ELEMENTARY ROW OPERATIONS
FORM OF A MATRIX—DEFINITIONS

Definitions:

Two matrices are said to be **row-equivalent** if one can be obtained from the other by a sequence (order may vary) of **elementary row operations** given below.

- Interchanging two rows
- Multiplying a row by a non zero constant
- Adding a multiple of a row to another row

Row-Echelon Form and Reduced Row-Echelon Form

A matrix in **row-echelon form** has the following properties.

- All rows consisting entirely of zeros occur at the bottom of the matrix.
- For each row that does not entirely consist of zeroes, the first non zero entry is 1—called a **leading 1**.
- For two successive non zero rows, the leading 1 in the higher row is farther to the left that the leading 1 in the lower row.

A matrix in *row-echelon form* is in **reduced row-echelon form** if every column that has a leading 1 has zeros in every position above and below its leading 1.

$$\text{Row-Echelon} \quad \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad \text{Reduced Row-Echelon} \quad \begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Augmented Matrix

A matrix derived from a system of linear equations is called the augmented matrix of the system.

System	Variables Lined Up	Augmented Matrix
$x + 3y = 9$ $-y + 4z = -2$ $x - 5z = 0$	$x + 3y \quad = 9$ $-y + 4z = -2$ $x \quad - 5z = 0$	$\left[\begin{array}{ccc:c} 1 & 3 & 0 & 9 \\ 0 & -1 & 4 & -2 \\ 1 & 0 & -5 & 0 \end{array}\right]$

Finding the Inverse of a Matrix

Let A be square matrix order n.

- Write a matrix that consists of the given matrix *A* on the left and the identity matrix
- *n* x *n* next to *A* on the right, separating the matrices *A* and *I* by a (vertical) dotted line to obtain [*A*:*I*].
- If possible, row reduce *A* to *I* using elementary row operations on the matrix [*A*:*I*]. The result will be the matrix $[I:A^{-1}]$.
  Note that for some matrices an inverse does not exist. If it is not possible to obtain $[I:A^{-1}]$, then the matrix *A* is not invertible.
- Check your work by multiplying to see that $AA^{-1} = I = A^{-1}A$.

Example: Find the inverse of the matrix:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 2 \\ 1 & 2 & -1 \end{bmatrix}$$

First, set up the matrix [$A:I$]: (Remember, the goal is [$I:A^{-1}$])

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

Next use row operations to get zeros in column 1:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & -2 & -2 & -2 & 1 & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} (-2)R1 + R2 \to R2 \\ (-1)R1 + R3 \to R3 \end{array}$$

Now multiply row 2 by -1/2 to obtain a 1 in column 2:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1/2 & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \end{array}\right] \qquad \left(-\frac{1}{2}\right)R2 \to R2$$

Next, get zeros in the second column:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 3/2 & 0 \\ 0 & 1 & 1 & 1 & -1/2 & 0 \\ 0 & 0 & -2 & 0 & -1/2 & 1 \end{array}\right] \qquad \begin{array}{l} (-3)R2 + R1 \to R1 \\ \\ R2 + R3 \to R3 \end{array}$$

Now Multiply row 3 by -1/2 to obtain a 1 in column 3:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 3/2 & 0 \\ 0 & 1 & 1 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 1/4 & -1/2 \end{array}\right] \qquad \left(-\frac{1}{2}\right)R3 \to R3$$

Finally, get zeros in the third column:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 7/4 & -1/2 \\ 0 & 1 & 0 & 1 & -3/4 & 1/2 \\ 0 & 0 & 1 & 0 & 1/4 & -1/2 \end{array}\right] \qquad \begin{array}{l} R1 + R3 \to R1 \\ (-1)R3 + R2 \to R2 \end{array}$$

The left side of the above matrix is the identity matrix and the matrix on the right is the inverse of the given matrix *A*. Thus,

$$A^{-1} = \begin{bmatrix} -2 & 7/4 & -1/2 \\ 1 & -3/4 & 1/2 \\ 0 & 1/4 & -1/2 \end{bmatrix}$$