spend the day together doing the same activity, what are the possible outcomes?

2. Buddy has 4 shirts, 3 pairs of pants and 2 hats, all color coordinated. How many different outfits can he wear? An outfit is different from another outfit if just one of the articles of clothing is different. Draw a tree for this sample space. Also list the sample space using symbols.

3. Imagine an experiment that involves flipping a coin several times.
   a. Draw a tree for the experiment of flipping a coin two times. How many branches are there?
   b. Use a different color pen or pencil to extend the tree to represent flipping the coin three times.
   c. Extend it again to represent a four-flip experiment and then a five-flip experiment.
   d. Describe the process of extending the tree to include one more flip. What is the pattern for the total number of branches at each stage?

4. Use the letters A, B, C, D and E, to make code words.
   a. How many of two-letter code words can you make?
   b. If you pick one of these words at random, what is the probability that it starts with the letter A?
   c. If you pick one of these words at random, what is the probability that it has a B or C?
5. In the experiment of flipping a coin four times, what is the probability of
   a. getting no heads?
   b. getting exactly one head?
   c. getting \{HTTT\}? Why is this different from part b?
   d. getting exactly 2 heads?
   e. getting at least 2 heads?

6. Suppose you have a bag with 4 red and 6 blue marbles. Determine if the
   compound events are independent or dependent.
   a. If you randomly draw one marble, what is the probability of getting a red?
   b. If you randomly draw one marble, what is the probability of getting a blue?
   c. If you draw a marble, replace it and draw another marble, what is the
      probability of getting two reds?
   d. If you draw two marbles, what is the probability of getting two red
      marbles?
   e. If you draw two marbles, what is the probability of getting two blue
      marbles?
   f. If you draw two marbles, what is the probability of getting a marble of
      each color?

7. John has a deck of cards.
   a. If he draws a card, what is the probability that it is a heart?
   b. If he draws a card, replaces it, then draws a card again, what is the
      probability that both cards are hearts?
   c. If he draws 2 cards, what is the probability that both are hearts?

8. You have only the following words to work with: \{m, a, t, h\}?
   a. How many 3-letter code words can be made from the four letters?
   b. What is the probability that the word starts with m?
   c. What is the probability that the word does not contain m?
   d. What is the probability that the word contains at least one m?
9. Given a four-letter alphabet, find the following:
   a. How many one-letter code words are possible?
   b. How many two-letter code words are possible?
   c. How many three-letter code words are possible?
   d. How many four-letter code words are possible?
   e. How many five-letter code words are possible?
   f. Use a pattern that you observe above to determine how many ten-letter code words are possible.

10. Think about the experiment of rolling a number cube or choosing a digit or choosing a letter from the alphabet. How many outcomes are possible?

11. Consider the experiment of rolling three number cubes: 1 green, 1 blue and 1 red. How many outcomes are there in the sample space?

12. Think about the experiment of rolling a number cube. What is the probability of getting an even number or a prime number when you roll the number cube? Is this answer different than you might expect? Explain your reasoning.

13. Kristen draws a card from a standard 52-card deck and then selects a letter from the alphabet. How many outcomes are possible?

14. Bill draws one card from a standard 52-card deck. How many ways can he draw a 10 or a face card? What is the probability of getting a ten or a face card?

15. **Ingenuity:**

   A short quiz has five multiple-choice questions. Each question has four choices with one correct and the other three incorrect.
   
   a. How many ways are there to answer the five questions?
   
   b. How many ways are there to miss every question?
   
   c. If you guess randomly for each question, what is the probability you will answer at least two correctly?
16. **Investigation:**

There are 4 students who volunteered to help their teacher on a project.

a. How many ways can the teacher choose 0 students to help?

b. How many ways can the teacher choose 1 student to help?

c. How many ways can the teacher choose 2 students to help?

d. How many ways can the teacher choose 3 students to help?

e. How many ways can the teacher choose 4 students to help?

f. How is the pattern in the answer above related to the sample space of flipping a coin 4 times?
1. Mr. Johnson’s class recorded the number of siblings each member of the class has in the bar graph below. Calculate the mean, median, and mode of the data.

![Bar Graph](image)

a. How many students were in the class?

b. Make an ordered list of the data parts.

c. What is the median?

d. What is the mode?

e. What is the mean?

f. Make a circle graph of this data.

2. Kate went to a family reunion. During her time there, she recorded which of her family members had blue eyes, which had brown hair, and which had green eyes. She recorded the data as a Venn diagram.

![Venn Diagram](image)
Review Problems

a. How many of Kate’s family members have green eyes?
b. How many of her family members have brown hair?
c. How many have green eyes AND brown hair?
d. How many have neither green eyes nor brown hair?
e. How many of Kate’s family members that have green eyes DON’T have brown hair?
f. What does the 10 on the outside of the circles represent? How many family members attended the reunion?

3. Mr. Greenstein asks his class who their favorite member of their immediate family is and records it in a pie chart. Graph this data as a bar graph. If you select a member of the class at random, what is the probability that a student’s favorite family member is a parent? What is the probability that the favorite family member is a female? Is it easier to figure this out from the pie chart or the bar graph?
4. The table shown lists 24 people and the day of the year they were born. Determine the mean, median, mode, and range of the data to determine the probability of the following:

   a. mean of the data set
   b. median of the data set
   c. mode of the data set
   d. the mean or the median of the data set
   e. the median and the mode of the data set
   f. the mean, the median, or the mode of the data set

5. If you roll two six-sided number cube, what is the probability that the sum of the two numbers rolled is 5? What is the probability the sum is 3?

6. If I roll two six-sided number cube, what is the probability that the product of the two numbers rolled is 5? What is the probability the product is 12?

7. Write out the set of all the possible outcomes when you flip a coin three times. How do you know that you’ve listed them all? What is the probability of getting exactly one head?
8. Suppose you have a bag with 4 red marbles and 6 blue marbles.
   a. If you pick a marble, what is the probability of getting a red marble?
   b. If you pick a marble, replace it in the bag and then pick another marble, what is the probability of getting a blue marble on each pick?
   c. If you pick 2 marbles (without replacement), what is the probability of getting 2 blue marbles?

9. Suppose we form 3-letter code words from the alphabet \{a, o, p, t\}. How many of these code words are there? How many of these code words start with p? What is the probability that one of these code words starts with p?
CHALLENGE PROBLEMS

Section 12.1:
Which represents a larger percentage: a slice of area 22 in a pie chart of radius 7 or a slice of area 2263 in a pie chart of radius 71?

Section 12.2:
A class of 11 students was given the following extra credit question on a test:

Pick a positive integer between 1 and 10, inclusive: _____

(The least integer greater than or equal to the nonnegative difference between the mean and the median of the answers given will be added to everyone’s score.)

What was the maximum bonus that the class could have earned?

Section 12.3:
Two bored statisticians play a coin-flipping game as follows: Each player chooses a sequence of 3 heads or tails, then a fair coin is flipped until one of the sequences arises in consecutive flips, making that player the winner. If player A chooses HHH and player B chooses optimally (he gets to see player A’s choice first, and hence cannot choose HHH), then what is the probability that player B wins?

Section 12.4:
A fair coin is tossed 8 times. What’s the probability of getting at most 5 heads?
Challenge Problems
In earlier chapters, you learned about percentages and percent discounts and explored different kinds of taxes. In this chapter, after a review of these basic ideas, you will begin to put all of the pieces together. The goal is to see how mathematics provides an essential tool for managing money. Using mathematics, you will learn to set up a system to organize your finances and make good financial decisions.

Begin by reviewing simple interest, then learn how banks extend the idea. First, what does it mean to invest a principal amount of money \( P \) in a bank at an interest rate \( r \)? Look at the following example:

**EXAMPLE 1**

Bobby invests $100 at a simple interest rate of 6%. How much money will he have after two months?

**SOLUTION**

The amount of money in Bobby’s account after two months is the original principal \( P = $100 \), plus the interest earned, \( I \). The 6% interest of any principal is paid after one year. In two months, Bobby will earn \( \frac{2}{12} \) of the yearly interest. The interest he earns is

\[
I = 100 \cdot (0.06) \cdot \left( \frac{2}{12} \right) = 100 \cdot \frac{0.12}{12} = 100 \cdot 0.01 = 1
\]

The amount in Bobby’s account after two months is the original principal plus the interest earned, or \( 100 + 1 = $101 \).
The simple interest earned is $I = P \cdot r \cdot t$ or $I = Prt$. Explain why.

**SIMPLE INTEREST FORMULA**

When a principal amount of money $P$ is invested at an interest rate in decimal form $r$ for $t$ years, the simple interest earned will be $I = Prt$.

If $t$ is less than one year, the fractional part of the year represents $t$, as above.

To compute the amount in an interest-bearing account at the end of $t$ years, called $A$, combine the original principal amount $P$ with the interest made to get the formula: $A = P + I = P + Prt = P(1 + rt)$

In the shortened form, to compute the amount $A$ in an interest-bearing account after $t$ years, using the inverse of the distributive property, multiply the original amount by $(1 + rt)$.

**EXAMPLE 2**

Maggie invests $500 in an account that earns 10% interest. How much will be in her account after 6 months?

**SOLUTION**

There are two ways to solve the problem:

A. First find the interest earned in six months. To do this, convert the time to years: $t = 6$ months $= 6$ months $\cdot \left( \frac{1 \text{ year}}{\frac{12}{12} \text{ months}} \right) = \frac{6}{\frac{12}{12}} \text{ year} = \frac{1}{2} \text{ year}$. The interest earned is $I = Prt = (500)(0.10)(\frac{1}{2}) = 25$. So the amount after six months will be $500 + 50 = 550$.

B. Alternatively, to compute the amount:

$A = P(1 + rt) = 500(1 + 0.10 \cdot \frac{1}{2}) = 500(1.05) = 550$.

Which way do you prefer?

Do banks usually use simple interest? The answer is no. Bank customers use them to build their savings or to get a loan. In both cases, banks use compound interest. Generally, they add or charge interest to an account at least four times a year, or quarterly, sometimes more often. Banks compute interest using a **compound interest** formula.
To do this, the banks divide the year into a number of periods $m$ per year. If they are computing interest quarterly, $m = 4$. If they are computing interest daily, $m = 365$. If they are computing interest monthly, $m = 12$. Each period, they adjust accounts or loans by adding the simple interest earned during the period. When the next period begins, the account or loan has more money than it did at the beginning of the given period. So in the next period, the account or loan will earn or be charged interest on both the original principal as well as any interest earned or charged the previous period.

This sounds complicated, but actually the idea is simple.

Step 1: At the beginning of a period, $P$ is the amount of money in an account. How much money will be in a bank account at the end of the period?

The amount at the end of the period will be $P + I$, where $I$ is the interest earned during the period. So what is $I$? In order to compute $I$, the bank uses two numbers:

1. The interest rate $r$.
2. The time period $t$. If there are $m$ periods per year, each period will be $\left(\frac{1}{m}\right)$ of a year. First, compute the interest earned during one period:

$$I = Pr \left(\frac{1}{m}\right) = P \cdot \left(\frac{r}{m}\right)$$

For short, use $I$ for the amount of interest and let $i = \frac{r}{m}$. Then $I = Pi$, so the amount at the end of the period is $P + I = P + Pi = P\left(1 + i\right)$.

To find the amount at the end of a period, multiply the amount at the beginning of the period by $(1 + i)$.

At the end of each period, multiply the amount at the beginning of the period by the factor $(1 + i)$.

<table>
<thead>
<tr>
<th># of Period</th>
<th>Amount in account at the beginning of period</th>
<th>Amount in account at end of period</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$P$</td>
<td>$P\left(1 + i\right)$</td>
</tr>
<tr>
<td>2</td>
<td>$P\left(1 + i\right)$</td>
<td>$P\left(1 + i\right)\left(1 + i\right)$</td>
</tr>
<tr>
<td>3</td>
<td>$P\left(1 + i\right)\left(1 + i\right)$</td>
<td>$P\left(1 + i\right)\left(1 + i\right)\left(1 + i\right)$</td>
</tr>
<tr>
<td>............</td>
<td>............</td>
<td>............</td>
</tr>
<tr>
<td>$n$</td>
<td>$P\left(1 + i\right)^{n-1}$</td>
<td>$P\left(1 + i\right)^n$</td>
</tr>
</tbody>
</table>
The amount at the end of period 1 is equal to the amount at the beginning of period 2. Each time a period passes, the amount at the beginning of the period is multiplied by the factor \((1 + i)\). Using exponential notation yields the compound interest formula:

### COMPOUND INTEREST FORMULA

If \(P\) is the principal amount, \(r\) is the interest rate, and \(m\) is the number of times the interest is compounded per year, then the amount \(A\) after \(n\) periods is

\[
A = P (1 + \frac{r}{m})^{mn}
\]

**EXAMPLE 3**

A $500 amount is invested at 10% rate of interest compounded monthly. How much will be in the account after two years?

**SOLUTION**

Use the compound interest formula, and find

\[
P = 500 \quad r = 10\% = 0.10 \quad m = 12 \quad \text{so} \quad i = \frac{r}{m} = \frac{0.10}{12}
\]

The next question is how many periods are there in two years? Because each period is one month, there will be 12 periods in a year, or 24 periods in two years. So \(n = 24\).

Substitute the values into the compound interest formula.

\[
A = 500 \left(1 + \frac{0.10}{12}\right)^{24}
\]

Computing this longhand is more than a nightmare. Using a calculator, the amount of money in the account after two years is \(A = 610.195 = \$610.20\).

**EXAMPLE 4**

Suppose $500 is invested in a bank at 10% simple interest rate. How much will be in the account after two years? Compare the earnings from a simple interest rate of 10% with a compound interest rate of 10% compounded monthly from Example 3.
SOLUTION

Use the simple interest formula, \( A = P (1 + rt) \)

\[ A = 500 \times (1 + 0.10 \times 2) = 600. \]

By compounding monthly, a customer will earn $10.20 more with compound interest than with simple interest. Practice using simple interest and compound interest formulas in the problems below.

EXERCISES

1. Sam invests $100 in the bank at a simple interest rate of 8%. How much will be in his account after
   a. 2 months?  
   b. 6 months?  
   c. 1 year?  
   d. 2 years?  
   e. 5 years?

2. How much money must be invested at a simple interest rate of 12% to have $1000 at the end of seven years?

3. Sue invests $500 at a simple interest rate of 12%. How much interest will she earn after
   a. 2 months?  
   b. 6 months?  
   c. 1 year?  
   d. 2 years?

4. A $100 amount is invested at an interest rate of 8% compounded monthly. How much will be in the account after
   a. 2 months?  
   b. 6 months?  
   c. 1 year?  
   d. 2 years?  
   e. 5 years?
5. How much money must be invested at an interest rate of 12% compounded monthly to have $10,00 at the end of six years?

6. If interest is compounded monthly, how many periods are there in
   a. 3 months?
   b. 1 year?
   c. 5 years?

7. Compare the difference in simple interest from money invested at 8% and interest compounded monthly at 8% after five years.

8. Explain to a fifth grader the difference between simple interest and compound interest.

9. **Exploration:**

   How long will it take money to double at a compound interest rate of 12% compounded monthly? At 8%? Research and explain the Banker’s Rule of 72.
SECTION 13.2 MAKING UP A PERSONAL BUDGET

In order to keep a good credit history, it’s necessary to pay all bills on time. This can be a challenge because often people do not have enough money to buy everything that they want. And even if people have enough money at a given time, they might need money to pay future expenses, like a house or college. For example, on payday Frank receives a monthly check of $2500. Does he have enough money to buy a $500 i-Pad? To know this, Frank must look at his upcoming expenses and develop a plan for how he will use his income so that he doesn’t run out of money he needs for essentials, like food.

To answer any financial question, it is necessary to examine the parts of a family budget. First, start with income, both from wages and savings. Family members might also have other sources of income, for example presents from a birthday or miscellaneous income from a part-time job. Ask your parents if they can think of other income sources.

Next, consider the expenses that a family needs to plan for. First, there are probably mortgage payments for a house or monthly rent. Second, budget for food. Next, might be car expenses, like monthly car payments, car and house insurance, emergency funds, savings for retirement, and local, state, and federal taxes. Your parents might also be saving money to send you to college, as well as for a vacation.

Some of these expenses are fixed expenses, they do not change each month. For example, rent is a fixed monthly expense. Other expenses are variable. A variable expense is the grocery bill. To figure variable expenses, just estimate the average amount figured from previous expenses. If the January grocery bill totaled $120, $150 in February, and $60 in March, then a good estimate of average monthly food expenses is

\[
\frac{120 + 150 + 60}{3} = 110
\]
EXAMPLE 1

Sally made a list of her expenses and income. Her table is given below:

<table>
<thead>
<tr>
<th>Monthly Income:</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Wages from job</td>
<td>2500</td>
</tr>
<tr>
<td>Interest on Savings account</td>
<td>25</td>
</tr>
<tr>
<td>Weekend typing – self-employment income</td>
<td>200</td>
</tr>
<tr>
<td>TOTAL</td>
<td>2725</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Monthly Expenses</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Rent</td>
<td>750</td>
</tr>
<tr>
<td>Food</td>
<td>500</td>
</tr>
<tr>
<td>Car insurance</td>
<td>120</td>
</tr>
<tr>
<td>Car payment</td>
<td>300</td>
</tr>
<tr>
<td>House insurance</td>
<td>85</td>
</tr>
<tr>
<td>Clothes allowance</td>
<td>100</td>
</tr>
<tr>
<td>Savings for college</td>
<td>100</td>
</tr>
<tr>
<td>Savings for emergencies</td>
<td>50</td>
</tr>
<tr>
<td>Savings for trips</td>
<td>100</td>
</tr>
<tr>
<td>Savings for taxes (see tax table using her yearly income)</td>
<td></td>
</tr>
<tr>
<td>Savings for retirement</td>
<td>200</td>
</tr>
<tr>
<td>TV Cable and Phone</td>
<td>100</td>
</tr>
<tr>
<td>Cell Phone</td>
<td>100</td>
</tr>
<tr>
<td>Savings for car repairs</td>
<td>50</td>
</tr>
<tr>
<td>Gas</td>
<td>75</td>
</tr>
<tr>
<td>Savings for presents for family and friends</td>
<td>20</td>
</tr>
</tbody>
</table>

EXAMPLE 2

a. What are her total monthly expenses?

b. What percentage is each category of the total budget for expenses? Use a pie chart to get a visual understanding of her expenses.
Section 13.2 Making Up a Personal Budget

SOLUTION

a. To find her total expenses, add up all of the expenses above.

b. In order to determine the percentage of each category, divide the expenses in that category by the total expenses. For example, the percentage of expenses for clothes is \( \frac{100}{2680} = 0.0371 = 3.7\% \)

When a person makes up a budget, the expenses will depend on the local situation. For example, if the person lives in a small city, the rent might be less than in a larger city. The family expenses also depend on several variables, like the number and age of the children, whether they need to go to day care, educational expenses if they are in school, and extracurricular expenses if they play music or sports.

PROBLEM 1

Sally family includes her father, mother, and two brothers aged 9 and 12. She lives in San Marcos, a small town in central Texas.

a. Create a family budget, and estimate the minimum household budget to meet her family’s basic needs.

b. Estimate the average hourly wage that Sally’s parents need to make, working 40 hours per week, to meet basic needs.

Sally has a friend, Victoria, who lives in San Antonio, a larger town south of San Marcos. She also has five members in her family: her father, mother, and two sisters aged 5 and 15.

a. Create a family budget, and estimate the minimum household budget to meet her family’s basic needs.

b. Estimate the average hourly wage that Victoria’s parents need to make, working 40 hours per week, to meet basic needs.

Now that you have an idea what a budget might look like, it’s time to look at the overall financial situation. To do this, look at all assets and liabilities. An asset is something owned, like a house. A liability is money owed, like a mortgage.
If a person’s house is worth $150,000 then she has an asset of $150,000. However, if she still owes $90,000 to the bank on the house, then her house is also a liability of $90,000. The net worth of the house, $150,000 - $90,000, is $60,000. Although the house is a $150,000 asset, the person really owns only $60,000 of the value of the house. The remaining $90,000 is considered a liability.

**EXAMPLE 3**

Sally from Example 1 made the following list of all of her assets and liabilities.

<table>
<thead>
<tr>
<th>Assets:</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Car</td>
<td>$18,500</td>
</tr>
<tr>
<td>Clothes</td>
<td>$800</td>
</tr>
<tr>
<td>Television</td>
<td>$150</td>
</tr>
<tr>
<td>Piano</td>
<td>$900</td>
</tr>
<tr>
<td>Savings account</td>
<td>$1,200</td>
</tr>
<tr>
<td>Savings for College</td>
<td>$4,000</td>
</tr>
<tr>
<td>Retirement account</td>
<td>$25,300</td>
</tr>
<tr>
<td><strong>Total Assets:</strong></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Liabilities</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Outstanding car loan</td>
<td>$14,300</td>
</tr>
<tr>
<td>Credit Card Balance</td>
<td>$2,540</td>
</tr>
<tr>
<td><strong>Total Liabilities</strong></td>
<td></td>
</tr>
</tbody>
</table>

Organize the above items into a spreadsheet, like Excel or a graphing calculator. Use the spreadsheet’s formulas to compute Sally’s net worth.

\[
\text{Net Worth} = \text{Total Assets} - \text{Total Liabilities}
\]
Section 13.2 Making Up a Personal Budget

EXERCISES

1. Make a list of items that might be part of a personal budget. List at least three possible sources of income and ten different expenses that you need to be budgeted.

2. Total the income and expenses. Calculate the percentage of the total budget in each category.

3. List ten examples of assets a family might own, and choose a value for each.

4. Use these to create a table of assets. Then group appropriate assets into categories, like savings.

5. List ten examples of liabilities a family might owe, and estimate a reasonable value for each.

6. Use these to create a table of liabilities. Group appropriate liabilities into an overall category, like credit card debt if there are several types of credit cards.

7. Create a net worth statement by combining the two tables above. How much is the family worth in this example?

8. Estimate the average hourly wage, assuming a 40-hour work week, needed for the family to meet its basic needs, using exercises 1-7.

9. Suppose the family moves to another city. Explain how this might affect the budget.

10. Develop your own sample budget for your city. Make another budget for a large nearby city. Compare the average hourly wage needed for your family to meet its basic needs in each city.
SECTION 13.3 TAXES

Taxes are financial charges made by a governing body such as a city, state, or federal government on an individual or property. One example is the sales tax on items that you purchase. Another example is the income tax on money that you earn.

1. What are some reasons governments charge a tax?
2. How are the tax revenues used?
3. What are some differences between a sales tax and an income tax?
4. Determine what is the sales tax rate in your city. Are all sales tax rates the same in your state? Find two other tax rates. Are the sales tax rates the same in Illinois? Compare the tax rate in Chicago with your city. How much would a shirt costing $29.99 cost with tax in Chicago. How much would the same shirt cost in your city?
5. Notice that the income tax is a tax on what you earn while a sales tax is a tax on what you buy. Investigate the income tax rates for a wage earner. You may need to consult the Internal Revenue Office Tax Table at:

Here is the expected 2012 Federal income tax brackets according to the Forbes website:


<table>
<thead>
<tr>
<th>Tax Bracket</th>
<th>Married Filing Jointly</th>
<th>Single</th>
</tr>
</thead>
<tbody>
<tr>
<td>10% Bracket</td>
<td>$0 - $17,400</td>
<td>$0 - $8,700</td>
</tr>
<tr>
<td>15% Bracket</td>
<td>$17,400 - $70,700</td>
<td>$8,700 - $35,350</td>
</tr>
<tr>
<td>25% Bracket</td>
<td>$70,700 - $142,700</td>
<td>$35,350 - $85,650</td>
</tr>
<tr>
<td>28% Bracket</td>
<td>$142,700 - $217,450</td>
<td>$85,650 - $178,650</td>
</tr>
<tr>
<td>33% Bracket</td>
<td>$217,450 - $388,350</td>
<td>$178,650 - $388,350</td>
</tr>
<tr>
<td>35% Bracket</td>
<td>Over $388,350</td>
<td>Over $388,350</td>
</tr>
</tbody>
</table>
a. Suppose you are a single worker and earned $34,500 in 2012. According to the information in the above table, how much can you expect to pay in taxes?

b. Suppose you are a married worker and earned $73,000 in 2012. According to the information in the above table, how much can you expect to pay in taxes, if you file jointly with your partner?

c. Your coworker is single and earned the same amount of $73,000 in 2012. According to the information in the above table, how much can she expect to pay in taxes?

Find a website, for example, http://www.irs.gov/pub/irs-pdf/i1040tt.pdf, you will find two ways that explains how to calculate taxes. Notice that tax filers can either use a tax table or a tax rate schedule. For either method, the first step involves deciding which category the filer is in: single, married filing jointly, married filing separately, or head of a household. The other component is the filer’s net income.

**EXAMPLE 1**

Susan has a net income of $69,500 and is single. Sarah and Bob are married and have a joint income of $69,500. Compute the tax they owe with a tax table or with a tax rate schedule. Which method is easier?

**SOLUTION 1**

Use a tax table. Find the row of the tax table that includes $69,500. Then locate the correct column to find the taxes owed. The table looks like:

<table>
<thead>
<tr>
<th>If line 43 is—</th>
<th>And you are —</th>
</tr>
</thead>
<tbody>
<tr>
<td>At least $69,000</td>
<td>But less than</td>
</tr>
<tr>
<td>Single</td>
<td>Married filing jointly</td>
</tr>
<tr>
<td>$69,000</td>
<td>13,286</td>
</tr>
<tr>
<td>69,050</td>
<td>13,299</td>
</tr>
<tr>
<td>69,100</td>
<td>13,311</td>
</tr>
<tr>
<td>69,150</td>
<td>13,324</td>
</tr>
<tr>
<td>69,200</td>
<td>13,336</td>
</tr>
</tbody>
</table>
If line 43 is—  
And you are —

<table>
<thead>
<tr>
<th>At least</th>
<th>But less than</th>
<th>Single</th>
<th>Married filing jointly</th>
<th>Married filing separately</th>
<th>Head of Household</th>
</tr>
</thead>
<tbody>
<tr>
<td>69,250</td>
<td>69,300</td>
<td>13,349</td>
<td>9,521</td>
<td>13,349</td>
<td>11,964</td>
</tr>
<tr>
<td>69,300</td>
<td>69,350</td>
<td>13,361</td>
<td>9,529</td>
<td>13,361</td>
<td>11,976</td>
</tr>
<tr>
<td>69,350</td>
<td>69,400</td>
<td>13,374</td>
<td>9,536</td>
<td>13,374</td>
<td>11,989</td>
</tr>
<tr>
<td>69,400</td>
<td>69,450</td>
<td>13,386</td>
<td>9,544</td>
<td>13,386</td>
<td>12,001</td>
</tr>
<tr>
<td>69,450</td>
<td>69,500</td>
<td>13,399</td>
<td>9,551</td>
<td>13,399</td>
<td>12,014</td>
</tr>
<tr>
<td>69,500</td>
<td>69,550</td>
<td>13,411</td>
<td>9,559</td>
<td>13,411</td>
<td>12,026</td>
</tr>
<tr>
<td>69,550</td>
<td>69,600</td>
<td>13,424</td>
<td>9,566</td>
<td>13,424</td>
<td>12,039</td>
</tr>
<tr>
<td>69,600</td>
<td>69,650</td>
<td>13,436</td>
<td>9,574</td>
<td>13,436</td>
<td>12,051</td>
</tr>
<tr>
<td>69,650</td>
<td>69,700</td>
<td>13,449</td>
<td>9,581</td>
<td>13,449</td>
<td>12,064</td>
</tr>
<tr>
<td>69,700</td>
<td>69,750</td>
<td>13,461</td>
<td>9,589</td>
<td>13,461</td>
<td>12,076</td>
</tr>
<tr>
<td>69,750</td>
<td>69,800</td>
<td>13,474</td>
<td>9,596</td>
<td>13,474</td>
<td>12,089</td>
</tr>
<tr>
<td>69,800</td>
<td>69,850</td>
<td>13,486</td>
<td>9,604</td>
<td>13,486</td>
<td>12,101</td>
</tr>
<tr>
<td>69,850</td>
<td>69,900</td>
<td>13,499</td>
<td>9,611</td>
<td>13,499</td>
<td>12,114</td>
</tr>
<tr>
<td>69,900</td>
<td>69,950</td>
<td>13,511</td>
<td>9,619</td>
<td>13,511</td>
<td>12,126</td>
</tr>
<tr>
<td>69,950</td>
<td>70,000</td>
<td>13,524</td>
<td>9,626</td>
<td>13,524</td>
<td>12,139</td>
</tr>
</tbody>
</table>

From this table and using the first column, Single, Susan should pay $13,411 in federal income taxes. Sarah and Bob use the married column to find that they owe $9,559.

Why do you think that Sarah and Bob have less to pay in taxes? The main reason is that married people together generally have more expenses than a single person, so they need more after-tax income so they’re charged a little less.
EXAMPLE 2

Using Schedule X and Schedule Y, explain how 1) Susan and 2) Sarah and Bob figure their taxes.

SOLUTION 2

Use a tax rate schedule. Susan will use Schedule X for filing status Singe, and Sarah and Bob will use Schedule Y-1.

<table>
<thead>
<tr>
<th>Schedule X</th>
</tr>
</thead>
<tbody>
<tr>
<td>If your taxable income is:</td>
</tr>
<tr>
<td>Over —</td>
</tr>
<tr>
<td>$0</td>
</tr>
<tr>
<td>8,700</td>
</tr>
<tr>
<td>35,350</td>
</tr>
<tr>
<td>85,650</td>
</tr>
<tr>
<td>178,650</td>
</tr>
<tr>
<td>388,350</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Schedule Y -1</th>
</tr>
</thead>
<tbody>
<tr>
<td>If your taxable income is:</td>
</tr>
<tr>
<td>Over —</td>
</tr>
<tr>
<td>$0</td>
</tr>
<tr>
<td>17,400</td>
</tr>
<tr>
<td>70,700</td>
</tr>
<tr>
<td>142,700</td>
</tr>
<tr>
<td>217,450</td>
</tr>
<tr>
<td>388,350</td>
</tr>
</tbody>
</table>

There are different income brackets in each schedule.

First compute Susan’s taxes with Schedule Y.
In the 10% bracket, for an income up to 8,700, 10% of net income.
In the 15% bracket, pay 10% of the first $8,700 of net income, then 15% of the amount above $8,700, but less than $35,350.
In the 25% bracket, pay the same taxes as the lower bracket on net income to $35,350, and then 25% of the income between $35,350 and $85,650.

Using Schedule Y, Susan will owe $4,867.50 + 25% of the amount between $35,350 and $69,500. The amount between $69,500 and $35,350 is $69,500 - $35,350 = $34,150.

Compute 25% of the amount to obtain

\[(0.25)(34,150) = 6,537.50.\]

Then add the original $4,867.50 to obtain $13,405, almost exactly the same amount as the tax table amount of $13,411.

Now compute Sarah and Bob’s joint taxes using Schedule Y-1.
In the 10% bracket, for an income up to 17,400, 10% of net earned income.

In the 15% bracket, pay 10% of the first $17,400, then 15% of the amount above $17,400, but less than $70,400. The amount that Sarah and Bob will pay 15% on is $69,500 - $17,400 = $52,100. Together the tax bill is $1,740 + (0.15) ($52,100) = $9,555. Again, this is almost exactly the same as the tax table result of $9,559.

EXERCISES

1. John earned $69,900 in 2012. How much did he owe, using the tax table, if he was single?

2. Compute the amount of tax John owed using Schedule Y. Show your work.

3. Sally and Bob filed a joint return and earned a net income of $69,900 in 2012. How much did they owe using the tax table?


5. If you earned $100,000 in 2012, what tax bracket will you be in? Using Schedule Y-1, how much did you owe in federal income taxes?

6. If you earned $100,000, would you pay 25% taxes on all of you net income? Explain.
7. Make a list of items that might be part of a personal budget. List at least three possible sources of income and ten different expenses that you need to be budgeted.

8. Total the income and expenses. Calculate the percentage of the total budget in each category.

9. List ten examples of assets a family might own, and choose a value for each.

10. Use these to create a table of assets. Then group appropriate assets into categories, like savings.

11. List ten examples of liabilities a family might owe, and estimate a reasonable value for each.

12. Use these to create a table of liabilities. Group appropriate liabilities into an overall category, like credit card debt if there are several types of credit cards.

13. Create a net worth statement by combining the two tables above. How much is the family worth in this example?

14. Estimate the average hourly wage, assuming a 40-hour work week, needed for the family to meet its basic needs, using exercises 1-7.

15. Suppose the family moves to another city. Explain how this might affect the budget.
Absolute value — 1. The absolute value of a number is its distance from zero.
   - 2. For any x, |x| is defined as follows: |x| = x, if x ≥ 0
       |x| = -x, if x < 0

Acute angle — An angle whose measure is greater than 0 degrees and less than 90 degrees.

Acute triangle — A triangle in which all three angles are acute angles.

Altitude of a triangle — A segment drawn from a vertex of the triangle perpendicular to the opposite side of the triangle, called the base, (or perpendicular to an extension of the base).

Angle — An angle is formed when two rays share a common vertex.

Area model — A mathematical model based on the area of a rectangle, used to represent multiplication or to represent fractional parts of a whole.

Arithmetic sequence — A sequence a₁, a₂, a₃, a₄,... is an arithmetic sequence if there is a number c such that for each n, aₙ₊₁ = aₙ + c, that is aₙ₊₁ − aₙ = c.

Attribute — A distinguishing characteristic of an object such as angles or sides of a triangle.

Axis — A number line in a plane. Plural form is axes. Also see: Coordinate Plane
Bar graph – A graph in which rectangular bars, either vertical or horizontal, are used to display data.

Base – 1. Any number \( x \) is raised to the \( n \)th power, written as \( x^n \), \( x \) is called the base of the expression; 2. Any side of a triangle; 3. Either of the parallel sides of a trapezoid. 4. Either of the parallel sides of a parallelogram.

Box and Whisker Plot – For data ordered smallest to largest the median, lower quartile and upper quartile are found and displayed in a box along a number line. Whiskers are added to the right and left and extended to the least and greatest values of the data.

Cartesian coordinate system – See: Coordinate Plane

Center of a circle – A point in the interior of the circle that is equidistant from all points of the circle.

Chord – A segment whose endpoints are points of a circle.

Circle – The set of points in a plane equidistant from a point in the plane.

Circumference – The distance around a circle. Its length is the product of the diameter of the circle and \( \pi \).
Coefficient – In the product of a constant and a variable the constant is the numerical coefficient of the variable and is frequently referred to simply as the coefficient.

Common Denominator – A common multiple of the denominators of two or more fractions. Also see: Least Common Denominator

Common Factor – A factor that two or more integers have in common. Also see: Greatest Common Factor.

Common Multiple – See: Least Common Multiple.

Complement – The complement of a set is a set of all the elements of the universal set that are not in the given set.

Complementary Angles – Two angles are complementary if the sum of their measures totals 90°.

Composite number – A prime number is an integer p greater than 1 with exactly two positive factors: 1 and p. A composite number is an integer greater than 1 that has more than two positive factors. The number 1 is neither a prime nor a composite number.

Concentric circles – Circles with the same center and in the same plane that have different radii.
**Cone** – A three-dimensional figure with a circular base joined to a point called the apex.

**Congruent** – Used to refer to angles or sides having the same measure and to polygons that have the same shape and size.

**Conjecture** – An assumption that is thought to be true based on observations.

**Constant** – A fixed value.

**Constant of Proportionality** – If a function has a rule in the form \( y = Kx \), then for any input \( x \neq 0 \), the quotient of \( \frac{y}{x} \) will always have the value \( K \). The number \( K \) is called **constant of proportionality**.

**Coordinate(s)** – A number assigned to each point on the number line which shows its position or location on the line. In a coordinate plane that ordered pair, \((x, y)\), assigned to each point of the plane showing its position in relation to the \( x \)-axis and \( y \)-axis.

**Coordinate Plane** – A plane that consists of a horizontal and vertical number line, intersecting at right angles at their origins. The number lines, called axes, divide the plane into four **quadrants**. The quadrants are numbered I, II, III, and IV beginning in the upper right quadrant and moving counterclockwise.
**Corresponding Angles** — 1. If two lines are cut by a transversal the angles on the same side of the transversal and on the same side of the two lines are corresponding angles. If the lines are parallel the pairs of angles will have equal measure. 2. If two polygons are similar the angles that are in the same relative position in the figures are corresponding angles and have equal measures.

**Corresponding Sides** — If two polygons are similar the sides of the polygons in the same relative positions are corresponding sides and the ratio of the lengths of each pair is the same.

**Counterclockwise** — A circular movement opposite to the direction of the movement of the hands of a clock.

**Counting numbers** — The *counting numbers* are the numbers in the following never-ending sequence: 1, 2, 3, 4, 5, 6, 7... We can also write this as +1, +2, +3, +4, +5, +6, +7,... These numbers are also called the *positive integers* or *natural numbers*.

**Cube** — 1. A three-dimensional shape having six congruent square faces. 2. The third power of a number.

**Cylinder** — A three-dimensional figure with parallel circular bases of equal size joined by a lateral surface whose net is a rectangle.

**Data** — A collection of information, frequently in the form of numbers.
Data analysis – The process of making sense of collected data.

Degree – 1. The circumference of a circle is divided into 360 equal parts or arcs. Radii drawn to both ends of the arc form an angle of 1 degree. 2. The degree of a term is the sum of the exponents of the variables. The degree of a polynomial is the highest degree of any of its terms.

Denominator – The denominator of a fraction indicates how many equal parts the whole is divided. The denominator appears beneath the fraction bar.

Diameter – A segment with endpoints on the circle that passes through its center.

Distance – For any two numbers \( x \) and \( y \), the distance between \( x \) and \( y \) is the absolute value of their difference; that is, \( \text{Distance} = |x - y| \).

Dividend – The quantity that is to be divided.

Divisibility – Suppose that \( n \) and \( d \) are integers, and that \( d \) is not 0. The number \( n \) is divisible by \( d \) if there is an integer \( q \) such that \( n = dq \). Equivalently, \( d \) is a factor of \( n \) or \( n \) is a multiple of \( d \).

Division Algorithm – Given two positive integers \( a \) and \( b \), we can always find unique integers \( q \) and \( r \) such that \( a = bq + r \) and \( 0 \leq r < b \). We call \( a \) the dividend, \( b \) the divisor, \( q \) the quotient, and \( r \) the remainder.
Divisor – The quantity by which the dividend is divided.

Domain – The set of input values in a function.

Edge – A segment that joins consecutive vertices of a polygon or a polyhedron.

Elements – Members of a set.

Empirical probability – Probability determined by real data collected from real experiments.

Empty set – Also called a Null Set. A set that has no elements.

Equation – A math sentence using the equal sign to state that two expressions represent the same number.

Equilateral triangle – An equilateral triangle is a triangle with three congruent sides. An equilateral triangle also has three congruent angles.

Equivalent – 1. A term used to describe equations or inequalities that have the same solution. 2. A term used to describe fractions or ratios that are equal. 3. A term used to describe fractions, decimals and percents that are equal.
**Event** – An event is any subset of the sample space. A **simple event** is a subset of the sample space containing only one possible outcome of an experiment. A **compound event** is a subset of the sample space containing two or more outcomes.

**Experiment** – A repeatable action with a set of outcomes.

**Exponent** – Suppose that n is a whole number. Then, for any number x, the n\textsuperscript{th} power of x, or x to the n\textsuperscript{th} power, is the product of n factors of the number x. This number is usually written x\textsuperscript{n}. The number x is usually called the base of the expression x\textsuperscript{n}, and n is called the **exponent**.

**Exponential Notation** – A notation that expresses a number in terms of a base and an exponent.

**Face** – Each of the surface polygons that form a polyhedron.

**Factor** – An integer that divides evenly into a dividend. Use interchangeably with divisor except in the **Division Algorithm**.

**Factorial** – The factorial of a non-negative number n is written n! and is the product of all positive integers less than or equal to n. By definition 0! = 1! = 1.

**Frequency** – The number of times a data point appears in a data set.
Function – A function is a rule which assigns to each member of a set of inputs, called the domain, a member of a set of outputs, called the range.

Graph of a function – The pictorial representation of a function.

Greater than, Less Than – Suppose that x and y are integers. We say that x is less than y, $x < y$, if x is to the left of y on the number line. We say that x is greater than y, $x > y$, if x is to the right of y on the number line.

Greatest common factor, GCF – Suppose m and n are positive integers. An integer d is a common factor of m and n if d is a factor of both m and n. The greatest common factor, or GCF, of m and n is the greatest positive integer that is a factor of both m and n. We write the GCF of m and n as GCF (m,n).

Height – The length of the perpendicular between the bases of a parallelogram or trapezoid; also the altitude of a triangle.

Horizontal axis – See Coordinate Plane.

Hypotenuse – The side opposite the right angle in a right triangle.

Improper fraction – A fraction in which the numerator is greater than or equal to the denominator.
Independent events – If the outcome of the first event does not affect the outcome of the second event.

Input values – The values of the domain of a function.

Integers – The collection of integers is composed of the counting numbers, their negatives, and zero; ... -4, -3, -2, -1, 0, 1, 2, 3, 4...

Intersection of sets – A set whose elements are all the elements that the given sets have in common, written $A \cap B$.

Irregular polygon – A polygon that is not a regular polygon.

Isosceles triangle – A triangle with at least two sides of equal length is called an isosceles triangle.

Lateral Area – The surface area of any three-dimensional figure excluding the area of any surface designated as a base of the figure.

Lattice point – A point of the coordinate plane, $(x,y)$, in which both $x$ and $y$ are integers.

Least Common Denominator – The least common denominator of the fractions $\frac{a}{n}$ and $\frac{b}{m}$ is $\text{LCM}(n, m)$. 
**Least common multiple, LCM** – The integers $a$ and $b$ are positive. An integer $m$ is a **common multiple** of $a$ and $b$ if $m$ is a multiple of both $a$ and $b$. The **least common multiple**, or **LCM**, of $a$ and $b$ is the smallest integer that is a common multiple of $a$ and $b$. We write the LCM of $a$ and $b$ as $\text{LCM}(a,b)$.

**Legs** – 1. The two sides of a right triangle that form the right angle. 2. The equal sides of an isosceles triangle or the non-parallel sides of a trapezoid.

**Less than** – See: **Greater Than**.

**Line graph** – A graph used to display data that occurs in a sequence. Consecutive points are connected by segments.

**Line Plot** – A graph that shows frequency of data along a number line.

**Line of symmetry** – Line $L$ is a line of symmetry for a figure if for every point $P$ of the figure there is a point $Q$ of the figure so that $L$ is the perpendicular bisector of segment $PQ$.

**Linear Model for Multiplication** – Skip counting on a number line

**Magnitude** – The **absolute value** of a number; its distance from zero.

**Mean** – The average of a set of data; sum of the data divided by the number of items.
Measures of central tendency – Generally measured by the mean, median or mode of the data set.

Median – The middle value of a set of data arranged in increasing or decreasing order. If the set has an even number of items the median is the average of the middle two items.

Missing Factor Model – A model for division in which the quotient of an indicated division is viewed as a missing factor of a related multiplication.

Mixed fraction – The sum of an integer and a proper fraction.

Mode – The value of the element that appears most frequently in a data set.

Multiplicity – The number of times a factor appears in a factorization.

Natural numbers – See: Counting Numbers

Negative integers – Integers less than zero.

Nets – One way to see the surface area of a three dimensional figure by cutting along its edges to produce a two dimensional figure.
Notation – A technical system of symbols used to convey mathematical information.

Null set – See: Empty Set.

Numerator – The expression written above the fraction bar in a common fraction to indicate the number of parts counted.

Obtuse Angle – An angle whose measure is greater than 90 degrees and less than 180 degrees.

Obtuse Triangle – A triangle that has one obtuse angle.

Ordered pair – A pair of numbers that represent the coordinates of a point in the coordinate plane with the first number measured along the horizontal scale and the second along the vertical scale.

Origin – The point with coordinate 0 on a number line; the point with coordinates (0,0) in the coordinate plane.

Outcomes – The set of possible results of an experiment.

Outlier – A term referring to a value that is drastically different from most of the other data values.
Output Values – The set of results obtained by applying a function rule to a set of input values.

Parallel lines – Two lines in a plane that never intersect.

Parallelogram – A parallelogram is a four-sided figure with opposite sides parallel.

Percent – A way of expressing a number as parts out of 100; the numerator of a ratio with a denominator of 100.

Perfect Cube – An integer \( n \) that can be written in the form \( n = k^3 \), where \( k \) is an integer.

Perfect Square – An integer \( n \) that can be written in the form \( n = k^2 \), where \( k \) is an integer.

Perimeter – The perimeter of a polygon is the sum of the lengths of its sides.

Perpendicular – Two lines or segments are perpendicular if they intersect to form a right angle.

Pi – The ratio of the circumference to the diameter of any circle, represented either by the symbol \( \pi \), or the approximation \( \frac{22}{7} \) or 3.1415926...
**Pie graph** – A graph using sectors of a circle that are proportional to the percent of the data represented.

**Polygon** – A polygon is a simple, closed, plain figure formed by three or more line segments.

**Polyhedron** – A three-dimensional figure with four or more faces all of which are polygons.

**Positive integers** – See: Counting Numbers

**Power** – See: Exponent

**Prime Number** – See: Composite Number

**Prime Factorization** – The process of finding the prime factors of an integer. The term is also used to refer to the result of the process.

**Prism** – A type of polyhedron that has two bases that are both congruent and parallel, and lateral faces which are parallelograms.

**Probability** – In an experiment in which each outcome is equally likely, the probability $P(A)$ of an event $A$ is $\frac{m}{n}$ where $m$ is the number of outcomes in the subset $A$ and $n$ is the total number of outcomes in the sample space $S$. 
Proper fraction – A fraction whose value is greater than 0 and less than 1.

Proportion – An equation of ratios in the form $\frac{a}{b} = \frac{c}{d}$, where $b$ and $d$ are not equal to zero.

Protractor – An instrument used to measure angles in degrees.

Pyramid – A type of polyhedron that has one face, called a base, and triangular lateral faces that meet at a point called the apex.

Pythagorean Theorem – The formula that states that if $a$ and $b$ are the lengths of the legs of a right triangle and $c$ is the length of the hypotenuse, then $a^2 + b^2 = c^2$.

Quadrant – See Coordinate Plane.

Quotient – The result obtained by doing division. See the Division Algorithm for a different use of quotient.

Radius – The distance from the center of a circle a point of the circle. Plural form is radii.

Range – The difference between the largest and smallest values of a data set. See: Function for another meaning of range.
Rate — A rate is a division comparison between two quantities with different units. Also see Unit Rate.

Ratio — A division comparison of two quantities with or without the same units. If the units are different they must be expressed to make the ratio meaningful.

Rational Number — A number that can be written as \( \frac{a}{b} \) where \( a \) is an integer and \( b \) is a natural number.

Ray — Part of a line that has a starting point and continues forever in only one direction.

Reciprocal — See: Multiplicative Inverse.

Reflection — The transformation that moves points or shapes by “flipping” them across a line or axis; a mirror image of the original set of points. If \( B \) is the reflection of \( A \) in line \( L \), then \( L \) is the perpendicular bisector of segment \( AB \).

Regular Polygon — A polygon with equal length sides and equal angle measures.

Relatively Prime — Two integers \( m \) and \( n \) are relatively prime if the GCF of \( m \) and \( n \) is 1.

Remainder — See: Division Algorithm.
Repeating decimal – A decimal in which a cycle of one or more digits is repeated infinitely.

Right Angle – An angle formed by the intersection perpendicular lines; an angle with a measure of 90°.

Right Triangle – A triangle that has a right angle.

Sample Space – The set of all possible outcomes of an experiment.

Scaffolding – A method of division in which partial quotients are computed, stacked and then combined.

Scale Factor – If polygons A and B are similar and s is a positive number so that for each side of A with length k there is a corresponding side of B with length sk, then s is the scale factor of A to B.

Scalene Triangle – A triangle with all three sides of different lengths is called a scalene triangle.

Scaling – 1. A process by which a shape is reduced or expanded proportionally. 2. Choosing the unit of measure to be used on a number line.

Sector – A region of a circle bounded by two radii and an arc of the circle which joins their endpoints.
Sequence – A list of terms ordered by the natural numbers. The outputs of a function whose domain is the natural numbers or whole numbers.

Set – A collection of objects or elements.

Similar polygons – Two polygons whose corresponding angles have equal measures and whose corresponding side lengths form equal ratios.

Simple event – See: Event.

Simplest Form of a Fraction – A form in which the greatest common factor of the numerator and denominator is 1.

Simplifying – The process of finding equivalent fractions to obtain its simplest form.

Skewed – An uneven representation of a set of data.

Slant Height – An altitude of a face of a pyramid or a cone.

Square Root – For non-negative numbers x and y, \( y = \sqrt{x} \), read “y is equal to the square root of x”, means \( y^2 = x \).
Stem and Leaf Plot – A method of showing the frequency of a certain data by sorting and ordering the values.

Straight Angle – An angle with a measure of 180 degrees formed by opposite rays.

Subset – Set B is a subset of set A if every element of set B is also an element of set A.

Supplementary Angles – Two angles are supplementary if the sum of their measures totals 180°.

Surface Area – The total area of all the faces of a polyhedron. The total of the lateral area and base area of a cone. The total of the lateral area and the two bases of a cylinder.

Term – 1. Each member of a sequence. 2. Each expression in a polynomial separated by addition and subtraction signs.

Terminating Decimal – If a and b are natural numbers with b ≠ 0 and a ÷ b yields a finite quotient, the decimal formed is a terminating decimal.

Theoretical probability – Probability based on mathematical law rather than a collection of data.
Translation – A transformation that moves a figure along a line in a plane but does not alter its size or shape.

Transversal – Any line that intersects two or more lines at different points.

Trapezoid – A four sided plane figure with exactly one set of parallel sides.

Tree diagram – 1. A process used to find the prime factors on an integer. 2. A method to organize the sample space of compound events.

Trichotomy – A property stating that exactly one of these statements is true for each real number: it is positive, negative or zero.

Union of Two Sets – A set that contains all of the elements that appear in either of the given sets, written A U B.

Unit Rate – A ratio of two unlike quantities that has a denominator of 1 unit.

Universal Set – A set containing all of the elements under consideration.

Variable – A letter or symbol that represents an unknown quantity.

Venn diagram – A diagram involving two or more overlapping circles that aids in organizing data.
**Vertex** — 1. The common endpoint of two rays forming an angle. 2. A point of a polygon or polyhedron where edges meet.

**Vertical angles** — If two straight lines intersect at a point, then each line is divided into two rays. The angles formed by using opposite rays from each line are called vertical angles.

**Vertical angle theorem** — If two lines intersect at a point $P$, then the vertical angles formed will always have the same measure.

**Vertical Axis** — See: Coordinate Plane.

**Volume** — A measure of space; the number of unit cubes needed to fill a three-dimensional shape.

**Whole numbers** — The whole numbers are the numbers in the following never-ending sequence: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ....

**x-axis** — The horizontal axis of a coordinate plane.

**y-axis** — The vertical axis of a coordinate plane.

**zero pair** — For any natural number $n$, $n + (-n)$ is called a zero pair because their sum is zero.
Summary of Important Ideas

Additive Property of Equality – If $A = B$, then $A + C = B + C$.

Additive Identity – For any number $x$, $x + 0 = x$.

Additive Inverses – For any number $x$, there exists a number $-x$, called the additive inverse of $x$, such that $x + (-x) = 0$.

Area of a Circle – The area of a circle with radius $r$ is $A = r^2$ square units.

Area of a Parallelogram – The area of a parallelogram with base $b$ and height $h$ is given by $A = bh$.

Area of a Triangle – The area of a triangle with base $b$ and height $h$ is given by $A = \frac{1}{2}bh$ or $A = \frac{bh}{2}$.

Associative Property of Addition – For any numbers $x$, $y$, and $z$, $(x + y) + z = x + (y + z)$.

Associative Property of Multiplication – For any numbers $x$, $y$, and $z$, $(xy)z = x(yz)$.

Commutative Property of Addition – For any numbers $x$ and $y$, $x + y = y + x$. 
Commutative Property of Multiplication – For any numbers A and B, $AB = BA$.

Corresponding Angle Postulate – If two parallel lines are cut by a transversal, then the corresponding angles have the same measure, and if two lines are cut by a transversal so that the corresponding angles have the same measure, then the two lines are parallel.

Distributive Property of Multiplication Over Addition – Given two positive integers $n$ and $d$, we can always find unique integers $q$ and $r$ such that $n = dq + r$ and $0 \leq r < b$. We call $n$ the dividend, $d$ the divisor, $q$ the quotient and $r$ the remainder.

Division Rules – 1) If dividend and divisor have different signs, (one positive, other negative), the quotient is negative. 2) If the dividend and the divisor have the same sign, (both positive or both negative), the quotient is positive.

Double Opposites Theorem – For any number $x$, $-(-x) = x$.

Division Algorithm – Given two positive integers $a$ and $b$, we can always find unique integers $q$ and $r$ such that $a = bq + r$ and $0 \leq r < b$. We call $a$ the dividend, $b$ the divisor, $q$ the quotient, and $r$ the remainder.

Equivalent Fraction Property – For any number $a$ and nonzero numbers $k$ and $b$, $\frac{a}{b} = \frac{ka}{kb} = \frac{ak}{bk}$.
Fractions and Division – For any number \( m \) and nonzero \( n \) the fraction \( \frac{m}{n} \) is equivalent to the quotient \( \frac{m}{n} \).

Fundamental Theorem of Arithmetic– If \( n \) is a positive integer, \( n > 1 \), then \( n \) is either prime or can be written as a product of primes \( n = p_1 \cdot p_2 \cdot \ldots \cdot p_k \) for some prime numbers \( p_1, p_2, \ldots, p_k \) such that \( p_1 \leq p_2 \leq \ldots \leq p_k \) where \( k \) is a natural number. In fact, there is only one way to write \( n \) in this form.

Multiplication of Powers – Suppose that \( x \) is a number and \( a \) and \( b \) are whole numbers, then \( x^a \cdot x^b = x^{a + b} \).

Multiplicative Identity – The number 1 is the multiplicative identity; that is, for any number \( n \), \( n \cdot 1 = n \).

Multiplicative Inverse – For every non-zero \( x \) there exists a number \( \frac{1}{x} \), called the multiplicative inverse or reciprocal of \( x \), such that \( x \cdot \frac{1}{x} = 1 \).

Multiplying Fractions – The product of two fractions \( \frac{a}{b} \) and \( \frac{c}{d} \), where \( b \) and \( d \) are nonzero, is \( \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \).

Subtraction Property of Equality – If \( A = B \), then \( A - C = B - C \).

Sums With Like Denominators – The sum of two fractions with like denominators, \( \frac{a}{n} \) and \( \frac{b}{n} \), is given by \( \frac{a}{n} + \frac{b}{n} = \frac{a + b}{n} \).
Surface Area of a Cube — The surface area, $SA$, of a cube is given by the formula, $SA = 6s^2$ where $s$ is the length of a side.

Surface Area of a Cylinder — The total surface, $SA$, of a cylinder is the sum of the areas of the bases and the middle section, (lateral area), given by: $SA = 2B + Ph = 2\pi r^2 + 2\pi rh$, where $B$ is the area of the base, $P$ is the perimeter, $h$ is the height, and $r$ is the radius.

Surface Area of a Rectangular Prism — The surface area, $SA$, of a rectangular prism is given by the formula, $SA = 2B + Ph$ where $B$ is the area of a base, $P$ is the perimeter of the base rectangle and $h$ is the height of the prism.

The Rule of Products — If one action can be performed in $m$ ways and a second independent action can be performed in $n$ ways, then there are $m \cdot n$ possible ways to perform both actions.

The Rule of Sums — If one action can be performed in $m$ ways and a second action can be performed in $n$ ways, then there are $(m + n)$ ways to perform one action or the other, but not both. This assumes that each action is equally likely and mutually exclusive.

Triangle Similarity Theorem — If two triangles have the same angle measures they are similar and the ratios of their corresponding sides are equal. Conversely, if two triangles have sides with the same ratio they are similar and their corresponding angles have equal measure.
Triangle Sum Theorem – The sum of the measures of the angles of a triangle equals $180^\circ$.

Unit Fraction – For any positive integer $n$, the multiplicative inverse or reciprocal of $n$ is the unit fraction $\frac{1}{n}$.

Vertical Angle Theorem – If two lines intersect at a point $P$, then the vertical angles will have the same measure.

Volume of a Cube – The volume of a cube with each side of length $s$ units is $s^3$ cubic units. $V = s^3$ or $V = Bh$ where $B$ is the base area of the three-dimensional figure and $h$ is the height.

Volume of a Cylinder – The volume of a cylinder with radius $r$ and height $h$ is given by $V = Bh = \pi r^2h$.

Volume of a Prism – The volume of a prism is the area of the base of the three-dimensional figure multiplied by the height of the prism. $V = Bh$ where $B$ is the base area.
**Absolute Value/ Valor Absoluto**: La distancia de un número a cero. Por cualquier valor \( x \), es definido como:

\[
|x| = \begin{cases} 
  x, & \text{si } x \geq 0 \\
  -x, & \text{si } x < 0 
\end{cases}
\]

**Acute Angle/ Angulo Agudo**: Un ángulo de medida mayor de 0 grados y menor de 90 grados.

**Acute Triangle/ Triángulo Agudo**: Un triángulo en el cual los tres ángulos son agudos.

**Altitude of a Triangle/ Altura de un Triángulo**: un segmento dibujado de la vertice de el triángulo perpendicular al lado opuesto, llamado la base, (o perpendicular a una extension de la base).

**Angle/ Angulo**: Un ángulo es formado cuando dos semirectas tienen el mismo extremo llamado vertice.

**Area Model/ Modelo Basado en la Area**: Un Modelo matemático basado en la area de un rectángulo, usado para representar dos números multiplicados ya sean enteros o quebrados (fracciones.)

**Arithmetic Sequence/ Sucesión Aritmetica**: Una sucesión \( a_1, a_2, a_3, a_4, \ldots \) es aritmética si existe un número \( c \) de tal manera que por cada \( n \), \( a_{n+1} = a_n + c \), y \( a_{n+1} - a_n = c \).
Attribute/ Atributo: Una característica de un objeto tal como angulos o lados de un triangulo.


Bar Graph: Grafica de Barras: Una grafica en la cual barras rectangulares, ya sean horizontales o verticales, se usan para mostrar los datos.

Base/ Base: 1. Por cualquier valor x, elevado a la potencia de n, escrito como xn, x es llamado la base de la expresión; 2. Cualquier lado de un triangulo; 3. Cualquiera de los lados paralelos de un trapezoide; 4. Cualquiera de los lados paralelos de un paralelogramo.

Box and Whisker Plot/ Digrarna de Cajas y Brazos: Para datos ordenados de mas chico a mas grande, la mediana, y el cuartile bajo y cuartile alto son mostrados en cajas sobre una recta numerica. Los brazos muestran los extremos de los datos.

Cartesian Plane/ Plano Cartesiano: ver Coordinate Plane

Center of a Circle/ Centro del Circulo: Un punto en el interior del circulo que queda a la misma distancia de cualquier punto el la periferia del circulo.

Chord/ Cuerda: Un segmento que une dos puntos el la periferia del circulo.
Circle/ Circulo: Un conjunto de puntos en un plano de igual distancia de un punto en el plano.

Circumference/ Circumferencia: La distancia de la periferia del círculo. La medida es el producto de el diámetro del círculo y π.

Coefficient/ Coeficiente: En el producto de un constante y un variable, el constante es el factor numérico de el término, y es referido como el coeficiente.

Common Denominator/ Denominador Comun: Un múltiple común de los denominadores de dos o mas fracciones. Ver Least Common Denominator.

Common Factor/ Factor Comun: Un factor que dos o mas enteros tienen en común. Ver Greatest Common Factor.

Common Multiple/ Multiple Comun: Ver Least Common Multiple.

Complement/ Complemento: El complemento de un conjunto es un conjunto de todos los elementos del conjunto universal que no estan en el conjunto dado.

Complementary Angles/ Angulos Complementarios: Dos ángulos son complementarios si la suma de sus medidas es 90°.
**Composite Number/Numero Compuesto:** Un número primo es un entero $p$ mayor que 1 con exactamente dos factores positivos; 1 y $p$. Un número compuesto es un entero mayor que 1 que tiene más de dos factores positivos. El número 1 no es primo ni compuesto.

**Compound Interest/Interes Compuesto:** Interés pagado sobre interés previamente ganado más el principal.

**Concentric Circles/Círculos Concentricos:** Círculos con el mismo centro en el mismo plano pero con diferentes radios.

**Cone/Cono:** Una figura de tres dimensiones con una base circular juntada a un punto llamado el ápice o cima.

**Congruent/Congruente:** Usado para referir a ángulos o lados teniendo la misma medida y para polígonos del mismo tamaño y figura.

**Conjecture/Conjetura:** Una afirmación basada en observaciones.

**Constant/Constante:** Un valor fijo.

**Coordinate(s)/Coordenado(s):** Un número asignando a un punto en la recta numérica. En el plano coordenado, el par ordenado $(x,y)$ sitúa cada punto y lo sitúa en respecto a los ejes de $x$, y $y$. 
Coordinate Plane/ Plano Coordenado: Un plano consistiendo de una recta numerica horizontal y una recta numerica vertical que se intersectan en el origen formando angulos rectos. Las rectas, llamadas ejes, dividen el plano en quatro quadrantes. Estos son numerados I, II, III, IV comensando con el quadrante en la esquina derecha de ariba y siguiendo contra-reloj.

Corresponding Angles/ Angulos correspondientes: 1. Si dos rectas son cortadas por una recta tranversal, los angulos del mismo lado del transversal y del mismo lado de las rectas son angulos correspondientes. Si las rectas son paralelas, las medidas de los angulos seran iguales. 2. Si dos poligonos son similares, los angulos en la misma posición relativamente seran angulos correspondientes y tendran medidas iguales.

Corresponding Sides/ Lados Correspondientes: Si dos poligonos son similares, los lados en la misma posición relativamente son lados correspondientes y la proporción de las medidas de cada par de lados es igual.

Counterclockwise/ Contra-reloj: un movimiento circular opuesto a la dirección de movimiento de las manecillas del reloj.

Counting Numbers/ Numeros de Conteo: Numeros usados para contar. Llamados numeros naturales o enteros, son en secuencia, 1, 2, 3, 4, 5, 6, 7…

Cylinder/ Cilindro: Una figura de tres dimensiones con bases circulares que son paralelas y del mismo tamaño unidas por una superficie lateral que tiene una plantilla rectangular.

Data (Data Set)/ Datos (Conjunto de Datos): Una colección de información frecuentemente el forma de numeros.

Data Analysis/ Analize de Datos: El proceso de entender los datos coleccionados.

Degree/ Grado: 1. La circumferencia de un circulo es dividida en 360 partes iguales or arcos. Radios dibujados a los terminos de uno de estos arcos forman un angulo de 1 grado. 2. El grado del termino es la suma de las potencias de los variables. El grado de un polinomio es la potencia mas grande de sus integrantes.

Denominator/ Denominador: El denominador de una fracción indica a que tantas partes iguales se ha dividido el entero. El denominador aparese en la parte de abajo de la fracción.

Diameter/ Diametro: El segmento que une dos puntos el la periferia del circulo y pasa por el centro del circulo.

Distance/ Distancia: Por cualquier dos numeros x y y, la distancia entre x y y es el valor absolute de sus diferencias; es decir, Distancia = .
Dividend/ Dividendo: La cantidad que se divide.

Divisibility/ Divisibilidad: Suponiendo que n y d son enteros, y que d ≠ 0. El número n es divisible por d si hay un entero q de tal manera que n = dq. Por igual, d es un factor de n o n es un múltiple de d.

Division Algorithm/ Algoritmo de División: Dados dos enteros positivos a y b, siempre podemos encontrar enteros únicos q y r de tal manera que a = bq + r y 0 ≤ r < b. Llamamos a el dividendo, b el divisor, q el cuociente, y r el restante.

Divisor/ Divisor: La cantidad por cual el dividendo se divide.

Domain/ Dominio: El conjunto de primeros números de una función.

Edge/ Lado: Un segmento que une vértices consecutivas de un polígono o un poliedro.

Elements/ Elementos: Miembros de un conjunto.

Empirical Probability/ Probabilidad Empírica: Probabilidad determinada por datos coleccionados de experimentos reales.

Empty Set/ Conjunto Vacío: Llamado conjunto nulo, es un conjunto que no tiene elementos.
**Equation/ Ecuación:** Una proposición matemática con igualdad señalando que dos expresiones son iguales.

**Equilateral Triangle/ Triángulo Equilátero:** Un triángulo con tres lados congruentes. Un triángulo equilátero también tiene tres ángulos congruentes.

**Equivalent/ Equivalente:** 1. Dos ecuaciones o desigualdades son iguales si tienen la misma solución o conjunto de soluciones. 2. Un término para describir fracciones o proporciones que son igual. 3. Un término para describir fracciones, decimales, y porciento que son iguales.

**Event/ Evento:** Un evento es un subconjunto del espacio muestral. Un evento simple es un subconjunto del espacio muestral conteniendo solamente un resultado del experimento. Un evento compuesto es un subconjunto del espacio muestral conteniendo dos o más resultados.

**Experiment/ Experimento:** Una acción que puede ser repetida con un conjunto de resultados.

**Exponent/ Exponente (Potencia):** Por cualquier valor x, elevado a la potencia de n, escrito como $x^n$, x es llamado la base de la expresión, y n es el exponente o potencia.

**Exponential Notation/ Notación Exponencial (Notación Potencial):** Notación para expresar un número en términos de base y potencia.
Face:/Cara: La superficie de cada polígono que forma un poliedro.

Factor/ Factor: Un entero que divide un dividendo exactamente. Intercambiable con el divisor excepto en el Algoritmo de División.

Factorial/ Factorial: El factorial de un número que no es negativo n es escrito como n! y es el producto de todos los números positivos menor o igual a n. Por definición 0! = 1! = 1.

Frequency/ Frecuencia: El número de veces que un dato aparece en un conjunto.

Function/ Función: Una función es una regla que asigna a cada número de un conjunto, llamado dominio, un número de salida, llamado el rango. La regla no permite que los números se repitan.

Graph of a Function/ Grafica de una Función: Una representación pictórica de una función graficando pares ordenados en el sistema coordinado.

Greater Than/ Mayor Que: ver Less Than.

Greatest Common Factor, GCF/ Máximo Común Factor, MCF: Si m y n son enteros positivos. Un entero d es un factor común de m y n si d es un factor de m y de n. El máximo común factor o MCF de m y n es el entero positivo más grande que es un factor de m y n. Escrito como MCF de m y n o como MCF(m,n).
**Height/ Altura:** La longitud de el segmento perpendicular que une las bases de un paralelogramo o un trapezoide. Lo alto de un triángulo.

**Horizontal Axis/ Eje Horizontal:** ver Coordinate Plane.

**Hypotenuse/ Hipotenusa:** El lado opuesto al ángulo recto en un triángulo recto.

**Improper Fraction/ Fracción Impropiamente:** Una fracción en la cual el numerador es mayor que o igual a el denominador.

**Independent Event/ Evento Independiente:** El resultado del primer evento no afecta el resultado del segundo evento.

**Input Values/ Primeros Valores:** Los valores del dominio de una función.

**Integers/ Enteros:** La colección de enteros es compuesta de números negativos, cero y los números positivos:…, -4, -3, -2, -1, 0, 1, 2, 3, 4,…

**Intersection of Sets/ Traslape de Conjuntos:** Un conjunto con elementos cuales son los elementos comunes de los conjuntos dados.

**Irregular Polygon/ Polígono Irregular:** Un polígono que no es regular.
**Isosceles Triangle/ Triángulo Isósceles:** Un triángulo con por lo menos dos lados de la misma medida es llamado un triángulo isósceles.

**Lateral Area/ Area Lateral:** La área de la superficie de cualquier figura de tres dimensiones no incluyendo la área de la superficie designada como la base.

**Lattice Points/ Puntos de Lattice:** Un punto en el plano coordinado (x,y) en el cual x y y son enteros.

**Least Common Denominator, LCD/ Mínimo Común Denominador, MCDn:** El mínimo común denominador de las fracciones \( \frac{a}{m} \) y \( \frac{b}{n} \) es MCM(n,m).

**Least Common Multiple, LCM/ Mínimo Común Múltiple, MCM:** Si a y b son enteros positivos, un entero m es un múltiple común de a y b si m es un múltiple de a y de b. El mínimo común múltiple o MCM de a y b es el entero positivo más chico que es un múltiple de a y b. Escrito como MCM de a y b o como MCM(a,b).

**Legs/ Catetos:** Los lados que forman el ángulo recto de un triángulo recto.

**Less Than, Greater Than/ Menor Que, Mayor Que:** El enunciado que un número a es menor que un número b, escrito \( a < b \) significa que hay un número positivo x de tal manera que \( b = a + x \). El número x tiene que ser \( b - a \). Si a es menor que b, b es mayor que a, escrito \( b > a \).
Line Graph/ Grafica de Rectas: Una grafica usada para enseñar datos que ocurren en secuencia. Puntos consecutivos van conectados por segmentos.

Line Plot/ Grafica de Rectas: Una grafica usada para mostrar la frecuencia de datos en una recta numerica.

Line of Symmetry/ Eje de Simetria: Recta L es un eje de simetría para la figura si por cada punto P en la figura hay un punto Q en la figura de tal manera que L es el mediatriz del segmento PQ.

Linear Model for Multiplication/ Modelo Lineal para Multiplicar: Contar en grupos usando la recta numerica.

Magnitude/ Magnitud: El valor absoluto de un numero; su distancia de cero.

Mean/ Media Aritmetica: El promedio de un conjunto de datos; la suma de los datos dividida por el numero de datos.

Measures of Central Tendency/ Medidas de Tendencia Central: Medidas generalmente por la media, la mediana y la moda de el conjunto de datos.

Median/ Mediana: El valor medio en un conjunto de datos ordenados. Si el conjunto tiene un numero par de datos, la mediana es el promedio de los dos numeros de enmedio.
**Missing Factor Model/ Modelo de Factor Faltante:** Un modelo de división en el cual el cuociente de la división indicada es visto como un factor faltante de una multiplicación relacionada.

**Mixed Fraction(Mixed Number)/ Numero Mixto:** La suma de un entero y una fracción propia.

**Mode/ Moda:** El valor del elemento que aparece más veces en un conjunto de datos.

**Multiplicity/ Multiplicidad:** El numero de veces que un factor aparece en una factorización.

**Natural Numbers/ Numeros Naturales:** ver Counting Numbers

**Negative Integers/ Enteros Negativos:** Enteros con valor menor que cero.

**Nets/ Plantillas:** Una manera de ver la area de la superficie de una figura de tres dimensiones cortando sobre los lados para producir una figura de dos dimensiones.

**Notation/ Anotación:** Un sistema tecnico de simbolos usado para mostrar información matematica.
Null Set/ Conjunto Nulo: ver Empty Set.

Numerator/ Numerador: La expresión en la parte de arriba de una fracción que indica el numero de partes contadas.

Obtuse Angle/ Angulo Obtuso: Un angulo con medida mayor que 90 grados y menor que 180 grados.

Obtuse Triangle/ Triangulo Obtuso: Un triangulo que tiene un angulo obtuso.

Ordered Pair/ Par Ordenado: Un par de numeros que representa los coordenados de un punto en el plano coordinado con el primer numero señalando la medida en el eje horizontal y el segundo señalando la medida en el eje vertical.

Origin/ Origen: El punto con el coordinado 0 el la recta numerica; el punto con los coordenados (0,0) en el plano coordinado.

Outcomes/ Numeros de Salida: El conjunto de resultados posibles de un experimento.

Outlier/ Valor Extremo: Un termino que se refiere a valor que es completamente diferente a los demas valores.

Output Values/ Valores de Salida: El conjunto de valores obtenidos al aplicar una función a un conjunto de primeros valores.
Parallel lines/ Lineas Paralelas: Dos lineas en un plano que nunca cruzan.

Parallelogram/ Paralelogramo: Es un cuádrilatero con los lados opuestos paralelos.

Percent/ Porcentaje: una manera de expresar un numero como partes de cien. El numerador de una proporción con 100 como denominador.

Perfect Cube/ Cubo Perfecto: Un entero n que puede ser escrito en la forma n = , y k es un entero.

Perfect Square/ Cuadrado Perfecto: Un entero n que puede ser escrito en la forma n = , y k es un entero.

Perimeter/ Perimetro: El perímetro de un polígono es la suma de la medida de los lados.

Perpendicular/ Perpendicular: Dos lineas o segmentos se dicen ser perpendiculares si cruzan a formar un ángulo recto.

Pi/ Pi: La proporción de la circunferencia a el diámetro de cualquier círculo, representado por el símbolo π, o la aproximación 3.1415926…
Pie Graph/ Grafica de Sectores: Una grafica usando sectores de un círculo que son proporcional al porciento de los datos representados.

Polygon/ Poligono: Un polígono es una figura simple plana formado por tres o más segmentos.

Polyhedron/ Poliedro: Una figura de tres dimensiones con cuatro o más caras todas cuales son polígonos.

Positive Integers/ Enteros Positivos: ver Counting Numbers

Power/ Potencia: ver Exponents

Prime Number/ Numero Primo: ver Composite Number

Prime Factorization/ Factorización en Primos: El proceso de encontrar los factores primos de un entero. El termino tambien se refiere al resultado de este proceso.

Prism/ Prisma: Un tipo de poliedro que tiene dos bases que son paralelas y congruentes, y caras laterales que son paralelogramos.
**Probability/ Probabilidad:** En un experimento en el cual cada resultado tiene la misma oportunidad, la probabilidad P(A) de un evento A es donde m es el número de resultados en el subconjunto A y n es el número total de resultados en el espacio muestral S.

**Proper Fraction/ Fracción Propia:** Una fracción con valor mayor que 0 y menor que 1.

**Proportion/ Proporción:** Una equación de razones en la forma , y b y d no son iguales a cero.

**Protractor/ Transportador:** Un instrumento para medir ángulos en grados.

**Pyramid/ Pirámide:** Un tipo de poliedro que tiene una cara llamada la base, y caras laterales triangulares que topan en un punto llamado el vértice.

**Pythagorean Theorem/ Teorema de Pitágoras:** Si a y b son las medidas de los catetos de un triángulo recto y c es la medida de la hipotenusa, entonces \( c^2 = a^2 + b^2 \).

**Quadrant/ Cuadrante:** ver Coordinate Plane

**Quotient/ Cociente:** El resultado obtenido al hacer división. Ver Division
Algorithm para encontrar diferentes usos del cociente.

Radius/ Radio: La distancia del centro de un círculo a un punto en la periferia.

Range/ Rango: La diferencia entre el valor más grande y el valor más chico en un conjunto de datos. Ver Function por otras definiciones de rango.

Rate/ Tasa de Variación: Una comparación por medio de un cociente, entre dos cantidades con diferentes unidades. Ver Unit Rates.

Ratio/ Razón: Una comparación por medio de un cociente. Si las unidades son diferentes, la razón tiene que tener sentido.

Ray/ Rayo: Parte de una recta con un extremo y se prolonga sin límite en una dirección.

Reciprocal/ Reciproco: ver Multiplicative Inverse

Reflection/ Reflexión: La transformación que mueve puntos o figuras al cruzar de una recta o eje; una imagen de espejo del conjunto original.

Regular Polygon/ Polígono Regular: Un polígono con lados de mismas medidas y ángulos de mismas medidas.
**Relatively Prime/ Numeros Primos Entre Si:** Dos enteros \( m \) y \( n \) son numeros primos entre si, si el MCF de \( m \) y \( n \) es 1.

**Remainder/ Restante:** Ver Division Algorithm.

**Repeating Decimal/ Decimal Periódico:** Un decimal en el que se repiten uno o mas digitos sin terminación.

**Right Angle/ Angulo Recto:** un angulo formado cuando cruzan dos lineas perpendiculares; un angulo que mide 90 grados.

**Right Triangle/ Triangulo Recto:** un triangulo que tiene un angulo recto.

**Sample Space/ Espacio Muestral:** El conjunto de todos los resultados posibles de un experimento.

**Scaffolding/ Andamiaje:** Un metodo de división en el cual cocientes parciales son computados, apilados, y combinados.

**Scale Factor/ Factor de Escala:** Si poligonos \( A \) y \( B \) son similares y \( s \) es un numero positive a que para cada lado de poligono \( A \) con medida \( k \) hay un lado correspondiente en \( B \) con medida \( sk \), entonces \( s \) es el factor de escala de \( A \) y \( B \).

**Scalene Triangle/ Triangulo Escaleno:** Un triangulo con los tres lados de diferentes medidas es llamado un triangulo escaleno.
Scaling/ Escalar: 1. El proceso en el cual una figura se reduce o aumenta proporcionalmente. Escojer la unidad de medida que sera usada en la recta numerica.

Sector/ Sector: Una región de un circulo rodeado por dos radios y un arco que une sus extremos.

Sequence/ Sucuencia: Un conjunto de terminos puestos en orden por los numeros naturales. Los numeros de salida de una función de dominio que incluye los numeros naturales o enteros.

Set/ Conjunto: una colección de objetos o elementos.

Similar Polygons/ Poligonos Similares: Dos poligonos cuales tienen angulos corespondientes de la misma medida y lados corespondientes de misma proporción.

Simple Event/ Evento Simple: Ver Event.

Simplest Form of a Fraction: Forma Mas Simple de una Fracción: Una forma en la cual el maximo comun factor del numerador y denominador es 1.

Simplifying/ Simplificar: El proceso de encontrar fracciones equivalentes para obtener la forma mas simple.

Skewed/ Sesgado: Una representación de un conjunto de datos desiguales.
**Stem and Leaf Plot/ Diagrama de Tallo y Hojas:** Un método para enseñar la frecuencia de ciertos datos al ordenarlos.

**Straight Angle/ Angulo Llano:** Un ángulo que mide 180 grados formado por rayos opuestos.

**Subset/ Subconjunto:** Conjunto B es un subconjunto de Conjunto A si todos los elementos de conjunto B son parte de los elementos de Conjunto A.

**Supplementary Angles/ Angulos Suplementarios:** Dos ángulos son suplementarios si la suma de sus medidas es 180°.

**Surface Area/ Area de la Superficie:** La área total de todas las caras en un poliedro. La área total de la lateral y la base en un cono. La área total de la lateral y las dos bases en un cilindro.

**Term/ Termino:** 1. un miembro de una sucesión. 2. Cada expresión en un polinomio separado por una suma o resta.

**Terminating Decimal/ Decimo Finito:** Si a y b son números naturales con b ≠ 0, y resulta en una cantidad finita, el número decimal que resulta es un decimal finito.

**Theoretical Probability/ Probabilidad Teórica:** Probabilidad basada en leyes matemáticas en vez de una colección de datos.
Translation/ Traslación: Una transformación que mueve una figura sobre el plano pero no altera el tamaño ni la forma.

Transversal/ Transversal: Cualquier recta que intersecta a dos o más rectas en distintos puntos.

Trapezoid/ Trapezoide: Un cuadrilátero con exactamente un par de lados paralelos.

Tree Diagram/ Diagrama de Árbol: 1. Un proceso para encontrar los factores primos de un entero. 2. Un método para organizar el espacio muestral de eventos compuestos.

Trichotomy/ Tricotomia: Una propiedad declarando que exactamente una de estas declaraciones es verdadera para cada número real: es un número positivo, negativo, o cero.

Union of Two Sets/ Unión de Dos Conjuntos: Un conjunto que contiene todos los elementos que aparecen en cualquiera de los conjuntos dados, escrito como A B.

Unit Rate/ Razon Unitaria: Una razón en la cual el denominador es una unidad.

Universal Set/ Conjunto Universal: Un conjunto conteniendo todos los elementos bajo consideración.
**Variable:** Una letra o símbolo que representa una cantidad desconocida.

**Venn Diagram:** Un diagrama con dos o más círculos que intersectan para ayudar en la organización de datos.

**Vertex:**
1. El punto común de los lados de un ángulo.
2. Un punto extremo de un polígono o poliedro donde se juntan los lados.

**Vertical Angles:** Si dos rectas cruzan en un punto, cada recta se divide en dos formando dos rayos. Los ángulos formados usando rayos opuestos de cada recta son llamados ángulos verticales.

**Vertical Angle Theorem:** Si dos rectas cruzan en un punto P, los ángulos verticales formados siempre tendrán la misma medida.

**Vertical Axis:** ver Coordinate Plane.

**Volume:** La medida de un espacio; El número de unidades cúbicas necesarias para llenar una figura de tres dimensiones.

**Whole Numbers:** Los números enteros son los números en la sucesión: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11…
x-axis/ eje de x: El eje horizontal en el plano coordinado.

y-axis/ eje de y: El eje vertical en el plano coordenado

Zero Pair/ Par Formando Cero: Por cualquier número natural n, n + ( -n ) es llamado un par formando cero porque su suma es cero.
Resumen de Ideas

Additive Property of Equality/ Propiedad Aditiva de la Igualdad: Si $A = B$, entonces $A + C = B + C$.

Additive Identity/ Identidad Aditiva: Por cualquier número $x$, $x + 0 = x$.

Additive Inverse/ Inverso Aditivo: Por cualquier número $x$, existe un número $-x$, llamado el inverso aditivo de $x$, de tal manera que $x + (-x) = 0$.

Area of a Circle/ Area de un Circulo: La área de un círculo con radio $r$ es $A = \pi r^2$ en unidades cuadradas.

Area of a Parallelogram/ Area de un Paralelogramo: La área de un paralelogramo con base $b$ y altura $h$ es $A = bh$ en unidades cuadradas.

Area of a Triangle/ Area de un Triángulo: La área de un triángulo con base $b$ y altura $h$ es $A = \frac{1}{2}bh$ o $A =$ en unidades cuadradas.

Associative Property of Addition/ Propiedad Asociativa de Adición: Por cualquier números $x$, $y$ and $z$, $(x + y) + z = x + (y + z)$.
Associative Property of Multiplication/ Propiedad Asociativa de Multiplicación: Por cualquier números x, y and z, \((x \cdot y) \cdot z = x \cdot (y \cdot z)\).

Commutative Property of Addition/ Propiedad Conmutativa de Adición: Por cualquier números x y y, \(x + y = y + x\).

Commutative Property of Multiplication/ Propiedad Conmutativa de Multiplicación: Por cualquier números A y B, \(A \cdot B = B \cdot A\).

Corresponding Angle Postulate/ Postulado de Angulos Correspondientes: Si dos rectas paralelas son intersectadas por un transversal, los angulos formados correspondientes tienen la misma medida, y si dos rectas son intersectadas por un transversal de tal manera que los angulos correspondientes tienen la misma medida, entonces las rectas son paralelas.

Distributive Property of Multiplication over Addition/ Propiedad Distributiva de Multiplicación sobre Adición: Por Cualquier números k, m y n, \(n \cdot (k + m) = n \cdot k + n \cdot m\).

Division Rules/ Reglas de División: 1. Si el dividendo y el divisor tienen diferentes signos, ( uno positivo y uno negativo), el cociente es negativo. 2. Si el dividendo y el divisor tienen el mismo signo, ( los dos positivos o los dos negativos), el cociente es positivo.

Double Opposites Theorem/ Teorema de Opuestos Dobles: Por cualquier número x, \(-(-x) = x\).
Equivalent Fraction Property/ Propiedad de Fracciones Equivalentes:
por cualquier numero a y numeros no igual a cero k y b.

Fractions and Division/ Fracciones y División: Por cualquier numero m y número no igual a cero n, la fracción es equivalente a el cociente.

Fundamental Theorem of Arithmetic/ Teorema Fundamental de la Aritmetica: Si n es un entero positivo, n > 1, entonces n es primo o puede ser escrito como el producto de primos n = p1· p2· p3·…· pk , para unos numeros primos p1, p2, p3,… pk de tal manera que p1 ≤ p2 ≤ p3 ≤…≤ pk. En realidad, nada mas hay una manera para escribir n en esta forma.

Multiplication of Powers/ Multiplicación con Potencias: Si x es un numero y a y b son numeros naturales, entonces (xa)(xb) = xa+b.

Multiplicative Identity/ Identidad Multiplicativa: El numero 1 es la identidad multiplicativa, es decir, por cualquier numero n, n · 1 = n.

Multiplicative Inverse/ Inverso Multiplicativo: Por cada x no igual a 0, existe un numero , llamado el inverso multiplicativo o reciproco de x de tal manera que

Multiplying Fractions/ Multiplicando Fracciones: El producto de dos fracciones en donde b y d son numeros no igual a cero, es.
Pythagorean Theorem/ Teorema de Pitagoras: Si a y b son las medidas de los catetos de un triángulo recto y c es la medida de la hipotenusa, entonces $c^2 = a^2 + b^2$.

Subtraction Property of Equality/ Propiedad Sustractiva de la Igualdad: Si $A = B$, entonces $A - C = B - C$.

Sums With Like Denominators/ Sumas de Fracciones con Denominadores Comunes: La suma de dos fracciones con denominadores iguales, $\frac{a}{b}$ y $\frac{c}{b}$, es dado por $\frac{a+c}{b}$.

Surface Area of a Cube/ Area de la Superficie de un Cubo: La área de la superficie, $SA$, de un cubo es dado por la fórmula, $SA = 6s^2$ donde $s$ es la medida de un lado.

Surface Area of a Cylinder/ Area de la Superficie de un Cilindro: La área de la superficie total $SA$ de un cilindro es la suma de la área de las bases y la área lateral, dado por la ecuación: $SA = 2B + Ph = 2\pi r^2 + 2\pi rh$, donde $B$ es la área de la base, $P$ es el perímetro de el círculo (circunferencia), $h$ es la altura y $r$ es el radio del círculo.

Surface Area of a Rectangular Prism/ Area de la Superficie de una Prisma rectangular: La área de la superficie $SA$ de una prisma rectangular es dado por la ecuación: $SA = 2B + Ph$, donde $B$ es la área de la base, $P$ es el perímetro de la base rectangular, y $h$ es la altura de la prisma.
The Rule of Products/ La Ley de Productos: Si una acción se puede hacer de \( m \) numero de maneras y una segunda acción se puede hacer en \( n \) numero de maneras, entonces hay \( m \cdot n \) numero de maneras para hacer las dos acciones.

The Rule of Sums/ La Ley de Sumas: Si una acción se puede hacer de \( m \) numero de maneras y una segunda acción se puede hacer en \( n \) numero de maneras, entonces hay \( m + n \) numero de maneras para hacer una o la otra acción pero no las dos. Esto supone que las dos acciones son exclusivas una de otra y tienen la misma oportunidad de acontecer.

Triangle Similarity Theorem/ Teorema de Semejanza de Triangulos: Si dos triangulos tienen angulos de mismas medidas, son semejantes o similares y las proporciones de los lados correspondientes son iguales. Por igual, si dos triangulos tienen lados con la misma proporción, los triangulos son similares y sus angulos correspondientes tienen las mismas medidas.

Triangle Sum Theorem/ Teorema de la Suma de los Angulos de un Triangulo: La suma de las medidas de los angulos de un triangulo es igual a 180°.

Unit Fraction/ Fracción Unitaria: Por cualquier entero positivo \( n \), el inverso multiplicativo o recíproco de \( n \) es la fracción unitaria \( \frac{1}{n} \).

Vertical Angle Theorem/ Teorema de Angulos Verticales: Si dos rectas intersectan en un punto \( P \), los angulos verticales formados tendrán la misma medida.
Volume of a Cube/ Volumen de un Cubo: El volumen de un cubo con lados de medida $s$ es $s^3$, $V = s^3$ o $V = Bh$ donde $B$ es la área de la base y $h$ es la altura.

Volume of a Cylinder/ Volumen de un Cilindro: El volumen de un cilindro con radio $r$ y altura $h$ es $V = Bh$ donde $B$ es la área del círculo o $V = \pi r^2 h$.

Volume of a Prism/ Volumen de una Prisma: El volumen de una prisma es la área de la base $B$ multiplicada por la altura $h$ de la prisma. Escrito $V = Bh$. 
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SECTION A.1 INTEREST

Let’s begin by looking at simple interest, and then see how banks extend this idea to everyday business. First, what does it mean to invest a principal amount of money $P$ in a bank at a simple interest rate $r$? The bank’s customers are letting the bank use their money for a period of time, and in return the bank is willing to return their original amount at the end of the period plus extra money called Interest. The amount of extra money depends on the interest rate that the bank pays and the period of time.

So if Mrs. Hamlisch puts $200 into a savings account at the beginning of the year at 12% interest rate, at the end of the year her account will have $200 plus the interest that she earned. This interest is 12% of $200, or, from the simple interest formula $I = Prt$,

$$I = (0.12)200 = 24.$$

So at the end of the year she will have $200 + $24 = $224 in her bank account.

EXAMPLE 1

Mr. Lablanc invests $500 at a simple interest rate of 6%. How much interest will he earn in one month? In six months? In one year?

Solution  The yearly interest he will earn is 6% of his original principal if he put his money in the bank for one year. However, in one month, he will only earn $\frac{1}{12}$ of this amount. The interest earned in one month is $I = (500)(0.06)(1/12) = $2.50. Similarly, in 6 months, the interest he will earn is $I = (500)(0.06)(6/12) = $15.
In one year the interest he will earn is $I = (500)(0.06)(1) = $30$. Compare the three amounts of interest to see if they are reasonable.

Each case above uses the Simple Interest Formula:

**SIMPLE INTEREST FORMULA**

If a principal amount $P$ is invested at an interest rate $r$ for $t$ years, then the simple interest earned will be $I = Prt$.

We can use the simple interest formula to find a formula for the amount of money $A$ that will be in a simple interest account after $t$ years. The amount $A$ is the original principal $P$ plus the interest $I$ earned over the period of time $t$. So:

$$A = P + I = P + Prt.$$  

To simplify the formula factor $P$ from the right-hand expression to obtain

$$A = P + Prt = P(1 + rt).$$

**PROBLEM 1**

Compute the total amount of money Mr. Lablanc will have in his account from Example 1. Do this in two ways:

1. Add the amount of interest earned to the initial amount. This is the total amount in the account after each time period.
2. Check your work by using the Simple Interest Formula for amount.

Most accounts, however, set more than one interest period a year. This is called compounding. Financial institutions divide the year into a certain number of periods and add simple interest to an
account after each period.

**EXAMPLE 2**

If you invest a principal $P$ at an interest rate $r$ compounded monthly, how much interest will $P$ earn in one period?

**Solution** Each period is one month long, so the length of time for one period is $\frac{1}{12}$ of a year. The interest earned for one period will be $I = P \times r \times (\frac{1}{12})$. This is the same as we found in Example 1.

In general, if there are $m$ periods in a year, the length of time for each period will be $(\frac{1}{m})$ of a year. The interest earned in one period will be $I = P \times r \times (\frac{1}{m})$.

The periodic interest rate, then, is $\frac{r}{m}$. We can $I$ the periodic interest amount and $i = \frac{r}{m}$. So the interest earned in one period is $I = Pi$. That means the amount of money in an interest-earning account at the end of a period is $P + Pi$. This looks just like the simple interest formula except the interest rate $r$ is replaced by the periodic interest rate $i = \frac{r}{m}$.

If an account earns interest compounded every six months, the periodic interest rate per each six-month period is $i = \frac{12\%}{2} = 6\%$. If the account earns interest compounded quarterly, or four times a year, the periodic interest rate is $i = \frac{12\%}{4} = 3\%$. Many accounts earn interest each month, so $i = \frac{r}{12}$.

**EXAMPLE 3**

Suppose we deposit $100 at 12\% per year compounded monthly. How much will be in the account after three months? Find a formula for the amount in the account at the end of $t$ months.

**Solution** Let’s make a list of the amount of money in your account at the end of each of the first 3 months. Since one month is $\frac{1}{12}$ of a year, the interest rate for one month is $i = \frac{12\%}{12} = .01$. The
amount in the account after one month would be $100 + (.01)(100) = 100 + 1 = 101$.

Let $A(t) =$ amount in the account at the end of $t$ months. So,

\[
A(0) = 100 = P = \text{amt. in account at beginning of first month}
\]

\[
A(1) = P + Pr = P(1 + i) = P(1.01) = \text{amt. in account at end of first month}
\]

Notice that we factor the $P$ out of the sum producing the factor $(1 + i)$. Computing the next two months, we get

\[
A(2) = A(1) + A(1)i = A(1)(1 + i)
\]

\[
= [P(1 + i)](1 + i) = P(1 + i)^2 = P(1.01)^2
\]

and similarly,

\[
A(3) = A(2) + A(2)r = A(2)(1 + i)
\]

\[
= [P(1 + i)^2](1 + i) = P(1 + i)^3 = P(1.01)^3
\]

Using this pattern, find the values for $A(4)$, $A(5)$, $A(12)$ and $A(t)$. Fill in the following table.
Of course this pattern works with different interest rates. The key idea is that for each period that passes, the amount at the end of the period is equal to the amount at the beginning of the same period multiplied by \((1 + \gamma)\), leading to the

**COMPUND INTEREST FORMULA**

If an initial principal \(P\) is invested at an interest rate \(r\) compounded \(m\) times per year, then the amount in an account after \(n\) periods is \(A(n) = P(1 + \frac{r}{m})^n\), where \(i = \frac{r}{m}\) is the interest earned each period.

**EXAMPLE 4**

Sally invests $1000 at 12% simple interest for three years. Ann invests $1000 at a rate of 12% compounded monthly for three years.

1. How much money will Sally have after three years?
2. How much money will Ann have?
3. Are you surprised by the results in any way?
4. If the time invested were ten years, by how much would the amounts differ?

**Solution**

1. The amount that Sally will have is \(A = 1000(1 + 0.12 \times 3) = \)
$1360.

2. First, find the periodic interest rate: \( i = \frac{0.12}{12} = 0.01 \). Second, find the number of periods in three years. Because \( m = 12 \), there are 12 periods per year. In three years, there are 36 periods. Finally, use the compound interest formula to find Ann’s amount, using a calculator:

\[
A = 1000\left(1 + \frac{0.12}{12}\right)^{36} = 1430.77.
\]

3. Ann will have \( 1430.77 - 1360 = 70.77 \) more than Sally after 3 years.

4. Performing these calculations for ten years, Sally will have \( A = 1000(1 + .12 \times 10) = 2200 \). Ann will have \( A = 1000(1 + .01)^{120} = 3300.39 \).

So after ten years, Ann will have $1100.39 more than Sally.

**PROBLEM 2**

Suppose we deposit $1000 at 12% per year compounded monthly. How much will be in the account after 6 months? Find a formula for the amount in the account at the end of \( t \) months. Graph the function using your graphing calculator. Using the graph, estimate how many months it will take before you have $1100.
EXERCISES

1. Sue invests $500 in the bank at a simple interest rate of 12%. How much interest will she earn after
   a. 2 months?
   b. 6 months?
   c. 1 year?
   d. 2 years?

2. Chris invests $100 in the Simple Bank of America at a simple interest rate of 8%. How much will be in his account after
   a. 2 months?
   b. 6 months?
   c. 1 year?
   d. 2 years?
   e. 5 years?

3. When interest is compounded monthly, how many periods are there in
   a. 3 months?
   b. 1 year?
   c. 5 years?

4. Chris discovers he could invest Texas Compound Bank. In that bank he can invest $100 in at an interest rate of 8% compounded monthly. How much will be in his account after
   a. 2 months?
   b. 6 months?
   c. 1 year?
   d. 2 years?
   e. 5 years?
   f. Compare with your answer from Exercise 2. Should he switch banks?

5. Jackie and Jennifer are sisters. On January 1, 1950, Jackie put $1000 in her safe and forgot about it. Her sister Jennifer only had $100 at that time. But she put her money into an account at the bank which paid 5% compounded yearly and forgot about it. On January 1, 2000, they decided to take out their money and throw a party. Who had more money to spend on the party? How much more?
6. A bank offers three types of accounts. In the Bronze account, you earn 12% annual interest compounded monthly. In the Silver account, you earn 12.2% compounded twice a year. Finally, in the Gold account you earn 12.4% compounded once a year. Which is the better deal? If you deposit $100 how much will you have at the end of the year?

7. How much money will Mr. Garza need to deposit into an account earning 12% per year (1% per month) compounded monthly in order that he have $500 at the end of 3 years?

8. John deposited $200 in the bank at a interest rate of 9% compounded monthly. How much will be in the account after 6 months?

9. Victoria invests $100 at a simple interest rate of 8%. Penelope invests $100 at a rate of 8% compounded monthly. How much will each girl have:
   a. After 6 months?
   b. After 1 year?
   c. After 10 years?
   Compare the amounts and summarize what you found.

10. Using the simple interest formula $A = P(1 + rt)$,
    a. Knowing $P$, $r$, and $t$, solve for $A$.
    b. Given $A$, $r$, $t$, solve for the principal $P$.
    c. Given $A$, $P$, $r$, solve for the time $t$. Using this information, find out how long will it take Sam to have $200 in his account, after he invests $100 at a simple interest rate of 8%.

11. Solve the compound interest formula $A = P(1 + i)^t$ for $P$.
    What does the new formula tell you? Make up a problem that you could use your new formula to solve.

12. Use your calculator to graph the following two functions. Then explain what information it gives: $A = 100(1 + 0.08t)$. $A = 100(1 + 0.08/12)^{12t}$. What difference do you notice in the two graphs? Which grows faster?

13. Compare the difference in simple interest from money invested at 8% and interest compounded monthly for money invested at 8% after five years.
14. Explain to a fifth grader the difference between simple interest and compound interest.

15. **Investigation:**
   How long will it take money to double at a compound interest rate of 12% compounded monthly? At 8%? Research and explain the Banker’s Rule of 72.

16. **Ingenuity:**
   Compute each of the products from a-d. Then speculate what the products are for parts e-f.
   
a. \((1 + x + x^2)(1 - x)\)

b. \((1 + x + x^2 + x^3)(1 - x)\)

c. \((1 + x + x^2 + x^3 + x^4)(1 - x)\)

d. \((1 + x + x^2 + x^3 + \cdots + x^8)(1 - x)\)

e. \((1 + x + x^2 + \cdots + x^{19} + x^{20})(1 - x)\)

f. \((1 + x + x^2 + \cdots + x^{(n-1)} + x^n)(1 - x)\)
SECTION A.2 COST OF CREDIT

What happens if instead of depositing money in the bank, we borrow money from the bank. For example, if we use a credit card. Do you think the interest rate will be the same, smaller, or larger? Now $A$ will be the amount we owe. How can we show that we owe the bank instead of the bank owing us?

Usually a traditional loan uses the compound interest formula. As with the simple interest formula, the principal $P$ is the amount of the loan, and the amount $A$ is the total amount to be repaid. Here is an example that uses the compound interest formula to understand loans.

EXAMPLE 1

You use a credit card to purchase a $100 MP3 player. The credit card company charges 24% per year compounded monthly. How much do you owe after $t$ months if you don’t pay the credit card company any money? How much do you owe at the end of the year?

Solution  You used the card to purchase a $100 MP3 player so the initial value is $100. To indicate that you owe the money, write $P = -100$. The annual rate is 24% or $i = 2\%$ per month. So the amount after $t$ months is

$$A = -100(1 + .02)^t = -100 \cdot 1.02^t$$

At the end of the year, the amount is

$$A = -100 \cdot 1.02^{12} = -126.8242$$

Hence you owe $126.82 to the credit card company.

EXPLORATION 1

Many Americans have money in savings accounts that pay interest and at the same time owe money to credit card companies.
Suppose we put $1000 dollars in the bank and the bank pays 12% compounded monthly. At the same time we use the credit card to purchase $1000 of clothes. The credit card company charges 24% compounded monthly. How much do we have in the bank after $t$ months? How much do we owe to the credit card company after $t$ months? Does this make sense to you? What function could we graph to investigate what happens? Graph the function and see what happens at the end of the year.

When you borrow money to purchase a very expensive item like a car or a house, the lender (usually a bank) asks you to pay something each month. After each payment is made, that much less is owed the bank. In this way, you slowly pay off your loan. The formulas in this situation are complicated, but we can compute a few steps to explore how this works. Then we can use an online calculator to analyze a more realistic situation.

**EXPLORATION 2**

Imagine you borrow $10000 to purchase a car. The bank says it will charge 10% annual interest rate.

1. Suppose you pay the bank $500 every 6 months.
   a. How long do you think it will take to repay the loan?
   b. Since you are paying each six months, the bank recomputes the interest and the money owed each six months. The money owed is called the principal of the loan. How much interest do you owe for the first 6 months? Adding the initial principal, $10000, and the interest, how much do you owe right before your first payment? How much do you owe after your first payment? What do you notice?
   c. Repeat the calculations above for the second 6 month period: How much interest do you owe for the second 6 months? Adding the amount owed after the first payment, and the interest, how much do you owe right before your second payment? How much do you owe after your second payment?
   d. At this rate, how long will it take to pay off the loan?
2. Now suppose you pay the bank $1000 every 6 months. You may use the table below to organize your calculations.
   a. Now how long do you think it will take to repay the loan?
   b. How much interest do you owe for the first 6 months? Adding the initial principal, $10000, and the interest, how much do you owe right before your first payment? How much do you owe after your first payment?
   c. Repeat the calculations above for the second 6 month period: How much interest do you owe for the second 6 months? Adding the amount owed after the first payment, and the interest, how much do you owe right before your second payment? How much do you owe after your second payment?
   d. Compare the first payment and the second payment. How much did the principal go down after the first payment? How much did it go down after the second payment? Why are these different?

<table>
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<th>Principal Amt. at Beginning</th>
<th>Interest Owed for Period</th>
<th>Amt. at End of Period</th>
<th>Payment</th>
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<td>10000</td>
<td></td>
<td></td>
<td>1000</td>
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If you continue the calculations in 2, you will see that it takes over 7 years to pay off the loan if you pay $1000 every six months. In the middle of 8th year, you will have to make a final payment of $210.71.

It is also possible to use an on-line calculator to make these computations. For example, use a search engine to find a loan calculator. What is the web address?

EXPLORATION 3

Use an online calculator to investigate how car loans work. Typically, when you buy a car you negotiate with the car dealer or bank the terms of the loan. Two important features are the annual interest rate and total length of the loan. For a one year loan, you must pay
back the whole loan in one year (12 monthly payments). For a 6
year loan, you pay the loan back over a longer period (72 monthly
payments). As you saw in Exploration 2, the amount you still owe
on the loan and the interest on the loan are calculated after each
payment.

Set the original amount of the loan to \( P = 10000 \). Use the
calculator to fill in the table.

1. Set the interest rate to 5%. Fill in the following table:

<table>
<thead>
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<th>Length of Loan</th>
<th>Monthly Payment</th>
<th>Total Interest</th>
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<td>1</td>
<td>856.07</td>
<td>272.9</td>
</tr>
<tr>
<td>2</td>
<td>438.71</td>
<td>529.13</td>
</tr>
<tr>
<td>3</td>
<td>299.71</td>
<td>789.52</td>
</tr>
<tr>
<td>4</td>
<td>230.29</td>
<td>1054.06</td>
</tr>
<tr>
<td>5</td>
<td>188.71</td>
<td>1322.74</td>
</tr>
<tr>
<td>6</td>
<td>161.05</td>
<td>1595.55</td>
</tr>
</tbody>
</table>

a. Using the data in the table, plot the Length of the Loan
versus the Monthly Payment.

b. What happens to the Monthly Payment as the length of
the loan increases? Why does this make sense?

c. Based on the graph, which type of loan would you prefer
and why?

d. On a separate coordinate plane, now plot the Length of
the Loan versus the Total Interest.

e. What happens to the Total Interest as the length of the
loan increases? Why does this make sense?

f. Now which loan do you prefer?

2. Set the interest rate to 10%. Fill in the following table:

<table>
<thead>
<tr>
<th>Length of Loan</th>
<th>Monthly Payment</th>
<th>Total Interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>879.16</td>
<td>549.91</td>
</tr>
<tr>
<td>2</td>
<td>461.45</td>
<td>1074.78</td>
</tr>
<tr>
<td>3</td>
<td>322.67</td>
<td>1616.19</td>
</tr>
<tr>
<td>4</td>
<td>253.63</td>
<td>2174.04</td>
</tr>
<tr>
<td>5</td>
<td>212.47</td>
<td>2748.23</td>
</tr>
<tr>
<td>6</td>
<td>185.26</td>
<td>3338.60</td>
</tr>
</tbody>
</table>

a. Add the data from this table to plot of the Length of the

Loan versus the Monthly Payment above. If possible, use a different symbol to tell the two data sets apart.

b. Add the data from this table to plot of the Length of the Loan versus the Total Interest above. If possible, use a different symbol to tell the two data sets apart.

c. Now which loan would you prefer?

3. Interest rates on credit cards tend be very high. Predict what the graphs would look like if the interest rate was 20%.

**PROBLEM 1**

John has a yearly income of $45,000. He is able to pay all of his bills on time each month, and has $425 after regular expenses. John is considering buying a new car. Assuming that John qualifies for a loan at 3% for 6 years, how expensive a car can he afford? John has three cars he is considering:

   Car 1: Costs $18,000 with other monthly expenses for insurance, gas and repairs of $120 per month.
   Car 2: Costs $24,000 with other monthly expenses for insurance, gas and repairs of $150 per month.
   Car 3: Costs $30,000 with other monthly expenses for insurance, gas and repairs of $180 per month.

Which of these three options can John afford to buy? Would it be financially wise to spend his entire $425 for the car? Explain.

Scientific and business calculators typically have options to compute monthly payments as well. In this example, we will show how one calculator can be used to compare loan options.

**EXAMPLE 2**

Janice has a credit card bill of $2500 that she needs to pay off. There are several options she is considering:

1. Repay the amount over a period of two years at an interest rate of 6% compounded monthly. If she chooses this method, what
will her monthly payment be, and what will the total cost of the loan be?

2. Repay the amount over a period of four years at an interest rate of 6% compounded monthly. What will her monthly payment be, and what will the total cost of the loan be?

3. Repay the amount over a period of six years, at an interest rate of 6% compounded monthly. What will her monthly payment be, and what will the total cost of the loan be?

In each of these options, Janice has the same interest rate, but a different length of time to make her payments. However, Janice needs to make sure that she makes her payments on time, or her interest rate will be increased to 18% compounded monthly.

4. How much is Janice’s monthly payment if she repays the credit card bill of $2500 over a two-year period at an interest rate of 18% compounded monthly. How much will the total cost of the loan be?

5. How much is Janice’s monthly payment if she repays the credit card bill of $2500 over a four-year period at an interest rate of 18% compounded monthly. How much will the total cost of the loan be?

6. How much is Janice’s monthly payment if she repays the credit card bill of $2500 over a six-year period at an interest rate of 18% compounded monthly. How much will the total cost of the loan be?

Solution The TI-83 Plus has special application to solve problems about loans and investments. To use the calculator

• Press APPS key. Choose 1: Finance.
• Choose 1: TVM Solver.
• A screen will appear where you enter the information. The following table explains each number you must enter.
When you are borrowing money, the present value is the amount you borrow. You get this money right away or at the present. You slowly pay off the loan until you owe no more money. So the future value is 0. Since we don’t know the monthly payment, we leave that blank. When you borrow money, your first payment is at the end of the first month, so we choose the END for the final option on the screen. To find the monthly payment for the Janice’s first loan option, we enter

<table>
<thead>
<tr>
<th>Variable on TI-83/84</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>24</td>
</tr>
<tr>
<td>$I%$</td>
<td>6</td>
</tr>
<tr>
<td>$PV$</td>
<td>2500</td>
</tr>
<tr>
<td>$PMT$</td>
<td></td>
</tr>
<tr>
<td>$FV$</td>
<td>0</td>
</tr>
<tr>
<td>$P/Y$</td>
<td>12</td>
</tr>
<tr>
<td>$C/Y$</td>
<td>12</td>
</tr>
<tr>
<td>$PMT$:</td>
<td>END</td>
</tr>
</tbody>
</table>

To have the calculator compute the monthly payment, press $\text{ALPHA}$ and then press $\text{SOLVE}$. The calculator should fill in the $PMT$ cell. In this case, we get $-110.80$. The number is negative because you pay that amount and the amount owed goes down.

The TVM solver does not report the total amount paid nor the interest paid. But these can be easily computed. To find the total amount paid multiply the monthly payment by the number of months. To find the interest paid, subtract the amount of the loan from the total amount paid.

The table below shows the difference between an interest rate of
Section A.2  COST OF CREDIT

6% and of 18% compounded monthly.

<table>
<thead>
<tr>
<th>Amount</th>
<th>Months</th>
<th>Rate</th>
<th>Payment</th>
<th>Total</th>
<th>Interest Paid</th>
</tr>
</thead>
<tbody>
<tr>
<td>2500</td>
<td>24</td>
<td>0.06</td>
<td>$110.80</td>
<td>$2,659.20</td>
<td>$159.20</td>
</tr>
<tr>
<td>2500</td>
<td>48</td>
<td>0.06</td>
<td>$58.71</td>
<td>$2,818.08</td>
<td>$318.08</td>
</tr>
<tr>
<td>2500</td>
<td>72</td>
<td>0.06</td>
<td>$41.43</td>
<td>$2,982.96</td>
<td>$482.96</td>
</tr>
<tr>
<td>2500</td>
<td>24</td>
<td>0.18</td>
<td>$134.81</td>
<td>$3,235.44</td>
<td>$735.44</td>
</tr>
<tr>
<td>2500</td>
<td>48</td>
<td>0.18</td>
<td>$73.44</td>
<td>$3,525.12</td>
<td>$1,025.12</td>
</tr>
<tr>
<td>2500</td>
<td>72</td>
<td>0.18</td>
<td>$57.02</td>
<td>$4,105.44</td>
<td>$1,605.44</td>
</tr>
</tbody>
</table>

**Different Payment Methods** Notice the significant difference that the higher interest rate makes both to monthly payments as well as the interest paid over the life of the loan. Also, the amount of interest increases with the time it takes to repay the loan. If you borrow at a high interest and pay the loan off over a long period of time, the amount of interest paid can be very large. In Example 2, when the interest rate was 18% and the loan was over 6 years, the interest paid was $1,605. This is 64.2% the original price of the car. However, some consumers cannot afford the monthly payment necessary to repay quickly, so they choose a longer term for the loan.

**The benefits and costs of financial responsibility** Credit Bureaus are organizations that collect information about individuals' borrowing and bill-paying habits. The credit bureau then uses this information to assign a credit score to everyone who wants to borrow money. Banks use this credit score to determine what interest rate they should charge a particular borrower. The lower the score, the higher the interest rate you will be charged. Hence, it is important to have a good credit history to qualify for a lower interest rate so borrowing is not so costly. In fact, with a good credit history, consumers may qualify for higher loan amounts than usual. Factors that influence a credit history include the timely payment of all bills are, a stable and adequate source of income, and the absence of bankruptcy history. Bankruptcy is a legal term used when a person cannot repay his debts.

It is financially beneficial to pay all credit card bills on time. It is absolutely financially necessary to make the minimum required payment. Not meeting the required payment usually results in a very high interest rate. The table above demonstrated that a higher
interest rate results in much higher monthly payments, and costs much more money over the life of the loan.

Credit Cards: The Real Story

Credit cards are a very convenient method of payment. But it is important to recognize some key facts:

- Eventually always must pay the bill.
- Interest rates on credit cards tend to be higher than other types of credit or loans.
- Typically, if you pay the entire balance of the credit card bill at the end of each month, you are not charged any interest.
- If you don’t pay the entire bill at the end of the month, the amount you owe can increase rapidly.

Let's explore a typical situation.

EXPLORATION 4

A credit card account charges no interest on purchases made during a month, if you pay the entire balance at the end of the month. If you don’t pay the entire bill at the end of the month, you are required to pay a minimum payment that is equal to 2% of the balance and you are charged 15% annual interest, compounded monthly on the unpaid charges. Make a table that identifies the advantages and disadvantages of paying the balance on a credit card monthly versus paying the minimum required. Use examples and calculations to support your claims.

One type of loan is called an easy-access loan. Many banks assess a penalty if a customer overdraws his account or charges more than his credit limit. To protect against these accidents, some people take out an "easy access" loan, or line of credit. In this case, when the bank account is overdrawn or the credit card is charge too much, the bank will not charge an overdraft penalty. Instead, the bank will loan the money through as an easy-access loan. This loan is like a regular loan and will need to be repaid.
Typically, easy-access loans are for much shorter duration than a regular loan. Usually these will be repaid almost immediately, but nonetheless the customer is charged interest. In Problem 2 below, you will calculate the cost of an easy access loan with different interest rates and for different time periods. Again, use a calculator to find the monthly payment.

PROBLEM 2

Sarah took out an easy-access loan of $750. The bank has two rates for easy-access loans. For customer with good credit, the rate is 6%. However, if a customer's credit is not so good, the rate is 12%. And if any payments are missed, the rate is 18%. There are different options for the length of time to repay the easy-access loan. The options for payment periods are one month, three months, six months, or one year. Fill out the table below to determine the monthly cost of repaying an easy-access loan, and the total cost of the loan.

<table>
<thead>
<tr>
<th>Amount</th>
<th>Months</th>
<th>Rate</th>
<th>Payment</th>
<th>Total</th>
<th>Interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>750</td>
<td>1</td>
<td>0.06</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>750</td>
<td>3</td>
<td>0.06</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>750</td>
<td>6</td>
<td>0.06</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>750</td>
<td>12</td>
<td>0.06</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>750</td>
<td>1</td>
<td>0.12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>750</td>
<td>3</td>
<td>0.12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>750</td>
<td>6</td>
<td>0.12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>750</td>
<td>12</td>
<td>0.12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>750</td>
<td>1</td>
<td>0.18</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>750</td>
<td>3</td>
<td>0.18</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>750</td>
<td>6</td>
<td>0.18</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>750</td>
<td>12</td>
<td>0.18</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
EXERCISES

Use an online loan calculator when necessary.

1. Explain the difference between a credit card loan and an easy-access loan.

2. Change the principal in Example 2 to $5000 and compute the costs.

3. Lisa wants to buy a new car that costs $18,500. There is an 8.25% sales tax, a charged based on a given percentage of the selling price, so that her total cost will be the cost of the car plus sales tax. She needs to save up for a 10% down payment of the total cost.
   a. How much is her down payment?
   b. How much will Lisa need to borrow?

4. Sue wants to purchase a new car that costs $20,000, including taxes. She will make a down payment of $2000 and will borrow the remaining $18,000. The interest rate she qualifies for depends on her credit score. The possible interest rates are 2% compounded monthly, 6% compounded monthly, and 12% compounded monthly. How does the interest rate affect her monthly payment?

5. Sue also needs to decide how long she should take to repay the loan. The options are three years, five years, or six years. If Sue only qualifies for the 6% interest rate, how does the loan option she chooses affect her monthly payment? How does the length of time for repaying the loan affect the amount of interest she will pay over the life of the loan?

6. Sarah has a credit card bill of $3500 that she cannot pay all at once. So she decides to repay it over time. The possible rates she will have to pay are 6%, 12%, and 18% compounded monthly. Calculate the total cost of repaying her credit card bill with each of these rates over a three-year period.

7. Sarah also needs to decide how long she should take to repay the loan. The options are three years, five years, or six years. If Sarah qualifies for the 6% interest rate, how does the loan option she chooses affect her monthly payment? How does the length of time to repay the credit card bill affect the amount
Section A.2  COST OF CREDIT

8. Janet has overdraft protection on her checking account and accidentally overdrafted her account by $500. She decided to take out an easy-access loan for $500. There are several possible interest rates she might be charged: 5%, 10%, or 15%. Use a calculator to calculate the total cost of repaying the loan with each of the above rates over a two-year period. Over a four-year period. Which costs more in interest: 5% over 3 years or 10% over 2 years? Explain.

9. Below are two payment options for buying a $250,000 house.
   a. Obtain a 6% loan for 30 years.
   b. Obtain a 4% loan for 15 years.
   Explain the advantages and disadvantages of each option.

10. Nancy is getting ready to buy a car but is not sure how expensive a car she can afford. With her credit score, she qualifies for a 6% loan compounded monthly for three years, or a 12% loan compounded monthly for six years. What are the advantages and disadvantages of the two different payment methods? Explain. Why might the type of payment method Nancy chooses determine how expensive a car she can afford?

11. Sandra and Bill have a combined yearly income of $115,000. They are able to pay all of their bills on time each month, and have $1250 after expenses. Their rent is $800 per month. Sandra and Bill are considering buying a new house. Assuming they qualify for a loan of 3% for 30 years, how expensive a house could they afford? There are three houses they are considering—one costs $100,000, one costs $150,000, and one costs $200,000. Which of these three options can they afford? Are there other house expenses they need to consider besides the monthly payment? Which of the three choices is financially responsible? What are some of the possible costs that could result from not being able to make their monthly payments?

12. Identify some of the benefits of financial responsibility. What does it mean to be financially responsible?

13. Identify some of the costs of financial irresponsibility. How can these costs rise and cause future problems?
## SECTION A.3 PLANNING FOR THE FUTURE

**Cost of College** One of the most important decisions you will make is your career. Once you decide, it may take many years to prepare for your chosen career. Many times preparing for a career involves getting a college education. Perhaps one of the best reasons to go to a two-year or four-year college is that this will give you more career options. There have been many studies about whether college graduates earn more money than their counterparts. While there might be some debate about how much more money they will earn, there is widespread agreement that college graduates have more options than their contemporaries.

Anyone interested in a career in Science, Technology, Engineering, or mathematics (STEM) will almost certainly need a college degree, and quite possibly even more advanced training like a masters or PhD. Many other higher-paying careers require a college education, or an associate two-year degree from a community or junior college. One advantage of going to a junior college is that this will let you try it out, and it provides a convenient stopping point after two years if you decide you don't want to continue. And if you do decide to continue, then you may transfer to a 4-year college and receive credit for your first two years at the junior college. However, there can be problems making this transfer as well since some of your courses taken at the junior college might not match the requirements of the degree program at the 4-year college.

As reported by cbsnews.com the highest-paying bachelor’s degrees for new graduates in 2012 were

- Electrical engineering $52,307
- Chemical engineering $51,823
- Mechanical engineering $51,625
- Computer engineering $50,375
- Computer programming $48,714
- Industrial engineering $48,566
- Computer science $47,561
Table A.1 The average cost of college: 2012-13

<table>
<thead>
<tr>
<th></th>
<th>Public 2-year (in-state)</th>
<th>Public 4-year (in state)</th>
<th>Private 4-year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tuition &amp; fees</td>
<td>$3,131</td>
<td>$8,655</td>
<td>$29,056</td>
</tr>
<tr>
<td>Room, board, books, etc.</td>
<td>$12,453</td>
<td>$13,606</td>
<td>$14,233</td>
</tr>
<tr>
<td>Total cost</td>
<td>$15,584</td>
<td>$22,261</td>
<td>$43,289</td>
</tr>
<tr>
<td>Net price (after scholarships, grants, aid)</td>
<td>$4,350</td>
<td>$5,750</td>
<td>$15,680</td>
</tr>
</tbody>
</table>

- Civil and environmental engineering $45,621

Notice that almost all of these are in engineering, and they all require a strong mathematics background.

Table A.1 shows the average cost of colleges from 2012. On average, for a public school, an education at a two-year college will cost $15,584 and at a four-year state college will cost $22,261 per year. The net price reflects the fact that most students obtain scholarships, grants, and financial aid to help pay for college. The net cost is what you and your family need to pay on average, assuming that you can obtain scholarships and other financial aid to cover the rest of the cost.

EXPLORATION 1

Work in groups. Have each group member explore one two-year college and one public four-year college. Compare their costs, including the family contribution at each. Discuss in your group how the costs vary from college to college. Is there anything that surprises you?

Saving for College It’s not too early to make a savings plan to save the money needed for at least the first year of college. Because the amount of scholarships and financial aid available is unknown, you should plan to save $22,261, an amount necessary to attend the first year of some four-year public colleges. This assumes that the costs of college will not rise over the next few years, which is
probably wishful thinking.

The savings plan will involve you (or your parents) making monthly payments into a savings account. It is hard to find a savings account that pays a high interest rate, but let’s assume that you can find an account that pays an interest rate of 3% compounded monthly.

**EXAMPLE 1**

How much must be invested each month into a savings account that pays 3% interest compounded monthly to have $22,261 after four years? How much to have $22,261 after eight years? After twelve years? What action do your answers suggest?

**Solution** To solve this problem we will use the TVM solver on the calculator. In this case, you want to invest enough money monthly in an savings account, so that at the end you have enough money for college. Since we start with no money, the present value is 0, but the desired future value is $22,261. The process doesn’t start until we invest our first payment, so the payments occur at the beginning of each month. To see how much you must invest monthly over 4 years, enter the following into the calculator:

<table>
<thead>
<tr>
<th>Variable on TI-83/84</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>48</td>
</tr>
<tr>
<td>( I% )</td>
<td>2</td>
</tr>
<tr>
<td>( PV )</td>
<td>0</td>
</tr>
<tr>
<td>( PMT )</td>
<td></td>
</tr>
<tr>
<td>( FV )</td>
<td>22261</td>
</tr>
<tr>
<td>( P/Y )</td>
<td>12</td>
</tr>
<tr>
<td>( C/Y )</td>
<td>12</td>
</tr>
</tbody>
</table>

To solve, press \( \text{ALPHA} \) and then press \( \text{SOLVE} \). The calculator should fill in the \( PMT \) cell. In this case, we get \(-435.99\). The number is negative because you pay that amount each month. If we change the number of years we have to invest and repeat the calculation, we can make a table:
Section A.3  PLANNING FOR THE FUTURE

<table>
<thead>
<tr>
<th>Years</th>
<th>Monthly Payment ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>435.99</td>
</tr>
<tr>
<td>8</td>
<td>204.95</td>
</tr>
<tr>
<td>12</td>
<td>128.30</td>
</tr>
</tbody>
</table>

Clearly, the longer you wait to save for college the more money you will need to invest each month. So it is a good idea to start early.

EXPLORATION 2

Unfortunately, the cost of going to college does not stay the same over time. Due to inflation, the costs increase each year. Suppose that the cost of college increases by 3% per year and it costs $22261 this year.

1. How much will college cost in 4 years?
2. How much more will you have to save each month in order to pay for college in four years? How does this compare to Example 1.
3. What if you plan to attend college 8 years from now?

To offset the cost of saving, plan to apply for scholarships and financial aid. In this way, your family contribution on average could be reduced to the net price in the table above, less than one-third of the total cost.

Father Off in the Future Just as small amounts of money invested monthly amount to significant savings for college, the same is true of a retirement plan. Investing or paying into an account at regular intervals, establishes an annuity or tax-free savings account.

EXAMPLE 2

The Ortizes are newly-weds and have figured that they need to save $50,000 to make the down payment on their dream house. To do this, they plan to make monthly deposits into an account that pays an interest rate of $r = 6\%$ compounded monthly. They can afford to save $600 dollars each month. How long must they wait until
they have money they need?

**Solution** We can use TVM solver to determine this. In this case we know the monthly payment, but do not know how many months are needed. The future value is $50000 and the present value is $0. The payment is $-600 because the Ortizes will pay this each month. Enter the following into the calculator:

<table>
<thead>
<tr>
<th>Variable on TI-83/84</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td></td>
</tr>
<tr>
<td>I%</td>
<td>6</td>
</tr>
<tr>
<td>PV</td>
<td>0</td>
</tr>
<tr>
<td>PMT</td>
<td>-600</td>
</tr>
<tr>
<td>FV</td>
<td>50000</td>
</tr>
<tr>
<td>P/Y</td>
<td>12</td>
</tr>
<tr>
<td>C/Y</td>
<td>12</td>
</tr>
<tr>
<td>PMT:</td>
<td>BEGIN</td>
</tr>
</tbody>
</table>

Leave the N = cell blank and press ALPHA and then SOLVE. The calculator should fill in $N = 69.54$. So the Ortizes need to save for 70 months or almost 6 years.

It is also important to start saving for retirement early as well. Advances in medicine has increased the number of years that Americans are living after retirement.

**EXPLORATION 3**

Planning for retirement is a complicated issue. There are even retirement specialists who help others make decisions about their retirement. However, there is a lot of useful information available on the web. Use the internet to explore the following questions.

1. How much money do experts suggest you need to save before you retire?
2. How much money should you invest each year in order to achieve this goal?
3. What is the effect of waiting 10 years before you start saving
for retirement? What about waiting 20 years?

4. There are many ways to save or invest for retirement. Describe at least three of these. How do they compare?

EXERCISES

1. Find the tuition and fees of two in-state junior colleges. Estimate the total cost of attending each. Find the commuting and residential cost for each.

2. Find the tuition and fees of one public in-state four-year college, one out-of-state public college, and one private four-year college. Estimate the total cost of attending each. Include travel costs, and assume that you will be living away from home.

3. Investigate three colleges that have programs you might be interested in attending. How much does each cost to attend per year?

4. Devise a savings plan to make a college education possible. Explain how small amounts of money invested each month could enable a future student to save enough to help pay for college.

5. Discuss retirement plans with your parents. Are they able to save money each month for retirement?
SECTION A.4  CHAPTER REVIEW

Key Terms
congruent  rotation
dilation  scale factor
irrational number  similar
radical  translation
reflection

Formulas
Simple Interest:  Compound Interest:
\[ I = Prt \]  \[ A(n) = P(1 + \frac{r}{m})^n \]

Practice Problems
1. Angelina invests $900 in the bank at a simple interest rate of 3%. How much interest will she earn after
   a. 1 month?
   b. 1 year?
   c. 2 years?
2. Valerie deposited $1200 into a bank account and left the account alone. The account collects 4% interest, compounded quarterly.
   a. How much will be in the account after 5 years? After 10 years? After 20 years?
   b. Write an equation to model the amount of money in the account.
3. Suppose that you use a new credit card to help furnish your home. After all the purchases, the total balance on your card is $12,457 (Wow!). The credit card charges 18.8% interest compounded monthly. If your were not to make a payment for 6 months, what would your balance be?
4. As a vehicle ages, its value decreases or depreciates. The function that models this is \( V(t) = I(0.082)^t \), where \( I \) is the original cost of the vehicle when it was new, \( t \) is the time in years from the date of purchase and \( V(t) \) is the value after \( t \) years. Patricia’s SUV cost $32,000 brand new.
   a. Find the value of Patricia’s vehicle 3 years after she purchased it.
   b. Instead of buying the car, Patricia could have leased the car. She would have paid $400 per month and then would have returned the car at the end of 3 years. Would this have been a better deal? Explain.

5. Susana Ormany is financial advisor on television. One day she says, it’s better to have compounding work for by investing, than against you by borrowing. Explain what she means using two examples.

6. Lalo and Lazaro are brothers. Lalo is very financially responsible and always pays his bills on time. Lazaro, on the other hand, often pays late. The brothers go to a bank separately to ask for a loan. Which brother is likely to have to pay the higher interest rate? What will the effect of this be on the monthly payments, each one has to pay?
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